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# CLT FOR RANDOM WALKS OF COMMUTING ENDOMORPHISMS ON COMPACT ABELIAN GROUPS

GUY COHEN AND JEAN-PIERRE CONZE

ABSTRACT. Let  $\mathcal{S}$  be an abelian group of automorphisms of a probability space  $(X, \mathcal{A}, \mu)$  with a finite system of generators  $(A_1, \dots, A_d)$ . Let  $A^\ell$  denote  $A_1^{\ell_1} \dots A_d^{\ell_d}$ , for  $\ell = (\ell_1, \dots, \ell_d)$ . If  $(Z_k)$  is a random walk on  $\mathbb{Z}^d$ , one can study the asymptotic distribution of the sums  $\sum_{k=0}^{n-1} f \circ A^{Z_k(\omega)}$  and  $\sum_{\ell \in \mathbb{Z}^d} \mathbb{P}(Z_n = \ell) A^\ell f$ , for a function  $f$  on  $X$ .

In particular, given a random walk on commuting matrices in  $SL(\rho, \mathbb{Z})$  or in  $\mathcal{M}^*(\rho, \mathbb{Z})$  acting on the torus  $\mathbb{T}^\rho$ ,  $\rho \geq 1$ , what is the asymptotic distribution of the associated ergodic sums along the random walk for a smooth function on  $\mathbb{T}^\rho$  after normalization?

In this paper, we prove a central limit theorem when  $X$  is a compact abelian connected group  $G$  endowed with its Haar measure (e.g. a torus or a connected extension of a torus),  $\mathcal{S}$  a totally ergodic  $d$ -dimensional group of commuting algebraic automorphisms of  $G$  and  $f$  a regular function on  $G$ . The proof is based on the cumulant method and on preliminary results on the spectral properties of the action of  $\mathcal{S}$ , on random walks and on the variance of the associated ergodic sums.

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## Introduction

Let  $\mathcal{S}$  be a finitely generated abelian group and let  $(T^s, s \in \mathcal{S})$  be a measure preserving action of  $\mathcal{S}$  on a probability space  $(X, \mathcal{A}, \mu)$ . A probability distribution  $(p_s, s \in \mathcal{S})$  on  $\mathcal{S}$  defines a random walk (r.w.)  $(Z_n)$  on  $\mathcal{S}$ . We obtain a Markov chain on  $X$  (and a random walk on the orbits of the action of  $\mathcal{S}$ ) by defining the transition probability from  $x \in X$  to  $T^s x$  as  $p_s$ . The Markov operator  $P$  of the corresponding Markov chain is  $Pf(x) = \sum_{s \in \mathcal{S}} p_s f(T^s x)$ . Limit theorems have been investigated for the associated random process. For instance, conditions on  $f \in L_0^2(\mu)$  are given in [11] for a “quenched” central limit theorem for the ergodic sums along the r.w.  $\sum_{k=0}^{n-1} f(T^{Z_k(\omega)} x)$  when  $\mathcal{S}$  is  $\mathbb{Z}$ . “Quenched”, there, is understood for a.e.  $x \in X$  and in law with respect to  $\omega$ . For other limit theorems in the quenched setting, see for instance [8] and references therein.

Another point of view, in the study of random dynamical systems, is to prove limiting laws for a.e. fixed  $\omega$  (cf. [21]). This is our framework: for a fixed  $\omega$ , we consider the asymptotic distribution of the above ergodic sums after normalization with respect to  $\mu$ .

More precisely, let  $(T_1, \dots, T_d)$  be a system of generators in  $\mathcal{S}$ . Every  $T \in \mathcal{S}$  can be represented as  $T = T^{\underline{\ell}} = T_1^{\ell_1} \dots T_d^{\ell_d}$ , for  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$ . (The elements of  $\mathbb{Z}^d$  are underlined to distinguish them from the scalars.) For a function  $f$  on  $X$  and  $T \in \mathcal{S}$ ,  $Tf$  denotes the composition  $f \circ T$ . The map  $f \rightarrow Tf$  defines an isometry on  $\mathcal{H} = L_0^2(\mu)$ , the space of square integrable functions  $f$  on  $X$  such that  $\mu(f) = 0$ .

Given a r.w.  $W = (Z_n)$  on  $\mathbb{Z}^d$ , we study the asymptotic distribution (for a fixed  $\omega$  and with respect to the measure  $\mu$  on  $X$ ) of  $\sum_{k=0}^{n-1} T^{Z_k(\omega)} f$ ,  $n \rightarrow \infty$ , after normalization. We consider also iterates of the Markov operator  $P$  introduced above (called below barycenter operator) and the asymptotic distribution of  $(P^n f)_{n \geq 1}$  after normalization.

Other sums for the random field  $(Tf, T \in \mathcal{S})$ ,  $f \in L_0^2(\mu)$ , can be considered. For instance, if  $(D_n) \subset \mathbb{N}^d$  is an increasing sequence of domains, a question is the asymptotic normality of  $|D_n|^{-\frac{1}{2}} \sum_{\ell \in D_n} T^\ell f$  and of the multidimensional “periodogram”  $|D_n|^{-\frac{1}{2}} \sum_{\ell \in D_n} e^{2\pi i \langle \ell, \theta \rangle} T^\ell f$ ,  $\theta \in \mathbb{R}^d$ . All above questions can be formulated in the framework of what is called below “summation sequences” and their associated kernel.

The specific model which is studied here is the following one:  $(X, \mathcal{A}, \mu)$  will be a compact abelian connected group  $G$  endowed with its Borel  $\sigma$ -algebra  $\mathcal{A}$  and its Haar measure  $\mu$  and  $\mathcal{S}$  will be a semigroup of commuting algebraic endomorphisms of  $G$ . This extends the classical situation of a CLT for a single endomorphism (R. Fortet (1940) and M. Kac (1946) for  $G = \mathbb{T}^1$ , V. Leonov (1960) for a general ergodic endomorphism of  $G$ ).

The main result (Theorem 5.1) is a quenched CLT for  $(\sum_{k=0}^{n-1} T^{Z_k(\omega)} f)$ , when  $f$  is a regular function on  $G$ ,  $(T^\ell)_{\ell \in \mathbb{Z}^d}$  a  $\mathbb{Z}^d$ -action on  $G$  by automorphisms and  $(Z_k)$  a r.w. on  $\mathbb{Z}^d$ . After reduction of the r.w., we examine three different cases: a) moment of order 2, centering, dimension 1; b) moment of order 2, centering, dimension 2; c) transient case. This covers all cases when there is a moment of order 2. As usual, the transient case is the easiest. The recurrent case requires to study the self-intersections of the r.w.

The paper is organized as follows. In Sect. 1 we discuss methods of summation and recall some facts on the spectral analysis of a finitely generated group of unitary operators with Lebesgue spectrum. The notion of “regular” summation sequence allows a unified treatment of different cases, including ergodic sums over sets or sums along random sequences generated by random walks. In Sect. 2 we prove auxiliary results on the regularity of quenched and barycenter summation sequences defined by a r.w. on  $\mathbb{Z}^d$ . These first two sections apply to a general  $\mathbb{Z}^d$ -action with Lebesgue spectrum.

In Sect. 3, the model of multidimensional actions by endomorphisms or automorphisms on a compact abelian group  $G$  is presented. We recall how to construct totally ergodic toral  $\mathbb{Z}^d$ -actions by automorphisms and we give explicit examples. The link between the regularity of a function  $f$  on  $G$  and its spectral density  $\varphi_f$  is established, with a specific treatment for tori for which the required regularity for  $f$  is weaker. A sufficient condition for the CLT on a compact abelian connected group  $G$  is given in Sect. 4.

The results of the first sections are applied in Sect. 5. A quenched CLT for the ergodic sums of regular functions  $f$  along a r.w. on  $G$  by commuting automorphisms is shown when  $G$  is connected. For a transient r.w. the variance, which is related to a measure on  $\mathbb{T}^d$  with an absolutely continuous part, does not vanish. Another summation method, iteration of barycenters, yields a polynomial decay of the iterates for regular functions and a CLT for the normalized sums, the nullity of the variance being characterized in terms of coboundary. An appendix (Sect. 6) is devoted to the self-intersections properties of a r.w. In Sect. 7, we recall the cumulant method used in the proof of the CLT.

To conclude this introduction, let us mention that in a previous work [5] we extended to ergodic sums of multidimensional actions by endomorphisms the CLT proved by

T. Fukuyama and B. Petit [14] for coprime integers acting on the circle. After completing it, we were informed of the results of M. Levin [26] showing the CLT for ergodic sums over rectangles for actions by endomorphisms on tori. The proof of the CLT for sums over rectangles is based in both approaches, as well as in [14], on results on  $S$ -units which imply mixing of all orders for connected groups (K. Schmidt and T. Ward [31]). The cumulant method used here is also based on this property of mixing of all orders.

## 1. Summation sequences

Let us consider first the general framework of an *abelian finitely generated group*  $\mathcal{S}$  (isomorphic to  $\mathbb{Z}^d$ ) of unitary operators on a Hilbert space  $\mathcal{H}$ . There exists a system of independent generators  $(T_1, \dots, T_d)$  in  $\mathcal{S}$  and each element of  $\mathcal{S}$  can be written in a unique way as  $T^{\underline{\ell}} = T_1^{\ell_1} \dots T_d^{\ell_d}$ , with  $\underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{Z}^d$ .

Our main example (Sect. 3) will be given by groups of commuting automorphisms (or extensions of semigroups of commuting surjective endomorphisms) of a compact abelian group  $G$ . They act by composition on  $\mathcal{H} = L_0^2(G, \mu)$  (with  $\mu$  the Haar measure of  $G$ ) and yield examples of  $\mathbb{Z}^d$ -actions by unitary operators.

Given  $f \in \mathcal{H}$ , there are various choices of summation sequences for the random field  $(T^{\underline{\ell}} f, \underline{\ell} \in \mathbb{Z}^d)$ . In the first part of this section, we discuss this point and the behavior of the kernels associated to summation sequences. The second part of the section is devoted to the spectral analysis of  $d$ -dimensional commuting actions with Lebesgue spectrum.

### 1.1. Summation sequences, kernels, examples.

**Definition 1.1.** We call *summation sequence* a sequence  $(R_n)_{n \geq 1}$  of functions from  $\mathbb{Z}^d$  to  $\mathbb{R}^+$  with  $0 < \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) < +\infty$ ,  $\forall n \geq 1$ . Given  $\mathcal{S} = \{T^{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d\}$  and  $f \in \mathcal{H}$ , the associated sums are  $\sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) T^{\underline{\ell}} f$ .

Let  $\check{R}_n(\underline{\ell}) := R_n(-\underline{\ell})$ . The normalized nonnegative kernel  $\check{R}_n$  (with  $\|\check{R}_n\|_{L^1(\mathbb{T}^d)} = 1$ ) is

$$(1) \quad \check{R}_n(\underline{t}) = \frac{|\sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle}|^2}{\sum_{\underline{\ell} \in \mathbb{Z}^d} |R_n(\underline{\ell})|^2} = \sum_{\underline{\ell} \in \mathbb{Z}^d} \frac{(R_n * \check{R}_n)(\underline{\ell})}{(R_n * \check{R}_n)(\underline{0})} e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle}, \quad \underline{t} \in \mathbb{T}^d.$$

**Definition 1.2.** We say that  $(R_n)$  is  $\zeta$ -regular, if  $(\check{R}_n)_{n \geq 1}$  weakly converges to a probability measure  $\zeta$  on  $\mathbb{T}^d$ , i.e.,  $\int_{\mathbb{T}^d} \check{R}_n \varphi d\underline{t} \xrightarrow{n \rightarrow \infty} \int_{\mathbb{T}^d} \varphi d\zeta$  for every continuous  $\varphi$  on  $\mathbb{T}^d$ .

The existence of the limit:  $L(\underline{p}) = \lim_{n \rightarrow \infty} \int \check{R}_n(\underline{t}) e^{-2\pi i \langle \underline{p}, \underline{t} \rangle} d\underline{t} = \lim_n \frac{(R_n * \check{R}_n)(\underline{p})}{(R_n * \check{R}_n)(\underline{0})}$ , for all  $\underline{p} \in \mathbb{Z}^d$ , is equivalent to  $\zeta$ -regularity with  $\hat{\zeta}(\underline{p}) = L(\underline{p})$ . Note that  $\check{R}_n$  and  $\zeta$  are even.

### Examples 1.3. Summation over sets

A class of summation sequences is given by summation over sets. If  $(D_n)_{n \geq 1}$  is a sequence of finite subsets of  $\mathbb{Z}^d$  and  $R_n = 1_{D_n}$ , we get the ‘‘ergodic’’ sums  $\sum_{\underline{\ell} \in D_n} T^{\underline{\ell}} f$ . The

simplest choice for  $(D_n)$  is an increasing family of  $d$ -dimensional rectangles. The sequence  $(R_n) = (1_{D_n})$  is  $\delta_0$ -regular if and only if  $(D_n)$  satisfies

$$(2) \quad \lim_{n \rightarrow \infty} |D_n|^{-1} |(D_n + \underline{p}) \cap D_n| = 1, \quad \forall \underline{p} \in \mathbb{Z}^d \text{ (Følner condition)}.$$

a) (Rectangles) For  $\underline{N} = (N_1, \dots, N_d) \in \mathbb{N}^d$  and  $D_{\underline{N}} := \{\underline{\ell} \in \mathbb{N}^d : \ell_i \leq N_i, i = 1, \dots, d\}$ , the kernels are the  $d$ -dimensional Fejér kernels  $K_{\underline{N}}(t_1, \dots, t_d) = K_{N_1}(t_1) \cdots K_{N_d}(t_d)$  on  $\mathbb{T}^d$ , where  $K_N(t)$  is the one-dimensional Fejér kernel  $\frac{1}{N} \left( \frac{\sin \pi N t}{\sin \pi t} \right)^2$ .

b) A family of examples satisfying (2) can be obtained by taking a non-empty domain  $D \subset \mathbb{R}^d$  with “smooth” boundary and finite area and putting  $D_n = \lambda_n D \cap \mathbb{Z}^d$ , where  $(\lambda_n)$  is an increasing sequence of real numbers tending to  $+\infty$ .

c) If  $(D_n)$  is a (non Følner) sequence of domains in  $\mathbb{Z}^d$  such that  $\lim_n \frac{|(D_n + \underline{p}) \cap D_n|}{|D_n|} = 0, \forall \underline{p} \neq 0$ , the associated kernel  $(\tilde{R}_n)$  is  $\zeta$ -regular, with  $\zeta$  the uniform measure on  $\mathbb{T}^d$ .

#### Examples 1.4. Sequential summation

Let  $(\underline{x}_k)_{k \geq 0}$  be a sequence in  $\mathbb{Z}^d$ . Putting  $\underline{z}_n := \sum_{k=0}^{n-1} \underline{x}_k$ ,  $n \geq 1$ , and  $R_n(\underline{\ell}) = \sum_{k=0}^{n-1} 1_{\underline{z}_k = \underline{\ell}}$  for  $\underline{\ell} \in \mathbb{Z}^d$ , we get a summation sequence of sequential type. Here,  $\sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) = n$ .

The associated sums are  $\sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) T^{\underline{\ell}} f = \sum_{k=0}^{n-1} T^{\underline{z}_k} f$ . Let us define

$$(3) \quad v_n = v_{n, \underline{0}} = \#\{0 \leq k' < k < n : \underline{z}_k = \underline{z}_{k'}\} = \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n^2(\underline{\ell}),$$

$$(4) \quad v_{n, \underline{p}} := \#\{0 \leq k' < k < n : \underline{z}_k - \underline{z}_{k'} = \underline{p}\} = \sum_{0 \leq k' < k < n} 1_{\underline{z}_k - \underline{z}_{k'} = \underline{p}}.$$

The quantity  $v_n$  is the number of “self-intersections” of  $(\underline{z}_k)_{k \geq 1}$  up to time  $n$ . We have

$$(5) \quad \tilde{R}_n(\underline{t}) = \sum_{\underline{p} \in \mathbb{Z}^d} \frac{v_{n, \underline{p}}}{v_{n, \underline{0}}} e^{2\pi i \langle \underline{p}, \underline{t} \rangle}, \quad \underline{t} \in \mathbb{T}^d,$$

and the  $\zeta$ -regularity of  $(\underline{z}_k)_{k \geq 1}$  is equivalent to  $\lim_n \frac{v_{n, \underline{p}}}{v_{n, \underline{0}}} = \hat{\zeta}(\underline{p}), \forall \underline{p} \in \mathbb{Z}^d$ . Remark that  $v_n - v_{n, \underline{p}} \geq 0$  by Cauchy-Schwarz inequality, since  $v_{n, \underline{p}} = \sum_{\underline{\ell}} R_n(\underline{\ell}) R_n(\underline{\ell} + \underline{p})$ .

In Sect. 2, we consider random sequential summation sequences defined by random walks.

##### 1.1.1. Reduction to the “genuine” dimension.

A reduction to the smallest lattice containing the support of the summation sequence is convenient and can be done using Minkowski lemma.

Below  $(e_1, \dots, e_d)$  is the canonical basis of  $\mathbb{R}^d$ . For  $r < d$ ,  $E_r$  (resp.  $\mathcal{E}_r$ ) is the lattice over  $\mathbb{Z}$  (resp. the vector space) generated by  $(e_1, \dots, e_r)$  and  $F_{d-r}$  the sublattice of  $\mathbb{Z}^d$  generated by  $(e_{r+1}, \dots, e_d)$ .

The set of non singular  $d \times d$ -matrices with coefficients in  $\mathbb{Z}$  is denoted by  $\mathcal{M}^*(d, \mathbb{Z})$ .

**Lemma 1.5.** a) Let  $L$  be a sublattice of  $\mathbb{Z}^d$ . If  $r \geq 1$  is the dimension of the  $\mathbb{R}$ -vector space  $\mathcal{L}$  spanned by  $L$ , there exists  $C \in SL(d, \mathbb{Z})$  such that  $\mathcal{L} = C\mathcal{E}_r$ .

b) If  $r = d$ , there exists  $B \in \mathcal{M}^*(d, \mathbb{Z})$ , such that  $L = B\mathbb{Z}^d$  and  $\text{Card}(\mathbb{Z}^d/L) = |\det(B)|$ .

*Proof.* a) The proof is by induction on  $r$ . By definition of  $\mathcal{L}$  we can find a non zero vector  $\underline{v} = (v_1, \dots, v_d)$  in  $\mathcal{L}$  with integral coordinates such that  $\text{gcd}(v_1, \dots, v_d) = 1$ .

Since a vector in  $\mathbb{Z}^d$  is extendable to a basis of  $\mathbb{Z}^d$  if and only if the gcd of its coordinates is 1, we can construct an integral matrix  $C_1$  such that  $\det C_1 = 1$  with  $\underline{v}$  as first column.

For  $\underline{w} \in L$ , let  $\tilde{\underline{w}} := \langle \underline{w}, \underline{w} \rangle \underline{v} - \langle \underline{v}, \underline{w} \rangle \underline{v}$ . The space  $\mathcal{L} \cap \underline{v}^\perp$  is generated by the lattice  $\{\tilde{\underline{w}}, \underline{w} \in L\}$  and has dimension  $r - 1$ . By the induction hypothesis, there is an integral matrix  $C_2$  with  $\det C_2 = 1$ , such that  $C_2 e_r = e_r$  and  $C_2 \mathcal{E}_{r-1} = C_1^{-1}(\mathcal{L} \cap \underline{v}^\perp)$ . Taking  $C = C_1 C_2$ , we have  $C\mathcal{E}_{r-1} = C_1 C_2 \mathcal{E}_{r-1} = \mathcal{L} \cap \underline{v}^\perp$ ,  $C e_r = C_1 C_2 e_r = C_1 e_r = \underline{v} \in \mathcal{L}$ .

b) The existence of  $B$  is equivalent to the existence in  $L$  of a set  $\{\underline{f}_1, \dots, \underline{f}_d\}$  of linearly independent vectors generating  $L$ . Let us construct such a set.

The set  $\{x_1 : \underline{x} = (x_1, \dots, x_d) \in L\}$  is an ideal, hence the set of multiples of some integer  $a$ . We can assume that  $a \neq 0$  (otherwise we permute the coordinates). Let  $\underline{f} = (f_1, \dots, f_d) \in L$  be such that  $f_1 = a$ . The set  $\{\underline{x} - a^{-1}x_1 \underline{f}, \underline{x} \in L\}$  is a sublattice  $L_0$  of  $L$  with 0 as first coordinate of every element. It is of dimension  $d - 1$  over  $\mathbb{R}$ , hence, by induction hypothesis, generated by linearly independent vectors  $\{\underline{f}_1, \dots, \underline{f}_{d-1}\}$  in  $L_0$ . Clearly  $\{\underline{f}_1, \dots, \underline{f}_{d-1}, \underline{f}\}$  is a set of linearly independent vectors generating  $L$ .

Let  $\mathcal{R}$  be a set of representatives of the classes of  $\mathbb{Z}^d$  modulo  $B\mathbb{Z}^d$  and let  $K$  be the unit cube in  $\mathbb{R}^d$ . Since if  $L$  is a cofinite lattice in  $\mathbb{Z}^d$ , two fundamental domains  $F_1, F_2$  in  $\mathbb{R}^d$  for  $L$  have the same measure:  $\text{Leb}(F_1) = \text{Leb}(F_2)$ . Taking  $F_1 = BK$  and  $F_2 = \bigcup_{\underline{r} \in \mathcal{R}} (K + \underline{r})$ , this implies  $|\det(B)| = \text{Leb}(BK) = \text{Leb}\left(\bigcup_{\underline{r} \in \mathcal{R}} (K + \underline{r})\right) = \text{Card}(\mathcal{R})$ .  $\square$

Let us consider the lattice  $L$  in  $\mathbb{Z}^d$  generated by the elements  $\underline{\ell} \in \mathbb{Z}^d$  such that  $R_n(\underline{\ell}) > 0$  for some  $n$ . For instance, for a sequential summation generated by  $(\underline{x}_k)_{k \geq 0}$ , the lattice  $L$  is the group generated in  $\mathbb{Z}^d$  by the vectors  $\underline{x}_k$ .

By the previous lemma, after a change of basis given by a matrix in  $SL(d, \mathbb{Z})$  and reduction, we can assume without loss of generality that  $d$  is the *genuine dimension*, i.e., that the vector space  $\mathcal{L}$  generated by  $L$  over  $\mathbb{R}$  has dimension  $d$ .

## 1.2. Spectral analysis, Lebesgue spectrum.

Recall that, if  $\mathcal{S}$  is an abelian group of unitary operators, for every  $f \in \mathcal{H}$  there is a positive finite measure  $\nu_f$  on  $\mathbb{T}^d$ , the spectral measure of  $f$ , with Fourier coefficients  $\hat{\nu}_f(\underline{\ell}) = \langle T^{\underline{\ell}} f, f \rangle$ ,  $\underline{\ell} \in \mathbb{Z}^d$ . When  $\nu_f$  is absolutely continuous, its density is denoted by  $\varphi_f$ .

