On the highest level's Stokes phenomenon of meromorphic linear differential systems

Pascal Remy

On the highest level's Stokes phenomenon of meromorphic linear differential systems

P. Remy
6 rue Chantal Mauduit
F-78 420 Carrières-sur-Seine, France
email: pascal.remy07@orange.fr

Abstract

Given a meromorphic linear differential system with several levels, we prove that the Borel transforms of its highest level's reduced formal solutions are summable-resurgent and we give a complete description of all their singularities. Then, as an application and under some convenient hypothesis on the geometric configuration of singular points, we state formulæ to express some highest level's Stokes multipliers of the initial system in terms of connection constants in the Borel plane, generalizing thus formulæ already displayed by M. Loday-Richaud and the author for systems with a single level. As an illustration, we develop three examples.

Keywords. Linear differential system, multisummability, Stokes phenomenon, Stokes multipliers, resurgence, singularities, connection constants

AMS subject classification. 34M03, 34M25, 34M30, 34M35, 34M40

1 Introduction

All along the article, we are given a positive integer $r \geq 1$ and a linear differential system (in short, a differential system or a system) of dimension $n \geq 2$ with meromorphic coefficients of order $r+1$ at the origin $0 \in \mathbb{C}$ of the form

$$(A) \qquad x^{r+1}\frac{dY}{dx} = A(x)Y \quad , \; A(x) \in M_n(\mathbb{C}\{x\}),\; A(0) \neq 0$$

together with a formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ at 0 where

- $\widetilde{F}(x) \in GL_n(\mathbb{C}[[x]][x^{-1}])$ is an invertible matrix with formal meromorphic entries in x,
- $L = \bigoplus_{j=1}^{J} L_j$ with J an integer ≥ 2 and $L_j := \lambda_j I_{n_j} + J_{n_j}$; I_{n_j} denotes the identity matrix of size n_j and

$$J_{n_j} = \begin{cases} 0 & \text{if } n_j = 1 \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } n_j \geq 2 \end{cases}$$

  is an irreductible Jordan block of size n_j,
- $Q\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J} q_j\left(\frac{1}{x}\right) I_{n_j}$ with the $q_j(1/x)$'s polynomials of maximal degree equal to r with respect to $1/x$.

For very general system (A), the $q_j(1/x)$'s may be polynomials in a fractional power in $1/x$. However, they can always be changed into polynomials in the variable $1/x$ itself by means of an adequate finite algebraic extension $x \mapsto x^\nu$, $\nu \geq 1$, of the variable x; henceforth, we suppose, without loss of generality, that the $q_j(1/x)$'s read as

$$q_j\left(\frac{1}{x}\right) = -\frac{a_{j,r}}{x^r} - \frac{a_{j,r-1}}{x^{r-1}} - \ldots - \frac{a_{j,1}}{x} \in \frac{1}{x}\mathbb{C}\left[\frac{1}{x}\right]$$

for all $j = 1, \ldots, J$. In addition, we suppose

$(C1)$ $\widetilde{F}(x) \in M_n(\mathbb{C}[[x]])$ is a formal power series in x satisfying

$$\widetilde{F}(x) = I_n + O(x^r),$$

$(C2)$ the eigenvalues λ_j satisfy $0 \leq \mathrm{Re}(\lambda_j) < 1$ for all $j = 1, \ldots, J$,

$(C3)$ $\lambda_1 = 0$ and $q_1 \equiv 0$.

Recall that these conditions are not restrictive since they can always be fulfilled by means of a convenient meromorphic gauge transformation $Y \mapsto T(x)x^{-\lambda_1}e^{-q_1(1/x)}Y$, where $T(x)$ has explicit computable polynomial entries in x and $1/x$ (*cf.* [4]). Recall also that conditions $(C1)$ and $(C2)$ guarantee the unicity of $\widetilde{F}(x)$ as formal series solutions of the homological system associated with system (A) (*cf.* [4]); condition $(C3)$ is for notational convenience.

Under the hypothesis that system (A) has the *unique* level $r \geq 1$ (see definition 2.1 below for the exact definition of levels), M. Loday-Richaud and the author proved in [14] (case $r = 1$) and [24] (case $r \geq 2$) that the formal Borel transforms $\widehat{\boldsymbol{F}}^{[u]}(\tau)$ of the *r-reduced series* $\widetilde{\boldsymbol{F}}^{[u]}(t)$, $u = 0, ..., r-1$ and $t = x^r$, of $\widetilde{F}(x)$ (= the sub-series of terms r by r of $\widetilde{F}(x)$) are *summable-resurgent*; then, they displayed *exact* formulæ relating *the Stokes multipliers of* $\widetilde{F}(x)$ and the so-called *connection constants* given, in the Borel plane, by *some convenient analytic continuations* of the $\widehat{\boldsymbol{F}}^{[u]}(\tau)$ at their various *singular points*, providing thus an efficient tool for the numerical calculation of the *Stokes-Ramis matrices* associated with $\widetilde{Y}(x)$.

These two results were generalized later by the author in [23] to the *first* (= *lowest*) level r_1 of any meromorphic linear differential system with several levels by considering the r_1-reduced series of $\widetilde{F}(x)$.

In the present paper, we suppose that system (A) has multi-levels, say $r_1 < ... < r_{p-1} < r_p$ with $p \geq 2$. Our aim is to extend the results above to the highest level r_p. To this end, we proceed in a similar way as [24] by considering the r_p-reduced system $(\boldsymbol{A})$ associated with system (A) to which the r_p-reduced series of $\widetilde{F}(x)$ are intimitely related (see [12] for instance).

The organization of the paper is as follows. In section 2, we briefly recall some basic definitions and properties about the multisummation theory we are needed in the sequel. In particular, we recall the factorization theorem of the Stokes-Ramis matrices due to M. Loday-Richaud and J.-P. Ramis ([11, 21, 22]) and its link with a convenient generalized multisummability. In section 3, we introduce the r_p-reduced system $(\boldsymbol{A})$ associated with system (A) and we give some relations between its formal solutions and the highest level's Stokes multipliers of $\widetilde{F}(x)$. Then, by adapting the method developed in [24], we prove in section 4 the *summable-resurgence* of the formal Borel transforms of the r_p-reduced series of $\widetilde{F}(x)$ (section 4.3, theorem 4.9) and we give a *complete* description of *all* their singularities (section 4.4, theorem 4.24). This we use in section 5 to state, in the case of a *Special Geometric Configuration* of singular points, some *highest level's connection-to-Stokes formulæ* which generalize, for highest level's Stokes multipliers of system (A), formulæ already given in [14, 23, 24]. As an illustration of all the results of this article, we develop in section 6 three examples.

In the whole paper, we denote by

- $\widetilde{\mathbb{C}}$ the Riemann surface of the logarithm,
- $\mathcal{O}(V)$ the space of analytic functions on an open set V of $\mathbb{C}$ or $\widetilde{\mathbb{C}}$; in particular, $\mathcal{O}(\mathbb{C})$ (resp. $\mathcal{O}(\widetilde{\mathbb{C}})$) denotes the space of entire functions on all $\mathbb{C}$ (resp. $\widetilde{\mathbb{C}}$),
- $\mathcal{O} := \mathbb{C}\{t\}$ the set of germs of analytic functions at $0 \in \mathbb{C}$ and $\widetilde{\mathcal{O}}$ the set of germs of analytic functions at $0 \in \widetilde{\mathbb{C}}$; recall that one has a natural injection $\mathcal{O} \hookrightarrow \widetilde{\mathcal{O}}$.

2 Multisummability and Stokes phenomenon

In this section, we recall, for the convenience of the reader, some basic definitions and results of the summation theory.

2.1 Levels and anti-Stokes directions

Split the matrix $\widetilde{F}(x)$ into J column-blocks fitting to the Jordan block-structure of L (for $\ell = 1, ..., J$, the matrix $\widetilde{F}^{\bullet;\ell}(x)$ has n_ℓ columns):

$$\widetilde{F}(x) = \left[\widetilde{F}^{\bullet;1}(x) \quad \widetilde{F}^{\bullet;2}(x) \quad \cdots \quad \widetilde{F}^{\bullet;J}(x)\right].$$

$\triangleleft$ **Levels.** Given a pair (q_j, q_ℓ) of polynomials of Q such that $q_j \neq q_\ell$, we denote

$$(q_j - q_\ell)\left(\frac{1}{x}\right) = -\frac{\alpha_{j,\ell}}{x^{r_{j,\ell}}} + o\left(\frac{1}{x^{r_{j,\ell}}}\right) \quad , \ \alpha_{j,\ell} \neq 0.$$

Definition 2.1 (Levels) We call

- *levels of system* (A) *associated with* $\widetilde{F}(x)$ (in short, *levels of* $\widetilde{F}(x)$) all the degrees $r_{j,\ell}$ for $j, \ell = 1, ..., J$ such that $q_j - q_\ell \not\equiv 0$;
- *levels of system* (A) *associated with* $\widetilde{F}^{\bullet;\ell}(x)$, $\ell \in \{1, ..., J\}$ (in short, *levels of* $\widetilde{F}^{\bullet;\ell}(x)$) all the degrees $r_{j,\ell}$ for $j = 1, ..., J$ such that $q_j - q_\ell \not\equiv 0$.

Note that, according to the normalizations of system (A), all the levels are integers. One sometimes refers to this case as the *unramified* case.

Let us now denote by $\mathcal{R} := \{r_1 < ... < r_p\}$ with $p \geq 1$, the set of all the levels of $\widetilde{F}(x)$. We have $r_1 \geq 1$ and $r_p \leq r$ the rank of system (A). Actually, if $r_p < r$, all the polynomials q_j, $j = 1, ..., J$, have the same degree

r and the terms of highest degree coincide; one then reduces this case to the case $r_p = r$ by means of a change of unknown vector of the form $Y = Ze^{q(1/x)}$ with a convenient polynomial $q(1/x) \in x^{-1}\mathbb{C}[x^{-1}]$. Recall that such a change does not affect levels or Stokes-Ramis matrices of system (A).

As we said in section 1, we suppose from now on that $p \geq 2$, *i.e.*, system (A) has at least two levels; otherwise, system (A) has the *unique level* $r_1 = r_p = r$ and we refer to [14, 24].

Let us also denote by $\mathcal{R}^{(\ell)} := \{r_1^{(\ell)} < ... < r_{p_\ell}^{(\ell)}\}$ with $p_\ell \geq 1$, the set of all the levels of $\widetilde{F}^{\bullet;\ell}(x)$, $\ell = 1, ..., J$. We clearly have $\mathcal{R}^{(\ell)} \subseteq \mathcal{R}$, $r_1^{(\ell)} \geq 1$ and $r_{p_\ell}^{(\ell)} = r_p = r$. Note that $\mathcal{R}^{(\ell)}$ may be the singleton $\mathcal{R}^{(\ell)} = \{r\}$, *i.e.*, $\widetilde{F}^{\bullet;\ell}(x)$ may have the unique level r. Note also that, since condition $(C3)$ above implies $q_1 \equiv 0$, the set $\mathcal{R}^{(1)}$ of all the levels of the first column-block $\widetilde{F}^{\bullet;1}(x)$ of $\widetilde{F}(x)$, is actually the set of all the degrees of polynomials $q_j \not\equiv 0$.

$\triangleleft$ **Anti-Stokes directions.** We are now able to define the *anti-Stokes directions* (= the singular directions) of system (A).

Definition 2.2 (Anti-Stokes directions) We call

- *anti-Stokes direction of system* (A) *associated with polynomial* $q_j - q_\ell \not\equiv 0$ (in short, *anti-Stokes direction of* $q_j - q_\ell \not\equiv 0$) any direction along which the exponential $e^{(q_j - q_\ell)(1/x)}$ has maximal decay, *i.e.*, any direction $\theta = \arg(\alpha_{j,\ell})/r_{j,\ell} \bmod \frac{2\pi}{r_{j,\ell}}$ along which $-\alpha_{j,\ell}/x^{r_{j,\ell}}$ is real negative; when $r_{j,\ell} = r_k$, such a direction is said to be *of k^{th} level (or of level r_k)*;

- *anti-Stokes direction of system* (A) *associated with* $\widetilde{F}(x)$ (in short, *anti-Stokes direction of* $\widetilde{F}(x)$) any anti-Stokes direction of polynomials $q_j - q_\ell \not\equiv 0$ for $j, \ell = 1, ..., J$;

- *anti-Stokes direction of system* (A) *associated with* $\widetilde{F}^{\bullet;\ell}(x)$, $\ell \in \{1, ..., J\}$, (in short, *anti-Stokes direction of* $\widetilde{F}^{\bullet;\ell}(x)$) any anti-Stokes direction of polynomials $q_j - q_\ell \not\equiv 0$ for $j = 1, ..., J$.

Note that a given anti-Stokes direction may be of several levels. Note also that the denomination "anti-Stokes directions" is not universal; indeed, such directions are called sometimes "Stokes directions".

Definition 2.3 (Stokes values) Let $k \in \{1, ..., p\}$. We call

- *k^{th} level's Stokes values (or of level r_k) of system* (A) *associated with* $\widetilde{F}(x)$ (in short, *k^{th} level's Stokes values of* $\widetilde{F}(x)$) all the $\alpha_{j,\ell}$ generating the k^{th} level's anti-Stokes directions of $\widetilde{F}(x)$;

- k^{th} *level's Stokes values (or of level* r_k*) of system* (A) *associated with* $\widetilde{F}^{\bullet;\ell}(x)$, $\ell \in \{1, ..., J\}$, (in short, k^{th} *level's Stokes values of* $\widetilde{F}^{\bullet;\ell}(x)$) all the $\alpha_{j,\ell}$, $j = 1, ..., J$, generating the k^{th} level's anti-Stokes directions of $\widetilde{F}^{\bullet;\ell}(x)$.

Note that, as anti-Stokes directions, Stokes values may be of several levels.

2.2 Multisummability

Recall that a formal power series $\widetilde{h}(t) \in \mathbb{C}[[t]]$ is said to be *Borel-Laplace summable of level* $k > 0$ (or simply k*-Borel-Laplace summable* or k*-summable*) in a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ if the following two conditions are satisfied:

1. the formal Borel transform $\mathcal{B}_k(\widetilde{h})(\tau)$ of level k of $\widetilde{h}(t)$ is convergent (*i.e.*, $\widetilde{h}(t)$ is $1/k$-Gevrey),

2. its sum can be analytically continued in a function $s_{\infty;\theta}(\mathcal{B}_k(\widetilde{h}))(\tau)$ [1] on a sector bisected by θ with an exponential growth of order $\leq k$ at infinity.

Then, the k-sum $s_{k;\theta}(\widetilde{h})(t)$ of $\widetilde{h}(t)$ in the direction θ is given by

$$\boxed{s_{k;\theta}(\widetilde{h}) = \mathcal{L}_{k;\theta}(s_{\infty;\theta}(\mathcal{B}_k(\widetilde{h})))}$$

and thus defined an analytic function $1/k$-Gevrey asymptotic to $\widetilde{h}$ on a germ of sector with vertex 0, bisected by θ and opening larger than π/k [2].

The notation $\mathcal{L}_{k;\theta}$ denotes the Laplace transformation of level k in direction θ. For precise definitions and properties of operators $\mathcal{B}_k$ and $\mathcal{L}_{k;\theta}$, we refer, for instance, to [19].

The summation of several levels $\underline{k} := (k_1 < ... < k_s)$, $s \geq 2$, is more complicated. It was investigated in great details by many authors and several equivalent definitions based on various methods such as asymptotic, cohomology, integral operators, ... were given. See for instance [1–3, 6, 11, 15, 19]. In this section, we focus on two of them −the so-called *accelero-summation* and *iterated Laplace method*− which "extend" the Borel-Laplace summation.

[1] Compare with the notation $s_{\underline{k};\theta}(\mathcal{B}_{k_s}(\widetilde{h}))$ of definition 2.6 below.

[2] When opening is $< 2\pi$, the sector can be seen as a sector of $\mathbb{C}\backslash\{0\}$; otherwise, it must be considered as sector of $\widetilde{\mathbb{C}}$.

$\triangleleft$ **Accelero-summation.** Historically, this approach was the first able to solve the problem of multisummation. First introduced by J. Écalle in a very general setting applying to series solutions of non-linear equations and more general functional equations, it was adapted by J. Martinet and J.-P. Ramis in [19] to the case of solutions of linear differential equations. The method proceeds by recursion on increasing levels and each step is performed with the use of special integral operators called *accelerators* or *Écalle's accelerators*.

Let $\mathcal{A}_{k',k;\theta}$ denote the accelerator of levels $0 < k < k'$ in direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. (see, for instance, [19] for its precise definition and properties). Recall however that $\mathcal{L}_{k;\theta}$ applies to any function with exponential growth of order $\leq k$ at infinity in direction θ and $\mathcal{A}_{k',k;\theta}$ applies to any function with exponential growth of order $\leq \kappa' := k'k/(k'-k)$ at infinity in direction θ.

Definition 2.4 ([19, Def. 2, p. 343]) Let $s \geq 2$ and $\underline{k} := (k_1 < \ldots < k_s)$ a s-tuple of positive real numbers. Denote

$$\kappa_j := \frac{k_{j+1}k_j}{k_{j+1}-k_j} \text{ for } j = 1, \ldots, s-1 \text{ and } \kappa_s := k_s.$$

A formal power series $\widetilde{h}(t) \in \mathbb{C}[[t]]$ is called *$\underline{k}$-summable in a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$* if the following two conditions are satisfied:

1. the formal Borel transformation $\mathcal{B}_{k_1}(\widetilde{h})(\tau)$ of level k_1 is convergent and its sum can be analytically continued along θ in a function $\mathcal{B}_{k_1;\theta}(\widetilde{h})(\tau)$ on a sector bisected by θ with an exponential growth of order $\leq \kappa_1$ at infinity,

2. for $j = 2, \ldots, s$, the functions $h_{j;\theta}$ recursively defined by

$$h_{j;\theta} := \mathcal{A}_{k_j,k_{j-1};\theta} h_{j-1;\theta} \qquad , \; h_{1;\theta} := \mathcal{B}_{k_1;\theta}(\widetilde{h})$$

   can be analytically continued along θ in a function, still denoted by $h_{j;\theta}$, on a sector bisected by θ with an exponential growth of order $\leq \kappa_j$ at infinity.

Then, the $\underline{k}$-sum $s_{\underline{k};\theta}(\widetilde{h})(t)$ of $\widetilde{h}(t)$ in the direction θ is given by

$$\boxed{s_{\underline{k};\theta}(\widetilde{h}) = \mathcal{L}_{k_s;\theta}\mathcal{A}_{k_s,k_{s-1};\theta} \ldots \mathcal{A}_{k_2,k_1;\theta}\mathcal{B}_{k_1;\theta}(\widetilde{h})}$$

and thus defined an analytic function $1/k_1$-Gevrey asymptotic to $\widetilde{h}$ on a germ of sector bisected by θ and opening larger than π/k_s.

Remark 2.5 If $\underline{k}' := (k'_1 < ... < k'_{s'})$ and $\underline{k} := (k_1 < ... < k_s)$ with $1 \leq s' < s$ satisfy $\{k'_1, ..., k'_{s'}\} \subseteq \{k_1, ..., k_s\}$, then $\underline{k}'$-summability implies $\underline{k}$-summability; furthermore, the two sums $s_{\underline{k}';\theta}(\widetilde{h})$ and $s_{\underline{k};\theta}(\widetilde{h})$ coincide.

$\triangleleft$ **Iterated Laplace method.** This method is due to W. Balser. It proceeds, unlike the accelero-summation, by recursion on decreasing levels and is based on the fact that a convenient formal Borel transformation of a formal series is itself (multi)-summable. Definition 2.6 below makes explicit this method:

Definition 2.6 ([1]) Let $s \geq 2$ and $\underline{k} := (k_1 < ... < k_s)$ a s-tuple of positive real numbers. Denote

$$\kappa_j := \frac{k_s k_j}{k_s - k_j} \text{ for } j = 1, ..., s-1.$$

A formal power series $\widetilde{h}(t) \in \mathbb{C}[[t]]$ is called *$\underline{k}$-summable in a direction* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ if the following two conditions are satisfied:

1. the formal Borel transformation $\mathcal{B}_{k_s}(\widetilde{h})$ of level k_s is $\underline{\kappa}$-summable in the direction θ with $\underline{\kappa} := (\kappa_1 < ... < \kappa_{s-1})$,
2. its $\underline{\kappa}$-sum $s_{\underline{\kappa};\theta}(\mathcal{B}_{k_s}(\widetilde{h}))$ can be analytically continued along θ in a function, still denoted $s_{\underline{\kappa};\theta}(\mathcal{B}_{k_s}(\widetilde{h}))$, on a sector bisected by θ with an exponential growth of order $\leq k_s$ at infinity.

Then, the $\underline{k}$-sum $s_{\underline{k};\theta}(\widetilde{h})(t)$ of $\widetilde{h}(t)$ in the direction θ is defined by

$$\boxed{s_{\underline{k};\theta}(\widetilde{h}) = \mathcal{L}_{k_s;\theta}(s_{\underline{\kappa};\theta}(\mathcal{B}_{k_s}(\widetilde{h}))).}$$

Remark 2.7 One can show that the two $\underline{k}$-sums given by definitions 2.4 and 2.6 coincide (see, for instance, [13]).

Lemma 2.8 below, which will be us useful later, relates the two methods of summation above by making explicit the sum $s_{\underline{\kappa};\theta}(\mathcal{B}_{k_s}(\widetilde{g}))$ in terms of accelerators.

Lemma 2.8 *With notations as above,*

$$\boxed{s_{\underline{\kappa};\theta}(\mathcal{B}_{k_s}(\widetilde{h})) = \mathcal{A}_{k_s,k_{s-1};\theta} ... \mathcal{A}_{k_2,k_1;\theta}\mathcal{B}_{k_1;\theta}(\widetilde{h}).}$$

Let us now turn to the formal factor series $\widetilde{F}(x)$ of $\widetilde{Y}(x)$.

◁ **Multisummability of $\widetilde{F}(x)$.** One has the following theorem:

Theorem 2.9 ([3, 6, 11, 15, 19])

1. *Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a non anti-Stokes direction of $\widetilde{F}(x)$.*
Let $\underline{r} := (r_1 < ... < r_{p-1} < r)$ be the p-tuple of all the levels of $\widetilde{F}(x)$.
Then, $\widetilde{F}(x)$ is $\underline{r}$-summable in the direction θ.

2. *More precisely, let $\ell \in \{1, ..., J\}$, $\theta^{(\ell)} \in \mathbb{R}/2\pi\mathbb{Z}$ a non anti-Stokes direction of $\widetilde{F}^{\bullet;\ell}(x)$ and $\underline{r}^{(\ell)} := (r_1^{(\ell)} < ... < r_{p_\ell-1}^{(\ell)} < r)$ the p_ℓ-tuple of all the levels of $\widetilde{F}^{\bullet;\ell}(x)$.*
Then, $\widetilde{F}^{\bullet;\ell}(x)$ is $\underline{r}^{(\ell)}$-summable in the direction $\theta^{(\ell)}$.

We are now able to define the sum of $\widetilde{Y}(x)$: let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a non anti-Stokes direction of $\widetilde{F}(x)$ and a choice of an argument of θ, say its principal determination $\theta^\star \in]-2\pi, 0]$ [3]. Then, *the sum $Y_\theta(x)$ of $\widetilde{Y}(x)$ in direction θ* is given by

$$\boxed{Y_\theta(x) := s_{\underline{r};\theta}(\widetilde{F})(x) Y_{0;\theta^\star}(x),}$$

where $Y_{0;\theta^\star}(x)$ is the actual analytic function $Y_{0;\theta^\star}(x) := x^L e^{Q(1/x)}$ defined by the choice $\arg(x)$ close to $\theta^\star$ (denoted below by $\arg(x) \simeq \theta^\star$).

2.3 Stokes phenomenon and Stokes-Ramis matrices

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be an anti-Stokes direction of $\widetilde{F}(x)$ and $s_{\underline{r};\theta^-}(\widetilde{F})$ and $s_{\underline{r};\theta^+}(\widetilde{F})$ the two *lateral sums* of $\widetilde{F}(x)$ at θ respectively obtained as analytic continuations of $s_{\underline{r};\theta-\eta}(\widetilde{F})$ and $s_{\underline{r};\theta+\eta}(\widetilde{F})$ to a sector with vertex 0, bisected by θ and opening π/r. Note that such analytic continuations exist without ambiguity when η is small enough.

◁ **Stokes phenomenon.** The *Stokes phenomenon* of system (A) stems from the fact that the sums $s_{\underline{r};\theta^-}(\widetilde{F})$ and $s_{\underline{r};\theta^+}(\widetilde{F})$ of $\widetilde{F}(x)$ are not analytic continuations from each other in general. This defect of analyticity is quantified by the collection of *Stokes-Ramis automorphisms* $St_{\theta^\star} : Y_{\theta^+} \longmapsto Y_{\theta^-}$ for all the anti-Stokes directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of $\widetilde{F}(x)$, where $Y_{\theta^\pm}$ denote the sums of $\widetilde{Y}(x)$ defined, for $\arg(x) \simeq \theta^\star$, by $Y_{\theta^\pm}(x) := s_{\underline{r};\theta^\pm}(\widetilde{F})(x) Y_{0;\theta^\star}(x)$.