In what follows, we assume that  $\mathcal{S}$  (isomorphic to  $\mathbb{Z}^d$ ) has the *Lebesgue spectrum property* for its action on  $\mathcal{H}$ , i.e., there exists a closed subspace  $\mathcal{K}_0$  such that  $\{T^{\underline{\ell}} \mathcal{K}_0, \underline{\ell} \in \mathbb{Z}^d\}$  is a family of pairwise orthogonal subspaces spanning a dense subspace in  $\mathcal{H}$ . If  $(\psi_j)_{j \in \mathcal{J}}$  is an orthonormal basis of  $\mathcal{K}_0$ ,  $\{T^{\underline{\ell}} \psi_j, j \in \mathcal{J}, \underline{\ell} \in \mathbb{Z}^d\}$  is an orthonormal basis of  $\mathcal{H}$ .

In Sect. 3, for algebraic automorphisms of compact abelian groups, the characters will provide a natural basis.

Let  $\mathcal{H}_j$  be the closed subspace generated by  $(T^\ell \psi_j)_{\ell \in \mathbb{Z}^d}$  and  $f_j$  the orthogonal projection of  $f$  on  $\mathcal{H}_j$ . We have  $\nu_f = \sum_j \nu_{f_j}$ . For  $j \in \mathcal{J}$ , we denote by  $\gamma_j$  an everywhere finite square integrable function on  $\mathbb{T}^d$  with Fourier coefficients  $a_{j,\ell} := \langle f, T^\ell \psi_j \rangle$ . A version of the density of the spectral measure corresponding to  $f_j$  is  $|\gamma_j|^2$ .

By orthogonality of the subspaces  $\mathcal{H}_j$ , it follows that, for every  $f \in \mathcal{H}$ ,  $\nu_f$  has a density  $\varphi_f$  in  $L^1(d\underline{t})$  which reads  $\varphi_f(\underline{t}) = \sum_{j \in \mathcal{J}} |\gamma_j(\underline{t})|^2$ .

Changing the system of independent generators induces composition of the spectral density by an automorphism acting on  $\mathbb{T}^d$ . After the reduction performed in 1.1.1 (Lemma 1.5) we obtain an action with Lebesgue spectrum with, possibly, a smaller dimension.

We have:  $\int_{\mathbb{T}^d} \sum_{j \in \mathcal{J}} |\gamma_j(\underline{t})|^2 d\underline{t} = \sum_{j \in \mathcal{J}} \sum_{\ell \in \mathbb{Z}^d} |a_{j,\ell}|^2 = \int_{\mathbb{T}^d} \varphi_f(\underline{t}) d\underline{t} = \|f\|_2^2 < \infty$ . For  $\underline{t}$  in  $\mathbb{T}^d$ , let  $M_{\underline{t}} f$  in  $\mathcal{K}_0$  be defined, when the series converges in  $\mathcal{H}$ , by:  $M_{\underline{t}} f := \sum_j \gamma_j(\underline{t}) \psi_j$ .

Under the condition  $\sum_{j \in \mathcal{J}} (\sum_{\ell} |a_{j,\ell}|)^2 < +\infty$ ,  $M_{\underline{t}} f$  is defined for every  $\underline{t}$ , the function  $\underline{t} \rightarrow \|M_{\underline{t}}\|_2^2$  is a continuous version of  $\varphi_f$ . For a general  $f \in \mathcal{H}$ , it is defined for  $\underline{t}$  in a set of full measure in  $\mathbb{T}^d$  (cf. [4]).

### Variance for summation sequences

Suppose that  $(R_n)$  is a  $\zeta$ -regular summation sequence. Let  $f$  be in  $L_0^2(\mu)$  with a continuous spectral density  $\varphi_f$ . By the spectral theorem, we have for  $\theta \in \mathbb{T}^d$ :

$$(6) \quad \left( \sum_{\underline{\ell}} R_n^2(\underline{\ell}) \right)^{-1} \left\| \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) e^{2\pi i \langle \underline{\ell}, \theta \rangle} T^{\underline{\ell}} f \right\|_2^2 = (\tilde{R}_n * \varphi_f)(\theta) \xrightarrow{n \rightarrow \infty} (\zeta * \varphi_f)(\theta).$$

*Examples.* 1) If  $(D_n)$  is a Følner sequence of sets in  $\mathbb{Z}^d$ , then (for  $\theta = 0$ )  $\zeta = \delta_0$  and we obtain the usual asymptotic variance  $\sigma^2(f) = \varphi_f(\underline{0})$ . More generally the usual asymptotic variance at  $\theta$  for the “rotated” ergodic sums  $R_n^\theta f$  is

$$\lim_n |D_n|^{-1} \left\| \sum_{\underline{\ell} \in D_n} e^{2\pi i \langle \underline{\ell}, \theta \rangle} T^{\underline{\ell}} f \right\|_2^2 = \lim_n (\tilde{R}_n * \varphi_f)(\theta) = (\delta_0 * \varphi_f)(\theta) = \varphi_f(\theta).$$

When  $(D_n)$  is a sequence of  $d$ -dimensional cubes, by the Fejér-Lebesgue theorem, for every  $f$  in  $\mathcal{H}$ , since  $\varphi_f \in L^1(\mathbb{T}^d)$ , for a.e.  $\theta$ , the variance at  $\theta$  exists and is equal to  $\varphi_f(\theta)$ .

2) Let  $(\underline{x}_k)_{k \geq 0}$  be a sequence in  $\mathbb{Z}^d$  and  $\underline{z}_n = \sum_{k=0}^{n-1} \underline{x}_k$ . If  $(\underline{z}_n)$  is  $\zeta$ -regular, then (cf. (5)):

$$\lim_n v_n^{-1} \left\| \sum_{k=0}^{n-1} T^{\underline{z}_k} f \right\|_2^2 = \lim_n \int \tilde{R}_n \varphi_f d\underline{t} = \zeta(\varphi_f).$$

## 2. Random walks

### 2.1. Random summation sequences (rws) defined by random walks.

For  $d \geq 1$  and  $J$  a set of indices, let  $\Sigma := \{\underline{\ell}_j, j \in J\}$  be a set of vectors in  $\mathbb{Z}^d$  and  $(p_j, j \in J)$  a probability vector with  $p_j > 0, \forall j \in J$ .



Let  $\nu$  denote the probability distribution with support  $\Sigma$  on  $\mathbb{Z}^d$  defined by  $\nu(\underline{\ell}) = p_j$  for  $\underline{\ell} = \underline{\ell}_j$ ,  $j \in J$ . The euclidian norm of  $\underline{\ell} \in \mathbb{Z}^d$  is denoted by  $|\underline{\ell}|$ .

Let  $(X_k)_{k \in \mathbb{Z}}$  be a sequence of i.i.d.  $\mathbb{Z}^d$ -valued random variables with common distribution  $\nu$ , i.e.,  $\mathbb{P}(X_k = \underline{\ell}_j) = p_j$ ,  $j \in J$ . The associated *random walk*  $W = (Z_n)$  in  $\mathbb{Z}^d$  (starting from  $\underline{0}$ ) is defined by  $Z_0 := \underline{0}$ ,  $Z_n := X_0 + \dots + X_{n-1}$ ,  $n \geq 1$ .

The characteristic function  $\Psi$  of  $X_0$  (computed with a coefficient  $2\pi$ ) is

$$(7) \quad \Psi(\underline{t}) = \mathbb{E}[e^{2\pi i \langle X_0, \underline{t} \rangle}] = \sum_{\underline{\ell} \in \mathbb{Z}^d} \mathbb{P}(X_0 = \underline{\ell}) e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle} = \sum_{j \in J} p_j e^{2\pi i \langle \underline{\ell}_j, \underline{t} \rangle}, \quad \underline{t} \in \mathbb{T}^d.$$

$$(8) \quad \text{It satisfies: } \mathbb{P}(Z_n = \underline{k}) = \int_{\mathbb{T}^d} \Psi^n(\underline{t}) e^{-2\pi i \langle \underline{k}, \underline{t} \rangle} d\underline{t}.$$

If  $(\Omega, \mathbb{P})$  denotes the product space  $(\mathbb{Z}^d)^{\mathbb{Z}}, \nu^{\otimes \mathbb{Z}}$  with coordinates  $(X_n)_{n \in \mathbb{Z}}$  and  $\tau$  the shift, then  $W$  can be viewed as the cocycle generated by the function  $\omega \rightarrow X_0(\omega)$  under the action of  $\tau$ , i.e.,  $Z_n = \sum_{k=0}^{n-1} X_0 \circ \tau^k$ ,  $n \geq 1$ .

Let  $T_1, \dots, T_d$  be  $d$  commuting unitary operators on a Hilbert space  $\mathcal{H}$  generating a representation of  $\mathbb{Z}^d$  in  $\mathcal{H}$  with Lebesgue spectrum. Let  $(Z_n)_{n \geq 1} = ((Z_n^1, \dots, Z_n^d))_{n \geq 1}$  be a r.w. with values in  $\mathbb{Z}^d$ . We consider the random sequence of unitary operators  $(T^{Z_n})_{n \geq 1}$ , with  $T^{Z_n} = T_1^{Z_n^1} \dots T_d^{Z_n^d}$ . For  $f \in \mathcal{H}$ , the “quenched” process (for  $\omega$  fixed) of the ergodic sums along the random walk is

$$(9) \quad \sum_{k=0}^{n-1} T^{Z_k(\omega)} f = \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\omega, \underline{\ell}) T^{\underline{\ell}} f, \quad \text{with } R_n(\omega, \underline{\ell}) = \sum_{k=0}^{n-1} 1_{Z_k(\omega) = \underline{\ell}}, \quad n \geq 1.$$

The sequence of “local times”  $(R_n(\omega, \underline{\ell}))$  will be called a *random walk sequential summation* (abbreviated in “r.w. summation” or in “rws”). It satisfies  $\sum_{\underline{\ell}} R_n(\omega, \underline{\ell}) = n$ .

We consider also the (Markovian) barycenter operator  $P : f \rightarrow \sum_{j \in J} p_j T^{\underline{\ell}_j} f = \mathbb{E}_{\mathbb{P}} [T^{X_0(\cdot)} f]$  and its powers:

$$(10) \quad P^n f = \mathbb{E}_{\mathbb{P}} [T^{Z_n(\cdot)} f] = \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) T^{\underline{\ell}} f, \quad \text{with } R_n(\underline{\ell}) = \mathbb{P}(Z_n = \underline{\ell}).$$

When the operators  $T_j$  are given by measure preserving transformations of a probability space  $(X, \mathcal{A}, \mu)$ , we call *quenched limit theorem* a limit theorem for (9) w.r.t.  $\mu$ . If the limit is taken w.r.t.  $\mathbb{P} \times \mu$ , it is called *annealed limit theorem*.

**Notations 2.1.** The random versions of (3) and (4) are

$$(11) \quad V_n(\omega) := \#\{0 \leq k' < k < n : Z_k(\omega) = Z_{k'}(\omega)\} = \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n^2(\omega, \underline{\ell}),$$

$$(12) \quad V_{n, \underline{p}}(\omega) := \#\{0 \leq k' < k < n : Z_k(\omega) - Z_{k'}(\omega) = \underline{p}\} = \sum_{0 \leq k' < k < n} 1_{Z_k(\omega) - Z_{k'}(\omega) = \underline{p}}.$$

$V_n(\omega) = V_{n, \underline{0}}(\omega)$  is the number of “self-intersections” of  $W$ .  $V_{n, \underline{p}}(\omega)$  is a sum of ergodic sums.

We have:

$$(13) \quad V_{n,\underline{p}}(\omega) = n1_{\underline{p}=\underline{0}} + \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} (1_{Z_k(\tau^j\omega)=\underline{p}} + 1_{Z_k(\tau^j\omega)=-\underline{p}});$$

$$(14) \quad \mathbb{E}V_{n,\underline{p}} = n1_{\underline{p}=\underline{0}} + \sum_{k=1}^{n-1} (n-k) [\mathbb{P}(Z_k = \underline{p}) + \mathbb{P}(Z_k = -\underline{p})].$$

### Variance for quenched processes

For a r.w. summation  $R_n(\omega, \underline{\ell}) = \sum_{k=0}^{n-1} 1_{Z_k(\omega)=\underline{\ell}}$ , let  $\underline{R}_n^\omega(t)$  and  $\tilde{R}_n^\omega(t)$  denote, respectively, the non normalized and normalized kernels (cf. (1))

$$(15) \quad \underline{R}_n^\omega(\underline{t}) = \left| \sum_{k=0}^{n-1} e^{2\pi i \langle Z_k(\omega), \underline{t} \rangle} \right|^2 = \left| \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\omega, \underline{\ell}) e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle} \right|^2, \quad \tilde{R}_n^\omega(\underline{t}) = \underline{R}_n^\omega(\underline{t}) / \int_{\mathbb{T}^d} \underline{R}_n^\omega dt.$$

Recall that, if  $\varphi_f$  is the spectral density of  $f \in \mathcal{H}$  for the action of  $\mathbb{Z}^d$ , we have

$$\left\| \sum_{k=1}^n T^{Z_k(\omega)} f \right\|^2 = \int_{\mathbb{T}^d} \underline{R}_n^\omega(t) \varphi_f(\underline{t}) dt$$
 by the spectral theorem (cf. (6)) and

$$(16) \quad \int_{\mathbb{T}^d} \frac{1}{n} \underline{R}_n^\omega(t) e^{-2\pi i \langle \underline{p}, \underline{t} \rangle} dt = \frac{1}{n} V_{n,\underline{p}}(\omega) = 1_{\underline{p}=\underline{0}} + \sum_{k=1}^{n-1} \frac{1}{n} \sum_{j=0}^{n-k-1} [1_{Z_k(\tau^j\omega)=\underline{p}} + 1_{Z_k(\tau^j\omega)=-\underline{p}}].$$

A rws defined by a r.w.  $(Z_n)$  is said to be  $\zeta$ -regular if  $(\tilde{R}_n^\omega)_{n \geq 1}$  weakly converges to a probability measure  $\zeta$  (not depending on  $\omega$ ) on  $\mathbb{T}^d$  for a.e.  $\omega$ .

This is equivalent to  $\lim_{n \rightarrow \infty} \frac{V_{n,\underline{p}}(\omega)}{V_n(\omega)} = \hat{\zeta}(\underline{p})$ , for a.e.  $\omega$  and all  $\underline{p} \in \mathbb{Z}^d$  (see Definition 1.2 and Example 1.4). After auxiliary results on r.w.'s and self-intersections, we will show that every rws is  $\zeta$ -regular for some measure  $\zeta$ .

## 2.2. Auxiliary results on random walks.

We recall now some general results on random walks in  $\mathbb{Z}^d$  (cf. [33]). Most of them are classical. Nevertheless, since we do not assume strict aperiodicity and in order to explicit the constants in the limit theorems, we include reminders and some proofs. We start with definitions and preliminary results.

### 2.2.1. Reduced form of a random walk .

Let  $L(W)$  (or simply  $L$ ) be the sublattice of  $\mathbb{Z}^d$  generated by the support  $\Sigma = \{\underline{\ell}_j, j \in J\}$  of  $\nu$ . From now on (without loss of generality, as shown by Lemma 1.5), we assume that  $L(W)$  is cofinite in  $\mathbb{Z}^d$  and we say that  $W$  is reduced (in other words ‘‘genuinely  $d$ -dimensional’’, cf. [33]). Therefore the vector space  $\mathcal{L}$  generated by  $L$  is  $\mathbb{R}^d$  and there is  $B \in \mathcal{M}^*(d, \mathbb{Z})$  such that  $L = B\mathbb{Z}^d$ . The rank  $d$  will be sometimes denoted by  $d(W)$ .

Let  $D = D(W)$  be the sublattice of  $\mathbb{Z}^d$  generated by  $\{\underline{\ell}_j - \underline{\ell}_{j'}, j, j' \in J\}$ ,  $\mathcal{D}$  the vector subspace of  $\mathbb{R}^d$  generated by  $D$  and  $\mathcal{D}^\perp$  its orthogonal supplementary in  $\mathbb{R}^d$ . We denote

$\dim(\mathcal{D})$  by  $d_0(W)$ . With the notations of 1.1.1, by Lemma 1.5, there is  $B_1 \in \mathcal{M}^*(d, \mathbb{Z})$  such that  $D = B_1 \mathbb{Z}^d$  if  $d_0(W) = d$  and  $D = B_1 F_{d-1}$  if  $d_0(W) = d - 1$ .

**Lemma 2.2.** *a) The quotient group  $L(W)/D(W)$  is cyclic (each  $\ell_j$  is a generator of the group). It is finite if and only if  $d_0(W) = d(W)$ .*

*b) If  $W$  is reduced, then  $d_0(W) = d(W)$  or  $d(W) - 1$ .*

*c) If  $W$  has a moment of order 1 and is centered, then  $d_0(W) = d(W)$  and  $D$  and  $L$  generate the same vector subspace.*

*Proof.* The point a) is clear, since  $\ell_j = \ell_1 \bmod D, \forall j \in J$ . For b), suppose  $\mathcal{D}^\perp \neq \{\underline{0}\}$ . There is  $v_0 \in \mathcal{D}^\perp$  such that  $\langle v_0, \ell_1 \rangle = 1$ . If  $v$  is in  $\mathcal{D}^\perp$ , then  $\langle v - \langle v, \ell_1 \rangle v_0, \ell_j \rangle = 0$ , which implies  $v = \langle v, \ell_1 \rangle v_0$ , since  $\mathcal{L} = \mathbb{R}^d$ .

c) If  $v \in \mathcal{D}^\perp$ , then  $\langle v, \ell_j \rangle = \langle v, \ell_1 \rangle = \sum_i p_i \langle v, \ell_i \rangle = \langle v, \sum_i p_i \ell_i \rangle = 0, \forall j$ ; hence  $v = \underline{0}$ .  $\square$

**Recurrence/transience.** Recall that a r.w.  $W = (Z_n)$  is recurrent if and only if  $\sum_{n=1}^\infty \mathbb{P}(Z_n = \underline{0}) = +\infty$ . If  $\sum p_j |\ell_j| < \infty$ , then  $W$  is transient if  $\sum p_j \ell_j \neq 0$  and, if  $d = 1$ , recurrent if  $\sum p_j \ell_j = 0$ . These last two results are special cases of a general result for cocycle over an ergodic dynamical system. A r.w.  $W$  is transient or recurrent according as  $\Re(\frac{1}{1-\Psi})$  is integrable on the  $d$ -dimensional unit cube or not ([33]).

The local limit theorem (LLT) (cf. Theorem 2.6) implies that a (reduced) r.w.  $W$  with finite variance is recurrent if and only if it is centered and  $d(W) = 1$  or 2.

For further references, let us introduce the following condition for a r.w.  $W$ :

(17)  *$W$  is reduced, has a moment of order 2 and is centered.*

If  $W$  (reduced) has a moment of order 2, either it is centered with  $d(W) \leq 2$  (hence recurrent) or transient. In the first case  $d_0(W) = d$ , in the second  $d_0(W) = d$  or  $d - 1$ .

**Annulators of  $L$  and  $D$  in  $\mathbb{T}^d$ .** The values of  $\underline{t} \in \mathbb{T}^d$  such that  $\Psi(\underline{t}) = 1$  or  $|\Psi(\underline{t})| = 1$  play an important role. They are characterized in the following lemma, where  $B$  and  $B_1$  are the matrices introduced at the beginning of 2.2.1. Recall that the annulator of a sublattice  $L$  in  $\mathbb{Z}^d$  is the closed subgroup  $\{\underline{t} \in \mathbb{T}^d : e^{2\pi i \langle \underline{r}, \underline{t} \rangle} = 1, \forall \underline{r} \in L\}$ .

**Notation 2.3.** We denote by  $\Gamma$  (resp.  $\Gamma_1$ ) the annulator of  $L$  (resp.  $D$ ), by  $d\gamma$  (resp.  $d\gamma_1$ ) the Haar probability measure of the group  $\Gamma$  (resp.  $\Gamma_1$ ).

**Lemma 2.4.** *1) We have  $\Gamma = \{\underline{t} \in \mathbb{T}^d : \Psi(\underline{t}) = 1\} = (B^t)^{-1} \mathbb{Z}^d \bmod \mathbb{Z}^d$ . The group  $\Gamma$  is finite and  $\text{Card}(\Gamma) = |\det B|$ .*

*2a) We have  $\Gamma_1 = \{\underline{t} \in \mathbb{T}^d : |\Psi(\underline{t})| = 1\} = \widetilde{\mathcal{D}}^\perp + (B_1^t)^{-1} \mathbb{Z}^d \bmod \mathbb{Z}^d$  with  $\widetilde{\mathcal{D}}^\perp := \mathcal{D}^\perp / \mathbb{Z}^d$ , which is either trivial or a 1-dimensional torus in  $\mathbb{T}^d$ . The quotient  $\Gamma_1 / \widetilde{\mathcal{D}}^\perp$  is finite and  $a_0(W) := \text{Card}(\Gamma_1 / \widetilde{\mathcal{D}}^\perp) = |\det B_1|$ .*

*2b) If  $\underline{p} \in D + n\ell_1$ , then the function  $F_{n,\underline{p}}(\underline{t}) := \Psi(\underline{t})^n e^{-2\pi i \langle \underline{p}, \underline{t} \rangle}$  is invariant by translation by elements of  $\Gamma_1$ . Its integral is 0 if  $\underline{p} \notin D + n\ell_1$ . We have  $|F_{n,\underline{p}}(\underline{t})| < 1, \forall \underline{t} \notin \Gamma_1$ , and  $F_{n,\underline{p}}(\underline{t}) = 1, \forall \underline{t} \in \Gamma_1$ .*

*Proof.* We prove 2). The proof of 1) is analogous. We have  $|\Psi(\underline{t})| \leq 1$  and by strict convexity,  $|\Psi(\underline{t})| = 1$  if and only if  $e^{2\pi i \langle \ell_j, \underline{t} \rangle} = e^{2\pi i \langle \ell_1, \underline{t} \rangle}, \forall j \in J$ , i.e., if and only if  $\underline{t} \in \Gamma_1$ .

Recall that  $D = B_1\mathbb{Z}^d$  or  $B_1F_{d-1}$ . Let us treat the second case. By orthogonality,  $\mathcal{D}^\perp = (B_1^t)^{-1}\mathcal{E}_1$ . If  $\underline{t} \in \Gamma_1$ ,  $e^{2\pi i\langle \underline{r}, \underline{t} \rangle} = 1$ ,  $\forall \underline{r} \in D$ , so that, for  $j = 2, \dots, d$ ,  $e^{2\pi i\langle B_1 e_j, \underline{t} \rangle} = 1$ , hence  $\langle e_j, B_1^t \underline{t} \rangle \in \mathbb{Z}$ . It follows that the  $d - 1$  last coordinates of  $B_1^t \underline{t}$  are integers. Therefore,  $B_1^t \underline{t} \in \mathcal{E}_1 \bmod \mathbb{Z}^d$ , i.e.,  $\underline{t} \in (B_1^t)^{-1}\mathcal{E}_1 + \Gamma_0 = \mathcal{D}^\perp + \Gamma_0 \bmod \mathbb{Z}^d$ , where  $\Gamma_0 := (B_1^t)^{-1}\mathbb{Z}^d \bmod \mathbb{Z}^d$  and  $\widetilde{\mathcal{D}}^\perp = \mathcal{D}^\perp$  modulo  $\mathbb{Z}^d$  is a closed 1-dimensional subtorus of  $\mathbb{T}^d$ .

By Lemma 1.5.b, we have  $\text{Card}(\Gamma_0) = \text{Card}(\mathbb{Z}^d/B_1^t\mathbb{Z}^d) = |\det(B_1)|$ . For a (reduced) centered r.w.,  $\Gamma_0 = \Gamma_1$  and  $\text{Card}(\Gamma_1) = |\det(B_1)|$ .

2b) Since  $F_{n,\underline{p}}(\underline{t} + \underline{t}_0) = e^{i\langle n\underline{\ell}_1 - \underline{p}, \underline{t}_0 \rangle} F_{n,\underline{p}}(\underline{t})$  for  $\underline{t}_0 \in \Gamma_1$ ,  $F_{n,\underline{p}}$  is invariant by all  $\underline{t}_0 \in \Gamma_1$  if  $\underline{p} \in D + n\underline{\ell}_1$ , and its integral is 0 if  $\underline{p} \notin D + n\underline{\ell}_1$ .  $\square$

For a reduced r.w.  $W$ , there are  $(\underline{r}_j, j \in J)$  in  $\mathbb{Z}^d$  such that  $\underline{\ell}_j = B \underline{r}_j$  and the lattice generated by  $(\underline{r}_j, j \in J)$  is  $\mathbb{Z}^d$ . The r.w. is said to be *aperiodic* if  $L = \mathbb{Z}^d$ , *strongly aperiodic* if  $D = \mathbb{Z}^d$ . If we replace  $W$  by the r.w.  $W'$  defined by the r.v.  $X'_0$  such that  $\mathbb{P}(X'_0 = \underline{r}_j) = p_j$ , we obtain an aperiodic r.w. This allows in several proofs to assume aperiodicity without loss of generality when the r.w. is reduced (for example see Theorem 2.10 in Sect. 6).

Strong aperiodicity is equivalent to  $|\Psi(\underline{t})| < 1$  for  $\underline{t} \neq \underline{0} \bmod \mathbb{Z}^d$  (cf. [33] and Lemma 2.4). It is also equivalent to: for every  $\underline{\ell} \in \mathbb{Z}^d$ , the additive group generated by  $\Sigma + \underline{\ell}$  is  $\mathbb{Z}^d$ . It implies  $d_0(W) = d(W)$  and  $a_0(W) = 1$ . Observe that, contrary to aperiodicity, it is not always possible to reduce proofs to the strictly aperiodic case.

A r.w. is “deterministic” if  $\mathbb{P}(X_0 = \underline{\ell}_0) = 1$  for some  $\underline{\ell}_0$ , so that  $|\Psi(\underline{t})| \equiv 1$  in this case.

## Quadratic form

**Lemma 2.5.** *Let  $W$  satisfy (17). Let  $Q$  be the quadratic form  $Q(\underline{u}) = \text{Var}(\langle X_0, \underline{u} \rangle) = \sum_{j \in J} p_j \langle \underline{\ell}_j, \underline{u} \rangle^2$  and  $\Lambda$  the corresponding symmetric matrix. Then  $Q$  is definite positive. If  $\text{Card}(J) = d + 1$ , then  $\det(\Lambda) = c \prod_{j=1}^{\text{Card}(J)} p_j$ , where  $c$  does not depend on the  $p_j$ 's.*

*Proof.* If  $Q(\underline{u}) = 0$ , then  $\langle X_0, \underline{u} \rangle$  is a.e. constant, i.e.,  $\langle \underline{\ell}_i, \underline{u} \rangle = \langle \underline{\ell}_j, \underline{u} \rangle$  for all  $i, j \in J$ ; hence  $\underline{u} = 0$ , since (17) implies  $\mathcal{D} = \mathbb{R}^d$ .

Now, if  $d' = \text{Card}(J)$  is finite, let  $J'$  be the set of indices  $\{2, \dots, d'\}$ . The quadratic form  $q(\underline{u}) := \sum_{j \in J'} p_j u_j^2 - (\sum_{j \in J'} p_j u_j)^2$ ,  $\underline{u} \in \mathbb{R}^{d'-1}$ , is defined by the symmetric matrix

$$A = D_{d'} M, \text{ where } D_{d'} := \text{diag}(p_2, \dots, p_{d'}) \text{ and } M := \begin{pmatrix} 1 - p_2 & -p_3 & \cdot & -p_{d'} \\ -p_2 & 1 - p_3 & \cdot & -p_{d'} \\ \cdot & \cdot & \cdot & \cdot \\ -p_2 & -p_3 & \cdot & 1 - p_{d'} \end{pmatrix}.$$

By subtracting line from line in the matrix  $M$ , we find  $\det(M) = 1 - \sum_{j \in J'} p_j$  and therefore  $\det(A) = \prod_{j=1}^{d'} p_j$ . The quadratic form  $q$  is positive definite since

$$\sum_{j \in J'} p_j u_j^2 - \left( \sum_{j \in J'} p_j u_j \right)^2 - \sum_{2 \leq j' < j \leq d'} p_j p_{j'} (u_j - u_{j'})^2 = \left( 1 - \sum_{j \in J'} p_j \right) \sum_{j \in J'} p_j u_j^2.$$

Let  $U$  be the map from  $\mathbb{R}^d$  to  $\mathbb{R}^{d'-1}$ :  $\underline{v} \rightarrow U\underline{v}$ , where  $U\underline{v}$  is the vector with coordinates  $\langle \underline{\ell}_j - \underline{\ell}_1, \underline{v} \rangle$ ,  $j \in J'$ . The quadratic form  $Q$  can be written  $Q(\underline{u}) = \sum_{j \in J'} p_j \langle \underline{\ell}_j - \underline{\ell}_1, \underline{u} \rangle^2 - (\sum_{j \in J'} p_j \langle \underline{\ell}_j - \underline{\ell}_1, \underline{u} \rangle)^2 = q(U\underline{u})$ . Since  $\mathcal{D} = \mathbb{R}^d$ ,  $U$  is injective, hence an isomorphism if and only if  $\dim \mathcal{D} = d = d' - 1$ .

If  $d = d' - 1$ , the determinant of  $\Lambda = U^t A U$  is  $c \prod_{j=1}^{d+1} p_j$ . The integer  $c$  does not depend on the probability vector  $(p_j)_{j \in J}$ . We have  $c = \det(\tilde{U})^2$  where, for  $j = 1, \dots, d$ , the matrix  $\tilde{U}$  representing  $U$  has as  $j$ -th row the  $d$  coordinates of the vector  $\underline{\ell}_{(j+1)} - \underline{\ell}_1$ .  $\square$

### 2.2.2. Local limit theorem.

Let  $W = (Z_n)$  be a reduced random walk in  $\mathbb{Z}^d$  associated to the distribution  $\mathbb{P}(X_0 = \underline{\ell}_j) = p_j$ ,  $j \in J$ , with a finite second moment. Recall that  $a_0(W)$  and  $\Lambda$  are defined in Lemmas 2.4 and 2.5. The local limit theorem (LLT) gives an equivalent of  $\mathbb{P}(Z_n = \underline{k})$  when  $n$  tends to infinity:

**Theorem 2.6.** *Suppose  $W$  centered. If  $\underline{k} \notin D + n\underline{\ell}_1$ , then  $\mathbb{P}(Z_n = \underline{k}) = 0$ . If  $\underline{k} \in D + n\underline{\ell}_1$ , then we have with  $\sup_{\underline{k}} |\varepsilon_n(\underline{k})| = o(1)$ :*

$$(18) \quad (2\pi n)^{\frac{d}{2}} \mathbb{P}(Z_n = \underline{k}) = a_0(W) \det(\Lambda)^{-\frac{1}{2}} e^{-\frac{1}{2n} \langle \Lambda^{-1} \underline{k}, \underline{k} \rangle} + \varepsilon_n(\underline{k}).$$

**Remarks 2.7.** 1) If  $(Z_n)$  is strongly aperiodic centered with finite second moment, then  $\lim_{n \rightarrow \infty} [(2\pi n)^{\frac{d}{2}} \mathbb{P}(Z_n = \underline{k})] = \det(\Lambda)^{-\frac{1}{2}}$  ([33], P.10). A version of the LLT in the centered case, extending a result of Pólya, was proved by van Kampen and Wintner (1939) in [18] without the strong aperiodicity assumption.

2) By Lemma 2.5, if  $J$  is finite and  $\text{Card}(J) = d + 1$ , then  $\det(\Lambda) = c \prod_{j=1}^{\text{Card}(J)} p_j$ , where  $c$  is an integer  $\geq 1$  not depending on the  $p_j$ 's.

3) It follows from (18) that, for every  $\underline{k}$ , there is  $N(\underline{k})$  such that, for  $n \geq N(\underline{k})$ ,  $\mathbb{P}(Z_n = \underline{k}) > 0$  if and only if  $\underline{k} \in D + n\underline{\ell}_1$ . It implies also  $\sup_{\underline{k} \in \mathbb{Z}^d} \mathbb{P}(Z_n = \underline{k}) = O(n^{-\frac{d}{2}})$ .

4) The condition  $\underline{k} \in D + n\underline{\ell}_1$  reads  $n\underline{\ell}_1 = \underline{k} \bmod D$ . Since here  $d_0(W) = d(W)$ ,  $L/D$  is a cyclic finite group by Lemma 2.2 and the set  $E(\underline{k}) := \{n \geq 1 : \underline{k} \in D + n\underline{\ell}_1\}$  for a given  $\underline{k}$  is an arithmetic progression.

5) Let us mention a general version of the LLT which can be proved under the assumption of finite second moment for a centered or non centered r.w.

The quadratic form  $Q$  reads now:  $Q(\underline{u}) = \sum_{j \in J} p_j \langle \underline{\ell}_j, \underline{u} \rangle^2 - (\sum_{j \in J} p_j \langle \underline{\ell}_j, \underline{u} \rangle)^2$ . If  $\Lambda$  is the self-adjoint operator such that  $Q(\underline{u}) = \langle \Lambda \underline{u}, \underline{u} \rangle$ , then  $\mathcal{D}$  is invariant by  $\Lambda$  and the restriction of  $Q$  to  $\mathcal{D}$  is definite positive. We denote by  $\Lambda_0$  the restriction of  $\Lambda$  to  $\mathcal{D}$ .

**Theorem 2.8.** *For  $\underline{k} \in \mathbb{Z}^d$ , let  $z_{n,\underline{k}} := \frac{\underline{k}}{\sqrt{n}} - \sqrt{n} \mathbb{E}(X_0) = \frac{\underline{k}}{\sqrt{n}} - \sqrt{n} \sum_j p_j \underline{\ell}_j$ . If  $\underline{k} \notin D + n\underline{\ell}_1$ , then  $\mathbb{P}(Z_n = \underline{k}) = 0$ . If  $\underline{k} \in D + n\underline{\ell}_1$ , then, with  $\sup_{\underline{k}} |\varepsilon_n(\underline{k})| = o(1)$ :*

$$(19) \quad (2\pi n)^{\frac{d_0(W)}{2}} \mathbb{P}(Z_n = \underline{k}) = a_0(W) \det(\Lambda_0)^{-\frac{1}{2}} e^{-\frac{1}{2} \langle \Lambda_0^{-1} z_{n,\underline{k}}, z_{n,\underline{k}} \rangle} + \varepsilon_n(\underline{k}).$$

*If the moment of order 3 is finite, then  $\sup_{\underline{k}} |\varepsilon_n(\underline{k})| = O(n^{-\frac{1}{2}})$ .*

In the non centered case, when  $z_{n,\underline{k}}$  is not too large, for example when  $e^{-\langle \Lambda_0^{-1} z_{n,\underline{k}}, z_{n,\underline{k}} \rangle} \geq 2 \sup_{\underline{k}} |\varepsilon_n(\underline{k})|$ , Equation (19) gives an information on how the r.w. spreads around the drift. With a finite third moment, it shows that, for any constant  $C$ , for  $n$  large, the r.w. takes with a positive probability the values  $\underline{k}$  such that  $\underline{k} \in D + n\underline{\ell}_1$  which belong to a ball of radius  $C\sqrt{n}$  centered at the drift  $n\mathbb{E}(X_0)$ .

**2.2.3. Upper bound for  $\Phi_n(\omega) := \sup_{\underline{\ell} \in \mathbb{Z}^d} R_n(\omega, \underline{\ell}) = \sup_{\underline{\ell} \in \mathbb{Z}^d} \sum_{k=0}^{n-1} 1_{Z_k(\omega)=\underline{\ell}}$ .**

For the quenched CLT, we need some results on the local times and on the number of self-intersections of a r.w.  $W$ .

**Proposition 2.9.** (cf. [3]) *a) If the r.w.  $W$  has a moment of order 2, then*

$$(20) \quad \text{for } d = 1 : \Phi_n(\omega) = o(n^{\frac{1}{2}+\varepsilon}); \quad \text{for } d = 2 : \Phi_n(\omega) = o(n^\varepsilon), \forall \varepsilon > 0.$$

*b) If  $W$  is transient with a moment of order  $\eta$  for some  $\eta > 0$ , then  $\Phi_n(\omega) = o(n^\varepsilon), \forall \varepsilon > 0$ .*

*Proof.* Let  $\mathcal{A}_n^r := \{\underline{\ell} \in \mathbb{Z}^d : |\underline{\ell}| \leq n^r\}$ . If  $\mathbb{E}(\|X_0\|^\eta) < \infty$  for  $\eta > 0$ , then for  $r\eta > 1$ ,  $\sum_{k=1}^\infty \mathbb{P}(|X_k| > k^r) \leq (\sum_{k=1}^\infty k^{-r\eta})\mathbb{E}(\|X_0\|^\eta) < \infty$ . By Borel-Cantelli lemma, it follows  $|X_k| \leq k^r$  a.s. for  $k$  big enough. Therefore there is  $N(\omega)$  a.e. finite such that:  $|X_0 + \dots + X_{n-1}| \leq n^{r+1}$  for  $n > N(\omega)$ . Hence  $\sup_{\underline{\ell} \in \mathbb{Z}^d} R_n^m(\omega, \underline{\ell}) = \sup_{\underline{\ell} \in \mathcal{A}_n^{r+1}} R_n^m(\omega, \underline{\ell})$  for  $n \geq N(\omega)$ .

a) We take  $r = 1$ . For all  $m \geq 1$  and for constants  $C_m, C'_m$  independent of  $\underline{\ell}$ , we have:

$$(21) \quad \mathbb{E}[R_n^m(\cdot, \underline{\ell})] = \mathbb{E}\left[\sum_{k=0}^{n-1} 1_{Z_k=\underline{\ell}}\right]^m \leq C_m n^{m/2}, \text{ for } d = 1, \text{ and } \leq C'_m (\text{Log } n)^m, \text{ for } d = 2.$$

To show (21), it suffices to bound  $\sum_{0 \leq k_1 < k_2 < \dots < k_m < n} \mathbb{P}(Z_{k_1} = \underline{\ell}, Z_{k_2} = \underline{\ell}, \dots, Z_{k_m} = \underline{\ell})$ . By independence and stationarity, with  $F_n(k) = 1_{[0, n-1](k)} \mathbb{P}(Z_k = \underline{\ell})$  the preceding sum is

$$\sum_{0 \leq k_1 < k_2 < \dots < k_m < n} \mathbb{P}(Z_{k_1} = \underline{\ell}) \mathbb{P}(Z_{k_2-k_1} = \underline{\ell}) \cdots \mathbb{P}(Z_{k_m-k_{m-1}} = \underline{\ell}) \leq C \sum_k (F_n * F_n * \dots * F_n)(k),$$

hence bounded by  $C(\sum_k F_n(k))^m$  and (21) follows from the bound given by the LLT.

By (21),  $\mathbb{E}[\sup_{\underline{\ell} \in \mathcal{A}_n^2} R_n^m(\omega, \underline{\ell})]$  is less than  $n^2 \sup_{\underline{\ell}} \mathbb{E}[R_n^m(\omega, \underline{\ell})] \leq C_m n^2 \cdot n^{m/2}$  for  $d = 1$  and less than  $n^4 \sup_{\underline{\ell}} \mathbb{E}[R_n^m(\omega, \underline{\ell})] \leq C_m n^4 (\text{Log } n)^m$  for  $d = 2$ . Hence, for all  $\varepsilon > 0$ ,

$$\text{for } d = 1, \quad \sum_{n=1}^\infty \mathbb{E}[n^{-(\frac{1}{2}+\varepsilon)} \sup_{\underline{\ell} \in \mathcal{A}_n^d} R_n(\omega, \underline{\ell})]^m \leq C_m \sum_{n=1}^\infty n^{2+m/2} / n^{(\frac{1}{2}+\varepsilon)m} < \infty, \text{ if } m > 3/\varepsilon.$$

$$\text{for } d = 2, \quad \sum_{n=1}^\infty \mathbb{E}[n^{-\varepsilon} \sup_{\underline{\ell} \in \mathcal{A}_n^d} R_n(\omega, \underline{\ell})]^m \leq C_m \sum_{n=1}^\infty n^4 (\text{Log } n)^m / n^{\varepsilon m} < \infty, \text{ if } m > 5/\varepsilon.$$

Since  $\sup_{\underline{\ell} \in \mathbb{Z}^d} R_n^m(\omega, \underline{\ell}) = \sup_{\underline{\ell} \in \mathcal{A}_n^2} R_n^m(\omega, \underline{\ell})$  for  $n \geq N(\varepsilon)$ , this proves a).

b) According to  $\sum_{k=0}^\infty \mathbb{P}(Z_k = \underline{p}) < +\infty$ , using the same method as above, it can be shown that there are constants  $C_m$  and  $M$  such that  $\mathbb{E}[R_n^m(\cdot, \underline{\ell})] \leq C_m M^m$ . From the existence of a moment of order  $\eta > 0$ , there is  $r$  and  $N(\omega) < +\infty$  a.e. such that  $\sup_{\underline{\ell} \in \mathbb{Z}^d} R_n^m(\omega, \underline{\ell}) = \sup_{\underline{\ell} \in \mathcal{A}_n^{r+1}} R_n^m(\omega, \underline{\ell})$  for  $n \geq N(\omega)$ .

In view of  $\mathbb{E}[\sup_{\underline{\ell} \in \mathcal{A}_r^n} R_n^m(\omega, \underline{\ell})] \leq n^{d(r+1)} \sup_{\underline{\ell}} \mathbb{E}[R_n^m(\omega, \underline{\ell})] \leq C_m n^{d(r+1)} M^m$ , for every  $\varepsilon > 0$  there is  $m$  such that  $\sum_{n=1}^{\infty} \mathbb{E}[n^{-\varepsilon} \sup_{\underline{\ell} \in \mathcal{A}_r^n} R_n(\omega, \underline{\ell})]^m < \infty$ . This implies *b*).  $\square$

#### 2.2.4. Self-intersections.

##### 1) Recurrent case

The proof of the following result is postponed to Sect. 6.

**Theorem 2.10.** *Let  $W$  satisfy (17) and let  $\underline{p}$  be in  $L(W)$ . If  $d(W) = 1$  or 2,  $\lim_n \frac{V_{n,\underline{p}}(\omega)}{V_n(\omega)} = 1$  a.e. If  $d(W) = 2$ ,  $V_{n,\underline{p}}$  satisfies a SLLN:  $\lim_n V_{n,\underline{p}}(\omega)/\mathbb{E}V_{n,\underline{p}} = 1$  a.e.*

##### 2) Transient case

In the transient case (without moment assumptions), a general argument is available:

**Lemma 2.11.** *If  $(\Omega, \mathbb{P}, \tau)$  is an ergodic dynamical system and  $(f_k)_{k \geq 1}$  a sequence of functions in  $L^1(\Omega, \mathbb{P})$  such that  $\sum_{k \geq 1} \|f_k\|_r < \infty$ , for some  $r > 1$ , then*

$$(22) \quad \lim_n \frac{1}{n} \sum_{k=1}^{n-1} \sum_{\ell=0}^{n-k-1} f_k(\tau^\ell \omega) = \sum_{k=1}^{\infty} \int f_k d\mathbb{P}, \text{ for a.e. } \omega.$$

*Proof.* We can assume  $f_k \geq 0$ . For the maximal function  $\tilde{f}_k(\omega) := \sup_{n \geq 1} \frac{1}{n} \sum_{\ell=0}^{n-1} f_k(\tau^\ell \omega)$ , by the ergodic maximal lemma, there is a finite constant  $C_r$  such that  $\|\tilde{f}_k\|_r \leq C_r \|f_k\|_r$ . Therefore  $\sum_{k=1}^{\infty} \tilde{f}_k \in L^r(\Omega, \mathbb{P})$ ; hence  $\sum_{k=1}^{\infty} \tilde{f}_k(\omega) < +\infty$ , for a.e.  $\omega$ .

Let  $\omega$  be such that  $\sum_{k=1}^{\infty} \tilde{f}_k(\omega) < +\infty$ . For  $\varepsilon > 0$ , there is  $L$  such that  $\sum_{k>L} \int f_k d\mathbb{P} \leq \varepsilon$  and  $\sum_{k>L} \|f_k\|_r \leq \varepsilon$ ; hence, uniformly in  $n$ ,  $\frac{1}{n} \sum_{k=L+1}^n \sum_{\ell=0}^{n-k-1} f_k(\tau^\ell \omega) \leq \varepsilon$ . By the ergodic theorem, we have  $\lim_n \frac{1}{n} \sum_{k=1}^L \sum_{\ell=0}^{n-k-1} f_k(\tau^\ell \omega) = \sum_{k=1}^L \int f_k d\mathbb{P}$ . Therefore, for  $n$  big enough:  $|\frac{1}{n} \sum_{k=1}^n \sum_{\ell=0}^{n-k-1} f_k(\tau^\ell \omega) - \sum_{k=1}^{\infty} \int f_k d\mathbb{P}| \leq 2\varepsilon$ .  $\square$

Recall that  $\Psi(\underline{t}) = \mathbb{E}[e^{2\pi i \langle X_0, \underline{t} \rangle}]$ ,  $\underline{t} \in \mathbb{T}^d$ . We use the notation:

$$(23) \quad w(\underline{t}) := \frac{1 - |\Psi(\underline{t})|^2}{|1 - \Psi(\underline{t})|^2}, \quad c_w = \int_{\mathbb{T}^d} w d\underline{t}, \text{ when } w \text{ is integrable.}$$

Outside the finite group  $\Gamma$ ,  $w$  is well defined, nonnegative and  $w(\underline{t}) = 0$  only on  $\Gamma_1 \setminus \Gamma$ . Hence it is positive for a.e.  $\underline{t}$ , excepted when the r.w. is “deterministic”.

**Proposition 2.12.** ([33]) *Let  $W = (Z_n)$  be a transient random walk in  $\mathbb{Z}^d$ .*

*a) The function  $w$  is integrable on  $\mathbb{T}^d$  and there is a nonnegative constant  $K$  such that, if  $W$  is aperiodic,*

$$(24) \quad I(\underline{p}) := 1_{\underline{p}=\underline{0}} + \sum_{k=1}^{\infty} [\mathbb{P}(Z_k = \underline{p}) + \mathbb{P}(Z_k = -\underline{p})] = \int_{\mathbb{T}^d} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t} + K.$$

*For a general reduced r.w.  $W$ , the sums  $I(\underline{p})$  are the Fourier coefficients of the measure  $w d\underline{t} + K d\gamma$ .*

b) If  $d > 1$ , then  $K = 0$ ; if  $d = 1$  and  $m(W) := \sum_{\ell \in \mathbb{Z}} \mathbb{P}(X_0 = \ell) |\ell| < +\infty$ , then  $K = |\sum_{\ell \in \mathbb{Z}} \mathbb{P}(X_0 = \ell) \ell|^{-1}$ ; if  $d = 1$  and  $m(W) = +\infty$ , then  $K = 0$ .