[3] Any choice is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leq \arg(z = 1/x) < 2\pi$ of the principal determination at infinity, we suggest to choose $-2\pi < \arg(x) \leq 0$ as principal determination about 0.

$\triangleleft$ **Stokes-Ramis matrices.** The *Stokes-Ramis matrices*[4] are then defined as matrix representations of the $St_{\theta^\star}$'s in $GL_n(\mathbb{C})$:

Definition 2.10 (Stokes-Ramis matrix) One calls *Stokes-Ramis matrix associated with* $\widetilde{Y}(x)$ *in direction* θ the matrix of $St_{\theta^\star}$ in the basis Y_{θ^+}. We still denote it by $St_{\theta^\star}$; it is uniquely determined by the relation

$$\boxed{Y_{\theta^-}(x) = Y_{\theta^+}(x) St_{\theta^\star} \quad \textit{for } \arg(x) \simeq \theta^\star.}$$

Let us now split $St_{\theta^\star} = [St_{\theta^\star}^{j;\ell}]$ into blocks fitting to the Jordan block-structure of L (for $j, \ell = 1, ..., J$, the matrix $St_{\theta^\star}^{j;\ell}$ has size $n_j \times n_\ell$). Then,

$$St_{\theta^\star}^{j;\ell} = \begin{cases} I_{n_j} & \text{if } j = \ell \\ 0 & \text{if } \theta \text{ is not an anti-Stokes direction of } q_j - q_\ell \end{cases}.$$

When θ is an anti-Stokes direction of $q_j - q_\ell$, the entries of $St_{\theta^\star}^{j;\ell}$ are called *Stokes multipliers of* $\widetilde{F}^{\bullet;\ell}(x)$ *in direction* θ.

$\triangleleft$ **Factorization of Stokes-Ramis matrices.** The factorization of $St_{\theta^\star}$ by levels was first proved by J.-P. Ramis in [21,22] by using the factorization theorem of $\widetilde{F}(x)$; a quite different proof based on Stokes cocycles and mainly algebraic was given later by M. Loday-Richaud in [11].

Theorem 2.11 (Factorization of $St_{\theta^\star}$, [11,21,22]) *With notations as above, the Stokes-Ramis matrix $St_{\theta^\star}$ can be written as*

$$St_{\theta^\star} = St_{1;\theta^\star} ... St_{p;\theta^\star} \quad , \; St_{k;\theta^\star} = [St_{k;\theta^\star}^{j;\ell}] \in GL_n(\mathbb{C})$$

where, for all $k = 1, ..., p$,

$$St_{k;\theta^\star}^{j;\ell} = \begin{cases} I_{n_j} & \textit{if } j = \ell \\ 0 & \textit{if } \theta \textit{ is not an anti-Stokes direction of } q_j - q_\ell \textit{ or } r_{j,\ell} \neq r_k \end{cases}.$$

Definition 2.12 (k^{th} level's Stokes multipliers) Let $k \in \{1, ..., p\}$.

1. The matrix $St_{k;\theta^\star}$ is called *k^{th} level's Stokes-Ramis matrix (or Stokes-Ramis matrix of level r_k) associated with $\widetilde{Y}(x)$ in direction θ.*

[4] In the literature, a Stokes matrix has a more general meaning where one allows to compare any two asymptotic solutions whose domains of definition overlap. According to the custom initiated by J.-P. Ramis [22] in the spirit of Stokes' work, we exclude this case here. We consider only matrices providing the transition between the sums on each side of a same anti-Stokes direction.

2. When θ is an anti-Stokes direction of $q_j - q_\ell$ and $r_{j,\ell} = r_k$, the entries of $St^{j;\ell}_{k;\theta^\star}$ are called *k^{th} level's Stokes multipliers (or Stokes multipliers of level r_k) of $\widetilde{F}^{\bullet;\ell}(x)$ in direction θ.*

Recall that the lowest level's (= first level's) Stokes multipliers of $St^{j;\ell}_{1;\theta^\star}$ coincide with the Stokes multipliers of $St^{j;\ell}_{\theta^\star}$.

Recall also that, in the present paper, we are interested in the highest level's (= p^{th} level's) Stokes multipliers of $St^{j;\ell}_{p;\theta^\star}$. To this end, we need to introduce the notion of *multisummability along a path* $(\theta, \underline{\varepsilon})$ with a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and a p-tuple $\underline{\varepsilon} = (\varepsilon_1, ..., \varepsilon_p)$ with $\varepsilon_i = \pm 1$ (see [19]).

2.4 Generalized multisummability

The notion of multisummability along a path $(\theta, \underline{\varepsilon})$, which is based on the accelero-summation (see definition 2.4), is given in the following definition:

Definition 2.13 ([19, Def. 3, p. 351]) Let $s \geq 2$ and $\underline{k} := (k_1 < ... < k_s)$ a s-tuple of positive real numbers.
Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction and $\widetilde{h}(t) \in \mathbb{C}[[t]]$ a $\underline{k}$-summable formal series in every direction $\theta' \in]\theta - \eta, \theta + \eta[\setminus\{\theta\}$ with $\eta > 0$ small enough.
Let $\underline{\varepsilon} := (\varepsilon_1, ..., \varepsilon_s)$ with $\varepsilon_i \in \{-1, +1\}$ for all $i = 1, ..., s$.
One says that $\widetilde{h}(t)$ is *$\underline{k}$-summable along the path* $(\theta, \underline{\varepsilon})$ if

$$\mathcal{L}_{k_s;\theta^{\varepsilon_s}} \mathcal{A}_{k_s,k_{s-1};\theta^{\varepsilon_{s-1}}} ... \mathcal{A}_{k_2,k_1;\theta^{\varepsilon_1}} \mathcal{B}_{k_1;\theta^{\varepsilon_1}}(\widetilde{h}) \tag{2.1}$$

exists. Then, the function (2.1) thus defined is analytic on a germ of sector with vertex 0 and bisected by θ; it is called *the sum of $\widetilde{h}(x)$ along the path* $(\theta, \underline{\varepsilon})$. We denote it by $s_{\underline{k};\theta,\underline{\varepsilon}}(\widetilde{h})$.

Remark 2.14 Laplace operator $\mathcal{L}_{k_s;\theta^{\varepsilon_s}}$, accelerators $\mathcal{A}_{k_{j+1},k_j;\theta^{\varepsilon_j}}$ and sum $\mathcal{B}_{k_1;\theta^{\varepsilon_1}}(\widetilde{h})$ are defined in the same way as the sums $s_{\underline{r};\theta^\pm}(\widetilde{F})$ (*cf.* section 2.3 above). Furthermore, to make sense in expression (2.1), the analytic continuations of the sums $\mathcal{A}_{k_{j+1},k_j;\theta^{\varepsilon_j}}(...)$ for $j = 1, ..., s$ are, of course, along the direction $\theta^{\varepsilon_{j+1}}$.

Back to $\widetilde{F}(x)$, we have the following theorem:

Theorem 2.15 ([19, Thm. 9, p. 366]) *Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be an anti-Stokes direction of $\widetilde{F}(x)$. Let $\underline{\varepsilon} = (\varepsilon_1, ..., \varepsilon_p)$ with $\varepsilon_i \in \{-1, +1\}$ for all $i = 1, ..., p$. Then, $\widetilde{F}(x)$ is $\underline{r}$-summable along the path $(\theta, \underline{\varepsilon})$.*

As in theorem 2.9, a more precise statement can be given for each column-block $\widetilde{F}^{\bullet;\ell}(x)$ of $\widetilde{F}(x)$.

For θ and $\underline{\varepsilon}$ as in theorem 2.15, we defined the sum $Y_{\theta,\underline{\varepsilon}}(x)$ of $\widetilde{Y}(x)$ along the path $(\theta,\underline{\varepsilon})$ by

$$Y_{\theta,\underline{\varepsilon}}(x) := s_{\underline{r};\theta,\underline{\varepsilon}}(\widetilde{F})(x)Y_{0;\theta^\star}(x) \quad \text{for } \arg(x)\simeq\theta^\star.$$

Note that for $\underline{\varepsilon}=(-1,...,-1)=-$ (resp. $\varepsilon=(+1,...,+1)=+$), the sums $Y_{\theta,-}$ and Y_{θ^-} (resp. $Y_{\theta,+}$ and Y_{θ^+}) coincide.

The comparison between sums $Y_{\theta,\underline{\varepsilon}}$ and $Y_{\theta,\underline{\varepsilon}'}$ for different $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ yields a *generalized Stokes phenomenon*: given $\underline{\varepsilon}\neq\underline{\varepsilon}'$, there exists a unique invertible matrix $St_{\theta^\star}^{\underline{\varepsilon},\underline{\varepsilon}'}\in GL_n(\mathbb{C})$ such that

$$Y_{\theta,\underline{\varepsilon}}(x) = Y_{\theta,\underline{\varepsilon}'}(x)St_{\theta^\star}^{\underline{\varepsilon},\underline{\varepsilon}'} \quad \text{for } \arg(x)\simeq\theta^\star. \tag{2.2}$$

Note that $St_{\theta^\star}^{-,+}=St_{\theta^\star}$ the Stokes-Ramis matrix associated with $\widetilde{Y}(x)$ in the direction θ. More generally, a convenient choice of $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ allows us to obtain *all* the k^{th} level's Stokes-Ramis matrices $St_{k;\theta^\star}$ for $k=1,...,p$. Theorem 2.16 below precises this point in the case of the highest level's Stokes-Ramis matrix $St_{p;\theta^\star}$ (recall that the aim of this paper is the calculation of this one).

Theorem 2.16 ([19, Thm. 9, p. 366]) *Let $\theta\in\mathbb{R}/2\pi\mathbb{Z}$ be an anti-Stokes direction of $\widetilde{F}(x)$.*
Let $\underline{\varepsilon}=(-1,...,-1)$ and $\underline{\varepsilon}'=(-1,...,-1,+1)$ with a "+1" only at index p.
Then, $St_{\theta^\star}^{\underline{\varepsilon},\underline{\varepsilon}'}=St_{p;\theta^\star}$ the highest level's Stokes-Ramis matrix associated with $\widetilde{Y}(x)$ in direction θ.

Theorem 2.16 will be us useful in next section 3.2 below.

3 Setting the problem

Any of the J column-blocks $\widetilde{F}^{\bullet;\ell}(x)$, $\ell=1,...,J$, of $\widetilde{F}(x)$ associated with the Jordan block-structure of L can be positioned at the first place by means of a convenient permutation P on the columns of $\widetilde{Y}(x)$. Furthermore, the same permutation P acting on the rows of $\widetilde{Y}(x)$ also allows to keep initial normalizations of $\widetilde{Y}(x)$; precisely, the new formal fundamental solution $P\widetilde{Y}(x)P$ reads

$$P\widetilde{Y}(x)P = P\widetilde{F}(x)Px^{P^{-1}LP}e^{P^{-1}Q(1/x)P} \text{ with } P\widetilde{F}(x)P = I_n + O(x^r).$$

Consequently, without loss of generality, we can restrict our study to the first column-block $\widetilde{F}^{\bullet;1}(x)$, which we denote below by $\widetilde{f}(x)$. Note that the size of $\widetilde{f}(x)$ is $n \times n_1$: $\widetilde{f}(x) \in M_{n,n_1}(\mathbb{C}[[x]])$.

The goal of this paper is double:

1. prove a *summable-resurgence theorem* for the formal Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ of the r-reduced series $\widetilde{\boldsymbol{f}}^{[u]}(t), u = 0, ..., r-1$, of $\widetilde{f}(x)$,
2. display *explicit* and *exact* formulæ relating the *highest level's Stokes multipliers* $st^{j;\bullet}_{p;\theta^\star} := St^{j;1}_{p;\theta^\star}$ of $\widetilde{f}(x)$ and the *connection constants* given by some convenient analytic continuations of the $\widehat{\boldsymbol{f}}^{[u]}(\tau)$'s at their various singular points.

Recall that the formal series $\widetilde{\boldsymbol{f}}^{[u]}(t)$ are intimitely related to the classical method of rank reduction and are uniquely determined by the relation

$$\widetilde{f}(x) = \widetilde{\boldsymbol{f}}^{[0]}(x^r) + x\widetilde{\boldsymbol{f}}^{[1]}(x^r) + ... + x^{r-1}\widetilde{\boldsymbol{f}}^{[r-1]}(x^r).$$

Before starting the calculations, let us begin by recalling some general results on the rank reduction.

3.1 Rank reduction

For the convenience of the reader, we briefly recall in this section some results on the rank reduction, such as the *r-reduced system associated with system* (A) and the structure of the *r-reduced formal fundamental solution associated with* $\widetilde{Y}(x)$, which will be used in next section 4. For more details, we refer, for instance, to [12].

$\triangleleft$ ***r*-reduced system.** The method of rank reduction is a procedure allowing to associate with system (A) a system with meromorphic entries, rank 1 and having as formal fundamental solution the matrix

$$\widetilde{\boldsymbol{Y}}(t) = \begin{bmatrix} \widetilde{Y}(t^{1/r}) & \widetilde{Y}(\rho t^{1/r}) & \cdots & \widetilde{Y}(\rho^{r-1}t^{1/r}) \\ (t^{1/r})^{-1}\widetilde{Y}(t^{1/r}) & (\rho t^{1/r})^{-1}\widetilde{Y}(\rho t^{1/r}) & \cdots & (\rho^{r-1}t^{1/r})^{-1}\widetilde{Y}(\rho^{r-1}t^{1/r}) \\ \vdots & \vdots & & \vdots \\ (t^{1/r})^{-(r-1)}\widetilde{Y}(t^{1/r}) & (\rho t^{1/r})^{-(r-1)}\widetilde{Y}(\rho t^{1/r}) & \cdots & (\rho^{r-1}t^{1/r})^{-(r-1)}\widetilde{Y}(\rho^{r-1}t^{1/r}) \end{bmatrix}$$

with $\rho := e^{-2i\pi/r}$, also called *r-reduced formal fundamental solution associated with* $\widetilde{Y}(x)$. It is well-known that this problem admits a unique solution: the so-called *r-reduced system*

$$(\boldsymbol{A}) \qquad rt^2\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{A}(t)\,\boldsymbol{Y}$$

associated with system (A), where $\boldsymbol{A}(t) \in M_{rn}(\mathbb{C}\{t\})$ is the $rn \times rn$-analytic matrix defined by

$$\boldsymbol{A}(t) = \begin{bmatrix} \boldsymbol{A}^{[0]}(t) & t\boldsymbol{A}^{[r-1]}(t) & \cdots & \cdots & t\boldsymbol{A}^{[1]}(t) \\ \boldsymbol{A}^{[1]}(t) & \boldsymbol{A}^{[0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \boldsymbol{A}^{[0]}(t) & t\boldsymbol{A}^{[r-1]}(t) \\ \boldsymbol{A}^{[r-1]}(t) & \cdots & \cdots & \boldsymbol{A}^{[1]}(t) & \boldsymbol{A}^{[0]}(t) \end{bmatrix} - \bigoplus_{u=0}^{r-1} utI_n$$

with $\boldsymbol{A}^{[0]}(t)$, ..., $\boldsymbol{A}^{[r-1]}(t)$ the r-reduced series of $A(x)$. Note that system $(\boldsymbol{A})$ has, by construction, levels ≤ 1.

Let us now precise the structure of the r-reduced formal fundamental solution $\widetilde{\boldsymbol{Y}}(t)$.

◃ ***r*-reduced formal fundamental solution.** As before, we denote by $\widetilde{\boldsymbol{F}}^{[0]}(t)$, ..., $\widetilde{\boldsymbol{F}}^{[r-1]}(t)$ the r-reduced series of $\widetilde{F}(x)$. Then, the r-reduced formal fundamental solution $\widetilde{\boldsymbol{Y}}(t)$ reads as $\widetilde{\boldsymbol{Y}}(t) = \widetilde{\boldsymbol{F}}(t)\widetilde{\boldsymbol{Y}}_0(t)$ where

$$\widetilde{\boldsymbol{F}}(t) = \begin{bmatrix} \widetilde{\boldsymbol{F}}^{[0]}(t) & t\widetilde{\boldsymbol{F}}^{[r-1]}(t) & \cdots & \cdots & t\widetilde{\boldsymbol{F}}^{[1]}(t) \\ \widetilde{\boldsymbol{F}}^{[1]}(t) & \widetilde{\boldsymbol{F}}^{[0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \widetilde{\boldsymbol{F}}^{[0]}(t) & t\widetilde{\boldsymbol{F}}^{[r-1]}(t) \\ \widetilde{\boldsymbol{F}}^{[r-1]}(t) & \cdots & \cdots & \widetilde{\boldsymbol{F}}^{[1]}(t) & \widetilde{\boldsymbol{F}}^{[0]}(t) \end{bmatrix}$$

and

$$\widetilde{\boldsymbol{Y}}_0(t) = \begin{bmatrix} (t^{\frac{1}{r}})^{\Lambda_0} e^{Q_0(t)} & (\rho t^{\frac{1}{r}})^{\Lambda_0} e^{Q_1(t)} & \cdots & (\rho^{r-1} t^{\frac{1}{r}})^{\Lambda_0} e^{Q_{r-1}(t)} \\ (t^{\frac{1}{r}})^{\Lambda_1} e^{Q_0(t)} & (\rho t^{\frac{1}{r}})^{\Lambda_1} e^{Q_1(t)} & \cdots & (\rho^{r-1} t^{\frac{1}{r}})^{\Lambda_1} e^{Q_{r-1}(t)} \\ \vdots & \vdots & \ddots & \vdots \\ (t^{\frac{1}{r}})^{\Lambda_{r-1}} e^{Q_0(t)} & (\rho t^{\frac{1}{r}})^{\Lambda_{r-1}} e^{Q_1(t)} & \cdots & (\rho^{r-1} t^{\frac{1}{r}})^{\Lambda_{r-1}} e^{Q_{r-1}(t)} \end{bmatrix}$$

with, for all $k = 0, ..., r-1$,

$$Q_k(t) := Q\left(\frac{1}{\rho^k t^{1/r}}\right) \quad , \ \Lambda_k := L - kI_n = \bigoplus_{j=1}^{J} \Lambda_{j,k} \quad , \ \Lambda_{j,k} := L_j - kI_{n_j}.$$

Note that initial condition $\widetilde{F}(x) = I_n + O(x^r)$ implies $\widetilde{\boldsymbol{F}}(t) = I_{rn} + O(t)$;

note also that the matrix of the first n_1 columns of $\widetilde{\boldsymbol{F}}(t)$ is the matrix

$$\widetilde{\boldsymbol{f}}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{[0]}(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}^{[r-1]}(t) \end{bmatrix} \tag{3.1}$$

formed by the r-reduced series of $\widetilde{f}(x)$. Thereby, it is equivalent to work with the r-reduced series $\widetilde{\boldsymbol{f}}^{[u]}(t)$ of with $\widetilde{\boldsymbol{f}}(t)$. In the rest of the article, we make the choice to work with $\widetilde{\boldsymbol{f}}(t)$ rather than with each $\widetilde{\boldsymbol{f}}^{[u]}(t)$. Of course, all the results which will be stated for $\widetilde{\boldsymbol{f}}(t)$ wil be immediately transposable to the $\widetilde{\boldsymbol{f}}^{[u]}(t)$'s.

To end this section, let us give some classical results about the multisummability of the formal factor series $\widetilde{\boldsymbol{F}}(t) \in M_{rn}(\mathbb{C}[[t]])$.

$\triangleleft$ **Multisummability of $\widetilde{\boldsymbol{F}}(t)$.** Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a non anti-Stokes direction of $\widetilde{F}(x)$. Then, the $\underline{r}$-summability of $\widetilde{F}(x)$ in direction θ (*cf.* theorem 2.9) implies the $\underline{\boldsymbol{r}}$-summability of $\widetilde{\boldsymbol{F}}(t)$ in direction $\boldsymbol{\theta} := r\theta$ with

$$\underline{\boldsymbol{r}} := (\boldsymbol{r}_1 < ... < \boldsymbol{r}_{p-1} < 1) \quad , \ \boldsymbol{r}_j := \frac{r_j}{r}.$$

More precisely, split

$$\widetilde{\boldsymbol{F}}(x) = \begin{bmatrix} \widetilde{\boldsymbol{F}}^{\bullet;1}(t) & \widetilde{\boldsymbol{F}}^{\bullet;2}(t) & \cdots & \widetilde{\boldsymbol{F}}^{\bullet;r}(t) \end{bmatrix}$$

into r column-blocks $\widetilde{\boldsymbol{F}}^{\bullet;v}(t)$ of size $rn \times n$; then, each

$$\widetilde{\boldsymbol{F}}^{\bullet;v}(t) = \begin{bmatrix} \widetilde{\boldsymbol{F}}^{\bullet;v,1}(t) & \widetilde{\boldsymbol{F}}^{\bullet;v,2}(t) & \cdots & \widetilde{\boldsymbol{F}}^{\bullet;v,J}(t) \end{bmatrix}$$

into J column-blocks $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ of size $rn \times n_\ell$ according to the Jordan block-structure of matrix L. Then, for all $v = 1, ..., r$, $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ is $\underline{\boldsymbol{r}}^{(\ell)}$-summable in direction $\boldsymbol{\theta}$ with

$$\underline{\boldsymbol{r}}^{(\ell)} := (\boldsymbol{r}_1^{(\ell)} < ... < \boldsymbol{r}_{p_\ell-1}^{(\ell)} < 1) \quad , \ \boldsymbol{r}_j^{(\ell)} := \frac{r_j^{(\ell)}}{r}.$$

In the same way, the $\underline{r}$-summability of $\widetilde{F}(x)$ along a path $(\theta, \underline{\varepsilon})$ (*cf.* theorem 2.15) implies the $\underline{\boldsymbol{r}}$-summability of $\widetilde{\boldsymbol{F}}(t)$ along the path $(\boldsymbol{\theta}, \underline{\varepsilon})$; the sum $\boldsymbol{Y}_{\boldsymbol{\theta},\underline{\varepsilon}}(t)$ of $\widetilde{\boldsymbol{Y}}(t)$ along the path $(\boldsymbol{\theta}, \underline{\varepsilon})$ is defined similarly as the sum $Y_{\theta,\underline{\varepsilon}}(x)$.

Notation 3.1 In the sequel, we shall use the following notations:

- Given a matrix M of size $m \times rn$ with $m \geq 1$, we split M into column-blocks in the same way as $\widetilde{\boldsymbol{F}}(t)$:
  - M is first split into r column-blocks $M^{\bullet;v}$, $v = 1, ..., r$, of size $m \times n$ according to the block-structure of matrix $\widetilde{\mathbf{Y}}(t)$,
  - each $M^{\bullet;v}$ is then split into J column-blocks $M^{\bullet;v,\ell}$, $\ell = 1, ..., J$, of size $m \times n_\ell$ according to the Jordan block-structure of matrix L.
- We shall also use a row-blocks splitting:
  - Given a matrix M of size $n \times m$ with $m \geq 1$, we split M into J row-blocks $M^{j;\bullet}$, $j = 1, ..., J$, of size $n_j \times m$ according to the Jordan block-structure of matrix L,
  - Given a matrix M of size $rn \times m$ with $m \geq 1$, we first split M into r row-blocks $M^{u;\bullet}$, $u = 1, ..., r$, of size $n \times m$ according to the block-structure of matrix $\widetilde{\mathbf{Y}}(t)$; then, each $M^{u;\bullet}$ into J row-blocks $M^{u,j;\bullet}$ of size $n_j \times m$ as above.

Let us now turn to the study of the highest level's Stokes multipliers of the first n_1 columns $\widetilde{f}(x)$ of $\widetilde{F}(x)$.

3.2 Highest level's Stokes multipliers and rank reduction

As we said at the beginning of section 3, we restrict our study to the highest level's Stokes multipliers $st^{j;\bullet}_{p;\theta^\star}$ of the first column-block $\widetilde{f}(x)$ of $\widetilde{F}(x)$.