*Proof.* For completeness we give a proof of a), following Spitzer (§9, P2 in [33]). For b) a more difficult result without assumption on the moments need to be used (see Spitzer §24, P5, P6, P8, T2 in [33]).

Let  $(Z_n)$  be a transient, reduced, aperiodic r.w. Since  $\sum_{k=1}^{\infty} \mathbb{P}(Z_k = \underline{p}) < +\infty$ , we have

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda^k \mathbb{P}(Z_k = \underline{p}) &= \int_{\mathbb{T}^d} e^{2\pi i \langle \underline{p}, \underline{t} \rangle} \frac{1}{1 - \lambda \Psi(\underline{t})} d\underline{t}, \quad \forall \lambda \in [0, 1[, \\ \sum_{k=0}^{\infty} [\mathbb{P}(Z_k = \underline{p}) + \mathbb{P}(Z_k = -\underline{p})] &= 2 \lim_{\lambda \uparrow 1} \int_{\mathbb{T}^d} \cos 2\pi \langle \underline{p}, \underline{t} \rangle \Re e \left( \frac{1}{1 - \lambda \Psi(\underline{t})} \right) d\underline{t}; \end{aligned}$$

therefore,  $I(\underline{p}) = \lim_{\lambda \uparrow 1} \int_{\mathbb{T}^d} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w_\lambda(\underline{t}) d\underline{t}$ , with  $w_\lambda(\underline{t}) := \frac{1 - \lambda^2 |\Psi(\underline{t})|^2}{|1 - \lambda \Psi(\underline{t})|^2}$ .

Taking  $\underline{p} = \underline{0}$  in the previous formula, we deduce, by Fatou's lemma, the integrability of  $w$  on  $\mathbb{T}^d$  and with a nonnegative constant  $K$  the equality:

$$I(\underline{0}) = 1 + 2 \sum_{k=1}^{\infty} \mathbb{P}(Z_k = \underline{0}) = \lim_{\lambda \uparrow 1} \int_{\mathbb{T}^d} w_\lambda(\underline{t}) d\underline{t} = \int_{\mathbb{T}^d} w(\underline{t}) d\underline{t} + K = c_w + K,$$

since  $w_\lambda(\underline{t}) \geq 0$ ,  $\lim_{\lambda \uparrow 1} w_\lambda(\underline{t}) = \frac{1 - |\Psi(\underline{t})|^2}{|1 - \Psi(\underline{t})|^2} = w(\underline{t})$ , for  $\underline{t} \in \mathbb{T}^d \setminus \{\underline{0}\}$ .

By aperiodicity of  $W$ ,  $\Psi(\underline{t}) \neq 1$  for  $\underline{t} \in U_\eta^c$ , where  $U_\eta^c$  is the complementary in  $\mathbb{T}^d$  of the ball  $U_\eta$  of radius  $\eta$  centered at  $\underline{0}$ , for  $\eta > 0$ . This implies  $\sup_{\underline{t} \in U_\eta^c} \sup_{\lambda < 1} w_\lambda(\underline{t}) < +\infty$ .

Therefore, we get:  $\lim_{\lambda \uparrow 1} \int_{U_\eta^c} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w_\lambda(\underline{t}) d\underline{t} = \int_{U_\eta^c} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t}$ , hence:

$$(25) \quad I(\underline{p}) = \int_{U_\eta^c} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t} + \lim_{\lambda \uparrow 1} \int_{U_\eta} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w_\lambda(\underline{t}) d\underline{t}, \quad \forall \eta > 0.$$

Let  $\varepsilon > 0$ . By positivity of  $w$ , we have, for  $\eta(\varepsilon)$  small enough:

$$\begin{aligned} (1 - \varepsilon) \int_{U_{\eta(\varepsilon)}} w_\lambda(\underline{t}) d\underline{t} &\leq \int_{U_{\eta(\varepsilon)}} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w_\lambda(\underline{t}) d\underline{t} \leq (1 + \varepsilon) \int_{U_{\eta(\varepsilon)}} w_\lambda(\underline{t}) d\underline{t}; \\ \text{hence, using (25): } &(1 - \varepsilon) \int_{U_{\eta(\varepsilon)}} w_\lambda(\underline{t}) d\underline{t} - \int_{U_{\eta(\varepsilon)}} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t} \\ &\leq I(\underline{p}) - \int_{\mathbb{T}^d} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t} \leq (1 + \varepsilon) \int_{U_{\eta(\varepsilon)}} w_\lambda(\underline{t}) d\underline{t} - \int_{U_{\eta(\varepsilon)}} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t}. \end{aligned}$$

For  $\varepsilon$  small enough,  $\int_{U_{\eta(\varepsilon)}} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t}$  can be made arbitrary small, since  $w$  is integrable, as well as  $\varepsilon \sup_{\lambda < 1} \int_{U_\eta} w_\lambda d\underline{t}$ , since  $\sup_{\lambda < 1} \int_{\mathbb{T}^d} w_\lambda d\underline{t} < \infty$ . Therefore the value of  $I(\underline{p}) - \int_{\mathbb{T}^d} \cos 2\pi \langle \underline{p}, \underline{t} \rangle w(\underline{t}) d\underline{t}$  does not depend on  $\underline{p}$ , hence is equal to  $K$ . This shows (24). The Riemann-Lebesgue Lemma implies  $K = \lim_{|\underline{p}| \rightarrow +\infty} \sum_{k=1}^{\infty} \mathbb{P}(Z_k = \pm \underline{p})$ .



In the aperiodic case, by (24) the sums  $I(\underline{p})$  are the Fourier coefficients of the measure  $w d\underline{t} + K\delta_0$ . In the general case, by replacing  $\underline{\ell}_j$  by  $\underline{r}_j$  with  $\underline{\ell}_j = B\underline{r}_j$  and  $\underline{t}$  by  $B^t\underline{t}$  (cf. Subsection 2.2), we get an aperiodic r.w. and we apply the previous result. The push forward of the measure  $\delta_0$  by  $B^t$  is the measure  $d\gamma$ , which shows a).  $\square$

### 2.3. $\zeta$ -regularity, variance.

#### 2.3.1. Regularity of summation sequences defined by r.w.

We have  $V_{n,\underline{p}} = 0$ , for  $\underline{p} \notin L(W)$ . If  $W$  satisfies (17), by the LLT (Theorem 2.6), for  $\underline{p} \in L(W)$ , with constants  $C_i$  independent of  $\underline{p}$ :

$$(26) \quad \mathbb{E}V_{n,\underline{p}} \sim C_1 n^{\frac{3}{2}} \text{ for } d = 1, \quad \mathbb{E}V_{n,\underline{p}} \sim C_2 n \text{ Log } n \text{ for } d = 2, \quad \mathbb{E}V_{n,\underline{p}} \sim C_d n \text{ for } d > 2.$$

It follows  $\lim_n \mathbb{E}V_{n,\underline{p}}/\mathbb{E}V_{n,\underline{0}} = 1, \forall \underline{p} \in L(W)$ . Let us check (26) for  $d = 2$ . Let  $C$  be the constant  $a_0(W) \det(\Lambda)^{-\frac{1}{2}} (2\pi)^{-1}$ . Recall that  $L/D$  is a finite cyclic group (each  $\ell_j$  is a generator of the cyclic group  $L/D$ , cf. Lemma 2.2).

By (18), for every  $\varepsilon > 0$ , there is  $K_\varepsilon$  such that:

$$\begin{aligned} & \left| \sum_{n=1}^N \mathbb{P}(Z_n = \underline{p}) - C \sum_{n=1}^N \frac{1_{n\underline{\ell}_1 = \underline{p} \bmod D}}{n} \right| \\ & \leq C \sum_{n=1}^N 1_{n\underline{\ell}_1 = \underline{p} \bmod D} \frac{|e^{-\frac{1}{2n} \langle \Lambda^{-1} \underline{p}, \underline{p} \rangle} - 1 + \varepsilon_n(\underline{p})|}{n} \leq C(K_\varepsilon + \varepsilon \text{Log } N). \end{aligned}$$

Since  $(\text{Log } N)^{-1} \sum_{n=1}^N \frac{1_{n\underline{\ell}_1 = \underline{p} \bmod D}}{n} \rightarrow (\text{Card } L/D)^{-1}$  we have:

$$(27) \quad \lim_N [(\text{Log } N)^{-1} \sum_{n=1}^N \mathbb{P}(Z_n = \underline{p})] = C \lim_N [(\text{Log } N)^{-1} \sum_{n=1}^N \frac{1_{n\underline{\ell}_1 = \underline{p} \bmod D}}{n}] = \frac{C}{\text{Card } L/D}.$$

From (27) and (14), we have in dimension 2, for every  $\underline{p} \in L$ :

$$\frac{\mathbb{E}V_{N,\underline{p}}}{N \text{Log } N} = \frac{1_{\underline{p}=0} + \sum_{k=1}^{N-1} (1 - \frac{k}{N}) [\mathbb{P}(Z_k = \underline{p}) + \mathbb{P}(Z_k = -\underline{p})]}{\text{Log } N} \rightarrow \frac{2C}{\text{Card } L/D}.$$

Recall that  $\zeta$ -regularity of a rws  $(R_n(\underline{\ell}, \omega))_{n \geq 1}$  is equivalent to  $\lim_n \frac{V_{n,\underline{p}}(\omega)}{V_{n,\underline{0}}(\omega)} = \hat{\zeta}(\underline{p}), \forall \underline{p} \in \mathbb{Z}^d$ , for a.e.  $\omega$ .

**Theorem 2.13.** *1) If the rws is defined by a r.w.  $W$  which satisfies (17) with  $d(W) \leq 2$  (hence centered and recurrent), then it is  $\zeta$ -regular with  $\zeta = d\gamma$  ( $= \delta_0$  if  $W$  is strictly aperiodic).*

- For  $d(W) = 1$ , the normalization  $V_{n,\underline{0}}(\omega)$  depends on  $\omega$ .

- For  $d(W) = 2$ , the normalization is  $C n \text{Log } n$ , with  $C = \pi^{-1} a_0(W) \det(\Lambda)^{-\frac{1}{2}}$ .

2) If the r.w. is defined by a transient r.w., then it is  $\zeta$ -regular and the normalization is  $Cn$ , with  $C = 1 + 2 \sum_{k=1}^{\infty} \mathbb{P}(Z_k = \underline{0})$ .

If  $d = 1$ , then  $C = c_w + K$  and  $d\zeta(\underline{t}) = (c_w + K)^{-1} (w(\underline{t}) d\underline{t} + d\gamma(\underline{t}))$ , where  $c_w$  is defined in (23) and  $K$  is the constant given by Proposition 2.12 ( $K \neq 0$  if  $m(W)$  is finite). If  $d \geq 2$ , then  $C = c_w$  and  $d\zeta(\underline{t}) = c_w^{-1} w(\underline{t}) d\underline{t}$ .

*Proof.* 1) The recurrent case follows from Theorem 2.10.

As  $V_n(\omega)/\mathbb{E}V_n \rightarrow 1$ , we choose the constant  $C$  such that  $\lim_n \mathbb{E}V_n/Cn \text{Log}n = 1$ , i.e., by the LLT:  $C = 2 \lim_n (\sum_{k=1}^{n-1} \mathbb{P}(Z_k = \underline{0})/\text{Log}n) = \pi^{-1} a_0(W) \det(\Lambda)^{-\frac{1}{2}}$ .

2) In the transient case, the Fourier coefficients of the kernel  $\frac{1}{n} R_n^\omega$  given by (16) converge by Lemma 2.11 to the finite sum of the series  $1_{\underline{p}=\underline{0}} + \sum_{k=1}^{\infty} [\mathbb{P}(Z_k = \underline{p}) + \mathbb{P}(Z_k = -\underline{p})]$ , which are the Fourier coefficients of the measure  $w d\underline{t} + K d\gamma$  according to Proposition 2.12.  $\square$

**Remarks 2.14.** 1) With Condition (17), if  $\text{Card}(J) = d + 1$ , then  $\det(\Lambda) = c \prod p_j$ , where  $c$  is an integer  $\geq 1$  depending only on the  $\underline{\ell}_j$ 's (cf. Lemma 2.5). In the strictly aperiodic case, we obtain  $C = \pi^{-1} (c \prod p_j)^{-\frac{1}{2}}$ .

2) For  $d \neq 1$ , the asymptotic variance w.r.t.  $\mathbb{P} \times \mu$  is the same as the quenched variance.

3) For a reduced r.w.,  $\zeta$  is a discrete measure only in the centered case when  $d \leq 2$ , and in the non centered case when  $d = 1$  and the r.w. is deterministic.

4) As an example, let us consider a r.w.  $W$  on  $\mathbb{Z}$  defined by  $\mathbb{P}(X_k = \ell_j) = p_j$ , with  $\ell_1 = 1$  and  $0 < p_1 < 1$ . Suppose that  $W$  is transient. Let  $S$  be a map with Lebesgue spectrum. The random sequence  $(S^{Z_k(\omega)})$  can be viewed as a random commutative perturbation of the iterates of  $S$ . If  $\varphi_f$  is continuous, the asymptotic variance of  $\frac{1}{\sqrt{n}} \|\sum_{k=0}^{n-1} f(S^{Z_k(\omega)})\|_2$  is  $\int \varphi_f d\zeta$ . By Theorem 2.13, the variance is  $\neq 0$  if  $f \not\equiv 0$ . A CLT for the quenched ergodic sums can be shown when  $S$  is of hyperbolic type.

### 2.3.2. Regularity of barycenter summations.

Let  $W = (Z_n)$  be a reduced r.w. in  $\mathbb{Z}^d$  and let  $\tilde{W} = (\tilde{Z}_n)$  be the symmetrized r.w. If  $P$  is the barycenter operator defined as in (10) by  $Pf = \sum_j p_j T^{\underline{\ell}_j} f$ , we have

$$P^n f = \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) T^{\underline{\ell}} f, \text{ with } R_n(\underline{\ell}) = \mathbb{P}(Z_n = \underline{\ell})$$

and by (1)  $(R_n * \tilde{R}_n)(\underline{\ell}) = \mathbb{P}(\tilde{Z}_n = \underline{\ell})$ . We have  $d_0(\tilde{W}) = d(W) - \delta = d_0(W)$  with  $\delta = 0$  or 1. The behaviour of  $P^n$  is given by the LLT applied to  $\tilde{W}$ . When  $\delta = 0$ ,  $\Gamma_1$  is finite and its probability Haar measure  $d\gamma_1$  is a discrete measure. If  $\delta = 1$ ,  $d\gamma_1$  is the product of a discrete measure by the uniform measure on a circle. The kernel  $\tilde{R}_n$  is  $\sum_{\underline{\ell}} \frac{\mathbb{P}(\tilde{Z}_n = \underline{\ell})}{\mathbb{P}(\tilde{Z}_n = 0)} e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle}$  and we have

$$\|P^n f\|_2^2 = \int_X |\mathbb{E}(f(A^{Z_n(\cdot)} x))|^2 d\mu(x) = \sum_{\underline{\ell} \in \mathbb{Z}^d} \mathbb{P}(\tilde{Z}_n = \underline{\ell}) \hat{\varphi}_f(\underline{\ell}).$$

**Proposition 2.15.** *The kernel  $(\tilde{R}_n)$  (with a normalization factor  $\sum_{\underline{\ell}} R_n(\underline{\ell})^2$  of order  $n^{d-\delta}$ ) weakly converges to the measure  $d\gamma_1$ .*

*Proof.* The characteristic function of  $\tilde{X}_0$  is  $|\Psi(\underline{t})|^2 = |\mathbb{E}(e^{2\pi i(X_0, \underline{t})})|^2$ . The LLT given by Theorem 2.8 ( $\tilde{W}$  is centered but non necessarily reduced) implies

$$\mathbb{P}(\tilde{Z}_n = \underline{\ell}) = (2\pi n)^{-\frac{d_0(\tilde{W})}{2}} [a_0 \det(\tilde{\Lambda}_0)^{-\frac{1}{2}} e^{-\frac{1}{2n}(\tilde{\Lambda}_0^{-1}\underline{\ell}, \underline{\ell})} + \varepsilon_n(\underline{\ell})].$$

For the r.w.  $(\tilde{Z}_n)$ , the condition  $\underline{k} \in D + n\underline{\ell}_1$  reduces to  $\underline{k} \in D$ , since  $\underline{0}$  belongs to the support of the symmetrized distribution. Therefore, for the symmetric r.w., we obtain:

$$(28) \quad \lim_n \frac{\int |\Psi(\underline{t})|^{2n} e^{-2\pi i(\underline{p}, \underline{t})} d\underline{t}}{\int |\Psi(\underline{t})|^{2n} d\underline{t}} = \lim_n \frac{\mathbb{P}(\tilde{Z}_n = \underline{p})}{\mathbb{P}(\tilde{Z}_n = \underline{0})} \rightarrow 1_D(\underline{p}), \forall \underline{p} \in \mathbb{Z}^d.$$

Remark that when  $W$  is strictly aperiodic, one can show that  $(|\Psi(\underline{t})|^{2n} / \int |\Psi(\underline{t})|^{2n} d\underline{t})_{n \geq 1}$  is an approximation of identity. This gives a direct proof of (28).

Since  $1_D$  is the Fourier transform of  $d\gamma_1$  viewed as a measure on the torus  $\mathbb{T}^d$ , the weak convergence of  $(\tilde{R}_n)$  to  $d\gamma_1$  follows from (28).  $\square$

### 3. $\mathbb{Z}^d$ -actions by commuting endomorphisms on $G$

#### 3.1. Endomorphisms of a compact abelian group $G$ .

Let  $G$  be a compact abelian group with Haar measure  $\mu$ . The group of characters of  $G$  is denoted by  $\hat{G}$  or  $H$  and the set of non trivial characters by  $\hat{G}^*$  or  $H^*$ . The Fourier coefficients of a function  $f$  in  $L^1(G, \mu)$  are  $c_f(\chi) := \int_G \bar{\chi} f d\mu$ ,  $\chi \in \hat{G}$ .

Every surjective endomorphism  $B$  of  $G$  defines a measure preserving transformation on  $(G, \mu)$  and a dual injective endomorphism on  $\hat{G}$ . For simplicity, we use the same notation for the actions on  $G$  and on  $\hat{G}$ . If  $f$  is function on  $G$ ,  $Bf$  stands for  $f \circ B$ .

We consider a semigroup  $\mathcal{S}$  of surjective commuting endomorphisms of  $G$ , for example the semigroup generated by commuting matrices on a torus. It will be useful to extend it to a group of automorphisms acting on a (possibly) bigger group  $\tilde{G}$  (a solenoidal group when  $G$  is a torus). Let us recall briefly the construction.

**Lemma 3.1.** *There is a smallest compact abelian group  $\tilde{G}$  (connected, if  $G$  is connected) such that  $G$  is a factor of  $\tilde{G}$  and  $\mathcal{S}$  is embedded in a group  $\tilde{\mathcal{S}}$  of automorphisms of  $\tilde{G}$ .*

*Proof.* On the set  $\{(\chi, A), \chi \in H, A \in \mathcal{S}\}$  we consider the law  $(\chi, A) + (\chi', A') = (A'\chi + A\chi', AA')$ . Let  $\tilde{H}$  be the quotient by the equivalence relation  $\mathcal{R}$  defined by  $(\chi, A) \mathcal{R} (\chi', A')$  if and only if  $A'\chi = A\chi'$ .

The transitivity of the relation  $\mathcal{R}$  follows from the injectivity of the dual action of each  $A \in \mathcal{S}$ . The map  $\chi \in \hat{G} \rightarrow (\chi, Id)/\mathcal{R}$  is injective. The elements  $A \in \mathcal{S}$  act on  $\tilde{H}$  by  $(\chi, B)/\mathcal{R} \rightarrow (A\chi, B)/\mathcal{R}$ . The equivalence classes are stable by this action. We can identify  $\mathcal{S}$  and its image. For  $A \in \mathcal{S}$ , the automorphism  $(\chi, B)/\mathcal{R} \rightarrow (\chi, AB)/\mathcal{R}$  is the inverse of  $(\chi, B)/\mathcal{R} \rightarrow (A\chi, B)/\mathcal{R}$ .

Let  $\tilde{G}$  be the compact abelian group (with Haar measure  $\tilde{\mu}$ ) dual of the group  $\tilde{H}$  endowed with the discrete topology. The group  $H = \hat{\tilde{G}}$  is isomorphic to a subgroup of  $\tilde{H}$  and  $G$  is a factor of  $\tilde{G}$ . We obtain an embedding of  $\mathcal{S}$  in a group  $\tilde{\mathcal{S}}$  of automorphisms of  $\tilde{H}$  and, by duality, in a group of automorphisms of  $\tilde{G}$ . If  $\hat{G}$  is torsion free, then  $\tilde{H}$  is also torsion free and its dual  $\tilde{G}$  is a connected compact abelian group.  $\square$

If  $(A_1, \dots, A_d)$  are  $d$  algebraically independent automorphisms of  $\tilde{G}$  generating  $\tilde{\mathcal{S}}$ , then  $\tilde{\mathcal{S}} = \{A^{\underline{\ell}} = A_1^{\ell_1} \dots A_d^{\ell_d}, \underline{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{Z}^d\}$  and  $\tilde{\mathcal{S}}$  is isomorphic to  $\mathbb{Z}^d$  if it is torsion free. The corresponding  $\mathbb{Z}^d$ -action on  $(\tilde{G}, \tilde{\mu})$  is denoted  $\mathbb{A}$ .

**Assumption 3.2.** *In what follows, we consider a set  $(B_j, j \in J)$  of commuting surjective endomorphisms of  $G$  and the generated semigroup  $\mathcal{S}$ . Denoting by  $\tilde{\mathcal{S}}$  the extension of  $\mathcal{S}$  to a group of automorphisms of  $\tilde{G}$  (if the  $B_j$ 's are invertible, then  $\tilde{G}$  is just  $G$ ), we assume that  $\tilde{\mathcal{S}}$  is torsion-free and non trivial, hence has a system of  $d \in [1, +\infty]$  algebraically independent generators  $A_1, \dots, A_d$  (not necessarily in  $\mathcal{S}$ ). We suppose  $d$  finite. In other words we consider a set  $(B_j, j \in J)$  of endomorphisms such that  $B_j = A^{\underline{\ell}_j}$ , where  $A_1, \dots, A_d$  are  $d$  algebraically independent commuting automorphisms of  $\tilde{G}$ .*

We begin with some spectral results. The measure preserving  $\mathbb{Z}^d$ -action  $\mathbb{A}$  is said to be *totally ergodic* if  $A^{\underline{\ell}}$  on  $(\tilde{G}, \tilde{\mu})$  is ergodic for every  $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$ .

One easily show that total ergodicity is equivalent to:  $A^{\underline{\ell}}\chi \neq \chi$  for any non trivial character  $\chi$  and  $\underline{\ell} \neq 0$  (free dual  $\mathbb{Z}^d$ -action on  $H^*$ ), to the Lebesgue spectrum property for  $\tilde{\mathcal{S}}$  acting on  $(\tilde{G}, \tilde{\mu})$ , as well as to 2-mixing, i.e.,  $\lim_{\|\underline{\ell}\| \rightarrow \infty} \mu(C_1 \cap A^{-\underline{\ell}} C_2) = \mu(C_1) \mu(C_2)$ , for all Borel sets  $C_1, C_2$  of  $G$ .

For a semigroup  $\mathcal{S} = \{A^{\underline{\ell}}, \underline{\ell} \in (\mathbb{Z}^+)^d\}$ , total ergodicity of the generated group is equivalent to the property (expressed on the dual of  $G$ ):  $A^{\underline{\ell}}\chi \neq A^{\underline{\ell}'}\chi$  for  $\chi \in H^*$  and  $\underline{\ell} \neq \underline{\ell}'$ .

Examples for  $G = \mathbb{T}^p$  are given in Subsection 3.2. What we call ‘‘totally ergodic’’ for a general compact abelian group  $G$  is also called ‘‘partially hyperbolic’’ for the torus.

### Spectral density $\varphi_f$ of a regular function $f$

For  $f$  in  $L^2(\tilde{G})$ , we have  $\hat{\varphi}_f(\underline{n}) = \langle f, A^{\underline{n}}f \rangle = \sum_{\chi \in \hat{\tilde{H}}} c_f(A^{\underline{n}}\chi) \overline{c_f(\chi)}$ ,  $\underline{n} \in \mathbb{Z}^d$ . Observe that, using the projection  $\Pi$  from  $\tilde{G}$  to  $G$ , a function  $f$  on the group  $G$  can be lifted to  $\tilde{G}$  and a character  $\chi \in \hat{G}$  viewed as a character on  $\tilde{G}$  via composition by  $\Pi$ . Putting  $\tilde{f}(x) = f(\Pi x)$ , we have  $\tilde{f} = f \circ \Pi = \sum_{\chi \in \hat{G}} c_f(\chi) \chi \circ \Pi$ , so that the only non zero Fourier coefficients of  $\tilde{f}$  correspond to characters on  $G$ .

If  $f$  is a function defined on  $G$ , the Fourier analysis can be done for the  $\mathbb{Z}^d$ -action  $\mathbb{A}$  in the group  $\tilde{G}$ , but expressed in terms of the Fourier coefficients of  $f$  computed in  $G$ . For the spectral density of  $f$ , viewed as the spectral density of  $\tilde{f}$  for the  $\mathbb{Z}^d$ -action  $\mathbb{A}$  on  $\tilde{G}$ , one checks that  $\hat{\varphi}_f(\underline{n}) = \sum_{\chi \in \hat{G}} c_f(A^{\underline{n}}\chi) \overline{c_f(\chi)}$ ,  $\underline{n} \in \mathbb{Z}^d$ , where  $c_f(A^{\underline{n}}\chi) = 0$  if  $A^{\underline{n}}\chi \notin \hat{G}$ .

For the action by endomorphisms of compact abelian groups, the family  $(\psi_j)$  (cf. notations in Subsection 1.2) is  $(\chi_j)$ . We have:

$a_{j,\underline{\ell}} = \langle f, A^{\underline{\ell}}\chi_j \rangle = c_f(A^{\underline{\ell}}\chi_j)$ ,  $\gamma_j(\underline{t}) = \sum_{\underline{\ell} \in \mathbb{Z}^d} c_f(A^{\underline{\ell}}\chi_j) e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle}$ ,  $M_{\underline{t}}f = \sum_{j \in J_0} \gamma_j(\underline{t}) \chi_j$ , where  $J_0$  denotes a *section* of the action of  $\tilde{\mathcal{S}}$  on  $\tilde{H}^*$ , i.e., a subset  $\{\chi_j\}_{j \in J_0} \subset \tilde{H}^* = \tilde{H} \setminus \{\underline{0}\}$  such that every  $\chi \in \tilde{H}^*$  can be written in a unique way as  $\chi = A_1^{n_1} \dots A_d^{n_d} \chi_j$ , with  $j \in J_0$  and  $(n_1, \dots, n_d) \in \mathbb{Z}^d$ .

Therefore,  $M_{\underline{t}}f$  is defined for every  $\underline{t}$ , if  $\sum_{j \in J_0} (\sum_{\underline{\ell} \in \mathbb{Z}^d} |c_f(A^{\underline{\ell}}\chi_j)|^2) < \infty$ .

We denote by  $AC_0(G)$  the class of real functions on  $G$  with *absolutely convergent Fourier series* and  $\mu(f) = 0$ , endowed with the norm:  $\|f\|_c := \sum_{\chi \in \tilde{G}} |c_f(\chi)| < +\infty$ .

**Proposition 3.3.** *If  $f$  is in  $AC_0(G)$ , then  $\sum_{\underline{\ell} \in \mathbb{Z}^d} |\langle A^{\underline{\ell}}f, f \rangle| < \infty$  and the spectral density  $\varphi_f$  of  $f$  is continuous. For any subset  $\mathcal{E}$  of  $\tilde{G}$ , if  $f_1(x) = \sum_{\chi \in \mathcal{E}} c_f(\chi) \chi$ , then*

$$(29) \quad \|\varphi_{f-f_1}\|_\infty \leq \|f - f_1\|_c^2.$$

*Proof.* (We work in  $\tilde{G}$ , but for simplicity we do not write  $\sim$ ). Since every  $\chi \in H^*$  can be written in an unique way as  $\chi = A^{\underline{\ell}}\chi_j$ , with  $j \in J_0$  and  $\underline{\ell} \in \mathbb{Z}^d$ , we have:  $\sum_{\underline{\ell} \in \mathbb{Z}^d} |c_f(A^{\underline{\ell}}\chi_j)| \leq \sum_{j \in J_0} \sum_{\underline{\ell} \in \mathbb{Z}^d} |c_f(A^{\underline{\ell}}\chi_j)| = \sum_{\chi \in \tilde{H}^*} |c_f(\chi)| = \|f\|_c$ .

Then, for every  $j \in J_0$ , the series defining  $\gamma_j$  is uniformly converging,  $\gamma_j$  is continuous and  $\sum_{j \in J_0} \|\gamma_j\|_\infty \leq \sum_{j \in J_0} \sum_{\underline{\ell} \in \mathbb{Z}^d} |c_f(A^{\underline{\ell}}\chi_j)| = \|f\|_c$ .

The spectral density of  $f$  is  $\varphi_f(\underline{t}) = \sum_{j \in J_0} |\sum_{\underline{\ell} \in \mathbb{Z}^d \setminus \{\underline{0}\}} c_f(A^{\underline{\ell}}\chi_j) e^{2\pi i \langle \underline{\ell}, \underline{t} \rangle}|^2$ . The function  $\sum_{j \in J_0} |\gamma_j|^2$  is a continuous version of the spectral density  $\varphi_f$  and  $\|\varphi_f\|_\infty$  is bounded by

$$\sum_{j \in J_0} (\sum_{\underline{\ell} \in \mathbb{Z}^d} |c_f(A^{\underline{\ell}}\chi_j)|)^2 \leq \sum_{j \in J_0} (\sum_{\underline{\ell} \in \mathbb{Z}^d} |c_f(A^{\underline{\ell}}\chi_j)|) (\sum_{\chi \in \tilde{G}} |c_f(\chi)|) \leq (\sum_{\chi \in \tilde{G}} |c_f(\chi)|)^2 = \|f\|_c^2.$$

Inequality (29) follows by replacing  $f$  by  $f - f_1$ .  $\square$

### 3.2. The torus case: $G = \mathbb{T}^\rho$ , examples of $\mathbb{Z}^d$ -actions.

Every  $B$  in the semigroup  $\mathcal{M}^*(\rho, \mathbb{Z})$  of non singular  $\rho \times \rho$  matrices with coefficients in  $\mathbb{Z}$  defines a surjective endomorphism of  $\mathbb{T}^\rho$  and a measure preserving transformation on  $(\mathbb{T}^\rho, \mu)$ . It defines also a dual endomorphism of the group of characters  $H = \widehat{\mathbb{T}^\rho}$  identified with  $\mathbb{Z}^\rho$  (action by the transposed of  $B$ ). Since we compose commuting matrices, for simplicity we do not write the transposition. When  $B$  is in the group  $GL(\rho, \mathbb{Z})$  of matrices with coefficients in  $\mathbb{Z}$  and determinant  $\pm 1$ , it defines an automorphism of  $\mathbb{T}^\rho$ .

For the torus, the construction in Lemma 3.1 reduces to the following. Let  $B_j, j \in J$ , be matrices in  $\mathcal{M}^*(\rho, \mathbb{Z})$  and  $q_j = |\det(B_j)|$ . Suppose for simplicity  $J$  finite. Then  $\tilde{G}$  is the compact group dual of the discrete group  $\tilde{H} := \{\prod_j q_j^{\ell_j} \underline{k}, \underline{k} \in \mathbb{Z}^\rho, \ell_j \in \mathbb{Z}^-\}$ ,  $\mathbb{Z}^\rho$  is a subgroup of  $\tilde{H}$  and  $\mathbb{T}^\rho$  is a factor of  $\tilde{G}$ .

It is well known that  $A \in \mathcal{M}^*(\rho, \mathbb{Z})$  acts ergodically on  $(\mathbb{T}^\rho, \mu)$  if and only if  $A$  has no eigenvalue root of unity. A  $\mathbb{Z}^d$ -action  $(A^{\underline{\ell}}, \underline{\ell} \in \mathbb{Z}^d)$  on  $(\mathbb{T}^\rho, \mu)$  is totally ergodic if and only if it is free on  $\mathbb{Z}^\rho \setminus \{\underline{0}\}$ , or equivalently if  $A^{\underline{\ell}}$  has no eigenvalue root of unity if  $\underline{\ell} \neq \underline{0}$ .

**Lemma 3.4.** *Let  $M \in \mathcal{M}^*(\rho, \mathbb{Z})$  be a matrix with irreducible (over  $\mathbb{Q}$ ) characteristic polynomial  $P$ . If  $\{B_j, j \in J\}$  are  $d$  matrices in  $\mathcal{M}^*(\rho, \mathbb{Z})$  commuting with  $M$ , they generate a commutative semigroup  $\mathcal{S}$  of endomorphisms on  $\mathbb{T}^\rho$  which is totally ergodic if and only if  $B^\underline{\ell} \neq B^{\underline{\ell}'}$ , for  $\underline{\ell} \neq \underline{\ell}'$  (where  $B^\underline{\ell} := B_1^{\ell_1} \dots B_d^{\ell_d}$ ). The  $\mathbb{Z}^d$ -action extending  $\mathcal{S}$  is the product of a totally ergodic  $\mathbb{Z}^{d'}$ -action, with  $d' \leq d$  by an action of finite order.*

*Proof.* Since  $P$  is irreducible, the eigenvalues of  $M$  are distinct. It follows that the matrices  $B_j$  are simultaneously diagonalizable on  $\mathbb{C}$ , hence are pairwise commuting. Now suppose that there are  $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$  and  $v \in \mathbb{Z}^\rho \setminus \{0\}$  such that  $B^{\underline{\ell}}v = v$ . Let  $\mathcal{E}$  be the subspace of  $\mathbb{R}^\rho$  generated by  $v$  and its images by  $M$ . The restriction of  $B^{\underline{\ell}}$  to  $\mathcal{E}$  is the identity.  $\mathcal{E}$  is  $M$ -invariant, the characteristic polynomial of the restriction of  $M$  to  $\mathcal{E}$  has rational coefficients and factorizes  $P$ . By the assumption of irreducibility over  $\mathbb{Q}$ , this implies  $\mathcal{E} = \mathbb{R}^\rho$ . Therefore  $B^{\underline{\ell}}$  is the identity.

Let  $\mathcal{K}$  be the kernel of the homomorphism  $h : \underline{\ell} \rightarrow B^{\underline{\ell}}$  and  $\tilde{h}$  the quotient of  $h$  in  $\mathbb{Z}^d/\mathcal{K}$ . The finitely generated group  $\mathbb{Z}^d/\mathcal{K}$  is isomorphic to  $\mathcal{U} \oplus \mathcal{T}$ , with  $\mathcal{U}$  isomorphic to  $\mathbb{Z}^{d'}$  for  $d' \in [0, d]$ , the restriction of  $\tilde{h}$  to  $\mathcal{U}$  a totally ergodic  $\mathbb{Z}^{d'}$ -action and  $\mathcal{T}$  a finite group.  $\square$

### Examples of $\mathbb{Z}^d$ -actions by automorphisms

In general proving total ergodicity and computing explicit independent generators is difficult. This may be easier with endomorphisms. For example, let  $(q_j, j \in J)$  be integers  $> 1$  and let  $x \rightarrow q_j x \bmod 1$  be the corresponding endomorphisms acting on  $\mathbb{T}^1$ . They generate a semigroup embedded in a group acting giving a totally ergodic  $\mathbb{Z}^d$ -action on an extension of  $\mathbb{T}^1$ , where  $d \in [1, +\infty]$  is the dimension of the vector space over  $\mathbb{Q}$  generated by  $\text{Log } q_j, j \in J$ . The construction extends to  $\rho \times \rho$  matrices  $B_j, j \in J$ , such that  $|\det(B_j)| > 1$  by replacing  $q_j$  by  $|\det(B_j)|$ , under the condition of Lemma 3.4. Without irreducibility condition, the action of commuting  $\rho \times \rho$  matrices  $B_j$ , when the numbers  $\text{Log } |\det(B_j)|$  are linearly independent over  $\mathbb{Q}$ , extends to a totally ergodic  $\mathbb{Z}^d$ -action with  $d = \text{Card}(J)$  on an extension of  $\mathbb{T}^\rho$ .

On the contrary, for automorphisms of  $G = \mathbb{T}^\rho$ , it can be difficult to compute independent generators of the generated group for  $\rho > 3$  or 4. We would like to discuss this point and recall some facts (see in particular [19] and [9]).

The construction of  $\mathbb{Z}^d$ -action by automorphisms on  $\mathbb{T}^\rho$  is related to the group of units in number fields (cf. [19]). To simplify let us consider a matrix  $M \in GL(\rho, \mathbb{Z})$  with an irreducible (over  $\mathbb{Q}$ ) characteristic polynomial  $P$ . The elements of the centralizer of  $M$  in  $GL(\rho, \mathbb{R})$  are simultaneously diagonalizable. The centralizer of  $M$  in  $\mathcal{M}(\rho, \mathbb{Q})$  can be identified with the ring of polynomials in  $M$  with rational coefficients modulo the principal ideal generated by the polynomial  $P$  and hence with the field  $\mathbb{Q}(\lambda)$ , where  $\lambda$  is an eigenvalue of  $M$ , by the map  $p(A) \rightarrow p(\lambda)$  with  $p \in \mathbb{Q}[X]$ .

By Dirichlet's theorem, if  $P$  has  $d_1$  real roots and  $d_2$  pairs of complex conjugate roots, there are  $d_1 + d_2 - 1$  fundamental units in the group of units in the ring of integers in the field  $K(P)$  associated to  $P$ . The centralizer  $\mathcal{C}(M)$  of  $M$  in  $GL(\rho, \mathbb{Z})$  provides a

totally ergodic  $\mathbb{Z}^{d_1+d_2-1}$ -action by automorphisms on  $\mathbb{T}^p$  (up to a product by a finite cyclic group consisting of roots of unity). The computation of the number of real roots of  $P$  gives the dimension  $d = d_1 + d_2 - 1$  of the  $\mathbb{Z}^d$  free action on  $\mathbb{T}^p$  generated by  $\mathcal{C}(M)$ .

Nevertheless it can be difficult to compute elements with determinant  $\pm 1$  in  $\mathcal{C}(M)$ . The explicit computation of fundamental units (hence of independent generators) relies on an algorithm (see H. Cohen's book [6]) which is in practice limited to low dimensions.

### Examples for $\mathbb{T}^3$

Let  $P(x) = -x^3 + qx + n$  be a polynomial with coefficients in  $\mathbb{Z}$ , irreducible over  $\mathbb{Q}$ .

Let  $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ n & q & 0 \end{pmatrix}$  be its companion matrix. Let  $\lambda$  be a root of  $P$ . If the field  $K(P)$

is listed in a table giving the characteristics of the first cubic real fields (see [6], [34]), we find a pair of fundamental units for the group of units in the ring of integers in  $K(P)$  of the form  $P_1(\lambda)$ ,  $P_2(\lambda)$ , with  $P_1, P_2 \in \mathbb{Z}[X]$ . The matrices  $A_1 = P_1(M)$  and  $A_2 = P_2(M)$  provide elements of  $\mathcal{C}(M)$  giving a totally ergodic  $\mathbb{Z}^2$ -action on  $\mathbb{T}^3$  by automorphisms.

#### 1) Explicit examples (from the table in [34])

a) Let us consider the polynomial  $P(x) = -x^3 + 12x + 10$  and its companion matrix

$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 12 & 0 \end{pmatrix}$ . Let  $\lambda$  be a root of  $P$ . The table gives a pair of fundamental units:

$P_1(\lambda) = \lambda^2 - 3\lambda - 3$ ,  $P_2(\lambda) = -\lambda^2 + \lambda + 11$ . Let  $A_1, A_2$  be the matrices

$$A_1 = P_1(M) = \begin{pmatrix} -3 & -3 & 1 \\ 10 & 9 & -3 \\ -30 & -26 & 9 \end{pmatrix}, \quad A_2 = P_2(M) = \begin{pmatrix} 11 & 1 & -1 \\ -10 & -1 & 1 \\ 10 & 2 & -1 \end{pmatrix}.$$

b) Consider now the polynomial  $P(x) = -x^3 + 9x + 2$  and its companion matrix  $M' =$

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 9 & 0 \end{pmatrix}$ . Let  $\lambda$  be a root of  $P$ . The table gives a pair of fundamental units for the

algebraic group associated to  $P$ :  $P_1(\lambda) = 85\lambda^2 - 245\lambda - 59$ ,  $P_2(\lambda) = -18\lambda^2 + 4\lambda + 161$ .

$$\text{Take } A_1 = P_1(M') = \begin{pmatrix} -59 & -245 & 85 \\ 170 & 706 & -245 \\ -490 & -2035 & 706 \end{pmatrix}, \quad A_2 = P_2(M') = \begin{pmatrix} 161 & 4 & -18 \\ -36 & -1 & 4 \\ 8 & 0 & -1 \end{pmatrix}.$$

In both cases a) and b), the matrices  $A_1$  and  $A_2$  are in  $GL(3, \mathbb{Z})$  and generate a totally ergodic actions of  $\mathbb{Z}^2$  by automorphisms on  $\mathbb{T}^3$ .

#### 2) A simple example on $\mathbb{T}^4$

If  $P(x) = x^4 + ax^3 + bx^2 + ax + 1$ , the polynomial  $P$  has two real roots:  $\lambda_0, \lambda_0^{-1}$  and two complex conjugate roots of modulus 1:  $\lambda_1, \bar{\lambda}_1$ .

Let  $\sigma_j = \lambda_j + \bar{\lambda}_j$ ,  $j = 0, 1$ . They are roots of  $Z^2 - aZ + b - 2 = 0$ .

If the conditions:  $a^2 - 4b + 8 > 0$ ,  $a > 4$ ,  $b > 2$ ,  $2a > b + 2$  are satisfied (i.e.,  $2 < b < 2a - 2$ ,  $a > 4$ , since  $2a - 2 \leq \frac{1}{4}a^2 + 2$ ), then  $\lambda_0, \lambda_0^{-1}$  are solutions of  $\lambda^2 - \sigma_0\lambda + 1 = 0$ , and  $\lambda_1, \bar{\lambda}_1$  are solutions of  $\lambda^2 - \sigma_1\lambda + 1 = 0$ , where

$$\sigma_0 = -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4b + 8}, \quad \sigma_1 = -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2 - 4b + 8}.$$

The polynomial  $P$  is not factorizable over  $\mathbb{Q}$ . Indeed, suppose that  $P = P_1P_2$  with  $P_1, P_2$  with rational coefficients and degree  $\geq 1$ . Since the roots of  $P$  are irrational, the degrees of  $P_1$  and  $P_2$  are 2. Necessarily their roots are, say,  $\lambda_1, \bar{\lambda}_1$  for  $P_1$ ,  $\lambda_0, \lambda_0^{-1}$  for  $P_2$ . The sum  $\lambda_1 + \bar{\lambda}_1$ , root of  $Z^2 - aZ + b - 2 = 0$ , is not rational and the coefficients of  $P_1$  are

not rational. Let us take  $A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -a & -b & -a \end{pmatrix}$ ,  $B = A + I$ . From the irreducibility

over  $\mathbb{Q}$ , it follows that, if there is a non zero fixed integral vector for  $A^k B^\ell$ , where  $k, \ell$  are in  $\mathbb{Z}$ , then we have  $A^k B^\ell = Id$ . This implies:  $\lambda_1^k (\lambda_1 - 1)^\ell = 1$ , hence, since we have  $|\lambda_1| = 1$ , it follows  $|\lambda_1 + 1| = 1$ , i.e.  $\lambda$  is also solution of  $z^2 - z + 1 = 0$ , which is not true.

An example is  $P(x) = x^4 + 5x^3 + 7x^2 + 5x + 1$ . If  $A$  is the companion matrix, then the matrices  $A$  and  $B = A + I$  generate a  $\mathbb{Z}^2$ -totally ergodic action on  $\mathbb{T}^4$ .

### 3) Construction by blocks

Let  $M_1, M_2$  be two ergodic matrices respectively of dimension  $d_1$  and  $d_2$ . Let  $(p_i, q_i)$ ,  $i = 1, 2$ , be two pairs of integers such that  $p_1q_2 - p_2q_1 \neq 0$ . On the torus  $\mathbb{T}^{d_1+d_2}$  we obtain a  $\mathbb{Z}^2$ -totally ergodic action by taking  $A_1, A_2$  of the following form:  $A_1 = \begin{pmatrix} M_1^{p_1} & 0 \\ 0 & M_2^{q_1} \end{pmatrix}$ ,  $A_2 = \begin{pmatrix} M_1^{p_2} & 0 \\ 0 & M_2^{q_2} \end{pmatrix}$ . Indeed, if there exists  $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{Z}^{d_1+d_2} \setminus \{0\}$  invariant by  $A_1^n A_2^\ell$ , then  $M_1^{np_1+\ell p_2} v_1 = v_1$ ,  $M_2^{nq_1+\ell q_2} v_2 = v_2$ , which implies  $np_1 + \ell p_2 = 0$ ,  $nq_1 + \ell q_2 = 0$ ; hence  $n = \ell = 0$ .

This method gives explicit free  $\mathbb{Z}^2$ -actions on  $\mathbb{T}^4$ . The same method gives free  $\mathbb{Z}^3$ -actions on  $\mathbb{T}^5$ . As discuss above, it is more difficult to explicit examples of full dimension, i.e., with 3 independent generators on  $\mathbb{T}^4$ , or with 4 independent generators on  $\mathbb{T}^5$ .

### 3.3. Spectral densities and Fourier series for tori.

The continuity of  $\varphi_f$  for a general compact abelian group follows from the absolute convergence of the Fourier series of  $f$ , hence, for  $G = \mathbb{T}^\rho$  from the condition:

$$(30) \quad |c_f(\underline{k})| = O(\|\underline{k}\|^{-\beta}), \quad \text{with } \beta > \rho.$$

Condition (30) implies a rate of approximation of  $f$  by the partial sums of its Fourier series, but for the torus a weaker regularity condition on  $f$  can be used. First, let us recall a result on the approximation of functions by trigonometric polynomials.