According to the normalization $q_1 \equiv 0$ and definitions 2.2–2.3, the highest level's anti-Stokes directions of $\widetilde{f}(x)$ are all the directions of maximal decay of exponentials $e^{q_j(1/x)}$ with polynomials q_j of degree r, *i.e.*, all the collections of the r directions θ_0, θ_1, ..., $\theta_{r-1} \in \mathbb{R}/2\pi\mathbb{Z}$ regularly distribued around the origin $x = 0$ which are given by the r^{th} roots of the nonzero highest level's Stokes values $a_{j,r} \neq 0$ of $\widetilde{f}(x)$. For such a collection (θ_k),

- we denote $\boldsymbol{\theta} := r\theta_0$ (hence, $\boldsymbol{\theta} = r\theta_k$ for any k) and

$$\Omega_{p;\boldsymbol{\theta}} := \{a_{j,r} \neq 0 \ ; \ \arg(a_{j,r}) = \boldsymbol{\theta}\}$$

  the set of all the highest level's Stokes values of $\widetilde{f}(x)$ generating (θ_k);

- we choose as principal determination $\theta_k^\star \in]-2\pi, 0]$ of θ_k the argument

$$\theta_k^\star := \frac{\arg^\star(a_{j,r})}{r} - \frac{2k\pi}{r} \quad , \ \arg^\star(a_{j,r}) \in]-2\pi, 0],$$

in order that the $\theta_k^\star$'s satisfy

$$-2\pi < \theta_{r-1}^\star < ... < \theta_1^\star < \theta_0^\star \leq 0.$$

In particular, identity (2.2) and theorem 2.16 imply that the Stokes-Ramis matrice $St_{p;\theta_k^\star}$ are uniquely determined, for all $k = 0, ..., r-1$, by relations

$$Y_{\theta_k,\underline{\varepsilon}}(\rho^k x) = Y_{\theta_k,\underline{\varepsilon}'}(\rho^k x) St_{p;\theta_k^\star} \quad \text{for } \arg(x) \simeq \theta_0^\star \text{ and } \rho = e^{-2i\pi/r}.$$

By definition of rank reduction, direction $\boldsymbol{\theta}$ is a highest level's anti-Stokes direction (*i.e.*, an anti-Stokes direction of level 1) of $\widetilde{\boldsymbol{f}}(t)$. Then, section 3.1 above and [12, Prop. 4.2] imply the following proposition:

Proposition 3.2 *Let $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ as in theorem 2.16. Then,*

$$\boxed{\mathbf{Y}_{\boldsymbol{\theta},\underline{\varepsilon}}(t) = \mathbf{Y}_{\boldsymbol{\theta},\underline{\varepsilon}'}(t) \left(\bigoplus_{k=0}^{r-1} St_{p;\theta_k^\star} \right) \quad \textit{for } \arg(t) \simeq \boldsymbol{\theta}^\star.} \tag{3.2}$$

Let us now write the Stokes-Ramis matrices $St_{p;\theta_k^\star}$ in the form $St_{p;\theta_k^\star} = I_n + C_{p;\theta_k^\star}$ (we have $C_{p;\theta_k^\star}^{j;\ell} = St_{p;\theta_k^\star}^{j;\ell}$ if $j \neq \ell$ and 0 otherwise). Identity (3.2) has the following "additive" form

$$s_{\underline{r};\boldsymbol{\theta},\underline{\varepsilon}}(\widetilde{\boldsymbol{F}})(t) - s_{\underline{r};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{F}})(t) = s_{\underline{r};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{F}})(t)\mathbf{Y}_{0;\boldsymbol{\theta}^\star}(t) \left(\bigoplus_{k=0}^{r-1} C_{p;\theta_k^\star} \right) \mathbf{Y}_{0;\boldsymbol{\theta}^\star}(t)^{-1}.$$

where $\mathbf{Y}_{0;\boldsymbol{\theta}^\star}(t)^{-1}$ is the matrix

$$\frac{1}{r}\begin{bmatrix} (t^{-\frac{1}{r}})^{\Lambda_0} e^{-Q_0(t)} & (t^{-\frac{1}{r}})^{\Lambda_1} e^{-Q_0(t)} & \cdots & (t^{-\frac{1}{r}})^{\Lambda_{r-1}} e^{-Q_0(t)} \\ (\overline{\rho} t^{-\frac{1}{r}})^{\Lambda_0} e^{-Q_1(t)} & (\overline{\rho} t^{-\frac{1}{r}})^{\Lambda_1} e^{-Q_1(t)} & \cdots & (\overline{\rho} t^{-\frac{1}{r}})^{\Lambda_{r-1}} e^{-Q_1(t)} \\ \vdots & \vdots & \ddots & \vdots \\ (\overline{\rho}^{r-1} t^{-\frac{1}{r}})^{\Lambda_0} e^{-Q_{r-1}(t)} & (\overline{\rho}^{r-1} t^{-\frac{1}{r}})^{\Lambda_1} e^{-Q_{r-1}(t)} & \cdots & (\overline{\rho}^{r-1} t^{-\frac{1}{r}})^{\Lambda_{r-1}} e^{-Q_{r-1}(t)} \end{bmatrix}.$$

Hence, in restriction to the first n_1 columns, the identity

$$\boxed{s_{\underline{r};\boldsymbol{\theta},\underline{\varepsilon}}(\widetilde{\boldsymbol{f}})(t) - s_{\underline{r};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{f}})(t) = s_{\underline{r};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{F}})(t)\boldsymbol{M}_{p;\boldsymbol{\theta}^\star}(t)} \tag{3.3}$$

where $\boldsymbol{M}_{p;\boldsymbol{\theta}^\star}(t)$ is the $rn \times n_j$-matrix defined by

$$\boxed{\boldsymbol{M}^{u,j;\bullet}_{p;\boldsymbol{\theta}^\star}(t) = \begin{cases} \dfrac{1}{r}\displaystyle\sum_{k=0}^{r-1}(\rho^k t^{\frac{1}{r}})^{\Lambda_{j,u-1}} st^{j;\bullet}_{p;\theta^\star_k}(\rho^k t^{\frac{1}{r}})^{-J_{n_1}} e^{q_j(1/(\rho^k t^{1/r}))} & \text{if } a_{j,r} \in \Omega_{p;\boldsymbol{\theta}} \\ 0 & \text{otherwise} \end{cases}}$$

for all $u = 1, ..., r$ and $j = 1, ..., J$. Recall that $\Lambda_{j,u-1} = L_j - (u-1)I_{n_j}$.

As we said in section 3.1.3, $\widetilde{\boldsymbol{f}}(t)$ is $\underline{\boldsymbol{r}}^{(1)}$-summable with

$$\underline{\boldsymbol{r}}^{(1)} := (\boldsymbol{r}^{(1)}_1 < ... < \boldsymbol{r}^{(1)}_{p_1-1} < 1) \quad , \ \boldsymbol{r}^{(1)}_j := \frac{r^{(1)}_j}{r}, \ p_1 \geq 1.$$

In particular, $\widetilde{\boldsymbol{f}}(t)$ is 1-summable when $p_1 = 1$ and multi-summable otherwise. Then, applying remark 2.5, lemma 2.8 and definition 2.13, this brings us to the following result:

Lemma 3.3 *Let $\underline{\varepsilon}$ and $\underline{\varepsilon}'$ as in theorem 2.16. Then, the sums $s_{\underline{\boldsymbol{r}};\boldsymbol{\theta},\underline{\varepsilon}}(\widetilde{\boldsymbol{f}})(t)$ and $s_{\underline{\boldsymbol{r}};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{f}})(t)$ read as the Laplace integrals*

$$\boxed{\begin{cases} s_{\underline{\boldsymbol{r}};\boldsymbol{\theta},\underline{\varepsilon}}(\widetilde{\boldsymbol{f}})(t) = \mathcal{L}_{1;\boldsymbol{\theta}^-}(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) \\ s_{\underline{\boldsymbol{r}};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{f}})(t) = \mathcal{L}_{1;\boldsymbol{\theta}^+}(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) \end{cases} , \ \widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-} := s_{\underline{\boldsymbol{\rho}}^{(1)};\boldsymbol{\theta}^-}(\widehat{\boldsymbol{f}})}$$

where $\widehat{\boldsymbol{f}} := \mathcal{B}_1(\widetilde{\boldsymbol{f}})$ and where $\underline{\boldsymbol{\rho}}^{(1)}$ is defined by

$$\boxed{\underline{\boldsymbol{\rho}}^{(1)} := \begin{cases} \infty & \textit{if } p_1 = 1 \\ (\boldsymbol{\rho}^{(1)}_1, ..., \boldsymbol{\rho}^{(1)}_{p_1-1}), \ \boldsymbol{\rho}^{(1)}_j := \dfrac{\boldsymbol{r}^{(1)}_j}{1-\boldsymbol{r}^{(1)}_j} = \dfrac{r^{(1)}_j}{r - r^{(1)}_j} & \textit{if } p_1 \geq 2 \end{cases}.}$$

Hence, according to identity (3.3), the following proposition:

Proposition 3.4 *Let $\underline{\varepsilon}'$ as in theorem 2.16. Then, for* $\arg(t) \simeq \boldsymbol{\theta}^\star$,

$$\boxed{(\mathcal{L}_{1;\boldsymbol{\theta}^-} - \mathcal{L}_{1;\boldsymbol{\theta}^+})(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) = s_{\underline{\boldsymbol{r}};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{F}})(t)\boldsymbol{M}_{p;\boldsymbol{\theta}^\star}(t)} \tag{3.4}$$

where the matrix $\boldsymbol{M}_{p;\boldsymbol{\theta}^\star}(t)$ is given by relations above.

Note that relation (3.4) characterizes all the highest level's Stokes multipliers of $\widetilde{f}(x)$ in terms of function $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$. This function (in fact, a more general function) is studied in great details in next section 4. In particular, we prove that it is *summable-resurgent* and we give a description of *all* its singularities.

4 Summable-resurgence and singularities

In this section, we fix a non anti-Stokes direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of $\widetilde{F}(x)$ and we set, as before, $\boldsymbol{\theta} := r\theta$. According to the properties of (multi)-summability of formal series $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ previously given, we can define, as in lemma 3.3, the functions $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}} := s_{\underline{\boldsymbol{\rho}}^{(\ell)};\boldsymbol{\theta}}(\widehat{\boldsymbol{F}}^{\bullet;v,\ell})$ where $\widehat{\boldsymbol{F}}^{\bullet;v,\ell} := \mathcal{B}_1(\widetilde{\boldsymbol{F}}^{\bullet;v,\ell})$ and where $\underline{\boldsymbol{\rho}}^{(\ell)}$ is defined by

$$\underline{\boldsymbol{\rho}}^{(\ell)} := \begin{cases} \infty & \text{if } p_\ell = 1 \\ (\boldsymbol{\rho}_1^{(\ell)}, ..., \boldsymbol{\rho}_{p_\ell - 1}^{(\ell)}),\ \boldsymbol{\rho}_j^{(\ell)} := \dfrac{\boldsymbol{r}_j^{(\ell)}}{1 - \boldsymbol{r}_j^{(\ell)}} = \dfrac{r_j^{(\ell)}}{r - r_j^{(\ell)}} & \text{if } p_\ell \geq 2 \end{cases}.$$

Recall that p_ℓ denotes the number of levels of $\widetilde{F}^{\bullet;,\ell}(x)$. Recall also that $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$ is analytic on a disc centered at $0 \in \mathbb{C}$ when $p_\ell = 1$ (indeed, $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ is 1-summable, hence, 1-Gevrey) and is analytic on a sector with vertex 0, bisected by $\boldsymbol{\theta}$ and opening larger than $\pi/\boldsymbol{\rho}^{(\ell)}_{p_\ell-1} = \pi(r - r^{(\ell)}_{p_\ell-1})/r^{(\ell)}_{p_\ell-1}$ otherwise.

The aim of this section is to study the analytic continuations of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}} := \widehat{\boldsymbol{F}}^{\bullet;1,1}_{\boldsymbol{\theta}}$ outside its domain of definition $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$. In particular, we shall prove that $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}(\tau)$ is *summable-resurgent* (theorem 4.9) and we shall give a complete description of *all* its singularities in the Borel plane (theorem 4.24)[5]. To this end, we shall proceed similarly as in [24]: we first reduce system $(\boldsymbol{A})$ into a convenient scalar linear differential equation with polynomial coefficients (section 4.2 below); then, we compare $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ with some "solutions" (precisely actual or micro-solutions) of its Borel transformed equation.

Before starting the calculations, let us begin by recalling some classical results about the Borel transformation, both formal and functional versions.

4.1 Borel transformation

$\triangleleft$ **Formal Borel transformation.** Let us begin by some recalls about the formal Borel transformation:

[5] Even if these two results are obtained for $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$, they can be obviously extended to the other column-blocks $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$. Indeed, when $v = 1$, the block $\widetilde{\boldsymbol{F}}^{\bullet;1,\ell}(t)$ is formed, for all $\ell \in \{2, ..., J\}$, by the r-reduced series of $\widetilde{F}^{\bullet;\ell}(x)$ (compare with the definition of $\widetilde{\boldsymbol{f}}(t)$) and, to normalize $\widetilde{F}^{\bullet;\ell}(x)$ as $\widetilde{f}(x)$, one has just to multiply by $x^{-\lambda_\ell} e^{-q_\ell(1/x)}$. As for the blocks $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$ with $v \in \{2, ..., r\}$, it clearly results from the definition of $\widetilde{\boldsymbol{F}}(t)$ that they have same properties as $\widehat{\boldsymbol{F}}^{\bullet;1,\ell}_{\boldsymbol{\theta}}$.

1. The formal Borel transformation
$$\mathcal{B}_1 : \widetilde{h}(t) = \sum_{m\geq 0} a_m t^m \longmapsto \widehat{h}(\tau) = \delta a_0 + \sum_{m\geq 1} a_m \frac{\tau^{m-1}}{(m-1)!}$$
is an isomorphism from the $\mathbb{C}$-differential algebra $\left(\mathbb{C}[[t]], +, \cdot, t^2\frac{d}{dt}\right)$ to the $\mathbb{C}$-differential algebra $(\delta\mathbb{C} \oplus \mathbb{C}[[\tau]], +, *, \tau\cdot)$ that changes ordinary product $\cdot$ into convolution product $*$ and derivation $t^2\frac{d}{dt}$ into multiplication by τ. It also changes multiplication by $\frac{1}{t}$ into derivation $\frac{d}{d\tau}$. In particular, it changes derivation $\frac{d^k}{dt^k}$ into $\frac{d^{k+1}}{d\tau^{k+1}}\left(\tau^k \frac{d^{k-1}}{d\tau^{k-1}}\right)$ for any $k \geq 1$.

2. If $\widetilde{h}(t) \in \mathcal{O}$ is analytic at the origin $0 \in \mathbb{C}$, then $\widehat{h}(\tau)$ defines an entire function on all $\mathbb{C}$ with exponential growth of order ≤ 1 at infinity: $\widehat{h}(\tau) \in \mathcal{O}^{\leq 1}(\mathbb{C})$.

$\triangleleft$ **Borel transformation.** Let us now consider the functional version of the Borel transformation. It is given, in each direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, by the integral
$$\mathcal{B}_{1;\theta}(h(t))(\tau) := \frac{1}{2\pi i}\int_{\gamma_\theta} h(t) e^{\tau/t}\frac{dt}{t^2}$$
where γ_θ denotes the image by $t \mapsto 1/t$ of a Hankel contour directed by direction θ and oriented positively[6]. Note that, using Hankel's formula for the inverse of gamma function, we obtain
$$\mathcal{B}_{1;\theta}(t^m)(\tau) = \frac{\tau^{m-1}}{(m-1)!} \quad \text{for all } m \geq 1 \text{ and } \theta$$
(hence, the coherence with the definition of the formal Borel transformation) and, more generally,
$$\mathcal{B}_{1;\theta}(t^\lambda)(\tau) = \frac{\tau^{\lambda-1}}{\Gamma(\lambda)} \quad \text{for all } \lambda \in \mathbb{C}\backslash(-\mathbb{N}) \text{ and } \theta.$$
Note also that the Borel transform $\mathcal{B}_{1;\theta}(h(t))$ may be integrable or not at 0:
$$\mathcal{B}_{1;\theta}(t^{1/2})(\tau) = \frac{\tau^{-1/2}}{\sqrt{\pi}} \quad \text{whereas} \quad \mathcal{B}_{1;\theta}(t^{-1/2})(\tau) = -2\sqrt{\pi}\tau^{-3/2}.$$

The operator $\mathcal{B}_{1;\theta}$ applies to any function with subexponential growth at the origin $0 \in \widetilde{\mathbb{C}}$ (in fact, to a more general class of functions defined near

[6] Observe that we need a contour that ends at 0 since the functions we consider are studied near the origin; if we worked at infinity, we would use a Hankel contour itself.

0; see, for instance, [18] for exact conditions). Recall that such a function $h(t) \in \widetilde{\mathcal{O}}$ satisfies

$$\overline{\lim_{|t|\to 0}} |t| \ln(|h(t)|) = 0 \text{ uniformly on any bounded sector with vertex 0.}$$

Let $\widetilde{\mathcal{O}}^{\leq \exp}$ denote the space of all the functions with subexponential growth at 0. For example,

- any power t^λ, $\lambda \in \mathbb{C}$, of t; hence, any analytic function $h(t) \in \mathcal{O}$,
- any power $(\ln t)^m$, $m \geq 1$, of the logarithm,
- any exponential $\exp(P(t^{-1/r}))$ with $P(t)$ polynomial in t of degree $< r$

belong to $\widetilde{\mathcal{O}}^{\leq \exp}$. Classical lemma 4.1 gives us some properties of $\mathcal{B}_{1;\theta}$.

Lemma 4.1 *Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $h, h_1, h_2 \in \widetilde{\mathcal{O}}^{\leq \exp}$. Then,*

1. *$\mathcal{B}_{1;\theta}(h) := \widehat{h}$ is holomorphic on all $\widetilde{\mathbb{C}}$ with exponential growth of order ≤ 1 on any bounded sector of infinity: $\widehat{h}(\tau) \in \mathcal{O}^{\leq 1}(\widetilde{\mathbb{C}})$.*
2. *$\mathcal{B}_{1;\theta}$ satisfies the same properties of $\mathcal{B}_1$:*
   - $\mathcal{B}_{1;\theta}\left(t^2 \dfrac{dh}{dt}\right) = \tau \widehat{h}$ *and* $\mathcal{B}_{1;\theta}\left(\dfrac{1}{t} h\right) = \dfrac{d\widehat{h}}{d\tau}$,
   - $\mathcal{B}_{1;\theta}(h_1 h_2) = \widehat{h}_1 * \widehat{h}_2$ *when $\widehat{h}_1$ and $\widehat{h}_2$ are integrable at* 0.
3. *$\mathcal{B}_{1;\theta}$ changes exponential $e^{-\omega/t}$ with $\omega \in \mathbb{C}^*$ into the translation by ω.*

Note that, when $h(t) \in \mathcal{O}$, point 1 coincides with the second recall given at the beginning of this section. Note also that the convolution product $*$ does not make sense when $\widehat{h}_1$ or $\widehat{h}_2$ are not integrable at 0. To consider such a case, we need to "extend" the definition of $\mathcal{B}_{1;\theta}$ (see section 4.4 below).

4.2 System $(\boldsymbol{A})$ vs scalar differential equation

The cyclic vector lemma due to P. Deligne ([7, Lemme II.1.3]) and the algebrisation theorem of G. Birkhoff (see [5] or [27, Thm. 3.3.1]) say us that there exists a meromorphic gauge transformation $\boldsymbol{Y} = \boldsymbol{M}(t)\boldsymbol{Z}$, $\boldsymbol{M}(t) \in GL_n(\mathbb{C}\{t\}[t^{-1}])$, which changes system $(\boldsymbol{A})$ into a system $({}^{\boldsymbol{M}}\boldsymbol{A})$ which is a companion form of a scalar linear differential equation with polynomial coefficients. Furthermore, multiplying the formal solutions of this equation by a convenient power of t if needed, we can always suppose that system $({}^{\boldsymbol{M}}\boldsymbol{A})$ has for formal fundamental solution a matrix of the form

$$\widetilde{\boldsymbol{Z}}(t) = \widetilde{\boldsymbol{G}}(t)\widetilde{\boldsymbol{Y}}_0(t) \quad \text{with } \widetilde{\boldsymbol{G}}(t) := \boldsymbol{M}^{-1}(t)\widetilde{\boldsymbol{F}}(t) \in M_{rn}(\mathbb{C}[[t]]). \tag{4.1}$$

Remark 4.2 Given $v \in \{1, ..., r\}$ and $\ell \in \{1, ..., J\}$, the two column-blocks $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ and $\widetilde{\boldsymbol{G}}^{\bullet;v,\ell}(t)$ are related by the relation $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t) = \boldsymbol{M}(t)\widetilde{\boldsymbol{G}}^{\bullet;v,\ell}(t)$. Thereby, they have same properties of (multi)-summability. In particular, writing $\boldsymbol{M}(t)$ on the form

$$\boldsymbol{M}(t) = \sum_{m=1}^{N} \frac{\alpha_m}{t^m} + \boldsymbol{M}'(t) \quad \text{with } N \geq 1,\ \alpha_m \in \mathbb{C} \text{ and } \boldsymbol{M}'(t) \in \mathcal{O},$$

identity $\widetilde{\boldsymbol{f}}(t) = \boldsymbol{M}(t)\widetilde{\boldsymbol{G}}^{\bullet;1,1}(t)$ implies, after Borel transformation, the following fundamental identity which will be us useful in the sequel

$$\text{(4.2)} \qquad \boxed{\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}(\tau) = \sum_{m=1}^{N} \alpha_m \frac{d^m \widehat{\boldsymbol{G}}^{\bullet;1,1}_{\boldsymbol{\theta}}}{d\tau^m} + \widehat{\boldsymbol{M}}' * \widehat{\boldsymbol{G}}^{\bullet;1,1}_{\boldsymbol{\theta}}}$$

where $\widehat{\boldsymbol{M}}'(\tau) \in \mathcal{O}^{\leq 1}(\mathbb{C})$ and where $\widehat{\boldsymbol{G}}^{\bullet;1,1}_{\boldsymbol{\theta}} = s_{\underline{\boldsymbol{\rho}}^{(1)};\boldsymbol{\theta}}(\widehat{\boldsymbol{G}}^{\bullet;1,1}) \in \mathcal{O}(V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}))$.

Let us now denote by $Dy(t) = 0$ the equation associated with $({}^{\boldsymbol{M}}\boldsymbol{A})$. It clearly has order rn and levels ≤ 1 at the origin (levels of D are levels of system $(\boldsymbol{A})$). Denote also by $\widetilde{Sol}_0(D)$ the space of formal solutions of D at 0. A basis of $\widetilde{Sol}_0(D)$ is obvious given by relation (4.1). More precisely, we have the following lemma:

Lemma 4.3 (Basis of $\widetilde{Sol}_0(D)$) *Denote by $\widetilde{\boldsymbol{g}}^{v,\ell,q}(t) \in \mathbb{C}[[t]]$ the entry at row 1 and column q of $\widetilde{\boldsymbol{G}}^{\bullet;v,\ell}(t)$. Then,*

$$\boxed{\widetilde{Sol}_0(D) = \text{vect}(\widetilde{\boldsymbol{z}}^{v,\ell,q}(t);\ v = 1, ..., r,\ \ell = 1, ..., J,\ q = 1, ..., n_\ell),}$$

where $\widetilde{\boldsymbol{z}}^{v,\ell,q}(t)$ is defined for all v, ℓ and q by

$$\boxed{\widetilde{\boldsymbol{z}}^{v,\ell,q}(t) := e^{q_\ell(1/(\rho^{v-1}t^{1/r}))} \sum_{u=1}^{r}\sum_{p=1}^{q} \rho^{(v-1)(\lambda_\ell - u+1)} \widetilde{\boldsymbol{g}}^{u,\ell,p}(t) t^{\frac{\lambda_\ell - u+1}{r}} \frac{\ln^{q-p}(\rho^{v-1} t^{\frac{1}{r}})}{(q-p)!}.}$$

The following result is a direct consequence of lemma 4.3 and of the fact that, by definition of a companion system, the first column-block of $\widetilde{\boldsymbol{Z}}(t)$ reads as

$$\begin{bmatrix} \widetilde{\boldsymbol{z}}^{1,1,1} & \widetilde{\boldsymbol{z}}^{1,1,2} & \cdots & \widetilde{\boldsymbol{z}}^{1,1,n_1} \\ \dfrac{d\widetilde{\boldsymbol{z}}^{1,1,1}}{dt} & \dfrac{d\widetilde{\boldsymbol{z}}^{1,1,2}}{dt} & \cdots & \dfrac{d\widetilde{\boldsymbol{z}}^{1,1,n_1}}{dt} \\ \vdots & \vdots & & \vdots \\ \dfrac{d^{rn-1}\widetilde{\boldsymbol{z}}^{1,1,1}}{dt^{rn-1}} & \dfrac{d^{rn-1}\widetilde{\boldsymbol{z}}^{1,1,2}}{dt^{rn-1}} & \cdots & \dfrac{d^{rn-1}\widetilde{\boldsymbol{z}}^{1,1,n_1}}{dt^{rn-1}} \end{bmatrix}.$$

In particular, it says us that all the entries of $\widetilde{\boldsymbol{G}}^{\bullet;1,1}(t)$ are expressed in terms of formal series $\widetilde{\boldsymbol{g}}^{1,1,q}(t)$'s with $q = 1, ..., n_1$.

Corollary 4.4 *Let $q \in \{1, ..., n_1\}$ and $m \in \{0, ..., rn-1\}$.*
Then, the $(m+1)^{th}$ entry of the q^{th} column of $\widetilde{\boldsymbol{G}}^{\bullet;1,1}(t)$ reads

$$\frac{d^m \widetilde{\boldsymbol{g}}^{1,1,q}}{dt^m} + \sum_{p=1}^{q-1} \sum_{k=q-p}^{m} \binom{m}{k} \frac{(-1)^{k-q+p}(k-q+p)!}{r^{q-p}t^{k-q+p+1}} \frac{d^{m-k}\widetilde{\boldsymbol{g}}^{1,1,p}}{dt^{m-k}} \tag{4.3}$$

with the classical convention $\binom{m}{k} = 0$ if $m < k$.