For  $f \in L^2(\mathbb{T}^\rho)$ , the Fourier partial sums of  $f$  over squares of sides  $N$  are denoted by  $s_N(f)$ . The *integral modulus of continuity* of  $f$  is defined as

$$\omega_2(\delta, f) = \sup_{|\tau_1| \leq \delta, \dots, |\tau_\rho| \leq \delta} \|f(\cdot + \tau_1, \dots, \cdot + \tau_\rho) - f\|_{L^2(\mathbb{T}^\rho)}.$$

**Lemma 3.5.** *The following condition on the modulus of continuity*

$$(31) \quad \exists \alpha > d \text{ and } C(f) < +\infty \text{ such that } \omega_2(\delta, f) \leq C(f) (\ln \frac{1}{\delta})^{-\alpha}, \forall \delta > 0,$$

*implies, for a constant  $R(f)$ :*

$$(32) \quad \|f - s_N(f)\|_2 \leq R(f) (\ln N)^{-\alpha}, \text{ with } \alpha > d.$$

*Proof.* Let  $K_{N_1, \dots, N_\rho}$  be the  $\rho$ -dimensional Fejér kernel, for  $N_1, \dots, N_\rho \geq 1$ , and let  $J_{N_1, \dots, N_\rho}(t_1, \dots, t_\rho) = K_{N_1, \dots, N_\rho}^2(t_1, \dots, t_\rho) / \|K_{N_1, \dots, N_\rho}\|_{L^2(\mathbb{T}^\rho)}^2$  be the  $\rho$ -dimensional Jackson's kernel. Using the moment inequalities  $\int_0^{\frac{1}{2}} t^k J_N(t) dt = O(N^{-k})$ ,  $\forall N \geq 1$ ,  $k = 0, 1, 2$ , satisfied by the 1-dimensional Jackson's kernel, we obtain that there exists a positive constant  $C_\rho$  such that, for every  $f \in L^2(\mathbb{T}^\rho)$ ,  $\|J_{N, \dots, N} * f - f\|_2 \leq C_\rho \omega_2(\frac{1}{N}, f)$ ,  $\forall N \geq 1$ . It follows:  $\|f - s_N(f)\|_2 \leq C_\rho \omega_2(\frac{1}{N}, f)$ ,  $\forall f \in L^2(\mathbb{T}^\rho)$ ,  $\forall N \geq 1$ , since  $\|f - s_N(f)\|_2 \leq \|f - P\|_2$  for every trigonometric polynomial  $P$  in  $\rho$  variables of degree at most  $N \times \dots \times N$ . Hence (32) follows from (31).  $\square$

The required regularity in the next theorem is weaker than in Proposition 3.3. The proof is like that of the analogous result in [25]. It uses the lemma below due to D. Damjanović and A. Katok [9], extended by M. Levin [26] to endomorphisms.

**Lemma 3.6.** [9] *If  $(A^{\underline{n}}, \underline{n} \in \mathbb{Z}^d)$  is a totally ergodic  $\mathbb{Z}^d$ -action on  $\mathbb{T}^\rho$  by automorphisms, there are  $\tau > 0$  and  $C > 0$ , such that for all  $(\underline{n}, \underline{k}) \in \mathbb{Z}^d \times (\mathbb{Z}^\rho \setminus \{\underline{0}\})$  for which  $A^{\underline{n}} \underline{k} \in \mathbb{Z}^\rho$ .*

$$(33) \quad \|A^{\underline{n}} \underline{k}\| \geq C e^{\tau \|\underline{n}\|} \|\underline{k}\|^{-\rho}.$$

**Theorem 3.7.** *Let  $\underline{\ell} \rightarrow A^{\underline{\ell}}$  be a totally ergodic  $d$ -dimensional action by commuting endomorphisms on  $\mathbb{T}^\rho$ . Let  $f$  be in  $L_0^2(\mathbb{T}^\rho)$  satisfying the regularity condition (31) or more generally (32). Let  $f_1(x) := \sum_{\underline{n} \in \mathcal{E}_1} c_{\underline{n}}(f) e^{2\pi i \langle \underline{n}, x \rangle}$ , where  $\mathcal{E}_1$  is a subset of  $\mathbb{Z}^\rho$ . Then there are finite constants  $B(f), C(f)$  depending only on  $R(f)$  such that*

$$(34) \quad |\langle A^{\underline{\ell}} f_1, f_1 \rangle| \leq B(f) \|f_1\|_2 \|\underline{\ell}\|^{-\alpha}, \quad \forall \underline{\ell} \neq \underline{0},$$

*the spectral density is continuous,  $\sum_{\underline{\ell} \in \mathbb{Z}^d} |\langle A^{\underline{\ell}} f, f \rangle| < \infty$  and  $\|\varphi_{f-f_1}\|_\infty \leq C(f) \|f - f_1\|_2$ .*

*Proof.* (Recall the convention  $c_{A^{\underline{\ell}} \underline{k}}(f) = 0$  if  $A^{\underline{\ell}} \underline{k} \notin \mathbb{Z}^\rho$ .) It suffices to prove the result for  $f$  since, by setting  $c_f(n) = 0$  outside  $\mathcal{E}_1$ , we obtain (34) with the same constant  $B(f)$  as shown by the proof. Let  $\lambda, b, h$  be such that  $1 < \lambda < e^\tau$ ,  $1 < b < \lambda^{\frac{1}{\rho}}$ ,  $h := \lambda b^{-\rho} > 1$  (where  $\tau$  is given by Lemma 3.6). We have for  $\underline{\ell} \in \mathbb{Z}^d$ :

$$(35) \quad \langle A^{\underline{\ell}} f, f \rangle = \sum_{\underline{k} \in \mathbb{Z}^\rho} c_{\underline{k}}(f) \bar{c}_{A^{\underline{\ell}} \underline{k}}(f) = \sum_{\|\underline{k}\| < b \|\underline{\ell}\|} + \sum_{\|\underline{k}\| \geq b \|\underline{\ell}\|} = (A) + (B).$$

From Inequality (33) of Lemma 3.6, we deduce that, if  $\|\underline{k}\| < b^{\|\underline{\ell}\|}$ , then  $\|A^{\underline{\ell}}\underline{k}\| \geq D\lambda^{\|\underline{\ell}\|} \|\underline{k}\|^{-\rho} \geq D\lambda^{\|\underline{\ell}\|} b^{-\rho\|\underline{\ell}\|} = Dh^{\|\underline{\ell}\|}$ ,  $\underline{\ell} \neq \underline{0}$ . It follows, for the sum (1):

$$|(A)| \leq \left( \sum_{\|\underline{k}\| < b^{\|\underline{\ell}\|}} |c_{\underline{k}}(f)|^2 \right)^{\frac{1}{2}} \left( \sum_{\|\underline{k}\| < b^{\|\underline{\ell}\|}} |c_{A^{\underline{\ell}}\underline{k}}(f)|^2 \right)^{\frac{1}{2}} \leq \|f\|_2 \sum_{\|\underline{m}\| > Dh^{\|\underline{\ell}\|}} |c_{\underline{m}}(f)|^2.$$

By Parseval inequality and (32), there is a finite constant  $B_1(f)$  such that, for  $\underline{\ell} \neq \underline{0}$ :

$$(36) \quad \left( \sum_{\|\underline{m}\| > Dh^{\|\underline{\ell}\|}} |c_{\underline{m}}(f)|^2 \right)^{\frac{1}{2}} \leq \|f - s_{[Dh^{\|\underline{\ell}\|}], \dots, [Dh^{\|\underline{\ell}\|}]}(f)\|_2 \leq \frac{R(f)}{(\ln[Dh^{\|\underline{\ell}\|}])^\alpha} \leq B_1(f) \|\underline{\ell}\|^{-\alpha}.$$

From the previous inequalities, it follows:  $|(A)| \leq B_1(f) \|f\|_2 \|\underline{\ell}\|^{-\alpha}$ ,  $\forall \|\underline{\ell}\| \neq 0$ . Analogously, we obtain  $|\sum_{\|\underline{k}\| \geq b^{\|\underline{\ell}\|}} c_{\underline{k}}(f) \bar{c}_{A^{\underline{\ell}}\underline{k}}(f)| \leq B_2(f) \|f\|_2 \|\underline{\ell}\|^{-\alpha}$ ,  $\underline{\ell} \neq \underline{0}$  for (B) in (35).

Taking  $B(f) = B_1(f) + B_2(f)$ , (34) follows from (35) and implies the last statements.  $\square$

## 4. CLT for summation sequences of endomorphisms on $G$

### 4.1. Criterium for the CLT on a compact abelian connected group $G$ .

For  $\mathbb{Z}^d$ -dynamical systems satisfying the  $K$ -property, a martingale-type property can be used to obtain a CLT. (For martingale methods applied to  $d$ -dimensional random fields, see for example [16], [35].) For  $\mathbb{Z}^d$ -action by automorphisms, the  $K$ -property is equivalent to mixing of all orders (cf. [31]) for zero-dimensional compact abelian groups but does not hold for instance for the model that we consider on tori. In the absence of  $K$ -property, we will use for abelian semigroups of endomorphisms of connected compact abelian groups the method of mixing of all orders applied by Leonov to a single ergodic automorphism.

#### Mixing actions by endomorphisms ( $G$ connected)

The proof of the CLT given by Leonov in [24] for a single ergodic endomorphism  $A$  of a compact abelian group  $G$  is based on the computation of the moments of the ergodic sums  $S_n f$  when  $f$  is a trigonometric polynomial. It uses the fact that  $A$  is mixing of all orders, which follows from the  $K$ -property for the  $\mathbb{Z}$ -action of a single ergodic automorphism ([29]). For  $\mathbb{Z}^d$ -actions by automorphisms on compact abelian groups, mixing of all orders is not always true (cf. [22], [32]), but it is satisfied for actions on connected compact abelian groups (Theorem 4.1 below) and the method of moments can be used.

In 1992, W. Philip in [28] and K. Schmidt and T. Ward in [31] applied results on the number of solutions of  $S$ -units equations (see ([30, 12]) to endomorphisms or automorphisms of compact abelian groups.

**Theorem 4.1.** ([31, Corollary 3.3]) *Every 2-mixing  $\mathbb{Z}^d$ -action by automorphisms on a compact connected abelian group  $G$  is mixing of all orders.*

With the notations of Lemma 3.1, if  $\mathcal{S}$  is a totally ergodic semigroup of endomorphisms on a compact connected abelian group  $G$ , then its extension  $\tilde{\mathcal{S}}$  to a group of automorphisms of  $\tilde{G}$  is mixing of all orders by Theorem 4.1.

From now on, we consider a totally ergodic  $\mathbb{Z}^d$ -action  $\underline{\ell} \rightarrow A^{\underline{\ell}}$  by commuting automorphisms on  $G$  (or on the extension  $\tilde{G}$ , but we will not write  $\sim$ ) which is mixing of all orders (an assumption satisfied when  $G$  is connected by Theorem 4.1).

Let  $(R_n)_{n \geq 1}$  be a summation sequence on  $\mathbb{Z}^d$ . We use the notations and the results of the appendix (Sect. 7) on cumulants. For  $f \in L^2(G)$ , we put  $\sigma_n(f) := \|\sum_{\underline{\ell}} R_n(\underline{\ell}) A^{\underline{\ell}} f\|_2$  and assume  $\sigma_n^2(f) \neq 0$ , for  $n$  big.

**Lemma 4.2.** *For a trigonometric polynomial  $f$  with zero mean, the condition*

$$(37) \quad \sum_{\underline{\ell}} \prod_{k=1}^r R_n(\underline{\ell} + \underline{j}_k) = o(\sigma_n^r(f)), \forall \{\underline{j}_1, \dots, \underline{j}_r\} \in \mathbb{Z}^d, \forall r \geq 3, \text{ implies}$$

$$(38) \quad \sigma_n(f)^{-1} \sum_{\underline{\ell}} R_n(\underline{\ell}) A^{\underline{\ell}} f \xrightarrow[n \rightarrow \infty]{\text{distrib}} \mathcal{N}(0, 1).$$

*Proof.* Let  $(\chi_{\underline{k}}, \underline{k} \in \Lambda)$  be a finite set of characters on  $G$ ,  $\chi_0$  the trivial character. If  $f = \sum_{\underline{k} \in \Lambda} c_{\underline{k}}(f) \chi_{\underline{k}}$ , the moments of the process  $(f(A^{\underline{n}}))_{\underline{n} \in \mathbb{Z}^d}$  are

$$m_f(\underline{n}_1, \dots, \underline{n}_r) = \int f(A^{\underline{n}_1} x) \dots f(A^{\underline{n}_r} x) dx = \sum_{\underline{k}_1, \dots, \underline{k}_r \in \Lambda} c_{\underline{k}_1} \dots c_{\underline{k}_r} 1_{A^{\underline{n}_1} \chi_{\underline{k}_1} \dots A^{\underline{n}_r} \chi_{\underline{k}_r} = \chi_0}.$$

For  $r$  fixed, the function  $(\underline{k}_1, \dots, \underline{k}_r) \rightarrow m_f(\underline{k}_1, \dots, \underline{k}_r)$  takes a finite number of values, since  $m_f$  is a sum with coefficients 0 or 1 of the products  $c_{\underline{k}_1} \dots c_{\underline{k}_r}$  with  $\underline{k}_j$  in a finite set. The cumulants of given order according to Identity (56) take also a finite number of values.

Therefore, since mixing of all orders implies  $\lim_{\max_{i,j} \|\underline{\ell}_i - \underline{\ell}_j\| \rightarrow \infty} s_f(\underline{\ell}_1, \dots, \underline{\ell}_r) = 0$  (cf. Notation (61)) by Lemma 7.6, there is  $M_r$  such that  $s_f(\underline{\ell}_1, \dots, \underline{\ell}_r) = 0$  if  $\max_{i,j} \|\underline{\ell}_i - \underline{\ell}_j\| > M_r$ .

We apply Theorem 7.2 (cf. appendix). Let us check (60), i.e.,

$$\sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, \dots, X_{\underline{\ell}_r}) R_n(\underline{\ell}_1) \dots R_n(\underline{\ell}_r) = o(\|Y^n\|_2^r), \forall r \geq 3.$$

Using (58), we obtain

$$\begin{aligned} & \left| \sum_{\underline{\ell}_1, \dots, \underline{\ell}_r} s_f(\underline{\ell}_1, \dots, \underline{\ell}_r) R_n(\underline{\ell}_1) \dots R_n(\underline{\ell}_r) \right| = \left| \sum_{\max_{i,j} \|\underline{\ell}_i - \underline{\ell}_j\| \leq M_r} s_f(\underline{\ell}_1, \underline{\ell}_2, \dots, \underline{\ell}_r) R_n(\underline{\ell}_1) \dots R_n(\underline{\ell}_r) \right| \\ & \leq \sum_{\underline{\ell}} \sum_{\|\underline{j}_2\|, \dots, \|\underline{j}_r\| \leq M_r, j_1=0} |s_f(\underline{\ell}, \underline{\ell} + \underline{j}_2, \dots, \underline{\ell} + \underline{j}_r)| \prod_{k=1}^r R_n(\underline{\ell} + \underline{j}_k) \\ & = \sum_{\underline{\ell}} \sum_{\|\underline{j}_2\|, \dots, \|\underline{j}_r\| \leq M_r, j_1=0} |s_f(\underline{j}_1, \underline{j}_2, \dots, \underline{j}_r)| \prod_{k=1}^r R_n(\underline{\ell} + \underline{j}_k). \end{aligned}$$

The right hand side is less than  $C \sum_{\underline{\ell}} \sum_{\|j_2\|, \dots, \|j_r\| \leq M_r, j_1 = \underline{0}} \prod_{k=1}^r R_n(\underline{\ell} + j_k)$ . Therefore (37) implies (60).  $\square$

**Theorem 4.3.** *Let  $(R_n)_{n \geq 1}$  be a summation sequence on  $\mathbb{Z}^d$  which is  $\zeta$ -regular (cf. Definition 1.2). Let  $f$  be a function in  $AC_0(G)$  with spectral density  $\varphi_f$ . The condition*

$$(39) \quad \left( \sup_{\underline{\ell}} R_n(\underline{\ell}) \right)^{r-1} \sum_{\underline{\ell}} R_n(\underline{\ell}) = o\left( \left( \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})^2 \right)^{r/2} \right), \text{ for every } r \geq 3,$$

*implies (with the convention that the limiting distribution is  $\delta_0$  if  $\sigma^2(f) = 0$ )*

$$(40) \quad \left( \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})^2 \right)^{-\frac{1}{2}} \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) f(A^{\underline{\ell}}) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \zeta(\varphi_f)).$$

*Proof.* We use (29) and the  $\zeta$ -regularity of  $(R_n)$ : for  $g$  in  $AC_0(G)$ ,

$$\left( \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})^2 \right)^{-1} \left\| \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) A^{\underline{\ell}} g \right\|_2^2 = \int_{\mathbb{T}^d} \tilde{R}_n \varphi_g dt \xrightarrow[n \rightarrow \infty]{} \zeta(\varphi_g).$$

Let  $(\mathcal{E}_s)_{s \geq 1}$  be an increasing sequence of finite sets in  $\hat{G}$  with union  $\hat{G} \setminus \{0\}$  and let  $f_s(x) := \sum_{\chi \in \mathcal{E}_s} c_f(\chi) \chi$  be the trigonometric polynomial obtained by restriction of the Fourier series of  $f$  to  $\mathcal{E}_s$ . Let us consider the processes defined respectively by

$$U_n^s := \left( \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})^2 \right)^{-\frac{1}{2}} \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) f_s(A^{\underline{\ell}}), \quad U_n := \left( \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})^2 \right)^{-\frac{1}{2}} \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) f(A^{\underline{\ell}}).$$

We can suppose  $\zeta(\varphi_f) > 0$ , since otherwise the limiting distribution is  $\delta_0$ . By Proposition 3.3 we have  $\zeta(\varphi_{f-f_s}) \leq \|f - f_s\|_c$ . It follows  $\zeta(\varphi_{f_s}) \neq 0$  for  $s$  big enough. We can apply Lemma 4.2 to the trigonometric polynomials  $f_s$ , since  $\sigma_n^2(f_s) \sim \left( \sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})^2 \right) \zeta(\varphi_{f_s})$  with  $\zeta(\varphi_{f_s}) > 0$  and since Condition (39) implies Condition (37) in Lemma 4.2.

It follows:  $U_n^s \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \zeta(\varphi_{f_s}))$  for every  $s$ . Moreover, since

$$\lim_n \int |U_n^s - U_n|_2^2 d\mu = \lim_n \int_{\mathbb{T}^d} \tilde{R}_n \varphi_{f-f_s} dt = \zeta(\varphi_{f-f_s}) \leq \|f - f_s\|_c,$$

we have  $\limsup_n \mu[|U_n^s - U_n| > \varepsilon] \leq \varepsilon^{-2} \limsup_n \int |U_n^s - U_n|_2^2 d\mu \xrightarrow[s \rightarrow \infty]{} 0$  for every  $\varepsilon > 0$ .

Therefore the condition  $\lim_s \limsup_n \mu[|U_n^s - U_n| > \varepsilon] = 0, \forall \varepsilon > 0$ , is satisfied and the conclusion  $U_n \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \zeta(\varphi_f))$  follows from Theorem 3.2 in [2].  $\square$

With the notations of Sect. 1, Theorem 4.3 implies for sequential summations:

**Corollary 4.4.** *If  $(\underline{x}_n)$  is a sequence in  $\mathbb{Z}^d$  such that  $\underline{z}_n = \sum_{k=0}^{n-1} \underline{x}_k$  is  $\zeta$ -regular, then the convergence  $v_n^{-\frac{1}{2}} \sum_{k=0}^{n-1} f(A^{z_k}) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \zeta(\varphi_f))$  follows from the condition*

$$(41) \quad n \left( \sup_{\underline{\ell}} \sum_{k=0}^{n-1} 1_{z_k = \underline{\ell}} \right)^{r-1} = o(v_n^{r/2}), \quad \forall r \geq 3.$$

Before considering random walks, let us apply Theorem 4.3 to summation over sets:

**Corollary 4.5.** *Let  $(D_n)_{n \geq 1}$  be a Følner sequence of sets in  $\mathbb{N}^d$  and let  $f$  be in  $AC_0(G)$ . We have  $\sigma^2(f) = \lim_n \|\sum_{\ell \in D_n} A^\ell f\|_2^2 / |D_n| = \varphi_f(0)$  and*

$$|D_n|^{-\frac{1}{2}} \sum_{\ell \in D_n} A^\ell f(\cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma^2(f)).$$

*Proof.* The sequence  $R_n(\ell) = 1_{D_n}(\ell)$  is  $\zeta$ -regular, with  $\zeta = \delta_0$ . Suppose that  $\varphi_f(0) \neq 0$ . We have  $\sigma_n^2(f) \sim |D_n| \varphi_f(0)$  and  $R_n(\ell + \underline{j}_k) = 0$  or 1. Therefore Condition (39) holds and the result follows from Theorem 4.3.  $\square$

**Remarks 4.6.** 1) The previous result is valid for the rotated sums: for  $f$  in  $AC_0(G)$ , for every  $\theta$ ,

$$(42) \quad \sigma_\theta^2(f) = \varphi_f(\theta), \quad |D_n|^{-\frac{1}{2}} \sum_{\ell \in D_n} e^{2\pi i \langle \ell, \theta \rangle} f(A^\ell \cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma_\theta^2(f)).$$

2) When  $G = \mathbb{T}^d$ , in view of Theorem 3.7, the conclusions of Theorem 4.3 and Corollary 4.5 are valid under the weaker regularity assumption (32), in particular under a logarithmic Hölderian regularity (condition (31)).

3) As mentioned in the introduction, the result of Corollary 4.5 for the sums over  $d$ -dimensional rectangles and regular functions was obtained by M. Levin ([26]).

*A CLT for the rotated sums for a.e.  $\theta$  without regularity assumption*

When  $(D_n)$  is a sequence of  $d$ -dimensional cubes in  $\mathbb{Z}^d$ , the following CLT for the rotated sums holds for a.e.  $\theta$  without regularity assumptions on  $f$ .

**Theorem 4.7.** *Let  $(D_n)_{n \geq 1}$  be a sequence of cubes in  $\mathbb{Z}^d$ . For  $f$  in  $L^2(G)$ , we have for a.e.  $\theta \in \mathbb{T}^d$ :  $\sigma_\theta^2(f) = \varphi_f(\theta)$  and  $|D_n|^{-\frac{1}{2}} \sum_{\ell \in D_n} e^{2\pi i \langle \ell, \theta \rangle} A^\ell f(\cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma_\theta^2(f))$ .*

*Proof.* As in [4], we use the relation  $\lim_n |D_n|^{-\frac{1}{2}} \|\sum_{\ell \in D_n} e^{2\pi i \langle \ell, \theta \rangle} T^\ell(f - M_\theta f)\|_2 = 0$ , which is satisfied, for any  $f \in L^2(G)$ , for  $\theta$  in a set of full measure (depending on  $f$ ).  $\square$

## 5. Application to r.w. of commuting endomorphisms on $G$

Now we apply the previous sections to random walks of commuting endomorphisms or automorphisms on a compact abelian group  $G$ .

Let us consider a family  $(B_j, j \in J)$  of commuting endomorphisms of  $G$  (extended if necessary to automorphisms of  $\tilde{G}$ ) and a probability vector  $\nu = (p_j, j \in J)$  such that  $p_j > 0, \forall j$ . These data define a random walk  $U_n := Y_0 \dots Y_{n-1}$  where  $(Y_k)_{k \geq 0}$  are i.i.d. r.v. with common distribution  $\mathbb{P}(Y_k = B_j) = p_j$ .

If the  $B_j$ 's are given a priori, we can try to express  $U_n$  via a r.w. on  $\mathbb{Z}^d$  for some  $d$ . This means that we have to find algebraically independent generators  $A_1, \dots, A_d$  for some  $d$ , in order to express the  $B_j$ 's as  $B_j = A^{\underline{\ell}_j}$ , for  $\underline{\ell}_j \in \mathbb{Z}^d$ . Another approach is to start from a totally ergodic  $\mathbb{Z}^d$ -action  $\mathbb{A}$  on  $G$  (or  $\tilde{G}$ ) and from a r.w.  $W$  on  $\mathbb{Z}^d$  and to transfer  $W$  to a r.w. on the group of automorphisms of  $\tilde{G}$ , thus getting the  $B_j$ 's a posteriori.

For example, when  $G$  is a torus, we can use the method described in Subsection 3.2, which gives an explicit construction of algebraically independent generators of a totally ergodic  $\mathbb{Z}^d$ -action by invertible matrices on a torus.

With  $(\Omega, \mathbb{P}) = ((\mathbb{Z}^d)^{\mathbb{Z}}, \nu^{\otimes \mathbb{Z}})$  and  $(X_n)$  the sequence of coordinates maps of  $\Omega$ , we obtain on  $(\Omega \times \tilde{G}, \mathbb{P} \times \tilde{\mu})$  a dynamical system  $(\omega, x) \rightarrow (\tau\omega, A^{X_0(\omega)}x)$ . The iterates are  $(\omega, x) \rightarrow (\tau^n\omega, A^{X_0(\omega)+\dots+X_{n-1}(\omega)}x) = (\tau^n\omega, A^{Z_n(\omega)}x)$ ,  $n \geq 1$ .

The random walk  $U_n = Y_0 \dots Y_{n-1}$  can be expressed as  $(A^{Z_n})$ , where  $(Z_n)$  is the r.w. in  $\mathbb{Z}^d$  (which can be and will be assumed reduced) with distribution  $\mathbb{P}(X_0 = \underline{\ell}_j) = p_j$ .

### 5.1. Random walks and quenched CLT.

Recall that the measure  $d\gamma$  is defined in Definition 2.3,  $w(\underline{t}) = \frac{1-|\Psi(\underline{t})|^2}{|1-\Psi(\underline{t})|^2}$  and  $V_n(\omega) = \#\{0 \leq k' < n : Z_k(\omega) = Z_{k'}(\omega)\}$ .

**Theorem 5.1.** *Let  $W$  be a reduced centered r.w. Let  $\underline{\ell} \rightarrow A^{\underline{\ell}}$  be a totally ergodic  $\mathbb{Z}^d$ -action by automorphisms on  $G$ . Let  $f$  be in  $AC_0(G)$  with spectral density  $\varphi_f$ .*

I) Suppose  $W$  with a finite moment of order 2 and centered.

a) If  $d = 1$ , then, for a.e.  $\omega$ ,  $V_n(\omega)^{-\frac{1}{2}} \sum_{k=0}^{n-1} A^{Z_k(\omega)} f(\cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \gamma(\varphi_f))$ .

b) If  $d = 2$ , then, for a.e.  $\omega$ ,  $(Cn \text{Log} n)^{-\frac{1}{2}} \sum_{k=0}^{n-1} A^{Z_k(\omega)} f(\cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \gamma(\varphi_f))$ , with

$$C = \pi^{-1} a_0(W) \det(\Lambda)^{-\frac{1}{2}}.$$

II) If  $W$  is transient with finite moment of order  $\eta$  for some positive  $\eta$ , then, for a.e.  $\omega$ ,

$(Cn)^{-\frac{1}{2}} \sum_{k=0}^{n-1} A^{Z_k(\omega)} f(\cdot) \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \zeta(\varphi_f))$ . We have for  $\zeta$  the following cases:

if  $d = 1$ , then  $C = c_w + K$  and  $d\zeta(\underline{t}) = (c_w + K)^{-1} (w(\underline{t}) d\underline{t} + d\gamma(t))$ , (cf. notations of Theorem 2.13) ( $K \neq 0$  if  $m(W)$  is finite);

if  $d \geq 2$ , then  $C = c_w$  and  $d\zeta(\underline{t}) = c_w^{-1} w(\underline{t}) d\underline{t}$  (the absolutely continuous part is non trivial if and only if the random walk is non deterministic).

*Proof.* Theorem 2.13 gives the  $\zeta$ -regularity for the r.w. summation  $(R_n(\omega, \underline{\ell}))_{n \geq 1} = (\sum_{k=0}^{n-1} 1_{Z_k(\omega)=\underline{\ell}})_{n \geq 1}$  and the expression of  $\zeta$ . This is the first step (variance). It remains to check Condition (41) of Corollary 4.4, which reads here:

$$(43) \quad n \left( \sup_{\underline{\ell}} \sum_{k=0}^{n-1} 1_{Z_k(\omega)=\underline{\ell}} \right)^{r-1} = o(V_n(\omega)^{r/2}), \text{ for every } r \geq 3.$$

a) For the recurrent 1-dimensional case, since  $\sum_{\underline{\ell}} R_n^2(\omega, \underline{\ell}) \geq Cn^{\frac{3}{2}}/\text{LogLog} n$  for a.e.  $\omega$  by (47), to have (43) it suffices that  $n \left( \sup_{\underline{\ell}} R_n(\omega, \underline{\ell}) \right)^{r-1} = o\left(\left(\frac{n^{\frac{3}{2}}}{\text{LogLog} n}\right)^{r/2}\right)$ ,  $\forall r \geq 3$ , i.e.,

$$\sup_{\underline{\ell}} R_n(\omega, \underline{\ell}) = o\left(n^{\frac{3r-4}{4r-4}} (\text{LogLog} n)^{-\frac{r}{2r-2}}\right), \forall r \geq 3.$$

The condition is satisfied, since the exponent  $\frac{3r-4}{4r-4}$  is bigger than  $\frac{5}{8}$  and  $\sup_{\underline{\ell}} R_n(\underline{\ell}) \leq n^{\frac{1}{2}+\varepsilon}$ .

b) For the recurrent 2-dimensional case, for a.e.  $\sum_{\underline{\ell}} R_n^2(\omega, \underline{\ell}) \sim \mathbb{E} \sum_{\underline{\ell}} R_n^2(\cdot, \underline{\ell}) \sim Cn \text{Log } n$ .

We need:  $n (\sup_{\underline{\ell}} R_n(\underline{\ell}))^{r-1} = o((n \text{Log } n)^{r/2}), \forall r \geq 3$ , i.e.,

$$\sup_{\underline{\ell}} R_n(\underline{\ell}) = o(n^{\frac{r-2}{2r-2}} (\text{Log } n)^{\frac{r}{2r-2}}), \forall r \geq 3.$$

The above condition is satisfied, since the exponent  $n^{\frac{r-2}{2r-2}}$  increases from  $\frac{1}{4}$  to  $\frac{1}{2}$  when  $r$  varies from 3 to  $+\infty$  and by Lemma 2.9  $\sup_{\underline{\ell}} R_n(\underline{\ell}) = o(n^\varepsilon), \forall \varepsilon > 0$ .

II) For the transient case, we have the same estimation without the logarithmic factor. Excepted in the deterministic case, the variance is  $> 0$  unless  $f \equiv 0$  a.e.  $\square$

**Remark 5.2.** If  $(\Phi_n(\omega, x))_{n \geq 1}$  is a process depending on two variables  $x$  and  $\omega$  such that, for a normalization by  $\sigma_n$  independent of  $\omega$ ,  $\int e^{it\sigma_n^{-1}\Phi_n(\omega, x)} d\mu(x) \rightarrow e^{-t^2/2}$ , for a.e.  $\omega$ , then the CLT holds for  $\Phi_n$  w.r.t.  $d\mu \times d\mathbb{P}$  holds, since by the dominated convergence theorem:  $\int_{\Omega} (\int_X e^{it\sigma_n^{-1}\Phi_n(\omega, x)} d\mu(x)) d\mathbb{P}(\omega) \rightarrow e^{-t^2/2}$ .

It follows that, for a r.w.  $W$  which is transient or with finite variance, centering and  $d(W) = 2$ , the annealed version of the CLT in Theorem 5.1.b is satisfied. This result can be viewed as a ‘‘toral’’ version of Bolthausen theorem in [3].

For  $W$  with finite variance, centering and  $d(W) = 1$ , the theorem of Kesten and Spitzer in [20] for a r.w. in random scenery gives an (annealed) convergence toward a distribution which is not the normal law. For the quenched process in their model or in the toral model of Theorem 5.1, convergence toward a normal law holds, but with a normalization depending on  $\omega$ . Let us mention that, in the toral model, the annealed theorem analogous to the result of [20] for the recurrent 1-dimensional r.w. holds (S. Le Borgne, personal communication). Let us mention other results of quenched type like for instance in [17].

### An example

Let us give an explicit example on  $\mathbb{T}^3$ . Consider the centered random walk on  $\mathbb{Z}^2$  with distribution  $\nu$  supported on  $\underline{\ell}_1 = (2, 1)$ ,  $\underline{\ell}_2 = (1, -2)$ ,  $\underline{\ell}_3 = (-3, 1)$ , such that  $\mathbb{P}(X_0 = \underline{\ell}_j) = \frac{1}{3}$ , for  $j = 1, 2, 3$ . With the notation of Subsection 2.2, here  $D$  is the sublattice generated by  $\{(1, 3), (4, -3)\}$  and we have  $\mathcal{D}^\perp = \{\underline{0}\}$  and  $a_0 = 15$ .

Let  $A_1$  and  $A_2$  be the commuting matrices computed in Subsection 3.2 (example 1.a).

$$\text{Let the matrices } B_j, j = 1, 2, 3, \text{ be defined by } B_1 = A_1^2 A_2 = \begin{pmatrix} 29 & 23 & -8 \\ -80 & -67 & 23 \\ 230 & 196 & -67 \end{pmatrix},$$

$$B_2 = A_1 A_2^{-2} = \begin{pmatrix} -13 & -11 & 4 \\ 40 & 35 & -11 \\ -110 & -92 & 35 \end{pmatrix}, B_3 = A_1^{-3} A_2 = \begin{pmatrix} 107 & 16 & -7 \\ -70 & 23 & 16 \\ 160 & 122 & 23 \end{pmatrix}.$$

The random walk on  $\mathbb{T}^3$  such that with equal probability we move from  $x \in \mathbb{T}^3$  to  $B_j x$ ,  $j = 1, 2, 3$ , gives rise to a random process  $(U_n(\omega)x)_{n \geq 1}$  on the torus such that for a regular

function  $f$  the limiting distribution of  $((n \operatorname{Log} n)^{-\frac{1}{2}} \sum_{k=0}^{n-1} f(V_k(\omega)))$  is a normal law, with variance  $\sigma_f(0) = c \varphi_f(0)$ , where the constant  $c$  is given by the LLT.

## 5.2. Powers of barycenter operators.

We show now that the iterates of the barycenter operators satisfy the condition of Theorem 4.3. Let  $(B_j, j \in J)$  be a set of endomorphisms satisfying Assumption 3.2, i.e., such that  $B_j = A^{\underline{\ell}_j}$ ,  $\underline{\ell}_j \in \mathbb{Z}^d$ , where  $A_1, \dots, A_d$  are  $d$  algebraically independent commuting automorphisms of  $\tilde{G}$ . We suppose that the corresponding  $\mathbb{Z}^d$ -action  $\mathbb{A}$  on  $(\tilde{G}, \tilde{\mu})$  is totally ergodic.

Let  $P$  be the barycenter operator  $Pf(x) := \sum_{j \in J} p_j f(B_j x)$ . The associated random walk  $(Z_n)$  on  $\mathbb{Z}^d$  is defined by  $\mathbb{P}(X_0 = \underline{\ell}_j) = p_j$ , for  $j \in J$ . The lattice  $L(\tilde{W})$  generated by the support of the distribution of  $\tilde{W}$  coincides with  $D(W)$ . The lattices  $D(W)$  and  $D(\tilde{W})$  are the same. We use the LLT (Theorem 2.8) for  $\tilde{W}$  (with the exponent  $d(\tilde{W}) = d_0(W) = d(W)$  or  $d_0(W) - 1$  and  $\Lambda$  replaced by  $\Lambda_0$ ).

We suppose that  $D(W)$  is not trivial, so that  $W$  is not deterministic and  $d_0 = d_0(W) \geq 1$ . The measure  $d\gamma_1$  was defined in Notation 2.3 (see also 2.3.2).

**Theorem 5.3.** *Let  $f$  be a function in  $AC_0(G)$  with spectral density  $\varphi_f$ . Then, for a constant  $C$  depending on the random walk, we have w.r.t. the Haar measure on  $G$ :*

$$C n^{\frac{d_0}{4}} P^n f \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma_P^2(f)), \text{ with } \sigma_P^2(f) = \int \varphi_f d\gamma_1.$$

In particular, if  $B_j = A_j, j = 1, \dots, d$ , then

$$(4\pi)^{\frac{d_0}{4}} (p_1 \dots p_d)^{\frac{1}{4}} n^{\frac{d_0}{4}} P^n f \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma_P^2(f)), \text{ with } \sigma_P^2(f) = \int_{\mathbb{T}^1} \varphi_f(u, u, \dots, u) du.$$

*Proof.* We apply Proposition 2.15. Here  $R_n(\underline{\ell}) = \mathbb{P}(Z_n = \underline{\ell})$ ,  $\sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell}) = 1$  and by Theorem 2.6,  $\sup_{\underline{\ell}} R_n(\underline{\ell}) = O(n^{-d_0/2})$ . If  $\sigma_P^2(f) \neq 0$ , the variance  $\sigma_n^2(f)$  is asymptotically like  $\sum_{\underline{\ell} \in \mathbb{Z}^d} R_n(\underline{\ell})^2 = \mathbb{P}(\tilde{Z}_n = \underline{0}) \sim C_0 n^{-d_0/2}$ , for a constant  $C_0 > 0$ .

Here Condition (39) reads  $(\sup_{\underline{\ell}} R_n(\underline{\ell}))^{r-1} = o(n^{-rd_0/4}), \forall r \geq 3$ . It is satisfied, since  $n^{-(r-1)d_0/2} = o(n^{-rd_0/4}), \forall r \geq 3$ . We conclude by Theorem 4.3.  $\square$

### Examples and remarks.

1) Let  $A_1, A_2$  be two commuting matrices with coefficients in  $\mathcal{M}(\rho, \mathbb{Z})$  generating a totally ergodic  $\mathbb{Z}^2$ -action on  $\mathbb{T}^\rho$ ,  $\rho \geq 3$ . Let  $P$  be the barycenter operator  $Pf(x) := p_1 f(A_1 x) + p_2 f(A_2 x)$ ,  $p_1, p_2 > 0, p_1 + p_2 = 1$ .

The symmetrized r.w.  $(\tilde{Z}_n)$  is 1-dimensional and the support of the distribution of  $\tilde{X}_0$  is  $\{(0, 0), (1, -1), (-1, 1)\}$ . If  $\varphi_f$  is continuous, then  $\lim_{n \rightarrow \infty} \sqrt{4p_1 p_2 \pi n} \|P^n f\|_2^2 = \int_{\mathbb{T}} \varphi_f(u, u) du$ . If  $f$  satisfies the regularity condition (31) on  $\mathbb{T}^\rho$ , we have, with  $\sigma_P^2(f) = \int_{\mathbb{T}} \varphi_f(u, u) du$ ,  $(4p_1 p_2 \pi)^{\frac{1}{4}} n^{\frac{1}{4}} P^n f \xrightarrow[n \rightarrow \infty]{\text{distr}} \mathcal{N}(0, \sigma_P^2(f))$ .



If  $f$  is in  $AC_0(G)$ , then  $\sigma_P(f) = 0$  if and only if  $\varphi_f(u, u) = 0$ , for every  $u \in \mathbb{T}^1$ . In particular, if  $f$  is not a mixed coboundary (cf. Theorem 5.4), then  $\sigma_P(f) \neq 0$  and the rate of convergence of  $\|P^n f\|_2$  to 0 is the polynomial rate given by Theorem 5.3.

2) Suppose now that  $A_1, A_2$  are automorphisms. We can take, for examples the matrices computed in Subsection 3.2. Let  $P$  be defined by  $Pf(x) := p_1 f(A_1 x) + p_2 f(A_2 x) + p_3 f(A_1^{-1} x) + p_4 f(A_2^{-1} x)$ ,  $p_j > 0$ ,  $\sum_j p_j = 1$ .

We have analogous result, excepted that here  $L(\tilde{W}) = D(\tilde{W}) = D(W) = \mathbb{Z}(1, 1) \oplus \mathbb{Z}(-1, 1)$ . The measure  $d\gamma_1$  is the barycenter of  $\delta_{(0,0)}$  and  $\delta_{(\frac{1}{2}, \frac{1}{2})}$ .

3) Observe that the previous example for the quenched limit theorem used for its computation the tables of units in a field number. As remarked in Sect. 3, for barycenters, it is easier to give examples which are endomorphisms. The maps on the torus are not necessarily invertible, but the analysis of the process uses a symmetrized r.w. on  $\mathbb{Z}^d$ . The difficulty is then to compute the dimension of the associated r.w. unless it is given by primality conditions of the determinants. Let us give a simple example.

Let  $P$  on  $L_0^2(\mathbb{T}^1)$  be defined by:  $(Pf)(x) = \frac{1}{8}f(2x) + \frac{2}{8}f(3x) + \frac{1}{8}f(5x) + \frac{3}{8}f(6x) + \frac{1}{8}f(15x)$ . The behavior of the iterates  $P^n$  is given by the LLT applied to the symmetrized r.w.  $(\tilde{Z}_n)$  (strictly aperiodic in  $\mathbb{Z}^3$ ) with distribution supported on  $\tilde{\Sigma} = \{(0, 0, 0), \pm(1, 0, 0), \pm(0, 1, 0), \pm(0, 0, 1), \pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1), \pm(1, 1, -1), \pm(1, -1, -1)\}$ . If  $f$  is regular on  $\mathbb{T}^1$ , the process  $(n^{-3/4}P^n f(\cdot))_{n \geq 1}$  converges in distribution to a normal law (non degenerate  $f$  is not a mixed coboundary).

4) For  $\nu$  a discrete measure on the semigroup  $\mathcal{T}$  of commuting endomorphisms of  $G$ , let us consider a barycenter of the form  $Pf(x) = \sum_{T \in \mathcal{T}} \nu(T)f(Tx)$ . When there is a finite moment of order 2 and  $d(W) < +\infty$ , the decay of  $P^n f$  is of order  $n^{-\frac{d(W)}{4}}$  when  $\varphi_f$  is continuous and  $\varphi_f(0) \neq 0$ . A question is to estimate the decay when  $\nu$  has an infinite support and  $d(W)$  is infinite and to study the asymptotic distribution of the normalized iterates (if there is a normalization).

For example if  $Pf(x) = \sum_{q \in \mathcal{P}} \nu(q)f(qx)$ , where  $\mathcal{P}$  is the set of prime numbers and  $(\nu(q), q \in \mathcal{P})$  a probability vector with  $\nu(q) > 0$  for every prime  $q$ , what is the decay to 0 of  $\|P^n f\|_2$ , when  $f$  is Hölderian on the circle?

When  $d_0(W)$  is infinite, a partial result is that the decay is faster than  $Cn^{-r}$ , for every  $r \geq 1$ . This follows from the following observation:

Let  $P_1$  and  $P_2$  be commuting contractions of  $L^2(G)$  such that  $\|P_1^n f\|_2 \leq Mn^{-r}$ . Let  $\alpha \in ]0, 1], \beta = 1 - \alpha$ . Then we have:  $\|(\alpha P_1 + \beta P_2)^n f\|_2 \leq \sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k} \|P_1^k f\|_2$ . There is  $c < 1$  such that  $\sum_{k \leq \frac{n}{2\alpha}} \binom{n}{k} \alpha^k \beta^{n-k} \leq c^n$ ; therefore:  $\sum_{k=0}^n \binom{n}{k} \alpha^k \beta^{n-k} \|P_1^k f\|_2 \leq M(\frac{n}{2\alpha})^{-r} + c^n \leq M'n^{-r}$ .

5) The case of commutative or amenable actions strongly differs from the case of non amenable actions for which a ‘‘spectral gap property’’ is available in certain cases ([15]), which implies a quenched CLT theorem (cf. ([7]).

For actions by algebraic *non commuting* automorphisms  $B_j, j = 1, \dots, d$ , on the torus, the existence of a spectral gap for  $P$  of the form  $Pf(x) := \sum_{j \in J} p_j f(B_j x)$  is related to the fact that the generated group has no factor torus on which it is virtually abelian (cf. [1]). In general, a question is to split the action into a (virtually) abelian action on the  $L^2$ -space of a factor torus and a supplementary subspace with a spectral gap, in order to get a full description of the quenched and the barycenter processes.

### Coboundary characterization

For a  $\mathbb{Z}^d$ -action  $\mathcal{S}$  with Lebesgue spectrum by automorphisms on  $\mathbb{T}^\rho$  and  $f$  on  $\mathbb{T}^\rho$ , let us give a characterization for  $\varphi_f(0) = 0$  in terms of coboundaries.