Remark 4.5 According to formula (4.3) above, the second entry of the q^{th} column of $\widetilde{\boldsymbol{G}}^{\bullet;1,1}$ is

$$\frac{d\widetilde{\boldsymbol{g}}^{1,1,1}}{dt} \text{ if } q = 1 \quad \text{and} \quad \frac{d\widetilde{\boldsymbol{g}}^{1,1,q}}{dt} + \frac{\widetilde{\boldsymbol{g}}^{1,1,q-1}}{rt} \text{ if } q \geq 2.$$

Consequently, since $\widetilde{\boldsymbol{G}}^{\bullet;1,1} \in \mathbb{C}[[t]]$, corollary 4.4 shows in particular that, for $n_1 \geq 2$, we have $\widetilde{\boldsymbol{g}}^{1,1,q}(t) \in t\mathbb{C}[[t]]$ for all $q \in \{1, ..., n_1 - 1\}$.

Corollary 4.4 and the study of the Borel transforms of the $\widetilde{\boldsymbol{z}}^{v,\ell,q}(t)$'s will allow us to investigate in next sections 4.3 and 4.4 the analytic continuations and the singularities of $\widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;1,1}$; hence, according to relation (4.2), the analytic continuations and the singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$.

4.3 Summable-resurgence theorem

4.3.1 Main result

Recall that $p_1 \geq 1$ denotes the number of levels of $\widetilde{f}(x)$. Recall also that $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ is analytic on a domain $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ which is:

- *Case $p_1 = 1$:* an open disc centered at $0 \in \mathbb{C}$ (indeed, $\widetilde{\boldsymbol{f}}$ is 1-summable, hence, 1-Gevrey),
- *Case $p_1 \geq 2$:* an open sector with vertex $0 \in \mathbb{C}$, bisected by $\boldsymbol{\theta}$ and opening larger than $\pi/\boldsymbol{\rho}_{p_1-1}^{(1)} = \pi(r - r_{p_1-1}^{(1)})/r_{p_1-1}^{(1)}$ (see definition 2.4).

Summable-resurgence theorem 4.9 below tells us that $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ can be analytically continued outside $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ on all a convenient Riemann surface; in particular, it says us that the only singular points of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ belong to the set

$\boldsymbol{\Omega}_p := \{a_{j,r}\ ;\ j = 1, ..., J\}$ of all the highest level's Stokes values of $\widetilde{f}(x)$. Note that, according to the possible two choices of domain $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$, we need definitions of resurgence and summable-resurgence more general than those used in [14, 23, 24]. Indeed, all the functions considered in these papers were analytic at the origin $0 \in \mathbb{C}$, whereas our functions are potentially singular at 0, possibly with multivalued analytic continuation around 0.

The adequate Riemann surface on which $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ lives is one of the following two surfaces:

- the Riemann surface $\mathcal{R}_{\boldsymbol{\Omega}_p}$ defined as (the termined end of) all homotopy classes in $\mathbb{C}\backslash\boldsymbol{\Omega}_p$ of path issuing from 0 and bypassing all points of $\boldsymbol{\Omega}_p$; only homotopically trivial paths are allowed to turn back to 0,
- the universal cover $\widetilde{\mathcal{R}}_{\boldsymbol{\Omega}_p} := \widetilde{\mathbb{C}\backslash\boldsymbol{\Omega}_p}$ of $\mathbb{C}\backslash\boldsymbol{\Omega}_p$.

Note that $\mathcal{R}_{\boldsymbol{\Omega}_p}$ is the Riemann surface used in [14, 23, 24]. Note also that the difference between $\mathcal{R}_{\boldsymbol{\Omega}_p}$ and $\widetilde{\mathcal{R}}_{\boldsymbol{\Omega}_p}$ just lies in the fact that $\mathcal{R}_{\boldsymbol{\Omega}_p}$ has no branch point at 0 in the first sheet. This brings us to extend definitions of resurgence and summable-resurgence given in [14, 23, 24] as follows:

Definition 4.6 (Resurgence)

- We call *resurgent function with singular support* $\boldsymbol{\Omega}_p, 0$ any function defined and analytic on all the Riemann surface $\mathcal{R}_{\boldsymbol{\Omega}_p}$.
- We call *resurgent function with singular support* $\boldsymbol{\Omega}_p, \widetilde{0}$ any function defined and analytic on all the Riemann surface $\widetilde{\mathcal{R}}_{\boldsymbol{\Omega}_p}$.

Let $\mathcal{R}es_{\boldsymbol{\Omega}_p,0}$ and $\mathcal{R}es_{\boldsymbol{\Omega}_p,\widetilde{0}}$ denote the sets of resurgent functions with singular support $\boldsymbol{\Omega}_p, 0$ and of resurgent functions with singular support $\boldsymbol{\Omega}_p, \widetilde{0}$. Note that we have a natural injection $\mathcal{R}es_{\boldsymbol{\Omega}_p,0} \hookrightarrow \mathcal{R}es_{\boldsymbol{\Omega}_p,\widetilde{0}}$.

Definition 4.7 (Summable-resurgence)

- A resurgent function of $\mathcal{R}es_{\boldsymbol{\Omega}_p,0}$ is said to be *summable-resurgent* if it grows at most exponentially on any bounded sector of infinity of $\mathcal{R}_{\boldsymbol{\Omega}_p}$.
- A resurgent function of $\mathcal{R}es_{\boldsymbol{\Omega}_p,\widetilde{0}}$ is said to be *summable-resurgent* if it grows at most exponentially on any bounded sector of infinity of $\widetilde{\mathcal{R}}_{\boldsymbol{\Omega}_p}$.

Let $\mathcal{R}es^{sum}_{\boldsymbol{\Omega}_p,0}$ and $\mathcal{R}es^{sum}_{\boldsymbol{\Omega}_p,\widetilde{0}}$ denote the sets of summable-resurgent functions with singular support $\boldsymbol{\Omega}_p, 0$ and of summable-resurgent functions with singular support $\boldsymbol{\Omega}_p, \widetilde{0}$. As before, we have a natural injection $\mathcal{R}es^{sum}_{\boldsymbol{\Omega}_p,0} \hookrightarrow \mathcal{R}es^{sum}_{\boldsymbol{\Omega}_p,\widetilde{0}}$.

The four sets of resurgent and summable-resurgent functions above have a natural structure of $\mathbb{C}$-algebra. Following lemma 4.8 gives us some other elementary stability properties.

Lemma 4.8 (Stability properties) *The sets of resurgent and summable-resurgent functions above have the following stability properties:*

- *Let* $\varphi_1 \in \mathcal{R}es_{\mathbf{\Omega}_p,0}$ *(resp.* $\mathcal{R}es^{sum}_{\mathbf{\Omega}_p,0}$*) and* $\varphi_2 \in \mathcal{O}^{\leq 1}(\mathbb{C})$*.*
Then, $\varphi_1 * \varphi_2 \in \mathcal{R}es_{\mathbf{\Omega}_p,0}$ *(resp.* $\mathcal{R}es^{sum}_{\mathbf{\Omega}_p,0}$*).*

- *Let* $\varphi_1 \in \mathcal{R}es_{\mathbf{\Omega}_p,\widetilde{0}}$ *(resp.* $\mathcal{R}es^{sum}_{\mathbf{\Omega}_p,\widetilde{0}}$*) and* $\varphi_2 \in \mathcal{O}^{\leq 1}(\widetilde{\mathbb{C}})$*.*
Suppose that φ_1 *and* φ_2 *are both integrable at* 0 *(in the first sheet).*
Then, $\varphi_1 * \varphi_2 \in \mathcal{R}es_{\mathbf{\Omega}_p,\widetilde{0}}$ *(resp.* $\mathcal{R}es^{sum}_{\mathbf{\Omega}_p,\widetilde{0}}$*).*

They are besides stable under the derivation $d/d\tau$*.*

We are now able to state the main result of this section:

Theorem 4.9 (Summable-resurgence theorem)

- Case $p_1 = 1$. $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ *is summable-resurgent with singular support* $\mathbf{\Omega}_p, 0$*:*

$$\boxed{\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}(\tau) \in \mathcal{R}es^{sum}_{\mathbf{\Omega}_p,0}}$$

- Case $p_1 \geq 2$. $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ *is summable-resurgent with singular support* $\mathbf{\Omega}_p, \widetilde{0}$*:*

$$\boxed{\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}(\tau) \in \mathcal{R}es^{sum}_{\mathbf{\Omega}_p,\widetilde{0}}}$$

The proof is developed in section 4.3.2 below. The following proposition, which will be us useful in the study of singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$, extends theorem 4.9 to the other functions $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}$ (see footnote 5).

Proposition 4.10 *Let* $v \in \{1, ..., r\}$ *and* $\ell \in \{1, ..., J\}$*.*

- Case $p_\ell = 1$. *Then,* $\boxed{\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau) \in \mathcal{R}es^{sum}_{\mathbf{\Omega}_p - a_{\ell,r},0}}$.

- Case $p_\ell \geq 2$. *Then,* $\boxed{\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau) \in \mathcal{R}es^{sum}_{\mathbf{\Omega}_p - a_{\ell,r},\widetilde{0}}}$.

Note that, since system $(\boldsymbol{A})$ has $p \geq 2$ levels, there exists at least one $\ell \in \{1, ..., r\}$ such that $\widehat{\boldsymbol{F}}^{\bullet;v,\ell}_{\boldsymbol{\theta}}(\tau) \in \mathcal{R}es^{sum}_{\mathbf{\Omega}_p - a_{\ell,r},\widetilde{0}}$.

4.3.2 Proof of theorem 4.9

According to relation (4.2) and lemma 4.8, it suffices to prove that theorem 4.9 is valid for $\widehat{\boldsymbol{G}}_{\boldsymbol{\theta}}^{\bullet;1,1}$ instead of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$. This stems obvious from corollary 4.4, properties of (formal) Borel transformation given in section 4.1 and following proposition 4.11.

Proposition 4.11 *Let $q \in \{1, ..., n_1\}$.*
Then, theorem 4.9 is valid for $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q}$ instead of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$.

The proof is essentially based on the following technical lemmas 4.12 and 4.13 which respectively provide some properties about the space $\widetilde{Sol}_0(D)$ and about the Borel transformed equation $\widehat{D}\widehat{y}(\tau) = 0$ of $Dy(t) = 0$. Recall that, multiplying D by a convenient power of $1/t$ if needed, this equation is again a linear differential equation with polynomial coefficients.

Lemma 4.12 *Let $q \in \{1, ..., n_1\}$. Then,*

$$\boxed{\sum_{p=1}^{q} \widetilde{\boldsymbol{g}}^{1,1,p}(t) \frac{\ln^{q-p}(t^{1/r})}{(q-p)!} \in \widetilde{Sol}_0(D)}$$

Proof. We shall prove in fact the following more general statement: for all $u \in \{1, ..., r\}$ and $q \in \{1, ..., n_1\}$, we have

$$h_{u,q}(t) := \sum_{p=1}^{q} \widetilde{\boldsymbol{g}}^{u,1,p}(t) t^{-\frac{u-1}{r}} \frac{\ln^{q-p}(t^{1/r})}{(q-p)!} \in \widetilde{Sol}_0(D).$$

To simplify notations and calculations below, we denote temporarily $g_{u,p}(t)$ for $\widetilde{\boldsymbol{g}}^{u,1,p}(t) t^{-\frac{u-1}{r}}$.

$\triangleleft$ Let us begin with the simplest case $n_1 = q = 1$. According to lemma 4.3, we have, for all $v = 1, ..., r$, the following equalities

$$\widetilde{\boldsymbol{z}}^{v,1,1}(t) = \sum_{u=1}^{r} \overline{\rho}^{(v-1)(u-1)} g_{u,1}(t) = \sum_{u=1}^{r} \overline{\rho}^{(v-1)(u-1)} h_{u,1}(t) \in \widetilde{Sol}_0(D)$$

which we can rewrite as the matrix identity

$$\begin{bmatrix} z_{1,1} \\ \vdots \\ z_{r,1} \end{bmatrix} = V \begin{bmatrix} h_{1,1} \\ \vdots \\ h_{r,1} \end{bmatrix} \quad \text{with } V := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \overline{\rho} & \cdots & \overline{\rho}^{r-1} \\ \vdots & \vdots & & \vdots \\ 1 & \overline{\rho}^{r-1} & \cdots & \overline{\rho}^{(r-1)^2} \end{bmatrix}.$$

Thereby, we deduce that all the $h_{u,1}$'s are linear combinations of the $\widetilde{\boldsymbol{z}}^{v,1,1}$'s (indeed, V is an invertible Van der Monde matrix). Hence, $h_{u,1}(t) \in \widetilde{Sol_0}(D)$ for all $u = 1, ..., r$ and the result follows.

$\triangleleft$ When $n_1 \geq 2$, we proceed by induction on q. Since the case $q = 1$ has been treated above, we now suppose that, for a certain $q \in \{1, ..., n_1 - 1\}$, $h_{u,p}(t) \in \widetilde{Sol_0}(D)$ for all $u = 1, ..., r$ and $p = 1, ..., q$. We must then prove that $h_{u,q+1}(t) \in \widetilde{Sol_0}(D)$ for all $u = 1, ..., r$. To do that, we apply again lemma 4.3 which says us that

$$\widetilde{\boldsymbol{z}}^{v,1,q+1}(t) = \sum_{u=1}^{r}\sum_{p=1}^{q+1}\overline{\rho}^{(v-1)(u-1)}g_{u,p}(t)\frac{\ln^{q+1-p}(\rho^{v-1}t^{\frac{1}{r}})}{(q+1-p)!} \in \widetilde{Sol_0}(D) \tag{4.4}$$

for all $v = 1, ..., r$. Let us temporarily denote

$$S_u := \sum_{p=1}^{q+1}\overline{\rho}^{(v-1)(u-1)}g_{u,p}(t)\frac{\ln^{q+1-p}(\rho^{v-1}t^{\frac{1}{r}})}{(q+1-p)!} \quad \text{for all } u \in \{1, ..., r\}$$

and apply Newton's formula

$$\frac{\ln^{q+1-p}(\rho^{v-1}t^{\frac{1}{r}})}{(q+1-p)!} = \frac{\ln^{q+1-p}(t^{\frac{1}{r}})}{(q+1-p)!} + A_{q,p}$$

with

$$A_{q,p} := \sum_{s=1}^{q+1-p}\frac{\ln^{s}(\rho^{v-1})}{s!}\frac{\ln^{q+1-p-s}(t^{\frac{1}{r}})}{(q+1-p-s)!}$$

for all $p = 1, ..., q$. We get

$$\begin{aligned}
S_u &= \overline{\rho}^{(v-1)(u-1)}\sum_{p=1}^{q}g_{u,p}\left(\frac{\ln^{q+1-p}(t^{\frac{1}{r}})}{(q+1-p)!} + A_{q,p}\right) + \overline{\rho}^{(v-1)(u-1)}g_{u,q+1} \\
&= \overline{\rho}^{(v-1)(u-1)}h_{u,q+1} + \overline{\rho}^{(v-1)(u-1)}\sum_{p=1}^{q}\left(g_{u,p}\sum_{s=1}^{q+1-p}\frac{\ln^{s}(\rho^{v-1})}{s!}\frac{\ln^{q+1-p-s}(t^{\frac{1}{r}})}{(q+1-p-s)!}\right) \\
&= \overline{\rho}^{(v-1)(u-1)}h_{u,q+1} + \overline{\rho}^{(v-1)(u-1)}\sum_{s=1}^{q}\left(\frac{\ln^{s}(\rho^{v-1})}{s!}\sum_{p=1}^{q+1-s}g_{u,p}\frac{\ln^{q+1-p-s}(t^{\frac{1}{r}})}{(q+1-p-s)!}\right) \\
&= \overline{\rho}^{(v-1)(u-1)}h_{u,q+1} + \overline{\rho}^{(v-1)(u-1)}\sum_{s=1}^{q}\frac{\ln^{s}(\rho^{v-1})}{s!}h_{u,q+1-s}.
\end{aligned}$$

Hence, using (4.4), the following identities

$$\widetilde{\boldsymbol{z}}^{v,1,q+1} - \sum_{u=1}^{r}\left(\overline{\rho}^{(v-1)(u-1)}\sum_{s=1}^{q}\frac{\ln^s(\rho^{v-1})}{s!}h_{u,q+1-s}\right) = \sum_{u=1}^{r}\overline{\rho}^{(v-1)(u-1)}h_{u,q+1}$$

hold for all $v = 1, ..., r$. Since the left-hand side belongs to $\widetilde{Sol}_0(D)$, we conclude, as in the case $q = 1$, that $h_{u,q+1}(t) \in \widetilde{Sol}_0(D)$ for all $u = 1, ..., r$. This ends the proof. ■

Lemma 4.13 *Let $\widehat{D}$ be the Borel transformed equation of D.*

1. *The singular points of $\widehat{D}$ are the highest level's Stokes values $a_{j,r} \in \boldsymbol{\Omega}_p$.*
2. *The levels of $\widehat{D}$ at infinity are ≤ 1.*

Proof. Point 1 can be seen as a consequence of Écalle's theorem on micro-solutions (see proposition 4.19 below). It can also be directly proved by using the Newton polygons of D and $\widehat{D}$ at 0 (adapt, for instance, the proof of [13, Lemma 6.3.16]). For point 2, it is a classical result and we refer, for instance, to [17, Thm. 1.4] or [13, Prop. 4.3.22]. ■

We are now able to prove proposition 4.11 and so theorem 4.9.

Proof of proposition 4.11. ◃ Let us first consider the case $n_1 = q = 1$. According to lemma 4.12 which says us that $\widetilde{\boldsymbol{g}}^{1,1,1}(t) \in \widetilde{Sol}_0(D)$, function $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,1}$ is an actual solution on $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ of $\widehat{D}\widehat{y}(\tau) = 0$. Proposition 4.11 follows then from lemma 4.13. Indeed, point 1 and Cauchy-Lipschitz's theorem show that $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,1}$ can be analytically continued along any path of $\mathbb{C}\backslash\boldsymbol{\Omega}_p$ originating from any point of $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})\backslash\{0\}$; hence, the resurgence of $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,1}$. As for the summable-resurgence, it stems from point 2 and Ramis index theorems [20].

◃ When $n_1 \geq 2$, we proceed by induction on q. Since the case $q = 1$ has been treated above, we now suppose that, for a certain $q \in \{1, ..., n_1 - 1\}$, theorem 4.9 is valid for any $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}$ with $p \in \{1, ..., q\}$. We must then prove that theorem 4.9 is still valid for $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q+1}$. According to lemma 4.12,

$$\widetilde{\boldsymbol{g}}^{1,1,q+1}(t) + \sum_{p=1}^{q}\widetilde{\boldsymbol{g}}^{1,1,p}(t)\frac{\ln^{q-p}(t^{1/r})}{(q-p)!} \in \widetilde{Sol}_0(D).$$

Since $\widetilde{\boldsymbol{g}}^{1,1,p}(t) \in t\mathbb{C}[[t]]$ for all $p = 1, ..., q$ (*cf.* remark 4.5), the terms $\widetilde{\boldsymbol{g}}^{1,1,p}(t)\ln^{q-p}(t^{1/r})$ can by written on the form

$$\widetilde{\boldsymbol{g}}^{1,1,p}(t)\ln^{q-p}(t^{1/r}) = \frac{\widetilde{\boldsymbol{g}}^{1,1,p}(t)}{t} \times t\ln^{q-p}(t^{1/r}) \quad \text{with } \frac{\widetilde{\boldsymbol{g}}^{1,1,p}(t)}{t} \in \mathbb{C}[[t]].$$

Hence, applying lemma 4.1, the function

$$\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q+1} + \sum_{p=1}^{q} \frac{1}{(q-p)!} \frac{d\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}}{d\tau} * \widehat{t \ln^{q-p}(t^{1/r})}$$

is an actual solution of $\widehat{D}$ and the same arguments as in the case $q = 1$ show it is summable-resurgent. Note that all the convolution products in the sum above make sense since $\frac{d\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}}{d\tau}$ and $\widehat{t \ln^{q-p}(t^{1/r})}$ are both integrable at 0. Indeed, $\frac{d\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}}{d\tau}$ admits an asymptotic expansion at 0 in $\mathbb{C}[[\tau]]$ and $\widehat{t \ln^{q-p}(t^{1/r})}(\tau) \in \mathbb{C}[\ln \tau]$ [7]. We are left to prove that $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,q+1}$ is summable-resurgent. To do that, it suffices to remark that, for all $p = 1, ..., q$,

1. our hypothesis and lemma 4.8 imply that all the functions $\frac{d\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{1,1,p}}{d\tau}$ are summable-resurgent;

2. the Borel transforms $\widehat{t \ln^{q-p}(t^{1/r})}$ belong to $\mathcal{O}^{\leq 1}(\widetilde{\mathbb{C}})$ (*cf.* lemma 4.1) and are integrable at the origin.

Hence, proposition 4.11 by applying once again lemma 4.8. ■

This ends the proof of theorem 4.9.

4.4 Description of singularities

Summable-resurgence theorem 4.9 above asserts that the only possible singular points of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ are the highest level's Stokes values $a_{j,r} \in \boldsymbol{\Omega}_p$ of $\widetilde{f}(x)$. The aim of this section is to give a *complete* description of singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ at the various nonzero points $a_{j,r} \neq 0$. Before starting the calculations, let us first recall some definitions and classical properties of singularities. For more precise details, we refer to [8, 16, 26].

4.4.1 The space $\mathcal{C}$ and the extended Borel transformation

◁ **The space $\mathcal{C}$.** Let $\mathcal{C}$ denote the space of singularities at the origin $0 \in \widetilde{\mathbb{C}}$. It is defined as the quotient $\mathcal{C} := \widetilde{\mathcal{O}}/\mathcal{O}$. Recall that $\mathcal{C}$ is also denoted by $\mathtt{SING}_0$ by J. Écalle and al. (*cf.* [26] for instance). Recall also that the elements of $\mathcal{C}$ are called *micro-functions* by B. Malgrange [17] by analogy with hyper- and micro-functions defined by Sato, Kawai and Kashiwara in higher dimensions.

[7] More generally, $\widehat{t^\lambda \ln^m(t)}(\tau) \in \tau^{\lambda-1}\mathbb{C}[\ln \tau]$ for all $\lambda \in \mathbb{C}$ and $m \geq 1$. For exact formulæ, we refer, for instance, to [16].

The elements of $\mathcal{C}$ are usually denoted with a nabla, like $\overset{\triangledown}{\varphi}$, for a singularity of the function φ. A representative of $\overset{\triangledown}{\varphi}$ in $\widetilde{\mathcal{O}}$ is often denoted by $\check{\varphi}$ and is called *a major* of φ.

It is worth to consider the two natural maps

$$\begin{array}{ll} \mathrm{can} : \widetilde{\mathcal{O}} \longrightarrow \mathcal{C} = \widetilde{\mathcal{O}}/\mathcal{O} & \text{the canonical map and} \\ \mathrm{var} : \mathcal{C} \longrightarrow \widetilde{\mathcal{O}} & \text{the variation map,} \end{array}$$

action of a positive turn around 0 defined by $\mathrm{var}\,\overset{\triangledown}{\varphi} = \check{\varphi}(\tau) - \check{\varphi}(\tau e^{-2i\pi})$, where $\check{\varphi}(\tau e^{-2i\pi})$ is the analytic continuation of $\check{\varphi}(\tau)$ along a path turning once clockwise around 0 and close enough to 0 for $\check{\varphi}$ to be defined all along (the result is independant of the choice of the major $\check{\varphi}$). The germ $\mathrm{var}\,\overset{\triangledown}{\varphi}$ is often denoted by $\widehat{\varphi}$ [8] and is called *the minor* of $\overset{\triangledown}{\varphi}$.

One can not multiply two elements of $\mathcal{C}$, but an element of $\mathcal{C}$ and an element of $\mathcal{O}$: $\alpha\overset{\triangledown}{\varphi} := \mathrm{can}(\alpha\check{\varphi}) = \overset{\triangledown}{\alpha\varphi}$ for all $\alpha \in \mathcal{O}$ and $\overset{\triangledown}{\varphi} \in \mathcal{C}$.

On the other hand, one can defined a convolution product $\circledast$ on $\mathcal{C}$ by setting $\overset{\triangledown}{\varphi}_1 \circledast \overset{\triangledown}{\varphi}_2 := \mathrm{can}(\check{\varphi}_1 *_u \check{\varphi}_2)$, where $\check{\varphi}_1 *_u \check{\varphi}_2$ is the truncated convolution product

$$(\check{\varphi}_1 *_u \check{\varphi}_2)(\tau) := \int_u^{\tau - u} \check{\varphi}_1(\tau - \eta)\check{\varphi}_2(\eta)d\eta \in \widetilde{\mathcal{O}}$$

with u arbitrarily close to 0 satisfying $\tau \in]0, u[$ and $\arg(\tau - u) = \arg(\tau) - \pi$. Note that $\overset{\triangledown}{\varphi}_1 \circledast \overset{\triangledown}{\varphi}_2$ makes sense since it does not depend on u, nor on the choice of the majors $\check{\varphi}_1$ and $\check{\varphi}_2$. The convolution product $\circledast$ is commutative and associative on $\mathcal{C}$ with unit $\delta := \mathrm{can}\left(\frac{1}{2i\pi\tau}\right)$.

Let $\delta^{(m)}$ denote the m^{th} derivative of δ. One has $\delta^{(m)} := \mathrm{can}\left(\frac{(-1)^m m!}{2i\pi\tau^{m+1}}\right)$ and the m^{th} derivative of a singularity $\overset{\triangledown}{\varphi}$ coincides with the convolution by $\delta^{(m)}$:

$$\frac{d^m}{d\tau^m}\overset{\triangledown}{\varphi} = \delta^{(m)} \circledast \overset{\triangledown}{\varphi}.$$

Note that $\frac{d}{d\tau}$ is not a $\circledast$-derivation; its action on $\circledast$ is actually given by

$$\frac{d^m}{d\tau^m}\left(\overset{\triangledown}{\varphi}_1 \circledast \overset{\triangledown}{\varphi}_2\right) = \left(\frac{d^m}{d\tau^m}\overset{\triangledown}{\varphi}_1\right) \circledast \overset{\triangledown}{\varphi}_2 = \overset{\triangledown}{\varphi}_1 \circledast \left(\frac{d^m}{d\tau^m}\overset{\triangledown}{\varphi}_2\right).$$

On the other hand, the multiplication by τ is an $\circledast$-derivation:

$$\tau\left(\overset{\triangledown}{\varphi}_1 \circledast \overset{\triangledown}{\varphi}_2\right) = \left(\tau\overset{\triangledown}{\varphi}_1\right) \circledast \overset{\triangledown}{\varphi}_2 + \overset{\triangledown}{\varphi}_1 \circledast \left(\tau\overset{\triangledown}{\varphi}_2\right).$$

[8] The fact that we use here the same notation $\widehat{\varphi}$ as the Borel transform of an element $\varphi \in \widetilde{\mathcal{O}}^{\leq \exp}$ will be justify below with the definition of the *extended* Borel transformation.

◁ **Micro-solutions.** Let Δ be a scalar linear differential operator with coefficients in $\mathcal{O}$. Recall that the solutions of $\Delta y(\tau) = 0$ in $\widetilde{\mathcal{O}}$ are always *of finite determination*, *i.e.*, they read as

$$\sum_{finite} \varphi_{\alpha,p}(\tau)\tau^{\alpha}(\ln\tau)^{p} \tag{4.5}$$

where $\alpha \in \mathbb{C}$, $p \in \mathbb{N}$ and $\varphi_{\alpha,p}(\tau)$ holomorphic on a punctured disc at 0; some of them may be *of Nilsson class*, *i.e.*, may be write on the form (4.5) with all the $\varphi_{\alpha,p}(\tau)$ in $\mathcal{O}$.

A *micro-solution of* Δ *at* 0 is any singularity $\overset{\triangledown}{\varphi} \in \mathcal{C}$ satisfying $\Delta\overset{\triangledown}{\varphi} = 0$ in $\mathcal{C}$, *i.e.*, any $\overset{\triangledown}{\varphi} \in \mathcal{C}$ whose some, hence all, majors $\breve{\varphi} \in \widetilde{\mathcal{O}}$ satisfy $\Delta\breve{\varphi} \in \mathcal{O}$.

Since any solution $\varphi \in \widetilde{\mathcal{O}}$ of $\Delta y(\tau) = 0$ defines a micro-solution $\overset{\triangledown}{\varphi} := \mathrm{can}(\varphi)$, it is natural to define the *singularities of finite determination* (resp. *of Nilsson class*), *i.e.*, the singularities for which some, hence all, majors are of finite determination (resp. of Nilsson class). In fact, as we shall see below, the majors of this type which occur in our study have besides a *summable-resurgence property*. This leads us then to consider the following two subspaces of $\mathcal{C}$:

Definition 4.14 (Summable-resurgent singularity)

- We call *summable-resurgent singularity of finite determination with singular support* $\boldsymbol{\Omega}_p, \widetilde{0}$ any singularity for which some, hence all, majors read on the form (4.5) with all the $\varphi_{\alpha,p}(\tau)$'s summable-resurgent with singular support $\boldsymbol{\Omega}_p, \widetilde{0}$.

- We call *summable-resurgent singularity of Nilsson class with singular support* $\boldsymbol{\Omega}_p, 0$ any singularity for which some, hence all, majors read on the form (4.5) with all the $\varphi_{\alpha,p}(\tau)$'s summable-resurgent with singular support $\boldsymbol{\Omega}_p, 0$.

Let $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p,\widetilde{0}}$ and $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p,0}$ denote the spaces of summable-resurgent singularities of finite determination with singular support $\boldsymbol{\Omega}_p, \widetilde{0}$ and of Nilsson class with singular support $\boldsymbol{\Omega}_p, 0$. Observe that these two spaces are stable under derivation $\frac{d}{d\tau}$ and under multiplication by an element of $\mathcal{O}^{\le 1}(\mathbb{C})$.

◁ **Extended Borel transformation.** Recall that the Borel transformation $\mathcal{B}_{1;\theta}$ defines, for any direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, an operator from the space $\widetilde{\mathcal{O}}^{\le \exp}$ of functions with subexponential growth at the origin to the space $\mathcal{O}^{\le 1}(\widetilde{\mathbb{C}})$ of holomorphic functions on all $\widetilde{\mathbb{C}}$ with exponential growth of order ≤ 1 on any bounded sector at infinity (see section 4.1). Let us now denote by $\mathcal{C}^{\leq 1}$ the subspace of singularities $\overset{\nabla}{\varphi} \in \mathcal{C}$ for which $\operatorname{var} \overset{\nabla}{\varphi} \in \mathcal{O}^{\leq 1}(\widetilde{\mathbb{C}})$. Recall that, for such a singularity, there always exists a major $\breve{\varphi} \in \mathcal{O}^{\leq 1}(\widetilde{\mathbb{C}})$ (see [8]); thereby, $\mathcal{C}^{\leq 1}$ is stable under the convolution product $\circledast$. One has the following classical result:

Proposition 4.15 (Écalle, [8, pp. 46-48]) *Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be a direction. The Borel transformation $\mathcal{B}_{1;\theta}$ can be extended into an isomorphism*

$$\mathcal{B}^{ext}_{1;\theta} : \left(\widetilde{\mathcal{O}}^{\leq \exp}, +, \cdot, t^2 \frac{d}{dt}\right) \longrightarrow \left(\mathcal{C}^{\leq 1}, +, \circledast, \tau \cdot\right)$$

of $\mathbb{C}$-differential algebras so that $\operatorname{var}(\mathcal{B}^{ext}_{1;\theta}\varphi) = \mathcal{B}_{1;\theta}(\varphi)$ for all $\varphi \in \widetilde{\mathcal{O}}^{\leq \exp}$. Its inverse is the extended Laplace transformation $\mathcal{L}^{ext}_{1;\theta}$ defines as follows: given $\overset{\nabla}{\varphi} \in \mathcal{C}^{\leq 1}$, $\breve{\varphi}$ a major of $\overset{\nabla}{\varphi}$ and $\widehat{\varphi} = \operatorname{var} \overset{\nabla}{\varphi}$ its variation,

$$\mathcal{L}^{ext}_{1;\theta}(\overset{\nabla}{\varphi})(t) := \int_{\gamma_{\theta,\varepsilon}} \breve{\varphi}(\tau) e^{-\tau/t} d\tau + \int_{\varepsilon e^{i\theta}}^{\infty e^{i\theta}} \widehat{\varphi}(\tau) e^{-\tau/t} d\tau$$

where $\gamma_{\theta,\varepsilon}$ denotes a circle centered at the origin and going from $\varepsilon e^{i(\theta-2\pi)}$ to $\varepsilon e^{i\theta}$, $\varepsilon > 0$ small enough.

Note that $\mathcal{L}^{ext}_{1;\theta}(\overset{\nabla}{\varphi})$ makes sense since it does not depend on the choice of ε nor on the chosen major $\breve{\varphi}$; in particular, for a choice $\breve{\varphi} \in \mathcal{O}^{\leq 1}(\widetilde{\mathbb{C}})$, one has

$$\mathcal{L}^{ext}_{1;\theta}(\overset{\nabla}{\varphi})(t) := \int_{\gamma_\theta} \breve{\varphi}(\tau) e^{-\tau/t} d\tau$$

where γ_θ denotes a Hankel path directed by direction θ and oriented positively. Note also that, if $\widehat{\varphi}$ is integrable at 0, then $\mathcal{L}^{ext}_{1;\theta}(\overset{\nabla}{\varphi})$ and $\mathcal{L}_{1;\theta}(\widehat{\varphi})$ coincide.

As $\mathcal{B}_{1;\theta}$, we omit to write θ and we simply denote $\overset{\nabla}{\varphi} := \operatorname{can}(\breve{\varphi})$ with $\breve{\varphi} \in \mathcal{O}^{\leq 1}(\widetilde{\mathbb{C}})$ for $\mathcal{B}^{ext}_{1;\theta}(\varphi)$; thus, using notation of lemma 4.1, $\operatorname{var}(\overset{\nabla}{\varphi}) = \widehat{\varphi}$ [9]. The following relations are essentially known: given $\lambda \in \mathbb{C}\backslash\mathbb{Z}$ and $m \in \mathbb{N}^*$,

$$\overset{\nabla}{t^\lambda} = \operatorname{can}\left(\frac{\tau^{\lambda-1}}{(1-e^{-2i\pi\lambda})\Gamma(\lambda)}\right), \quad \overset{\nabla}{t^\lambda \ln t} = \operatorname{can}\left(\frac{d}{d\lambda}\left(\frac{\tau^{\lambda-1}}{(1-e^{-2i\pi\lambda})\Gamma(\lambda)}\right)\right)$$

$$\overset{\nabla}{t^m} = \operatorname{can}\left(\frac{\tau^{m-1}\ln\tau}{(m-1)!}\right), \quad \overset{\nabla}{1} = \operatorname{can}\left(\frac{1}{2i\pi\tau}\right), \quad \overset{\nabla}{t^{-m}} = \operatorname{can}\left(\frac{(-1)^m m!}{2i\pi\tau^{m+1}}\right)$$

[9] This relation justifies the notation of the variation of any singularity of $\mathcal{C}$; see note 8.

More generally, let $\mathbb{C}[t^\lambda,(\ln t)^p]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$ denote the subspace of $\varphi\in\widetilde{\mathcal{O}}^{\leq\exp}$ of the form

$$\sum_{finite}\alpha_{\lambda,p}t^\lambda(\ln t)^p\quad\text{with }\alpha_{\lambda,p}\in\mathbb{C},\ \lambda\in\mathbb{C},\ p\in\mathbb{N}.$$

Let us also denote by $\overset{\triangledown}{\mathbb{C}}[t^\lambda,(\ln t)^p]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$ its image by $\mathcal{B}^{ext}_{1;\theta}$. Then, for any $\overset{\triangledown}{\varphi}\in\overset{\triangledown}{\mathbb{C}}[t^\lambda,(\ln t)^p]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$, there exits a major $\breve{\varphi}\in\mathbb{C}[\tau^\mu,(\ln\tau)^q]_{\mu\in\mathbb{C},q\in\mathbb{N}}$.

Moreover, the spaces $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\mathbf{\Omega}_p,\widetilde{0}}$ and $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\mathbf{\Omega}_p,0}$ are stable under the $\circledast$-convolution by an element of $\overset{\triangledown}{\mathbb{C}}[t^\lambda,(\ln t)^p]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$.

In the same way as $\mathcal{B}_{1;\theta}$, the formal Borel transformation $\mathcal{B}_1$ can be extended to formal expansions of the form

$$\widetilde{h}(t)=\sum_{m\geq 0}h_m(t)\quad\text{with }h_m(t)\in\widetilde{\mathcal{O}}^{\leq\exp}$$

by applying separately $\mathcal{B}^{ext}_{1;\theta}$ for any θ on each term $h_m(t)$. As previously, we denote by $\overset{\triangledown}{h}$ the extended formal Borel transform $\mathcal{B}^{ext}_1(\widetilde{h})$. Note that, when $\widetilde{h}(t)\in\mathbb{C}[[t]]$, the variation $\operatorname{var}\overset{\triangledown}{h}$ coincides with the formal Borel transformation $\widehat{h}=\mathcal{B}_1(\widetilde{h})$. In particular, when $\widetilde{h}(t)\in\mathbb{C}[[t]]_1$ is a 1-Gevrey formal series, one has

$$\widehat{h}(\tau)=\mathcal{B}_1(\widetilde{h})\in\mathcal{O}\quad\text{and}\quad\overset{\triangledown}{h}=\mathcal{B}^{ext}_1(\widetilde{h})=\operatorname{can}\left(\frac{\widehat{h}(\tau)\ln\tau}{2i\pi}\right).$$

More generally, one has the following classical result which will be us useful later:

Proposition 4.16 ([14, 16, 26]) *With notations as above:*

1. *Let $\lambda\in\mathbb{C}$, $p\in\mathbb{N}$ and $\widetilde{h}(t)\in\mathbb{C}[[t]]$. Suppose that the formal Borel transform $\widehat{h}(\tau)$ of $\widetilde{h}(t)$ is summable-resurgent with singular support $\mathbf{\Omega}_p,0$: $\widehat{h}(\tau)\in\mathcal{R}es^{sum}_{\mathbf{\Omega}_p,0}$. Then,*

$$\boxed{\widetilde{h}(t)\overset{\triangledown}{t^\lambda(\ln t)^p}\in\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\mathbf{\Omega}_p,0}.}$$

2. Reciprocally, let $\overset{\triangledown}{h} = \mathrm{can}(h_{\lambda,p}(\tau)\tau^{\lambda}(\ln\tau)^{p}) \in \overset{\triangledown}{\mathcal{N}}il^{s\text{-}res}_{\mathbf{\Omega}_p,0}$ *with*

$$h_{\lambda,p}(\tau) = \sum_{m\geq 0} h_{\lambda,p;m}\tau^{m} \in \mathcal{O}$$

in a neighborhood of $0 \in \mathbb{C}$*. Then, for any direction* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ *such that* $\mathbf{\Omega}_p \cap]0, \infty e^{i\theta}[= \emptyset$*,*

$$\boxed{\mathcal{L}^{ext}_{1;\theta}(\overset{\triangledown}{h}) = \sum_{k=0}^{p}\binom{p}{k} s_{1;\theta}(\widetilde{h}_{\lambda,p-k})(t)t^{\lambda+1}(\ln t)^{k}}$$

where, for all $\ell = 0, ..., p$*,*

$$\widetilde{h}_{\lambda,\ell}(t) = 2i\pi\sum_{m\geq 0}\frac{d^{\ell}}{dz^{\ell}}\left(\frac{e^{-i\pi z}}{\Gamma(1-z)}\right)_{|z=m+1+\lambda} h_{\lambda,p;m}t^{m}$$

and $\widehat{h}_{\lambda,\ell}(\tau) \in \mathcal{R}es^{sum}_{\mathbf{\Omega}_p,0}$*. In particular,* $\widetilde{h}_{\lambda,\ell}(t) \in \mathbb{C}[[t]]_1$*.*

$\triangleleft$ **Singularity at ω.** For any $\omega \in \mathbb{C}^*$, we denote by $\mathcal{C}_{|\omega}$ the space of singularities at ω, *i.e.*, the space $\mathcal{C}$ translated from 0 to ω. A function $\breve{\varphi}$ is a major of a singularity at ω if $\breve{\varphi}(\omega+\tau)$ is a major of a singularity at 0. In the same way, we define the spaces $\overset{\triangledown}{\mathcal{D}}et^{s\text{-}res}_{\mathbf{\Omega}_p,\widetilde{0}}{}_{|\omega}$ and $\overset{\triangledown}{\mathcal{N}}il^{s\text{-}res}_{\mathbf{\Omega}_p,0}{}_{|\omega}$ as the translated of the spaces $\overset{\triangledown}{\mathcal{D}}et^{s\text{-}res}_{\mathbf{\Omega}_p,\widetilde{0}}$ and $\overset{\triangledown}{\mathcal{N}}il^{s\text{-}res}_{\mathbf{\Omega}_p,0}$ to ω.

Let us now turn to the description of singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$.

4.4.2 Structure of singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$

Theorem 4.9 tells us that the only singular points of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ are the highest level's Stokes values $a_{j,r} \in \mathbf{\Omega}_p$ of $\widetilde{f}(x)$. The behavior of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ at any of these points ω depends, of course, on the "homotopic class" of the path γ of analytic continuation followed from any point $a \neq 0$ of $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ [10] to a neighborhood of ω. In particular, "homotopic class" implies that the behavior of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ does not depend on the choice of a. We denote below by

- $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$ the analytic continuation of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ along the path γ,
- $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma} := \mathrm{can}(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma})$ the singularity of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ at ω defined by $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$.

[10] See page 23 for the exact definition of $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$.

To investigate the singularities $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$, our approach is similar to the one developed in [24] for the study of singularities of systems with a unique level and is based on the same arguments as those detailed in section 4.3.2 for the proof of summable-resurgence of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$. Precisely,

1. we first study the singularities $\overset{\triangledown}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$ of $\widehat{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta}}$ (see page 22 for notation) at ω defined by the analytic continuation $\widehat{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$ of $\widehat{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta}}$ along γ,

2. next, we "extend" the structure of $\overset{\triangledown}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$ to $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$ by means of relation

$$\widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}(\tau) = \sum_{m=1}^{N} \alpha_m \frac{d^m \widehat{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}}{d\tau^m} + \widehat{\boldsymbol{M}}' * \widehat{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$$

derived from (4.2).

Before starting the calculations, let us introduce the key notion of *front of a singularity.*

$\lhd$ **Front of a singularity.** Let $\omega \in \boldsymbol{\Omega}_p\backslash\{0\}$. Following [24], we call *front of* ω the set

$$Fr(\omega) := \left\{ q_\ell\left(\frac{1}{x}\right) \ ; \ a_{\ell,r} = \omega \right\}.$$

of polynomials $q_\ell(1/x)$ of $Q(1/x)$ with leading term $-\omega/x^r$. Note that, contrarily to the case of systems with a unique level considered in reference above, the front $Fr(\omega)$ may be here not a singleton. This brings us to the following definition:

Definition 4.17 (Singularity with good/bad front)

- A singular point $\omega \in \boldsymbol{\Omega}_p\backslash\{0\}$ is said to be *with good front* when $Fr(\omega)$ is a singleton. The corresponding singularity $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$ is then called *singularity with good front.*

- When $Fr(\omega)$ is not a singleton, ω (hence, its corresponding singularity $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$) is said to be *with bad front.*

Remark 4.18 The denomination *good/bad front* is due to the following fact: when the front $Fr(\omega)$ is (resp. is not) a singleton, the column-blocks $\widetilde{F}^{\bullet;\ell}(x)$ of $\widetilde{F}(x)$, with $\ell = 1, ..., J$ such that $q_\ell(1/x) \in Fr(\omega)$, have all the *unique* level

r (resp. at least *two* levels $r' < r$). Thereby, when $Fr(\omega)$ is a singleton, the r-reduced series $\widetilde{\boldsymbol{F}}^{[u]\bullet;\ell}(t)$ of $\widetilde{F}^{\bullet;\ell}(x)$ are always 1-Gevrey (more precisely, 1-summable) formal series and, consequently, according to proposition 4.16 and Écalle's theorem below (see proposition 4.19), they always yield singularities of Nilsson class at ω. On the other hand, when $Fr(\omega)$ is not a singleton, the $\widetilde{\boldsymbol{F}}^{[u]\bullet;\ell}(t)$'s yield in general more complicated singularities at ω, no longer of Nilsson class, but of finite determination.

Let us now consider a singular point ω with good front. Then,

$$Fr(\omega) = \left\{ -\frac{\omega}{x^r} + \dot{q}_\omega\left(\frac{1}{x}\right) \right\}$$

where $\dot{q}_\omega(1/x) \in x^{-1}\mathbb{C}[x^{-1}]$ is a polynomial in $1/x$ with degree $< r$ and without constant term. By analogy with [24], ω (hence, its corresponding singularity) is said to be with *good monomial front* (resp. *good nonmonomial front*) when $\dot{q}_\omega \equiv 0$ (resp. $\dot{q}_\omega \not\equiv 0$).

$\triangleleft$ **Structure of singularities $\overset{\triangledown}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$.** The study of singularities $\overset{\triangledown}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$ is essentially based on Écalle's theorem, as stated and proved by B. Malgrange in [17, Thm. 2.2], which asserts that the space $\widetilde{Sol}_0(D)$ of formal solutions of D at 0 and the space $\overset{\triangledown}{\mathcal{M}}(\widehat{D})$ of micro-solutions of $\widehat{D}$ are isomorphic[11]. In our case, this theorem reads as follows:

Proposition 4.19 (Écalle) *With notations as lemma 4.3.*

1. *Let $v \in \{1,...,r\}$, $\ell \in \{1,...,J\}$ and $q \in \{1,...,n_\ell\}$. Then, the extended formal Borel transformation $\overset{\triangledown}{\boldsymbol{z}}{}^{v,\ell,q} := \mathcal{B}_1^{ext}(\widetilde{\boldsymbol{z}}^{v,\ell,q})$ of $\widetilde{\boldsymbol{z}}^{v,\ell,q}$ is a micro-solution of $\widehat{D}$ at $a_{\ell,r}$.*

2. *Denote by $\overset{\triangledown}{\mathcal{M}}_\omega(\widehat{D})$ the space of micro-solutions of $\widehat{D}$ at $\omega \in \boldsymbol{\Omega}_p$. Then,*

$$\boxed{\overset{\triangledown}{\mathcal{M}}_\omega(\widehat{D}) = \mathrm{vect}(\overset{\triangledown}{\boldsymbol{z}}{}^{v,\ell,q}\ ;\ v=1,...,r,\ q=1,...,n_\ell)_{\ell;a_{\ell,r}=\omega}.}$$

Following lemma 4.20 precises the structure of singularities $\overset{\triangledown}{\boldsymbol{z}}{}^{v,\ell,q}$.

Lemma 4.20 *Let $v \in \{1,...,r\}$, $\ell \in \{1,...,J\}$ and $q \in \{1,...,n_\ell\}$.*

[11] In [17], B. Malgrange states actually this theorem not in terms of Borel transformation, but in terms of Fourier (= Laplace) transformation.

1. *Suppose that $a_{\ell,r}$ has a* good *front. Then,*

$$\boxed{\overset{\nabla}{\boldsymbol{z}}{}^{v,\ell,q} \in \overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p - a_{\ell,r},0} \circledast \overset{\nabla}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}{}_{|a_{\ell,r}}.}$$

2. *Suppose that $a_{\ell,r}$ has a* bad *front. Then,*

$$\boxed{\overset{\nabla}{\boldsymbol{z}}{}^{v,\ell,q} \in \overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p - a_{\ell,r},\widetilde{0}} \circledast \overset{\nabla}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}{}_{|a_{\ell,r}}.}$$

Notation $\overset{\nabla}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}$ *denotes the singularity* $\mathcal{B}_1^{ext}(e^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))})$*, where* $\dot{q}_{a_{\ell,r}}(1/x)$ *is the polynomial in* $1/x$ *and degree* $< r$ *defined by*

$$\dot{q}_{a_{\ell,r}}\left(\frac{1}{x}\right) := q_\ell\left(\frac{1}{x}\right) - \frac{a_{\ell,r}}{x^r} = -\frac{a_{\ell,r-1}}{x^{r-1}} - \dots - \frac{a_{\ell,1}}{x}.$$

Proof. Following lemma 4.3, $\widetilde{\boldsymbol{z}}^{v,\ell,q}$ reads as

$$\widetilde{\boldsymbol{z}}^{v,\ell,q}(t) = \widetilde{\boldsymbol{\varphi}}^{v,\ell,q}(t)e^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}e^{-a_{\ell,r}/t}$$

with

$$\widetilde{\boldsymbol{\varphi}}^{v,\ell,q}(t) := \sum_{u=1}^{r}\sum_{p=1}^{q}\rho^{(v-1)(\lambda_\ell-u+1)}\widetilde{\boldsymbol{g}}^{u,\ell,p}(t)t^{\frac{\lambda_\ell-u+1}{r}}\frac{\ln^{q-p}(\rho^{v-1}t^{\frac{1}{r}})}{(q-p)!}.$$

Recall that $\widehat{\boldsymbol{g}}_{\boldsymbol{\theta}}^{v,\ell,q}$ and $\widehat{\boldsymbol{F}}_{\boldsymbol{\theta}}^{\bullet;v,\ell}$ have the same summable-resurgence properties. In particular, they are analytic at 0 as soon as $a_{\ell,r}$ has a good front (see remark 4.18). This brings us to the following discussion.

◁ *First case : $a_{\ell,r}$ has a good front.* In this case, all the $\widetilde{\boldsymbol{g}}^{v,\ell,q}(t)$'s are 1-Gevrey and, consequently, propositions 4.10 and 4.16 imply that

$$\overset{\nabla}{\boldsymbol{\varphi}}{}^{v,\ell,q} := \mathcal{B}_1^{ext}(\widetilde{\boldsymbol{\varphi}}^{v,\ell,q}) \in \overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p - a_{\ell,r},0}.$$

◁ *Second case : $a_{\ell,r}$ has a bad front.* In this case, let us begin by observing that, since $\widetilde{\boldsymbol{\varphi}}^{v,\ell,q}$ is also a formal solution at 0 of a convenient scalar linear differential equation, we necessarily have $\overset{\nabla}{\boldsymbol{\varphi}}{}^{v,\ell,q}$ of finite determination (see page 31). We are then left to prove that $\overset{\nabla}{\boldsymbol{\varphi}}{}^{v,\ell,q} \in \overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p - a_{\ell,r},\widetilde{0}}$. To do that, it suffices to remark that $\overset{\nabla}{\boldsymbol{\varphi}}{}^{v,\ell,q}$ can be written as the sum

$$\sum_{u=1}^{r}\sum_{p=1}^{q}\frac{\rho^{(v-1)(\lambda_\ell-u+1)}}{(q-p)!}\overset{\nabla}{\boldsymbol{g}}{}_{\boldsymbol{\theta}}^{u,\ell,p} \circledast \overset{\nabla}{h} \quad ; \ h(t) = t^{\frac{\lambda_\ell-u+1}{r}}\ln^{q-p}(\rho^{v-1}t^{\frac{1}{r}})$$

where $\overset{\nabla}{h} \in \overset{\nabla}{\mathbb{C}}[t^\lambda, (\ln t)^p]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$ and where $\overset{\nabla}{\boldsymbol{g}}{}_{\boldsymbol{\theta}}^{u,\ell,p} \in \overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-a_{\ell,r},\widetilde{0}}$ is the singularity at 0 defined by $\widehat{\boldsymbol{g}}{}_{\boldsymbol{\theta}}^{u,\ell,q}$; then, we conclude by properties of stability.

◁ This ends the proof of lemma 4.20 since $\mathcal{B}_1^{ext}(e^{-a_{\ell,r}/t})$ is the translation by $a_{\ell,r}$ and since $\mathcal{B}_1^{ext}(\widetilde{\boldsymbol{\varphi}}^{v,\ell,q}e^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}) = \overset{\nabla}{\boldsymbol{\varphi}}{}^{v,\ell,q} \circledast \overset{\nabla}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}$. ■

Proposition 4.19 and lemma 4.20 above lead us to the following result which makes explicit the stucture of micro-solutions of $\widehat{D}$.

Corollary 4.21 *Let $\omega \in \boldsymbol{\Omega}_p$ and $\overset{\nabla}{\boldsymbol{\varphi}}_\omega \in \overset{\nabla}{\mathcal{M}}_\omega(\widehat{D})$ a micro-solution of $\widehat{D}$ at ω.*

1. *Suppose that ω has a* good *front. Then,*

$$\boxed{\overset{\nabla}{\boldsymbol{\varphi}}_\omega \in \sum_{v=1}^{r} \overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0} \circledast \overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}{}_{|\omega}.}$$

2. *Suppose that ω has a* bad *front. Then,*

$$\boxed{\overset{\nabla}{\boldsymbol{\varphi}}_\omega \in \sum_{\ell;a_{\ell,r}=\omega}\sum_{v=1}^{r} \overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}} \circledast \overset{\nabla}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}{}_{|\omega}.}$$

Remark 4.22 As spaces $\overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0}$ and $\overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}}$, the spaces of corollary 4.21 above are stable under derivation $\frac{d}{d\tau}$ and under $\circledast$-convolution by an element of $\overset{\nabla}{\mathbb{C}}[t^\lambda, (\ln t)^p]_{\lambda\in\mathbb{C},p\in\mathbb{N}}$ (use the associativity of $\circledast$). They are also stable by multiplication by τ: given $\overset{\nabla}{\varphi} \in \overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0}$ (resp. $\overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}}$), we have

$$\tau(\overset{\nabla}{\varphi} \circledast \overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}) = (\tau\overset{\nabla}{\varphi}) \circledast \overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))} + \overset{\nabla}{\varphi} \circledast (\tau\overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))})$$

with

$$\tau\overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))} = \mathcal{B}_1^{ext}\left(t^2\frac{de^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}}{dt}\right) = \overset{\nabla}{P}_v \circledast \overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}$$

where

$$P_v(t) = -\frac{1}{r\rho^{v-1}}t^{1-\frac{1}{r}}\dot{q}'_\omega\left(\frac{1}{\rho^{v-1}t^{1/r}}\right) \in t\mathbb{C}[t^{-1/r}].$$

Hence,

$$\tau(\overset{\nabla}{\varphi} \circledast \overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}) = (\tau\overset{\nabla}{\varphi} + \overset{\nabla}{\varphi} \circledast \overset{\nabla}{P}_v) \circledast \overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}$$

with $\tau\overset{\nabla}{\varphi} + \overset{\nabla}{\varphi} \circledast \overset{\nabla}{P}_v \in \overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0}$(resp. $\overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}}$).

We are now able to display the structure of singularities $\overset{\nabla}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$.

Proposition 4.23 (Description of singularities $\overset{\nabla}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$) *Let $\omega \in \boldsymbol{\Omega}_p\backslash\{0\}$ and γ a path on $\mathbb{C}\backslash\boldsymbol{\Omega}_p$ starting from a point of $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ [12] and ending in a neighborhood of ω.*

1. *Suppose that ω has a* good *front. Then,*

$$\boxed{\overset{\nabla}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma} \in \sum_{v=1}^{r} \overset{\nabla}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0} \circledast \overset{\nabla}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}{}_{|\omega}.}$$

2. *Suppose that ω has a* bad *front. Then,*

$$\boxed{\overset{\nabla}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma} \in \sum_{\ell;a_{\ell,r}=\omega} \sum_{v=1}^{r} \overset{\nabla}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}} \circledast \overset{\nabla}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}{}_{|\omega}.}$$

Proof. $\triangleleft$ Suppose for the moment that proposition 4.23 holds for all the first entry $\overset{\nabla}{\boldsymbol{g}}{}^{1,1,q}_{\boldsymbol{\theta};\omega,\gamma}$ of all the n_1 columns of $\overset{\nabla}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$. Then, since the Borel transformation changes the multiplication by $\frac{1}{t}$ into the derivation $\frac{d}{d\tau}$ and the derivation $\frac{d^k}{dt^k}$ into $\frac{d^{k+1}}{d\tau^{k+1}}\left(\tau^k\frac{d^{k-1}}{d\tau^{k-1}}\right)$, corollary 4.4 and remark 4.22 show that proposition 4.23 still holds for all the other entries of $\overset{\nabla}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$.
$\triangleleft$ We are left to prove the result for $\overset{\nabla}{\boldsymbol{g}}{}^{1,1,q}_{\boldsymbol{\theta};\omega,\gamma}$. To do that, we adapt the arguments of the proof of proposition 4.11:

– Let us first suppose that $n_1 = q = 1$. As we have already seen, the function $\widehat{\boldsymbol{g}}{}^{1,1,1}_{\boldsymbol{\theta}}$ is an actual solution on $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ of $\widehat{D}\widehat{y}(\tau) = 0$. Then, the singularity $\overset{\nabla}{\boldsymbol{g}}{}^{1,1,1}_{\boldsymbol{\theta};\omega,\gamma} = \mathrm{can}(\widehat{\boldsymbol{g}}{}^{1,1,1}_{\boldsymbol{\theta};\omega,\gamma})$ defines a micro-solution of $\widehat{D}$ at ω and, consequently, the structure of $\overset{\nabla}{\boldsymbol{g}}{}^{1,1,1}_{\boldsymbol{\theta};\omega,\gamma}$ follows from corollary 4.21.

– When $n_1 \geq 2$, we proceed by induction on q and we now suppose that, for a certain $q \in \{1, ..., n_1\}$, proposition 4.23 is valid for any $\overset{\nabla}{\boldsymbol{g}}{}^{1,1,p}_{\boldsymbol{\theta};\omega,\gamma}$ with $p \in \{1, ..., q\}$. As in the case $q = 1$, we derive from the proof of proposition 4.11 that

$$\widehat{\boldsymbol{h}}_{\boldsymbol{\theta};\omega,\gamma} := \widehat{\boldsymbol{g}}{}^{1,1,q+1}_{\boldsymbol{\theta};\omega,\gamma} + \sum_{p=1}^{q} \frac{1}{(q-p)!} \frac{d\widehat{\boldsymbol{g}}{}^{1,1,p}_{\boldsymbol{\theta};\omega,\gamma}}{d\tau} * t\,\widehat{\ln^{q-p}(t^{1/r})}$$

[12] See page 23 for the exact definition of $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$.

defines a micro-solution $\overset{\triangledown}{\boldsymbol{h}}_{\boldsymbol{\theta};\omega,\gamma}$ of $\widehat{D}$ at ω which belongs either to

$$\sum_{v=1}^{r} \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0} \circledast \overset{\triangledown}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}{}_{|\omega} \quad \text{(case } \omega \text{ with good front)}$$

or to

$$\sum_{\ell; a_{\ell,r}=\omega} \sum_{v=1}^{r} \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}} \circledast \overset{\triangledown}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}{}_{|\omega} \quad \text{(case } \omega \text{ with bad front)} .$$

Since singularities $\overset{\triangledown}{\boldsymbol{g}}{}^{1,1,p}_{\boldsymbol{\theta};\omega,\gamma}$ also belongs, for all $p \in \{1, ..., q\}$, to these spaces and since $t\,\widehat{\ln^{q-p}(t^{1/r})} \in \mathbb{C}[t^\mu, (\ln t)^q]_{\mu\in\mathbb{C}, q\in\mathbb{N}}$, remark 4.22 implies that $\overset{\triangledown}{\boldsymbol{g}}{}^{1,1,q+1}_{\boldsymbol{\theta};\omega,\gamma}$ still belongs to these spaces.
◁ Hence, proposition 4.23. ■

We are now able to state the main result of section 4.4.

◁ **Structure of singularities $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$.** According to relation

$$\widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}(\tau) = \sum_{m=1}^{N} \alpha_m \frac{d^m \widehat{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}}{d\tau^m} + \widehat{\boldsymbol{M}}' * \widehat{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$$

and properties of stability of spaces $\overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0} \circledast \overset{\triangledown}{e}{}^{\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))}$ and $\overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}} \circledast \overset{\triangledown}{e}{}^{\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))}$ previously given, it is clear that proposition 4.23 is still valid when we replace $\overset{\triangledown}{\boldsymbol{G}}{}^{\bullet;1,1}_{\boldsymbol{\theta};\omega,\gamma}$ by $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$. In fact, this result can be improved by observing that some polynomials $\dot{q}_\omega(1/(\rho^{v-1}t^{1/r}))$ with $v = 1, ..., r$ (or some polynomials $\dot{q}_{a_{\ell,r}}(1/(\rho^{v-1}t^{1/r}))$ with $a_{\ell,r} = \omega$ and $v = 1, ..., r$) may be equal.

This brings us to the following main result:

Theorem 4.24 (Description of singularities $\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma}$) *Let $\omega \in \boldsymbol{\Omega}_p\backslash\{0\}$ and γ a path on $\mathbb{C}\backslash\boldsymbol{\Omega}_p$ starting from a point of $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ and ending in a neighborhood of ω.*

1. *Suppose that ω has a* good *front. Let*

$$\dot{Q}_\omega = \left\{ \dot{q}_\omega \left(\frac{1}{\rho^{v-1}t^{1/r}} \right) \ ; \ v = 1, ..., r \right\}.$$

Then,

$$\boxed{\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma} \in \sum_{q\in\dot{Q}_\omega} \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0} \circledast \overset{\triangledown}{e}{}^{q}{}_{|\omega}.}$$

In particular, if ω has besides a monomial *front, then*

$$\boxed{\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma} \in \overset{\triangledown}{\mathcal{N}il}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,0\,|\omega}.}$$

2. *Suppose that ω has a* bad *front. Let*

$$\dot{Q}_\omega = \left\{ \dot{q}_{a_{\ell,r}}\left(\frac{1}{\rho^{v-1}t^{1/r}}\right) \ ; \ v = 1, \ldots, r \text{ and } \ell \text{ such that } a_{\ell,r} = \omega \right\}.$$

Then,

$$\boxed{\overset{\triangledown}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma} \in \sum_{q\in\dot{Q}_\omega} \overset{\triangledown}{\mathcal{D}et}{}^{s\text{-}res}_{\boldsymbol{\Omega}_p-\omega,\widetilde{0}} \circledast \overset{\triangledown}{e}{}^{q}{}_{|\omega}.}$$

A more precise description of singularities with good monomial front will be given in next section 5 in the case of some special geometric configurations of singular points of $\boldsymbol{\Omega}_p\backslash\{0\}$.

4.4.3 Principal singularities of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$

As said at the beginning of section 4.4.2, the singularity of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ at $\omega \in \boldsymbol{\Omega}_p\backslash\{0\}$ depends on the path γ of analytic continuation and meanwhile, on the chosen determination of the argument around ω.

Denote by d_α the half-line $[0, \infty e^{i\alpha}[$ issuing from $0 \in \mathbb{C}$ with argument $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ and suppose that $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}) \cap d_{\arg(\omega)} \neq \emptyset$. Then, we can always make the following choices:

- τ_0 is a point of $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}) \cap d_{\arg(\omega)}$ lied in the first sheet of $\mathcal{R}_{\boldsymbol{\Omega}_p}$ or $\widetilde{\mathcal{R}}_{\Omega_p}$ [13],
- $\gamma^+_{\tau_0,\omega}$ is a path starting from τ_0, going along the straight line $[0,\omega]$ to a point τ close to ω and avoiding all singular points of $\boldsymbol{\Omega}_p\cap]0,\omega]$ to the right (see figure 4.1 below),
- we choose the principal determination of the variable τ around ω, say $\arg(\tau) \in]-2\pi, 0]$ as in section 2.2.2 (*cf.* note 3).

[13] This last condition is, of course, always fulfilled when $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ is a disc or a sector with opening $< 2\pi$.

The analytic continuation $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,+} := \widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,\gamma^+_{\tau_0,\omega}}$ is called *right analytic continuation of* $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ *at* ω. Note that it does not depend on the choice of τ_0.

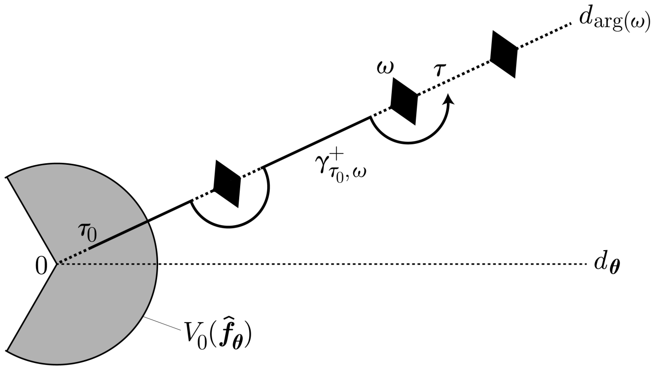

Figure 4.1 – A path $\gamma^+_{\tau_0,\omega}$ in the case of a sector $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}})$ with opening $< 2\pi$

Definition 4.25 (Principal singularity) Let $\omega \in \boldsymbol{\Omega}_p \backslash \{0\}$ and suppose that $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}) \cap d_{\arg(\omega)} \neq \emptyset$. We call *principal singularity of* $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ *at* ω the singularity $\overset{\nabla}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,+}$ defined by the right analytic continuation $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,+}$ of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}}$ at ω. A major $\breve{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,+}$ of $\overset{\nabla}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,+}$ is then called *principal major.*

As we shall see in section 5 below, the principal singularities $\overset{\nabla}{\boldsymbol{f}}_{\boldsymbol{\theta};\omega,+}$ will be play a key role in the calculation of highest level's Stokes multipliers.

5 Highest level's connection-to-Stokes formulæ

Let us now fix a collection $(\theta_k)_{k=0,\ldots,r-1} \in (\mathbb{R}/2\pi\mathbb{Z})^r$ of highest level's anti-Stokes directions of $\widetilde{f}(x)$ so that the principal determinations $\theta_k^\star \in]-2\pi, 0]$ of the θ_k's satisfy

$$-2\pi < \theta_{r-1}^\star < \ldots < \theta_1^\star < \theta_0^\star \leq 0.$$

Recall that such a collection is generating by the nonzero highest level's Stokes values $a_{j,r} \in \boldsymbol{\Omega}_p \backslash \{0\}$ of $\widetilde{f}(x)$.

As in section 3.2, we denote $\boldsymbol{\theta} := r\theta_0$ and $\boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$ the set of all the elements of $\boldsymbol{\Omega}_p \backslash \{0\}$ with argument $\boldsymbol{\theta}$. Recall that the highest level's Stokes multipliers of $\widetilde{f}(x)$ in direction θ_k are all the entries of the matrices $st^{j;\bullet}_{p;\theta_k^\star}$ with j such that $a_{j,r} \in \boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$. Recall also that these matrices are completely determined by identity (3.4) given in proposition 3.4.

In the rest of this section, we restrict our study to the following Special Geometric Configuration (denoted below by *SG-Configuration*) of $\mathbf{\Omega}_{p;\boldsymbol{\theta}}$:

- all the elements of $\mathbf{\Omega}_{p;\boldsymbol{\theta}}$ have a *good* front,
- there exists (at least) one element of $\mathbf{\Omega}_{p;\boldsymbol{\theta}}$ with a *good monomial* front.

Note that this last condition can always be fulfilled by means of a convenient change of the variable x in the initial system (A). Precisely, one has the following classical result:

Lemma 5.1 (M. Loday-Richaud, [10]) *Let* $\omega \in \mathbf{\Omega}_p \backslash \{0\}$ *with a good front and* $q_\omega(1/x)$ *the unique element of* $Fr(\omega)$*.*

1. *There exists a change of the variable* x *of the form*

$$x = \frac{y}{1 + \alpha_1 y + ... + \alpha_{r-1} y^{r-1}} \quad , \ \alpha_1, ..., \alpha_{r-1} \in \mathbb{C} \tag{5.1}$$

   such that the polar part of $q_\omega(1/x(y))$ *reads* $-\omega/y^r$*.*

2. *The Stokes-Ramis matrices of system* (A) *are preserved by the change of variable (5.1).*

Observe that, although lemma 5.1 be proved in [10] in the case of systems of dimension 2 (hence, with a single level), it can be extended to any system of dimension ≥ 3. Indeed, the change of variable (5.1) being tangent to identity, it "preserves" levels, Stokes values and summation operators.

Observe also that lemma 5.1 has already used in [23,24] to display connection-to-Stokes formulæ for systems with a unique level and for the first level of systems with several levels.

5.1 Singularities vs highest level's Stokes multipliers for the SG-Configuration's case

$\triangleleft$ The left-hand side of identity (3.4) can be seen as the Laplace integral

$$(\mathcal{L}_{1;\boldsymbol{\theta}^-} - \mathcal{L}_{1;\boldsymbol{\theta}^+})(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) = \int_{\gamma'_{\boldsymbol{\theta}}} \widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau) e^{-\tau/t} d\tau,$$

where $\gamma'_{\boldsymbol{\theta}}$ is a Hankel type path going along the straight line $d_{\boldsymbol{\theta}} := [0, \infty e^{i\boldsymbol{\theta}}[$ from infinity to 0 and back to infinity passing positively all singular points of $\mathbf{\Omega}_{p;\boldsymbol{\theta}}$ on both ways (recall that $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ is integrable at 0 in the first sheet). Without changing the value of this integral (use here the summable-resurgence of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$; see theorem 4.9), the path $\gamma'_{\boldsymbol{\theta}}$ can be deformed into a union $\gamma'_{\boldsymbol{\theta}} = \bigcup_{\omega\in\mathbf{\Omega}_{p;\boldsymbol{\theta}}} \gamma'_{\boldsymbol{\theta}}(\omega)$ of Hankel type paths $\gamma'_{\boldsymbol{\theta}}(\omega)$ with asymptotic direction $\boldsymbol{\theta}$ around each singular point $\omega \in \mathbf{\Omega}_{p;\boldsymbol{\theta}}$. Then, using the fact that direction $\boldsymbol{\theta}^-$ is actually a direction $\boldsymbol{\theta} - \eta$ (with $\eta > 0$ small enough) satisfying $V_0(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}) \cap d_{\boldsymbol{\theta}} \neq \emptyset$, we can replace $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ by one of its principal majors $\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ at each ω, obtaining so, after translation from ω to 0:

$$(5.2) \qquad (\mathcal{L}_{1;\boldsymbol{\theta}^-} - \mathcal{L}_{1;\boldsymbol{\theta}^+})(\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-})(t) = \sum_{\omega\in\mathbf{\Omega}_{p;\boldsymbol{\theta}}} e^{-\omega/t} \int_{\gamma_{\boldsymbol{\theta}}} \check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}(\omega+\tau) e^{-\tau/t} d\tau,$$

where $\gamma_{\boldsymbol{\theta}}$ is, as shown on figure 5.1, a classical Hankel path directed by direction $\boldsymbol{\theta}$ and oriented positively around 0.

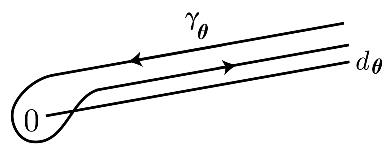

Figure 5.1 – A Hankel path $\gamma_{\boldsymbol{\theta}}$

$\triangleleft$ On the other hand, the right-hand side of identity (3.4) can be written on a similar form:

$$(5.3) \qquad s_{\underline{\boldsymbol{r}};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{F}})(t)\boldsymbol{M}_{p;\boldsymbol{\theta}^\star}(t) = \sum_{\omega\in\mathbf{\Omega}_{p;\boldsymbol{\theta}}} e^{-\omega/t}\boldsymbol{M}_{p;\boldsymbol{\theta}^\star,\omega}(t),$$

where

$$(5.4) \qquad \boldsymbol{M}_{p;\boldsymbol{\theta}^\star,\omega}(t) := \sum_{v=1}^{r} \sum_{\ell;a_{\ell,r}=\omega} s_{\underline{\boldsymbol{r}};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{F}}^{\bullet;v,\ell})(t)\dot{\boldsymbol{M}}^{v,\ell}_{p;\boldsymbol{\theta}^\star,\omega}(t)$$

with

$$\dot{\boldsymbol{M}}^{v,\ell}_{p;\boldsymbol{\theta}^\star,\omega}(t) := e^{\omega/t}\boldsymbol{M}^{v,\ell;\bullet}_{p;\boldsymbol{\theta}^\star}(t) = \frac{1}{r}\sum_{k=0}^{r-1}(\rho^k t^{\frac{1}{r}})^{\Lambda_{\ell,v-1}} st^{\ell;\bullet}_{p;\theta^\star_k}(\rho^k t^{\frac{1}{r}})^{-J_{n_1}} e^{\dot{q}_\omega(1/(\rho^k t^{1/r}))}.$$

Following key lemma 5.2 stems obvious from the SG-Configuration considered here and from remark 4.18.

Lemma 5.2 *Let* $\omega \in \boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$, $v \in \{1, ..., r\}$ *and* $\ell \in \{1, ..., J\}$ *such that* $a_{\ell,r} = \omega$. *Then,* $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ *is* 1*-summable in any direction* $\boldsymbol{\theta} + \eta$ *with* $\eta > 0$ *small enough. In particular,* $s_{\underline{r};\boldsymbol{\theta},\underline{\varepsilon}'}(\widetilde{\boldsymbol{F}}^{\bullet;v,\ell})(t) = s_{1;\boldsymbol{\theta}^+}(\widetilde{\boldsymbol{F}}^{\bullet;v,\ell})(t)$.

$\triangleleft$ We are now able to state the main two results of this section.

Proposition 5.3 *Given* $\omega \in \boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$, *the following identity*

$$\boxed{\int_{\gamma_{\boldsymbol{\theta}}} \check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}(\omega + \tau) e^{-\tau/t} d\tau = \boldsymbol{M}_{p;\boldsymbol{\theta}^\star,\omega}(t)} \tag{5.5}$$

holds for $\arg(t) \simeq \boldsymbol{\theta}^\star$.

The proof of proposition 5.3 is similar to the one of [14, Prop. 4.1] (see also [24, Sect. 4.3]) and stems from the structure of singularities with good front (*cf.* theorem 4.24), identities (5.2) and (5.3) and from lemma 5.2 above. Note in particular the importance of the 1-summability of formal series $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t)$ of identity (5.4).

Proposition 5.4 (Structure of principal singularities with good monomial front) *Let* $\omega \in \boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$ *a singular point with good monomial front. Then, the principal singularity* $\overset{\nabla}{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ *admits a major* $\check{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}$ *of the form*

$$\boxed{\check{\boldsymbol{f}}^{u,j;\bullet}_{\boldsymbol{\theta}^-;\omega,+}(\omega + \tau) = \tau^{\frac{\lambda_j - u + 1}{r} - 1} \tau^{\frac{J_{n_j}}{r}} \boldsymbol{K}^{u,j;\bullet}_{\omega^\star,+} \tau^{-\frac{J_{n_1}}{r}} + rem^{u,j;\bullet}_{\omega^\star,+}(\tau)}$$

for all $u = 1, ..., r$ *and* $j = 1, ..., J$ *with a remainder*

$$rem^{u,j;\bullet}_{\omega^\star,+}(\tau) := \sum_{\lambda_\ell; a_{\ell,r} = \omega} \sum_{v=1}^{r} \tau^{\frac{\lambda_\ell - v + 1}{r}} \boldsymbol{R}^{u,j;\bullet}_{\lambda_\ell,v;\omega^\star,+}(\ln \tau)$$

where

- $\boldsymbol{K}^{u,j;\bullet}_{\omega^\star,+}$ *denotes a constant* $n_j \times n_1$*-matrix such that* $\boldsymbol{K}^{u,j;\bullet}_{\omega^\star,+} = 0$ *as soon as* $a_{j,r} \neq \omega$,
- $\boldsymbol{R}^{u,j;\bullet}_{\lambda_\ell,v;\omega^\star,+}(X)$ *denotes a* $n_j \times n_1$*-polynomial matrix with coefficients in* $\mathcal{R}es^{sum}_{\boldsymbol{\Omega}_p - \omega,0}$ *whose the columns are of log-degree*

$$N[\ell] = \begin{cases} \big[(n_\ell - 1) \quad (n_\ell - 1) + 1 \quad \cdots \quad (n_\ell - 1) + (n_1 - 1)\big] & \text{if } \lambda_\ell \neq 0 \\[2ex] \big[n_\ell \quad n_\ell + 1 \quad \cdots \quad n_\ell + (n_1 - 1)\big] & \text{if } \lambda_\ell = 0. \end{cases}$$

Proof. It suffices to apply the extended Borel transformation to identity (5.5) and to remark (see conditions $(C1)$ and $(C2)$ page 2) that

1. $\widetilde{\boldsymbol{F}}^{\bullet;v,\ell}(t) = I_{rn}^{\bullet;v,\ell} + O(t)$ with I_{rn} the identity matrix of size rn,

2. the eigenvalues λ_ℓ of L satisfy $0 \leq \mathrm{Re}(\lambda_\ell) < 1$.

The calculations are left to the reader. ∎

Definition 5.5 (Connection constants) Let $\omega \in \boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$ a singular point with good monomial front. The nontrivial entries of matrices $\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet}$ are called *connection constants of* $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ *at* ω.

Note that, in practice, the matrix $\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet}$ can be determined as the coefficient of the monomial $\tau^{(\lambda_j-u+1)/r-1}$ in the major $\breve{\boldsymbol{f}}_{\boldsymbol{\theta}^-;\omega,+}^{u,j;\bullet}(\omega+\tau)$.

Remark 5.6 When $\boldsymbol{\Omega}_{\boldsymbol{\theta}}$ admits (at least) a singular point with *bad* front, it seems that propositions 5.3 and 5.4 are still valid. Nevertheless, we will not treat this case in this article because calculations become much more complicated due to singularities of finite determination which occur. This will be investigate in great details in a further article.

5.2 Highest level's connection-to-Stokes formulæ for the SG-Configuration's case

◁ Suppose for the moment that $\omega \in \boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$ is a singular point with *good monomial* front. Then, using the same arguments as those detailed in [14, Sect. 4.3] and [24, Sect. 4.3], we derive from propositions 5.3 and 5.4 above the following main result which displays explicit formulæ between the highest level's Stokes multipliers $st_{p;\theta_k^\star}^{j;\bullet}$ of $\widetilde{f}(x)$ in direction θ_k, $k = 0, ..., r-1$ and j such that $a_{j,r} = \omega$, and the connection constants $\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet}$ of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω.

Theorem 5.7 (Highest level's connection-to-Stokes formulæ) *Let $j \in \{1, ..., J\}$ such that $a_{j,r} = \omega$. Then, the data of the highest level's Stokes multipliers $(st_{p;\theta_k^\star}^{j;\bullet})_{k=0,...,r-1}$ of $\widetilde{f}(x)$ and the data of the connection constants $(\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet})_{u=0,...,r-1}$ of $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}(\tau)$ at ω are equivalent and are related, for all $k = 0, ..., r-1$, by the relations*

$$\boxed{st_{p;\theta_k^\star}^{j;\bullet} = \sum_{u=1}^{r} \rho^{k((u-1)I_{n_j}-L_j)} \boldsymbol{I}_{\omega^\star}^{u,j;\bullet} \rho^{kJ_{n_1}}} \tag{5.6}$$

where

$$\boxed{\boldsymbol{I}_{\omega^\star}^{u,j;\bullet} := \int_{\gamma_0} \tau^{\frac{\lambda_j-u+1}{r}-1}\tau^{\frac{J_{n_j}}{r}}\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet}\tau^{-\frac{J_{n_1}}{r}}e^{-\tau}d\tau} \tag{5.7}$$

and where γ_0 is a Hankel type path around the nonnegative real axis $\mathbb{R}^+$ with argument from -2π to 0.

Note that relation (5.6) is similar to the one obtained in [24] for systems with a unique level. In particular, an expanded form providing each entry of formula (5.6) can be found in [24, Cor. 4.6]. This can be useful for effective numerical calculations.

Here below, we recall this expanded form in the special case where the matrix L of exponents of formal monodromy is diagonal: $L = \mathrm{diag}(\lambda_1, ..., \lambda_n)$. In this case, the matrices $st_{p;\theta_k^\star}^{j;\bullet}$ and $\boldsymbol{K}_{\omega^\star,+}^{u,j;\bullet}$ are reduced to just one entry which we respectively denote $st_{p;\theta_k^\star}^{j}$ and $\boldsymbol{K}_{\omega^\star,+}^{u,j}$. Then, identity (5.7) becomes

$$\int_{\gamma_0} \tau^{\frac{\lambda_j-u+1}{r}-1}\boldsymbol{K}_{\omega^\star,+}^{u,j}e^{-\tau}d\tau = 2i\pi\frac{e^{-i\pi\frac{\lambda_j-u+1}{r}}}{\Gamma\left(1-\frac{\lambda_j-u+1}{r}\right)}\boldsymbol{K}_{\omega^\star,+}^{u,j}$$

and the highest level's connection-to-Stokes formulæ (5.6) becomes

$$\boxed{st_{p;\theta_k^\star}^{j} = 2i\pi\sum_{u=1}^{r}\rho^{k(u-1-\lambda_j)}\frac{e^{-i\pi\frac{\lambda_j-u+1}{r}}}{\Gamma\left(1-\frac{\lambda_j-u+1}{r}\right)}\boldsymbol{K}_{\omega^\star,+}^{u,j}.} \tag{5.8}$$

$\lhd$ When $\omega \in \boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$ is a singular point with *good nonmonomial* front, a result of the same type exists too, but requires to first reduce ω into a singular point with monomial front. To do that, we apply lemma 5.1 and we construct a *new* system, denoted below by (A') and satisfying the following properties:

- (A') has same levels and satisfies same normalizations as system (A),
- (A') has the same (highest level's) Stokes values (hence, (highest level's) anti-Stokes directions) as system (A),
- $\boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$ has still the SG-Configuration,
- ω still belongs to $\boldsymbol{\Omega}_{p;\boldsymbol{\theta}}$, but has now a good *monomial* front,
- (A') has the same Stokes-Ramis matrices (hence, the same highest level's Stokes-Ramis matrices) as system (A).

Then, applying theorem 5.7 to system (A'), we can again express the highest level's Stokes multipliers of $\widetilde{f}(x)$ associated with ω in terms of connection constants. Note however that these constants are calculated from system (A') and not from system (A).

5.3 Effective calculation of the highest level's Stokes multipliers for the SG-Configuration's case

When $\mathbf{\Omega}_{p;\boldsymbol{\theta}}$ has the SG-Configuration, theorem 5.7 tells us that the effective calculation of the highest level's Stokes multipliers associated with $\omega \in \mathbf{\Omega}_{p;\boldsymbol{\theta}}$ is reduced, after applying lemma 5.1 if needed, to the effective calculation of the connection constants at ω.

In section 6 below, we treat in detail some typical examples to both illustrate the structure of singularities and the highest level's connection-to-Stokes formulæ .

Recall that, according to initial normalizations $(C1) - (C3)$ (see page 2), the matrix $\widetilde{\boldsymbol{f}}(t)$ is uniquely determined by the first n_1 columns

$$(\boldsymbol{A_H}) \qquad rt^2\frac{d\boldsymbol{f}}{dt} = \boldsymbol{A}(t)\boldsymbol{f} - t\boldsymbol{f}\,J_{n_1}$$

of the homological system of the r-reduced system $(\boldsymbol{A})$ jointly with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{rn,n_1}$ = the first n_1 columns of the identity matrix of size rn (see [4]). Thereby, the sum $\widehat{\boldsymbol{f}}_{\boldsymbol{\theta}^-}$ itself is completely determined by the convolution system $(\boldsymbol{A^*_H})$ deduced from $(\boldsymbol{A_H})$ by Borel transformation. Note that, in the special case where matrix $A(x)$ of initial system (A) has rational coefficients, convolution system $(\boldsymbol{A^*_H})$ can actually be always replaced by a convenient linear differential system.

6 Examples

To end this article, we develop in this section three typical examples in which, for a full effectivity, systems are chosen with *rational* coefficients.

In the first one, we consider a SG-Configuration and we choose a simple enough system to allow the exact calculation of the connection constants and so of the highest level's Stokes multipliers. Of course, such a case is anecdotal, but it is worth to be treated.

In the second example, we consider once again a SG-Configuration, but we choose this time a more general system for which no exact calculations are possible. We then show how the connection constants can be related, through a convenient linear differential systems, to "special values" which can be numerically computed.

As for the third example, it deals with a singularity with bad front.

6.1 Example 1

In this first example, we consider the system

$$x^3\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x^3 & ix & 0 & 0 \\ x^2 & 0 & 2+\frac{x^2}{2} & 0 \\ x^2 & -x^2 & x^2 & 4 \end{bmatrix} Y \tag{6.1}$$

and its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ where

- $Q\left(\frac{1}{x}\right) = \mathrm{diag}\left(0, -\frac{i}{x}, -\frac{1}{x^2}, -\frac{2}{x^2}\right), \qquad L = \mathrm{diag}\left(0, 0, \frac{1}{2}, 0\right),$
- $\widetilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \widetilde{f}_2(x) & 1 & 0 & 0 \\ \widetilde{f}_3(x) & 0 & 1 & 0 \\ \widetilde{f}_4(x) & * & * & 1 \end{bmatrix}$ verifies $\widetilde{F}(x) = I_4 + O(x^2)$. More precisely,

$$\left\{ \begin{array}{ll} \widetilde{f}_2(x) = ix^2 + 2x^3 + O(x^4) & \in x^2\mathbb{C}[[x]] \\ \widetilde{f}_3(x) = -\frac{1}{2}x^2 + O(x^4) & \in x^2\mathbb{C}[[x^2]] \\ \widetilde{f}_4(x) = -\frac{1}{4}x^2 + O(x^4) & \in x^2\mathbb{C}[[x]] \end{array} \right. . \tag{6.2}$$

System (6.1) has levels $(1, 2)$ and the set of highest level's Stokes values of the first column $\widetilde{f}(x)$ of $\widetilde{F}(x)$ is $\mathbf{\Omega}_2 = \{0, 1, 2\}$. In particular, the highest level's anti-Stokes directions of $\widetilde{f}(x)$ are given by the unique collection $(\theta_0 = 0, \theta_1 = -\pi)$ generated by $\tau = 1$ and $\tau = 2$. Obviously, the corresponding highest level's Stokes-Ramis matrices $St_{2;\theta_k}$ read as

$$St_{2;\theta_k} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ st^3_{2;\theta_k} & 0 & 1 & 0 \\ st^4_{2;\theta_k} & * & * & 1 \end{bmatrix} \quad , \ k = 0, 1$$

and, since the set $\mathbf{\Omega}_{2;0} = \{1, 2\}$ has the SG-Configuration (the highest level's Stokes values $\tau = 1$ and $\tau = 2$ have indeed both a good monomial front), theorem 5.7 tells us that the two highest level's Stokes multipliers $st^3_{2;0}$ and

$st^3_{2;-\pi}$ (resp. $st^4_{2;0}$ and $st^4_{2;-\pi}$) are expressed in terms of the connection constants of $\widehat{\boldsymbol{f}}_{0^-}(\tau)$ at $\tau = 1$ (resp. $\tau = 2$). We are then left to determine these constants, what is the purpose of the following calculations.

According to relations (3.1) (see page 15) and (6.2), the matrix $\widetilde{\boldsymbol{f}}(t) \in M_{8,1}(\mathbb{C}[[t]])$ reads on the form

$$\widetilde{\boldsymbol{f}}(t) = \begin{bmatrix} \widetilde{\boldsymbol{f}}^1(t) \\ \widetilde{\boldsymbol{f}}^2(t) \end{bmatrix} \quad \text{with } \widetilde{\boldsymbol{f}}^1(t) = \begin{bmatrix} 1 \\ \widetilde{\boldsymbol{f}}^{1,2}(t) \\ \widetilde{\boldsymbol{f}}^{1,3}(t) \\ \widetilde{\boldsymbol{f}}^{1,4}(t) \end{bmatrix}, \widetilde{\boldsymbol{f}}^2(t) = \begin{bmatrix} 0 \\ \widetilde{\boldsymbol{f}}^{2,2}(t) \\ \widetilde{\boldsymbol{f}}^{2,3}(t) \\ \widetilde{\boldsymbol{f}}^{2,4}(t) \end{bmatrix}$$

and the $\widetilde{\boldsymbol{f}}^{u,j}(t)$'s satisfying

$$\begin{cases} \widetilde{\boldsymbol{f}}^{1,2}(t) = it + O(t^2), & \widetilde{\boldsymbol{f}}^{1,3}(t) = -\frac{1}{2}t + O(t^2), & \widetilde{\boldsymbol{f}}^{1,4}(t) = -\frac{1}{4}t + O(t^2), \\ \widetilde{\boldsymbol{f}}^{2,2}(t) = 2t + O(t^2), & \widetilde{\boldsymbol{f}}^{2,3}(t) = 0, & \widetilde{\boldsymbol{f}}^{2,4}(t) = O(t^2). \end{cases}$$

Following relation $(\boldsymbol{A_H})$, $\widetilde{\boldsymbol{f}}(t)$ is uniquely determined by the system

$$2t^2\frac{d\boldsymbol{f}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^2 & it & 0 & 0 \\ t & 0 & 2+\frac{t}{2} & 0 & 0 & 0 & 0 & 0 \\ t & -t & t & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t & 0 & 0 & 0 \\ t & i & 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 2-\frac{t}{2} & 0 \\ 0 & 0 & 0 & 0 & t & -t & t & 4-t \end{bmatrix} \boldsymbol{f}$$

jointly with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{8,1}$; in particular, the $\widetilde{\boldsymbol{f}}^{u,j}(t)$'s are uniquely determined as formal series solutions of the following system

$$2t^2\frac{d\widetilde{\boldsymbol{f}}^{1,2}}{dt} = it\widetilde{\boldsymbol{f}}^{2,2}, \qquad 2t^2\frac{d\widetilde{\boldsymbol{f}}^{2,2}}{dt} = t + i\widetilde{\boldsymbol{f}}^{1,2} - t\widetilde{\boldsymbol{f}}^{2,2},$$

$$2t^2\frac{d\widetilde{\boldsymbol{f}}^{1,3}}{dt} = t + \left(2 + \frac{t}{2}\right)\widetilde{\boldsymbol{f}}^{1,3},$$

$$2t^2\frac{d\widetilde{\boldsymbol{f}}^{1,4}}{dt} = t - t\widetilde{\boldsymbol{f}}^{1,2} + t\widetilde{\boldsymbol{f}}^{1,3} + 4\widetilde{\boldsymbol{f}}^{1,4}, \qquad 2t^2\frac{d\widetilde{\boldsymbol{f}}^{2,4}}{dt} = -t\widetilde{\boldsymbol{f}}^{2,2} + (4-t)\widetilde{\boldsymbol{f}}^{2,4}$$

satisfying $\widetilde{\boldsymbol{f}}^{u,j}(t) = O(t)$. As a result, their formal Borel transforms $\widehat{\boldsymbol{f}}^{u,j}(\tau)$'s verify the following equations:

$$
(6.3)\qquad \begin{cases}
4\tau^2 \dfrac{d^2\widehat{\boldsymbol{f}}^{1,2}}{d\tau^2} + (14\tau+1)\widehat{\boldsymbol{f}}^{1,2} + 6\widehat{\boldsymbol{f}}^{1,2} = 0 \qquad (*) \\[2ex]
\widehat{\boldsymbol{f}}^{2,2} = -2i\dfrac{d}{d\tau}\left(\tau\widehat{\boldsymbol{f}}^{1,2}\right) \qquad (**) \\[2ex]
2(\tau-1)\dfrac{d\widehat{\boldsymbol{f}}^{1,3}}{d\tau} + \dfrac{3}{2}\widehat{\boldsymbol{f}}^{1,3} = 0 \quad , \ \widehat{\boldsymbol{f}}^{1,3}(0) = -\dfrac{1}{2} \\[2ex]
2(\tau-2)\dfrac{d\widehat{\boldsymbol{f}}^{1,4}}{d\tau} + 2\widehat{\boldsymbol{f}}^{1,4} = -\widehat{\boldsymbol{f}}^{1,2} + \widehat{\boldsymbol{f}}^{1,3} \quad , \ \widehat{\boldsymbol{f}}^{1,4}(0) = -\dfrac{1}{4} \\[2ex]
2(\tau-2)\dfrac{d\widehat{\boldsymbol{f}}^{2,4}}{d\tau} + 3\widehat{\boldsymbol{f}}^{2,4} = -\widehat{\boldsymbol{f}}^{2,2} \quad , \ \widehat{\boldsymbol{f}}^{2,4}(0) = 0
\end{cases}
$$

Note that, according to the Newton polygon at 0 of equation $(*)$, the formal series $\widehat{\boldsymbol{f}}^{1,2}(\tau)$ (hence, $\widehat{\boldsymbol{f}}^{2,2}(\tau)$) is 1-summable in any direction $\theta \neq -\pi$. In particular, it is 1-summable in direction 0^- (or 0). Note also that its 1-sum in direction θ is 1-Gevrey asymptotic to $\widehat{\boldsymbol{f}}^{1,2}(\tau)$ on a germ of sector bisected by θ and opening $> \pi$ and can be analytically continued on all $\widetilde{\mathbb{C}}$ since 0 is the unique singular point of $(*)$.

More precisely, by considering the formal series solutions in $\mathbb{C}[[\tau]]$ of equation $(*)$, one can easily check that

$$
\widehat{\boldsymbol{f}}^{1,2}(\tau) = \frac{2i}{\sqrt{\pi}} \sum_{m\geq 0} (-4)^m \Gamma\left(m + \frac{3}{2}\right) \tau^m \quad \in \mathbb{C}[[\tau]].
$$

Consequently, its formal Borel transform is given by

$$
\mathcal{B}_1(\widehat{\boldsymbol{f}}^{1,2})(u) = i\delta + \frac{2i}{\sqrt{\pi}} \sum_{m\geq 1} (-4)^m \Gamma\left(m + \frac{3}{2}\right) \frac{u^{m-1}}{(m-1)!} = i\delta - 6i(4u+1)^{-\frac{5}{2}}
$$

and its 1-sum in direction $\theta = 0$ is given by

$$
\widehat{\boldsymbol{f}}^{1,2}_0(\tau) = -\frac{i\sqrt{\pi}}{4}\tau^{-3/2}e^{1/(4\tau)} + \frac{i}{2\tau}\,{}_1F_1\left(1; \frac{1}{2}; \frac{1}{4\tau}\right)
$$

where ${}_1F_1\big(1; \frac{1}{2}; \tau\big)$ denotes the confluent hypergeometric function with parameters 1 and $\frac{1}{2}$. Note that this relation also defines the analytic continuation of $\widehat{\boldsymbol{f}}_0^{1,2}(\tau)$ on all $\widetilde{\mathbb{C}}$. Relation $(**)$ above provides then us the exact expression of the sum $\widehat{\boldsymbol{f}}_0^{2,2}(\tau)$.

Note that the fact we can make explicit the sums $\widehat{\boldsymbol{f}}_0^{1,2}(\tau)$ and $\widehat{\boldsymbol{f}}_0^{2,2}(\tau)$ is only due to the great simplicity of system (6.1). Of course, for more general systems, such exact calculations are not possible anymore.

Let us now turn to the calculation of the connection constants of $\widehat{\boldsymbol{f}}_{0^-}$ at 1 and 2, and so of the highest level's Stokes multipliers of $\widetilde{f}(x)$. According to the last three equations of (6.3), following equalities hold for all $|\tau| < 1$:

$$\begin{cases} \widehat{\boldsymbol{f}}_{0^-}^{1,3}(\tau) = -\dfrac{1}{2}(1-\tau)^{-3/4}, & \widehat{\boldsymbol{f}}_{0^-}^{2,3}(\tau) = 0, \\ \widehat{\boldsymbol{f}}_{0^-}^{1,4}(\tau) = \dfrac{1}{\tau-2}\left(-\dfrac{3}{2} + 2(1-\tau)^{1/4} - \displaystyle\int_0^\tau \widehat{\boldsymbol{f}}_0^{1,2}(\eta)d\eta\right), & \\ \widehat{\boldsymbol{f}}_{0^-}^{2,4}(\tau) = \dfrac{1}{2}(2-\tau)^{-3/2}\displaystyle\int_0^\tau \widehat{\boldsymbol{f}}_0^{2,2}(\eta)(2-\eta)^{1/2}d\eta. & \end{cases}$$

Hence, setting

$$\int_0^\tau \widehat{\boldsymbol{f}}_0^{2,2}(\eta)(2-\eta)^{1/2}d\eta = \alpha + (2-\tau)^{3/2}g(\tau) \quad , \ \alpha \in \mathbb{C},\ g(\tau) \in \mathbb{C}\{\tau - 2\}$$

and choosing a determination of the logarithm such that $\ln(\tau) \in \mathbb{R}$ for $\tau > 0$, the analytic continuations $\widehat{\boldsymbol{f}}_{0^-;\omega,+}^{u,j}$'s of the $\widehat{\boldsymbol{f}}_{0^-}^{u,j}$'s at $\omega \in \{1,2\}$ verify

$$\boxed{\begin{array}{ll} \widehat{\boldsymbol{f}}_{0^-;1,+}^{1,3}(1+\tau) = \dfrac{1+i}{2\sqrt{2}}\tau^{-3/4}, & \widehat{\boldsymbol{f}}_{0^-;1,+}^{2,3}(1+\tau) = 0, \\[2ex] \widehat{\boldsymbol{f}}_{0^-;2,+}^{1,4}(2+\tau) = \dfrac{\beta}{\tau} + h_1(\tau), & \widehat{\boldsymbol{f}}_{0^-;2,+}^{2,4}(2+\tau) = \dfrac{i\alpha}{2}\tau^{-3/2} + h_2(\tau). \end{array}}$$

with

$$\boxed{\beta = -\frac{3}{2} + \sqrt{2} + i\sqrt{2} - \int_0^2 \widehat{\boldsymbol{f}}_0^{1,2}(\eta)d\eta \quad \text{and} \quad h_1(\tau), h_2(\tau) \in \mathbb{C}\{\tau\}.}$$

Consequently, the connection constants of $\widehat{\boldsymbol{f}}_{0^-}(\tau)$ at the points $\tau = 1$ and $\tau = 2$ are given by

$$\boxed{\boldsymbol{K}_{1,+}^{1,3} = \frac{1+i}{\sqrt{2}} \qquad \boldsymbol{K}_{1,+}^{2,3} = 0 \qquad \boldsymbol{K}_{2,+}^{1,4} = \beta \qquad \boldsymbol{K}_{2,+}^{2,4} = \frac{i\alpha}{2}}$$

We are now able to determine the highest level's Stokes multipliers of $\widetilde{f}(x)$. Since the matrix L of exponents of formal monodromy is diagonal, it results from (5.8) that $st^3_{2;0}$ and $st^3_{2;-\pi}$ (resp. $st^4_{2;0}$ and $st^4_{2;-\pi}$) are related to the connection constants $\boldsymbol{K}^{1,3}_{1,+}$ and $\boldsymbol{K}^{2,3}_{1,+}$ (resp. $\boldsymbol{K}^{1,4}_{2,+}$ and $\boldsymbol{K}^{2,4}_{2,+}$) above by relations

$$\boxed{\begin{array}{l} st^3_{2;0} = \dfrac{(1+i)\pi\sqrt{2}}{\Gamma\left(\dfrac{3}{4}\right)}\boldsymbol{K}^{1,3}_{1,+} - (4-4i)\Gamma\left(\dfrac{3}{4}\right)\boldsymbol{K}^{2,3}_{1,+}, \\[3ex] st^3_{2;-\pi} = \dfrac{(-1+i)\pi\sqrt{2}}{\Gamma\left(\dfrac{3}{4}\right)}\boldsymbol{K}^{1,3}_{1,+} + (4+4i)\Gamma\left(\dfrac{3}{4}\right)\boldsymbol{K}^{2,3}_{1,+}, \\[3ex] st^4_{2;0} = 2i\pi\boldsymbol{K}^{1,4}_{2,+} - 4\sqrt{\pi}\boldsymbol{K}^{2,4}_{2,+}, \qquad st^4_{2;-\pi} = 2i\pi\boldsymbol{K}^{1,4}_{2,+} + 4\sqrt{\pi}\boldsymbol{K}^{2,4}_{2,+} \end{array}}$$

(recall that $\rho = e^{-i\pi}$ since $r=2$). Hence,

$$\boxed{\begin{array}{ll} st^3_{2;0} = \dfrac{i\pi}{\Gamma\left(\dfrac{3}{4}\right)} & st^3_{2;-\pi} = -\dfrac{\pi}{\Gamma\left(\dfrac{3}{4}\right)} \\[3ex] st^4_{2;0} = 2i\sqrt{\pi}(\beta\sqrt{\pi}-\alpha) & st^4_{2;-\pi} = 2i\sqrt{\pi}(\beta\sqrt{\pi}+\alpha) \end{array}}$$

6.2 Example 2

In this second example, we consider the system

$$x^3\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & x^2 \\ x^3 & x & 0 \\ x^4 & x^2+x^3 & 2+\frac{x^2}{3} \end{bmatrix} Y \tag{6.4}$$

and its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ where

- $Q\left(\dfrac{1}{x}\right) = \operatorname{diag}\left(0, -\dfrac{1}{x}, -\dfrac{1}{x^2}\right), \qquad L = \operatorname{diag}\left(0,0,\dfrac{1}{3}\right),$
- $\widetilde{F}(x) \in M_3(\mathbb{C}[[x]])$ satisfies $\widetilde{F}(x) = I_3 + O(x^2)$.

As in previous example, system (6.4) has levels $(1,2)$ and we denote by $\widetilde{f}(x)$ the first column of $\widetilde{F}(x)$. Since the set of its highest level's Stokes values is

$\mathbf{\Omega}_2 = \{0,1\}$, the highest level's anti-Stokes directions of $\widetilde{f}(x)$ are given by the unique collection $(\theta_0 = 0, \theta_1 = -\pi)$ generated by $\tau = 1$. The corresponding highest level's Stokes-Ramis matrices $St_{2;\theta_k}$ with $k = 0,1$ read then as

$$St_{2;\theta_k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ st^3_{2;\theta_k} & * & 1 \end{bmatrix}.$$

and, since $\mathbf{\Omega}_{2;0} = \{1\}$ has a SG-Configuration (1 has a good monomial front), theorem 5.7 applies once again and allows to express the highest level's Stokes multipliers $st^3_{2;0}$ and $st^3_{2;-\pi}$ in terms of the connection constants $\boldsymbol{K}^{1,3}_{1,+}$ and $\boldsymbol{K}^{2,3}_{1,+}$ of $\widehat{\boldsymbol{f}}_{0^-}(\tau)$ at $\tau = 1$. More precisely, due to the fact that matrix L is diagonal, we deduce from (5.8) the following relations:

$$\boxed{\begin{aligned} st^3_{2;0} &= \frac{i\pi(\sqrt{3}-i)}{\Gamma\left(\frac{5}{6}\right)} \boldsymbol{K}^{1,3}_{1,+} + \frac{3i\sqrt{3}}{2}(1+i\sqrt{3})\Gamma\left(\frac{2}{3}\right)\boldsymbol{K}^{2,3}_{1,+} \\ st^3_{2;-\pi} &= \frac{i\pi(\sqrt{3}+i)}{\Gamma\left(\frac{5}{6}\right)} \boldsymbol{K}^{1,3}_{1,+} + \frac{3i\sqrt{3}}{2}(1-i\sqrt{3})\Gamma\left(\frac{2}{3}\right)\boldsymbol{K}^{2,3}_{1,+} \end{aligned}}$$

Contrarily to previous example, the exact calculation of $\boldsymbol{K}^{1,3}_{1,+}$ and $\boldsymbol{K}^{2,3}_{1,+}$ is no longer possible. Nevertheless, we can determine *approximate values* of these constants. To do that, we can proceed as follows.

Following relation $(\boldsymbol{A_H})$, the matrix $\widetilde{\boldsymbol{f}}(t) \in M_{6,1}(\mathbb{C}[[t]])$ is a formal solution of system

$$2t^2\frac{d\boldsymbol{f}}{dt} = \begin{bmatrix} 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & t^2 & t & 0 \\ t^2 & t & 2+\frac{t}{3} & 0 & t^2 & 0 \\ 0 & 0 & 0 & -t & 0 & t \\ t & 1 & 0 & 0 & -t & 0 \\ 0 & t & 0 & t^2 & t & 2-\frac{2t}{3} \end{bmatrix}\boldsymbol{f}$$

As in previous example, this system provides relations on each entry of $\widetilde{\boldsymbol{f}}(t)$. Then, multiplying each on these relations by $1/t^2$ and applying a Borel transformation, one can check that the matrix

$$\mathcal{F}_{0^-} := \begin{bmatrix} \widehat{\boldsymbol{f}}_{0^-} \\ \widehat{\boldsymbol{f}}'_{0^-} \end{bmatrix}$$

is an analytic solution on $V_0(\widehat{\boldsymbol{f}}_{0^-})$ [14] of the system

$$\frac{d\boldsymbol{Z}}{d\tau} = C(\tau)\boldsymbol{Z} \tag{6.5}$$

where $C(\tau)$ is the following 12×12-matrix

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-2}{\tau} & 0 & \frac{1}{2\tau} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2\tau} & 0 & 0 & 0 & \frac{-2}{\tau} & 0 & 0 & \frac{1}{2\tau} & 0 \\
\frac{1}{2\tau-2} & 0 & 0 & 0 & \frac{1}{2\tau-2} & 0 & 0 & \frac{1}{2\tau-2} & \frac{-11}{6\tau-6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-5}{2\tau} & 0 & \frac{1}{2\tau} \\
0 & 0 & 0 & \frac{1}{4\tau^2} & 0 & 0 & \frac{1}{2\tau} & \frac{-1}{\tau^2} & 0 & 0 & \frac{1-10\tau}{4\tau^2} & 0 \\
0 & 0 & 0 & \frac{1}{2\tau-2} & 0 & 0 & 0 & \frac{1}{2\tau-2} & 0 & 0 & \frac{1}{2\tau-2} & \frac{-7}{3\tau-3}
\end{bmatrix}$$

Note that system (6.5) has an irregular singular point at $\tau = 0$ (due to the level 1 of $\widetilde{f}(x)$) and a regular singular point at $\tau = 1$ (due to the fact that 1 has a good monomial front). More precisely, it reads near $\tau = 1$ as

$$(\tau - 1)\frac{d\boldsymbol{Z}}{d\tau} = C_1(\tau)\boldsymbol{Z} \tag{6.6}$$

where $C_1(\tau) := (\tau - 1)C(\tau)$ is diagonalizable and analytic on the open disc $D(1, 1)$ with center 1 and radius 1. Following Wasow [28], let the matrix

$$D_1 := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{3}{11} & 0 & 0 & 0 & \frac{3}{11} & 0 & 0 & \frac{3}{11} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{3}{14} & 0 & 0 & 0 & \frac{3}{14} & 0 & 0 & \frac{3}{14} & 1
\end{bmatrix}$$

[14] Here, $V_0(\widehat{\boldsymbol{f}}_{0^-})$ is a sector with opening $> \pi$. Indeed, $\widetilde{f}(x)$ has levels $(1, 2)$; hence, $\widetilde{\boldsymbol{f}}(t)$ is $(\frac{1}{2}, 1)$-summable (*cf.* section 3.1) and so $\widehat{\boldsymbol{f}}(\tau)$ is 1-summable (*cf.* def. 2.6).

so that

$$M_1 := D_1^{-1}C_1(1)D_1 = \mathrm{diag}\left(0,0,0,0,0,0,0,0,-\frac{11}{6},0,0,-\frac{7}{3}\right).$$

Hence, choosing as previously a determination of the logarithm so that $\ln(\tau) \in \mathbb{R}$ for $\tau > 0$, system (6.6) has for fundamental solution at $\tau = 1$ a matrix of the form $\boldsymbol{Z}_1(\tau) = D_1 G_1(\tau)(\tau-1)^{M_1}$ where $G_1(\tau) \in M_{12}(\mathbb{C}\{\tau-1\})$ is analytic on $D(1,1)$ and is completely determined by relations

$$(\tau-1)\frac{dG_1}{d\tau} = D_1^{-1}C_1(\tau)D_1G_1 - G_1M_1 \quad , \ G_1(1) = I_{12}.$$

In particular, the ninth and twelfth columns of $\boldsymbol{Z}_1(\tau)$ read respectively as

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{6}{5}(\tau-1)^{-\frac{5}{6}} \\ 0 \\ 0 \\ 0 \\ -\frac{3}{5}(\tau-1)^{-\frac{5}{6}} \\ 0 \\ (\tau-1)^{-\frac{11}{6}} \\ 0 \\ 0 \\ 0 \end{bmatrix} + (\tau-1)^{\frac{1}{6}}z_9(\tau) \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{3}{4}(\tau-1)^{-\frac{4}{3}} \\ 0 \\ 0 \\ 0 \\ -\frac{3}{8}(\tau-1)^{-\frac{4}{3}} \\ 0 \\ (\tau-1)^{-\frac{7}{3}} \end{bmatrix} + (\tau-1)^{-\frac{1}{3}}z_{12}(\tau)$$

with $z_9(\tau)$ and $z_{12}(\tau)$ analytic on $D(1,1)$; as for the other columns of $\boldsymbol{Z}_1(\tau)$, they are analytic on $D(1,1)$.

Let us now apply Cauchy-Lipschitz's theorem: the analytic continuation $\mathcal{F}_{0^-;1,+}$ of $\mathcal{F}_{0^-}$ at $\tau = 1$ is a solution of system (6.6); thereby, there exists a unique matrix $\Sigma_1 \in M_{12,1}(\mathbb{C})$ such that $\mathcal{F}_{0^-}(\tau) = \boldsymbol{Z}_1(\tau)\Sigma_1$ for all $\tau \in D(1,1)\backslash\{1\}$. In particular, denoting by $\Sigma_1 := [\sigma_1^1, \sigma_1^2, ..., \sigma_1^{12}]$, calculations above show that connection constant $\boldsymbol{K}_{1,+}^{1,3}$ (resp. $\boldsymbol{K}_{1,+}^{2,3}$) is equal to $-\frac{6}{5}\sigma_1^9$ (resp. $-\frac{3}{4}\sigma_1^{12}$). Hence, the following relations:

$$\boxed{\begin{aligned} st_{2;0}^3 &= -\frac{6i\pi(\sqrt{3}-i)}{5\Gamma\left(\frac{5}{6}\right)}\sigma_1^9 - \frac{9i\sqrt{3}}{8}(1+i\sqrt{3})\Gamma\left(\frac{2}{3}\right)\sigma_1^{12} \\ st_{2;-\pi}^3 &= -\frac{6i\pi(\sqrt{3}+i)}{5\Gamma\left(\frac{5}{6}\right)}\sigma_1^9 - \frac{9i\sqrt{3}}{8}(1-i\sqrt{3})\Gamma\left(\frac{2}{3}\right)\sigma_1^{12} \end{aligned}}$$

It then remains to numerically evaluate σ_1^9 and σ_1^{12}. To do that, one can adapt the method detailed in [25] by considering a point $\alpha \in V_0(\widehat{\boldsymbol{f}}_{0^-}) \cap \mathbb{R}^+$ so that $\Sigma_1 = \boldsymbol{Z}_1(\alpha)^{-1}\mathcal{F}_{0^-}(\alpha)$. Note that, by definition of the right analytic continuation, $\boldsymbol{Z}_1(\alpha)$ is evaluated at a point such that $\arg(\alpha - 1) = -\pi$. Note also that the evaluation of $\mathcal{F}_{0^-}(\alpha)$ requires methods for the effective calculation of multi-sums of formal series (see, for instance, [9]).

6.3 Example 3

In this last example, we consider the system

$$x^3 \frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 \\ x^2 & 2+x & 0 \\ x^2 & x^3 & 2 \end{bmatrix} Y \tag{6.7}$$

together with the formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)e^{Q(1/x)}$ where

- $Q\left(\dfrac{1}{x}\right) = \operatorname{diag}\left(0, -\dfrac{1}{x^2} - \dfrac{1}{x}, -\dfrac{1}{x^2}\right)$ (hence, system (6.7) has, once again, levels $(1,2)$),
- $\widetilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2(x) & 1 & 0 \\ \widetilde{f}_3(x) & * & 1 \end{bmatrix}$ satisfies $\widetilde{F}(x) = I_3 + O(x^2)$. More precisely,

$$\begin{cases} \widetilde{f}_2(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^3 - \frac{5}{8}x^4 + \frac{11}{16}x^5 + O(x^6) \\ \widetilde{f}_3(x) = -\frac{1}{2}x^2 - \frac{1}{2}x^4 + \frac{1}{4}x^5 + O(x^6) \end{cases} \tag{6.8}$$

As before, we denote by $\widetilde{f}(x)$ the first column of $\widetilde{F}(x)$. Note that $\widetilde{f}(x)$ has just the level 2; thereby, $\widetilde{\boldsymbol{f}}(t)$ is 1-Gevrey and $\widehat{\boldsymbol{f}}(\tau)$ is analytic at $0 \in \mathbb{C}$.

The set of highest level's Stokes values of $\widetilde{f}(x)$ is $\boldsymbol{\Omega}_2 = \{0, 1\}$. Contrarily to previous examples, our present aim is not to calculate the highest level's Stokes multipliers of $\widetilde{f}(x)$, but just to display the structure of the singularity of $\widehat{\boldsymbol{f}}(\tau)$ at 1 in order to illustrate theorem 4.24 in the case of a singularity with a bad front. Indeed, we have here

$$Fr(1) = \left\{-\frac{1}{x^2} - \frac{1}{x}, -\frac{1}{x^2}\right\}.$$

As in section 6.1, system (6.7) is simple enough to allow exact calculations.

According to relations (3.1) (see page 15) and (6.8), the matrix $\widetilde{\boldsymbol{f}}(t) \in M_{6,1}(\mathbb{C}[[t]])$ reads on the form

$$\widetilde{\boldsymbol{f}}(t) = \begin{bmatrix} \widetilde{\boldsymbol{f}}^{1}(t) \\ \widetilde{\boldsymbol{f}}^{2}(t) \end{bmatrix} \quad \text{with } \widetilde{\boldsymbol{f}}^{1}(t) = \begin{bmatrix} 1 \\ \widetilde{\boldsymbol{f}}^{1,2}(t) \\ \widetilde{\boldsymbol{f}}^{1,3}(t) \end{bmatrix}, \widetilde{\boldsymbol{f}}^{2}(t) = \begin{bmatrix} 0 \\ \widetilde{\boldsymbol{f}}^{2,2}(t) \\ \widetilde{\boldsymbol{f}}^{2,3}(t) \end{bmatrix}$$

and the $\widetilde{\boldsymbol{f}}^{u,j}(t)$'s satisfying

$$\begin{cases} \widetilde{\boldsymbol{f}}^{1,2}(t) = -\frac{1}{2}t - \frac{5}{8}t^2 + O(t^3), & \widetilde{\boldsymbol{f}}^{1,3}(t) = -\frac{1}{2}t - \frac{1}{2}t^2 + O(t^3), \\ \widetilde{\boldsymbol{f}}^{2,2}(t) = \frac{1}{4}t + \frac{11}{16}t^2 + O(t^3), & \widetilde{\boldsymbol{f}}^{2,3}(t) = \frac{1}{4}t^2 + O(t^3). \end{cases}$$

Following relation $(\boldsymbol{A_H})$, $\widetilde{\boldsymbol{f}}(t)$ is uniquely determined by the system

$$2t^2\frac{d\boldsymbol{f}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ t & 2 & 0 & 0 & t & 0 \\ t & 0 & 2 & 0 & t^2 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 1 & 0 & t & 2-t & 0 \\ 0 & t & 0 & t & 0 & 2-t \end{bmatrix} \boldsymbol{f}$$

jointly with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{6,1}$. Then, adapting calculations already made in section 6.1, one can check that the Borel transforms $\widehat{\boldsymbol{f}}^{u,j}(\tau)$, $u = 1, 2$ and $j = 2, 3$, verify the following equations:

$$(6.9) \quad \begin{cases} 2(\tau-1)\dfrac{d\widehat{\boldsymbol{f}}^{1,2}}{d\tau} + 2\widehat{\boldsymbol{f}}^{1,2} = \widehat{\boldsymbol{f}}^{2,2} & , \ \widehat{\boldsymbol{f}}^{1,2}(0) = -\dfrac{1}{2} \qquad (*) \\[2ex] 2(\tau-1)\dfrac{d\widehat{\boldsymbol{f}}^{2,2}}{d\tau} - \dfrac{d\widehat{\boldsymbol{f}}^{1,2}}{d\tau} + 3\widehat{\boldsymbol{f}}^{2,2} = 0 & , \ \widehat{\boldsymbol{f}}^{2,2}(0) = \dfrac{1}{4} \\[2ex] 2(\tau-1)\dfrac{d\widehat{\boldsymbol{f}}^{1,3}}{d\tau} + 2\widehat{\boldsymbol{f}}^{1,3} = 1 * \widehat{\boldsymbol{f}}^{2,2} & , \ \widehat{\boldsymbol{f}}^{1,3}(0) = -\dfrac{1}{2} \\[2ex] 2(\tau-1)\dfrac{d\widehat{\boldsymbol{f}}^{2,3}}{d\tau} + 3\widehat{\boldsymbol{f}}^{2,3} = \widehat{\boldsymbol{f}}^{1,2} & , \ \widehat{\boldsymbol{f}}^{2,3}(0) = 0 \end{cases}$$

Since $(*)$ implies $\widehat{\boldsymbol{f}}^{2,2} = 2\frac{d}{d\tau}\left((\tau-1)\widehat{\boldsymbol{f}}^{1,2}\right)$, we deduce from the first two equations of (6.9) and calculations above that $\widehat{\boldsymbol{f}}^{1,2}$ is the unique solution of the differential equation

$$\begin{cases} 4(\tau-1)^2\dfrac{d^2\widehat{\boldsymbol{f}}^{\,1,2}}{d\tau^2} + (14\tau-15)\dfrac{d\widehat{\boldsymbol{f}}^{\,1,2}}{d\tau} + 6\widehat{\boldsymbol{f}}^{\,1,2} = 0 \\ \\ \widehat{\boldsymbol{f}}^{\,1,2}(0) = -\dfrac{1}{2},\ \dfrac{d\widehat{\boldsymbol{f}}^{\,1,2}}{d\tau}(0) = -\dfrac{5}{8} \end{cases}$$

Hence, for all $|\tau| < 1$,

$$\widehat{\boldsymbol{f}}^{\,1,2}(\tau) = \frac{e^{\frac{1}{4(1-\tau)}}}{1-\tau}\left[\frac{{}_1F_1\left(\frac{1}{2};\frac{3}{2};-\frac{1}{4}\right)}{4}(1-\tau)^{-\frac{1}{2}} - \frac{1}{2}{}_1F_1\left(-\frac{1}{2};\frac{1}{2};-\frac{1}{4(1-\tau)}\right)\right]$$

where ${}_1F_1(a;b;\bullet)$ denotes the confluent hypergeometric function with parameters a and b. This relation shows then us that the singularity at $\tau = 1$ is strongly irregular and, thereby, much more complicated than the singularities met in previous examples (see sections 6.1 and 6.2) for singular points with good monomial front.

Note that explicit formulæ can also be displayed for $\widehat{\boldsymbol{f}}^{\,2,2}$, $\widehat{\boldsymbol{f}}^{\,1,3}$ and $\widehat{\boldsymbol{f}}^{\,2,3}$ and provide yet much "harder" singularities. These formulæ stem from the following relations which result from $(*)$ and from the last two equations of (6.9): for all $|\tau| < 1$,

$$\begin{cases} \widehat{\boldsymbol{f}}^{\,2,2}(\tau) = 2(\tau-1)\dfrac{d\widehat{\boldsymbol{f}}^{\,1,2}}{d\tau}(\tau) + 2\widehat{\boldsymbol{f}}^{\,1,2}(\tau), \\ \\ \widehat{\boldsymbol{f}}^{\,1,3}(\tau) = -\dfrac{1}{2} + \dfrac{1}{1-\tau}\displaystyle\int_0^\tau (1-\eta)\widehat{\boldsymbol{f}}^{\,1,2}(\eta)d\eta, \\ \\ \widehat{\boldsymbol{f}}^{\,2,3}(\tau) = -\dfrac{1}{2}(1-\tau)^{-3/2}\displaystyle\int_0^\tau (1-\eta)^{1/2}\widehat{\boldsymbol{f}}^{\,1,2}(\eta)d\eta. \end{cases}$$

References

[1] W. Balser. Summation of formal power series through iterated Laplace integrals. *Math. Scand.*, 70(2):161–171, 1992.

[2] W. Balser. *Formal power series and linear systems of meromorphic ordinary differential equations.* Universitext. Springer-Verlag, New-York, 2000.

[3] W. Balser, B. J. L. Braaksma, J.-P. Ramis, and Y. Sibuya. Multisummability of formal power series solutions of linear ordinary differential equations. *Asymptot. Anal.*, 5(1):27–45, 1991.

[4] W. Balser, W. B. Jurkat, and D. A. Lutz. A general theory of invariants for meromorphic differential equations; Part I, formal invariants. *Funkcial. Ekvac.*, 22:197–221, 1979.

[5] G. Birkhoff. Singular points of ordinary linear differential equations. *Trans. Amer. Math. Soc.*, 10:436–470, 1909.

[6] B. L. J. Braaksma. Multisummability and Stokes multipliers of linear meromorphic differential equations. *J. Differential Equations*, 92:45–75, 1991.

[7] P. Deligne. *Equations différentielles à points singuliers réguliers*, volume 163 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1970.

[8] J. Écalle. Les fonctions résurgentes, tome III : l'équation du pont et la classification analytique des objets locaux. *Publ. Math. Orsay*, 85-05, 1985.

[9] F. Jung, F. Naegelé, and J. Thomann. An algorithm of multisummation of formal power series solutions of linear odes. *Math. Comput. Simulation*, 42:409–425, 1996.

[10] M. Loday-Richaud. Calcul des invariants de Birkhoff des systèmes d'ordre deux. *Funkcial. Ekvac.*, 33:161–225, 1990.

[11] M. Loday-Richaud. Stokes phenomenon, multisummability and differential Galois groups. *Ann. Inst. Fourier (Grenoble)*, 44(3):849–906, 1994.

[12] M. Loday-Richaud. Rank reduction, normal forms and Stokes matrices. *Expo. Math.*, 19:229–250, 2001.

[13] M. Loday-Richaud. Divergent series and differential equations. *Prépublication du LAREMA*, 375, 2014.

[14] M. Loday-Richaud and P. Remy. Resurgence, Stokes phenomenon and alien derivatives for level-one linear differential systems. *J. Differential Equations*, 250:1591–1630, 2011.

[15] B. Malgrande and J.-P. Ramis. Fonctions multisommables. *Ann. Inst. Fourier (Grenoble)*, 42:353–368, 1992.

[16] B. Malgrange. Introduction aux travaux de J. Écalle. *Enseign. Math.*, 31:261–282, 1985.

[17] B. Malgrange. Fourier transform and differential equations. In *Recent Developments in Quantum Mechanics*, volume 12 of *Mathematical Physics Studies*, pages 33–48. Kluwer Academic Publishers, 1991.

[18] J. Martinet and J.-P. Ramis. Théorie de Galois différentielle et resommation. In *Computer algebra and differential equations*, pages 117–214. Academic Press, 1989.

[19] J. Martinet and J.-P. Ramis. Elementary acceleration and multisummability. *Ann. Inst. H. Poincaré Phys. Théor.*, 54(4):331–401, 1991.

[20] J.-P. Ramis. Théorèmes d'indices Gevrey pour les équations différentielles ordinaires. *Mem. Amer. Math. Soc.*, 48:viii+95, 1984.

[21] J.-P. Ramis. Phénomène de Stokes et resommation. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(4):99–102, 1985.

[22] J.-P. Ramis. Filtration de Gevrey sur le groupe de Picard-Vessiot d'une équation différentielle irrégulière (juin 1985). In *P. Deligne, B. Malgrange, J.-P. Ramis, Singularités irrégulières*, volume 5 of *Documents Mathématiques (Paris) (Mathematical Documents (Paris))*. Société Mathématique de France, Paris, 2007.

[23] P. Remy. First level's connection-to-stokes formulae for meromorphic linear differential systems. *Adv. Dyn. Syst. Appl.*, 7(2):249–285, 2012.

[24] P. Remy. Matrices de Stokes-Ramis et constantes de connexion pour les systèmes différentiels linéaires de niveau unique. *Ann. Fac. Sci. Toulouse*, 21(1):93–150, 2012.

[25] P. Remy. Stokes phenomenon for single-level linear differential systems: a perturbative approach. *Funkcial. Ekvac.*, to appear.

[26] D. Sauzin. Resurgent functions and splitting problems. *RIMS Kôkyûroku*, 1493:48–117, 2005.

[27] Y. Sibuya. *Linear differential equations in the complex domain: problems of analytic continuation*, volume 82 of *Translation of Mathematical Monographs*. Amer. Math. Soc., 1990.

[28] W. Wasow. *Asymptotic expansions for ordinary differential equations*. Dover Publications, Inc., New-York, 1965.