For the dual action of  $\mathbb{Z}^d$  on  $\mathbb{Z}^\rho$ , we construct the section  $J_0$  introduced before Proposition 3.3 in the following way. For a fixed  $\underline{\ell}$ , the set  $\{A^{\underline{k}\underline{\ell}}, \underline{k} \in \mathbb{Z}^d\}$  is discrete and  $\lim_{\|\underline{k}\| \rightarrow \infty} \|A^{\underline{k}\underline{\ell}}\| = +\infty$ . Therefore we can choose an element  $\underline{j}$  in each class modulo the action of  $\mathcal{S}$  on  $\mathbb{Z}^\rho$  which achieves the minimum of the norm. By this choice, we have

$$(44) \quad \|\underline{j}\| \leq \|A^{\underline{k}\underline{j}}\|, \forall \underline{j} \in J_0, \underline{k} \in \mathbb{Z}^d.$$

**Theorem 5.4.** *If  $|c_f(\underline{k})| = O(\|\underline{k}\|^{-\beta})$ , with  $\beta > \rho$ , then  $\varphi_f(0) = 0$  if and only if  $f$  satisfies the following mixed coboundary condition: there are continuous functions  $u_i$ ,  $1 \leq i \leq d$  such that*

$$(45) \quad f = \sum_{i=1}^d (I - A_i)u_i.$$

*Proof.* Let  $\varepsilon \in ]0, \beta - \rho[$ . If  $\delta = (\beta - \rho - \varepsilon)/(\beta(1 + \rho))$ , we have  $\delta\beta\rho - \beta(1 - \delta) = -(\rho + \varepsilon)$ . There is a constant  $C_1$  such that  $\|\underline{k}\|^d e^{-\delta\beta\tau\|\underline{k}\|} \leq C_1, \forall \underline{k} \in \mathbb{Z}^d$ . According to (33), we have  $|c_f(A^{\underline{k}\underline{j}})| \leq C \|A^{\underline{k}\underline{j}}\|^\beta \leq C e^{-\beta\tau\|\underline{k}\|} \|\underline{j}\|^{\beta\rho}$ ; hence

$$(46) \quad e^{\delta\beta\tau\|\underline{k}\|} |c_f(A^{\underline{k}\underline{j}})|^\delta \leq C \|\underline{j}\|^{\delta\beta\rho}.$$

For every  $\underline{\ell} \in \mathbb{Z}^\rho \setminus \{0\}$ , there is a unique  $(\underline{k}, \underline{j}) \in \mathbb{Z}^d \times J_0$  such that  $A^{\underline{k}\underline{j}} = \underline{\ell}$ ; hence by Inequality (44):

$$\begin{aligned} \sum_{\underline{j} \in J_0} \sum_{\underline{k} \in \mathbb{Z}^d} \|\underline{k}\|^d |c_f(A^{\underline{k}\underline{j}})| &= \sum_{\underline{j} \in J_0} \sum_{\underline{k} \in \mathbb{Z}^d} \|\underline{k}\|^d |c_f(A^{\underline{k}\underline{j}})|^\delta |c_f(A^{\underline{k}\underline{j}})|^{1-\delta} \\ &\leq C_1 \sum_{\underline{j} \in J_0} \sum_{\underline{k} \in \mathbb{Z}^d} e^{\delta\beta\tau\|\underline{k}\|} |c_f(A^{\underline{k}\underline{j}})|^\delta |c_f(A^{\underline{k}\underline{j}})|^{1-\delta} \leq C_2 \sum_{\underline{j} \in J_0} \sum_{\underline{k} \in \mathbb{Z}^d} \|\underline{j}\|^{\delta\beta\rho} |c_f(A^{\underline{k}\underline{j}})|^{1-\delta}. \end{aligned}$$

According to (46) and (44), the right hand side is less than

$$\begin{aligned} C_2 \sum_{\underline{j} \in J_0} \sum_{\underline{k} \in \mathbb{Z}^d} \|A^{\underline{k}\underline{j}}\|^{\delta\beta\rho} |c_f(A^{\underline{k}\underline{j}})|^{1-\delta} &= C_2 \sum_{\underline{\ell} \in \mathbb{Z}^\rho \setminus \{0\}} \|\underline{\ell}\|^{\delta\beta\rho} |c_f(\underline{\ell})|^{1-\delta} \\ &\leq C_3 \sum_{\underline{\ell} \in \mathbb{Z}^\rho \setminus \{0\}} \|\underline{\ell}\|^{\delta\beta\rho - \beta(1-\delta)} \leq C_3 \sum_{\underline{\ell} \in \mathbb{Z}^\rho \setminus \{0\}} \|\underline{\ell}\|^{-(\rho+\varepsilon)} < +\infty. \end{aligned}$$

The sufficient condition for (45) given in [5] reads here:  $\sum_{\underline{j} \in J_0} \sum_{\underline{k} \in \mathbb{Z}^d} (1 + \|\underline{k}\|^d) |c_f(A^{\underline{k}\underline{j}})| < \infty$ . This condition holds by the previous inequality. Since here the functions involved

in the proof of the coboundary characterization are characters, hence continuous and uniformly bounded, the functions  $u_i$  in (45) are continuous.  $\square$

## 6. Appendix I: self-intersections of a centered r.w.

In this appendix, we prove Theorem 2.10. *As already noticed, we can assume aperiodicity in the proofs.*

### 6.1. $d=1$ : a.s. convergence of $V_{n,p}/V_{n,0}$ .

We need the following lemmas.

**Lemma 6.1.** *If  $W$  is a 1-dimensional r.w. with finite variance and centered, then, for a.e.  $\omega$ , there is  $C(\omega) > 0$  such that*

$$(47) \quad V_n(\omega) \geq C(\omega) n^{\frac{3}{2}} (\text{LogLog } n)^{-\frac{1}{2}}.$$

*Proof.* By the law of iterated logarithm, there is a constant  $c > 0$  such that, for a.e.  $\omega$ , the inequality  $|Z_n(\omega)| > c(n \text{LogLog } n)^{\frac{1}{2}}$  is satisfied only for finitely many values of  $n$ . This implies that, for a.e.  $\omega$ , there is  $N(\omega)$  such that  $|Z_n(\omega)| \leq (cn \text{LogLog } n)^{\frac{1}{2}}$ , for  $n \geq N(\omega)$ . It follows, for  $N(\omega) \leq k < n$ ,  $|Z_k(\omega)| \leq (ck \text{LogLog } k)^{\frac{1}{2}} \leq (cn \text{LogLog } n)^{\frac{1}{2}}$ .

Therefore, if  $\mathcal{R}_n(\omega)$  is the set of points visited by the random walk up to time  $n$ , we have  $\text{Card}(\mathcal{R}_n(\omega)) \leq 2(c_1(\omega) n \text{LogLog } n)^{\frac{1}{2}}$ , with an a.e. finite constant  $c_1(\omega)$ .

We have  $n = \sum_{\ell \in \mathbb{Z}} \sum_{k=0}^{n-1} 1_{Z_k(\omega)=\ell} \leq (\sum_{\ell \in \mathcal{R}_n(\omega)} (\sum_{k=0}^{n-1} 1_{Z_k(\omega)=\ell})^2)^{\frac{1}{2}} \text{Card}(\mathcal{R}_n(\omega))^{\frac{1}{2}}$  by Cauchy-Schwarz inequality; hence:  $V_n(\omega) \geq n^2 / \text{Card}(\mathcal{R}_n(\omega))$ , which implies (47).  $\square$

**Lemma 6.2.** *For an aperiodic r.w. in dimension 1 with finite variance and centered, we have:*

$$(48) \quad \sup_n \left| \sum_{j=1}^n [2\mathbb{P}(Z_j = 0) - \mathbb{P}(Z_j = \underline{p}) - \mathbb{P}(Z_j = -\underline{p})] \right| < +\infty, \quad \forall \underline{p} \in L(W).$$

*Proof.* Since  $1 - \Psi(t)$  vanishes on the torus only at  $t = \underline{0}$  with an order 2, we have:

$$\begin{aligned} & \left| \sum_{j=1}^{N-1} [2\mathbb{P}(Z_j = 0) - \mathbb{P}(Z_j = \underline{p}) - \mathbb{P}(Z_j = -\underline{p})] \right| = 4 \left| \int_{\mathbb{T}} \sin^2 \pi \langle \underline{p}, t \rangle \Re e \left( \frac{1 - \Psi^N(t)}{1 - \Psi(t)} \right) dt \right| \\ & \leq 4 \int_{\mathbb{T}} \sin^2 \pi \langle \underline{p}, t \rangle \left| \frac{1 - \Psi^N(t)}{1 - \Psi(t)} \right| dt \leq 8 \int_{\mathbb{T}} \frac{\sin^2 \pi \langle \underline{p}, t \rangle}{|1 - \Psi(t)|} dt < +\infty. \end{aligned}$$

$\square$

**Proof of Theorem 2.10** (for  $d = 1$ :  $\lim_n \frac{V_{n,p}(\omega)}{V_n(\omega)} = 1$ , a.e. for every  $p \in L(W)$ .)

Recall that  $V_n(\omega) - V_{n,p}(\omega) \geq 0$  (cf. Ex. 1.4). By (13) we can bound  $n^{-1} \mathbb{E}[V_n - V_{n,p}]$  by

$$1 + n^{-1} \sum_{k=1}^{n-1} \left| \sum_{j=0}^{n-k-1} (\mathbb{P}(Z_j = \underline{p}) - \mathbb{P}(Z_j = 0)) + \sum_{j=0}^{n-k-1} (\mathbb{P}(Z_j = -\underline{p}) - \mathbb{P}(Z_j = 0)) \right|$$

which is bounded according to (48).

Let  $\delta > 0$ . The bound  $\mathbb{E}\left(\frac{V_n(\omega) - V_{n,p}}{n^{\frac{3}{2}}}\right) = O(n^{-\frac{1}{2}})$  implies by the Borel-Cantelli lemma that  $\left(\frac{V_n(\omega) - V_{n,p}}{n^{\frac{3}{2}}}\right)$  tends to 0 a.e. along the sequence  $k_n = \lceil n^{2+\delta} \rceil$ ,  $n \geq 1$ .

Since  $V_n \geq c(\omega) n^{\frac{3}{2}} (\text{LogLog } n)^{-\frac{1}{2}}$  by (47), we obtain

$$0 \leq 1 - \frac{V_{n,p}(\omega)}{V_n(\omega)} \leq \frac{V_n(\omega) - V_{n,p}(\omega)}{V_n(\omega)} < \frac{V_n(\omega) - V_{n,p}(\omega)}{c(\omega) n^{\frac{3}{2}} (\text{LogLog } n)^{-\frac{1}{2}}}.$$

Therefore  $\left(\frac{V_{n,p}(\omega)}{V_n(\omega)}\right)_{n \geq 1}$  converges to 1 a.e. along the sequence  $(k_n)$ .

To complete the proof, it suffices to prove that a.s.  $\lim_n \max_{k_n \leq j < k_{n+1}} |V_{j,p}/V_j - V_{k_n,p}/V_{k_n}| = 0$ . By monotonicity of  $V_{j,p}$  and  $V_j$  with respect to  $j$ , we have

$$\frac{V_{k_n,p}}{V_{k_n}} \frac{V_{k_n}}{V_{k_{n+1}}} \leq \frac{V_{j,p}}{V_j} \leq \frac{V_{k_{n+1},p}}{V_{k_{n+1}}} \frac{V_{k_{n+1}}}{V_{k_n}}, \quad k_n \leq j < k_{n+1}.$$

Therefore, since the first factors in the left and right terms of the inequality tend to 1, it is enough to prove that  $\frac{V_{k_{n+1},p} - V_{k_n,p}}{V_{k_n}} \rightarrow 0$  a.s.

By (20), for  $d = 1$ , for each  $\varepsilon > 0$ , there is an a.e. finite constant  $c_\varepsilon(\omega)$  such that  $\sup_{\underline{\ell} \in \mathbb{Z}} R_n(\omega, \underline{\ell}) = \sup_{\underline{\ell} \in \mathbb{Z}} \sum_{k=0}^{n-1} 1_{Z_k(\omega) = \underline{\ell}} = c_\varepsilon(\omega) n^{\frac{1}{2} + \varepsilon}$ . This implies

$$\begin{aligned} V_{k_{n+1}} - V_{k_n} &= \sum_{\ell, j=0}^{k_{n+1}} 1_{\{Z_\ell = Z_j\}} - \sum_{\ell, j=0}^{k_n} 1_{\{Z_\ell = Z_j\}} \leq \sum_{\ell=k_{n+1}}^{k_{n+1}} \sum_{j=1}^{k_{n+1}} 1_{\{Z_\ell = Z_j\}} + \sum_{j=k_{n+1}}^{k_{n+1}} \sum_{\ell=1}^{k_{n+1}} 1_{\{Z_\ell = Z_j\}} \\ &\leq 2(k_{n+1} - k_n) \sup_{p \in \mathbb{Z}} \sum_{j=1}^{k_{n+1}} 1_{\{Z_j = p\}} \leq 2c_\varepsilon(\omega) (k_{n+1} - k_n) k_{n+1}^{\frac{1}{2} + \varepsilon} \leq K(\omega) n^{1+\delta + (2+\delta)(\frac{1}{2} + \varepsilon)}. \end{aligned}$$

Therefore,  $V_{k_{n+1}} - V_{k_n} \leq K(\omega) n^{2 + \frac{3}{2}\delta + 2\varepsilon + \varepsilon\delta}$  and  $\frac{V_{k_{n+1}} - V_{k_n}}{V_{k_n}}$  is a.s. bounded by

$$\begin{aligned} \frac{V_{k_{n+1}} - V_{k_n}}{k_n^{3/2} (\text{LogLog } k_n)^{-\frac{1}{2}}} &\leq 2K(\omega) \frac{n^{2 + \frac{3}{2}\delta + 2\varepsilon + \varepsilon\delta}}{(n^{2+\delta})^{3/2} (\text{LogLog } (n^{2+\delta}))^{-\frac{1}{2}}} \\ &\leq 2K(\omega) n^{2 + \frac{3}{2}\delta + 2\varepsilon + \varepsilon\delta} n^{-(3 + \frac{3}{2}\delta)} ((\text{LogLog } (n^{2+\delta}))^{\frac{1}{2}})^{-1} = 2K(\omega) n^{-1 + 2\varepsilon + \varepsilon\delta} (\text{LogLog } (n^{2+\delta}))^{\frac{1}{2}}, \end{aligned}$$

which tends to 0, if  $2\varepsilon + \varepsilon\delta < 1$ .  $\square$

## 6.2. $d=2$ : variance and SLLN for $V_{n,p}$ .

For  $d = 2$ , in the centered case with finite variance, the a.s. convergence  $V_{n,p}/V_{n,\underline{0}} \rightarrow 1$  for  $p \in L(W)$  follows from the strong law of large numbers (SLLN):  $\lim_n V_{n,p}/\mathbb{E}V_{n,p} = 1$ , a.s. We adapt the method of [27] to the case  $\underline{p} \neq \underline{0}$  in the estimation of  $\text{Var}(V_{n,p})$ . See also [10] for the computation of the variance of  $V_{n,\underline{0}}$ . We need two auxiliary results.

**Lemma 6.3.** *There is  $C$  such that, if  $\underline{p}, \underline{q}$  are in  $L$  and  $n, k$  are such that  $n\underline{\ell}_1 = \underline{p} \bmod D$  and  $(n+k)\underline{\ell}_1 = \underline{q} \bmod D$ , then*

$$(49) \quad |\mathbb{P}(Z_{n+k} = \underline{q}) - \mathbb{P}(Z_n = \underline{p})| \leq C \left( \frac{1}{(n+k)^{\frac{3}{2}}} + \frac{k}{n(n+k)} \right), \forall n, k \geq 1.$$

*Proof.* We have  $\mathbb{P}(Z_{n+r} = \underline{q}) - \mathbb{P}(Z_n = \underline{p}) = \int_{\mathbb{T}^2} G_{n,r}(\underline{t}) d\underline{t}$ , with

$$G_{n,r}(\underline{t}) := \Re e [e^{-2\pi i \langle \underline{q}, \underline{t} \rangle} \Psi(\underline{t})^{n+r} - e^{-2\pi i \langle \underline{p}, \underline{t} \rangle} \Psi(\underline{t})^n].$$

The functions  $e^{-2\pi i \langle \underline{q}, \underline{t} \rangle} \Psi(\underline{t})^{n+r}$  and  $e^{-2\pi i \langle \underline{p}, \underline{t} \rangle} \Psi(\underline{t})^n$  are invariant by translation by the elements of  $\Gamma_1$  and have a modulus  $< 1$ , except for  $\underline{t} \in \Gamma_1$  (cf. Lemma 2.4). To bound the integral of  $G_{n,r}$ , it suffices to bound its integral  $I_n^0 := \int_{U_0} G_{n,r}(\underline{t}) d\underline{t}$  restricted to a fundamental domain  $U_0$  of  $\Gamma_1$  acting on  $\mathbb{T}^2$ .

Denote by  $B(\eta)$  the ball with a small radius  $\eta$  and center  $\underline{0}$  in  $\mathbb{T}^2$ . If  $\eta > 0$  is small enough, on  $U_0 \setminus B(\eta)$ , we have  $|\Psi(\underline{t})| \leq \lambda(\eta)$  with  $\lambda(\eta) < 1$ , which implies:

$$|I_n^0| \leq 2C\lambda^n(\eta) + \int_{B(\eta)} |G_{n,r}(\underline{t})| d\underline{t}.$$

and we have

$$|G_{n,r}(\underline{t})| := |\Re e [e^{-2\pi i \langle \underline{q}, \underline{t} \rangle} \Psi(\underline{t})^{n+r} - e^{-2\pi i \langle \underline{p}, \underline{t} \rangle} \Psi(\underline{t})^n]| \leq |\Psi(\underline{t})|^n |\Psi(\underline{t})^r - e^{2\pi i \langle \underline{q} - \underline{p}, \underline{t} \rangle}|.$$

Since the distribution  $\nu$  of the centered r.w.  $W$  is assumed to have a moment of order 2, by Lemma 2.5, for  $\eta$  sufficiently small, there are constants  $a, b > 0$  such that

$$|\Psi(t)| < 1 - a\|t\|^2, \quad |1 - \Psi(t)| < b\|t\|^2, \quad \forall t \in B(\eta).$$

We distinguish two cases:

if  $\underline{p} = \underline{q}$ , then  $|G_{n,r}(\underline{t})| \leq C(r)(1 - a\|t\|^2)^n \|t\|^2$ ;

if  $\underline{p} \neq \underline{q}$ ,  $|G_{n,r}(\underline{t})| \leq C(r)|\Psi(\underline{t})|^n \|t\| \leq C(r)(1 - a\|t\|^2)^n \|t\|$ .

Now we bound the integral  $\int_{B(\eta)} (1 - a\|t\|^2)^n \|t\|^2 dt_1 dt_2$ . By the change of variable  $(t_1, t_2) \rightarrow (\frac{s_1}{\sqrt{n}}, \frac{s_2}{\sqrt{n}})$ , then  $(s_1, s_2) \rightarrow (\rho \cos \theta, \rho \sin \theta)$ , it becomes successively:

$$\frac{1}{n^2} \int_{B(\eta)} \left(1 - a \frac{\|s\|^2}{n}\right)^n \|s\|^2 ds_1 ds_2 \leq \frac{C}{n^2} \int_{\mathbb{R}} e^{-a\rho^2} \rho^2 d\rho = O\left(\frac{1}{n^2}\right).$$

So for  $\underline{p} = \underline{q}$ , we get the bound  $O(n^2)$ . Likewise, if  $\underline{p} \neq \underline{q}$ , we get the bound  $O(n^{3/2})$ .

Recall the  $L/D$  is a cyclic group (Lemma 2.2). If  $n, k$  satisfy the condition of the lemma, we write:  $k = r + uv$ , where  $r$  and  $v$  ( $= |L/D|$ ) are the smallest positive integers such that respectively:  $r\underline{\ell}_1 = \underline{q} - \underline{p} \bmod D$ ,  $v\underline{\ell}_1 = \underline{0} \bmod D$ . Then, using the previous bounds and writing the difference  $\mathbb{P}(Z_{n+k} = \underline{q}) - \mathbb{P}(Z_n = \underline{p})$  as

$$\begin{aligned} & \mathbb{P}(Z_{n+k} = \underline{q}) - \mathbb{P}(Z_{n+r+(u-1)v} = \underline{q}) + \mathbb{P}(Z_{n+r+(u-1)v} = \underline{q}) - \mathbb{P}(Z_{n+r+(u-2)v} = \underline{q}) \\ & + \dots + \mathbb{P}(Z_{n+r+v} = \underline{q}) - \mathbb{P}(Z_{n+r} = \underline{q}) + \mathbb{P}(Z_{n+r} = \underline{q}) - \mathbb{P}(Z_n = \underline{p}), \end{aligned}$$

the telescoping argument gives (49). □

**Lemma 6.4.** For  $d \geq 1$ , let  $M_n^{(d)} := \{(m_1, \dots, m_d) \in \mathbb{N}^d : \sum_i m_i = n\}$ . Let  $\tilde{M}_n^{(5)} := \{(m_1, \dots, m_5) \in M_n^{(5)}, m_3, m_4 > 0\}$ . If  $|\lambda|, |\alpha|, |\beta|, |\gamma| < 1$ , then

$$(50) \quad \sum_{n=0}^{\infty} \lambda^n \sum_{(m_1, \dots, m_5) \in \tilde{M}_n^{(5)}} \alpha^{m_2} \beta^{m_3} \gamma^{m_4} = \frac{1}{(1-\lambda)^2} \frac{\lambda^2 \beta \gamma}{(1-\lambda\alpha)(1-\lambda\beta)(1-\lambda\gamma)}.$$

*Proof.* We have for  $\lambda, \alpha_1, \dots, \alpha_d$  such that  $|\lambda\alpha_1|, \dots, |\lambda\alpha_d| \in [0, 1[$ :

$$(51) \quad \sum_{n=0}^{\infty} \lambda^n \sum_{(m_1, \dots, m_d) \in M_n^{(d)}} \alpha_1^{m_1} \alpha_2^{m_2} \dots \alpha_d^{m_d} = \prod_{i=1}^d \frac{1}{(1-\lambda\alpha_i)}.$$

Indeed, the left hand of (51) is the sum over  $\mathbb{Z}$  of the discrete convolution product of the functions  $G_i$  defined on  $\mathbb{Z}$  by  $G_i(k) = 1_{[0, \infty[}(k)(\lambda\alpha_i)^k$ , hence is equal to

$$\sum_{k \in \mathbb{Z}} (G_1 * \dots * G_d)(k) = \prod_{i=1}^d \left( \sum_{k \in \mathbb{Z}} G_i(k) \right) = \prod_{i=1}^d \frac{1}{(1-\lambda\alpha_i)}.$$

For  $d = 5$  we find

$$\sum_{n=0}^{\infty} \lambda^n \sum_{(m_1, \dots, m_5) \in M_n^{(5)}} \alpha^{m_2} \beta^{m_3} \gamma^{m_4} = \frac{1}{(1-\lambda)^2} \frac{1}{(1-\lambda\alpha)(1-\lambda\beta)(1-\lambda\gamma)}.$$

The left hand of (50) reads

$$\begin{aligned} & \sum_{n=0}^{\infty} \lambda^n \sum_{(m_1, \dots, m_5) \in M_n^{(5)}} \alpha^{m_2} \beta^{m_3} \gamma^{m_4} - \sum_{n=0}^{\infty} \lambda^n \sum_{(m_1, m_2, m_4, m_5) \in M_n^{(4)}} \alpha^{m_2} \gamma^{m_4} \\ & - \sum_{n=0}^{\infty} \lambda^n \sum_{(m_1, m_2, m_3, m_5) \in M_n^{(4)}} \alpha^{m_2} \beta^{m_3} + \sum_{n=0}^{\infty} \lambda^n \sum_{(m_1, m_2, m_5) \in M_n^{(3)}} \alpha^{m_2} \\ & = \frac{1}{(1-\lambda)^2} \left[ \frac{1}{(1-\lambda\alpha)(1-\lambda\beta)(1-\lambda\gamma)} - \frac{1}{(1-\lambda\alpha)(1-\lambda\gamma)} \right. \\ & \quad \left. - \frac{1}{(1-\lambda\alpha)(1-\lambda\beta)} + \frac{1}{(1-\lambda\alpha)} \right] = \frac{1}{(1-\lambda)^2} \frac{\lambda^2 \beta \gamma}{(1-\lambda\alpha)(1-\lambda\beta)(1-\lambda\gamma)}. \quad \square \end{aligned}$$

**Proof of Theorem 2.10** (Case  $d = 2$ )

A) Below we consider:

$$(52) \quad \sum_{0 \leq i_1 < j_1 < n; 0 \leq i_2 < j_2 < n} \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}, Z_{j_2} - Z_{i_2} = \underline{s}).$$

The (disjoint) sets of possible configurations for  $(i_1, j_1, i_2, j_2)$  are the following:

- (1a)  $i_1 \leq i_2 < j_1 < j_2$ , (1b)  $i_2 \leq i_1 < j_2 < j_1$ ,  
(2a)  $i_1 < i_2 < j_2 \leq j_1$ , (2b)  $i_2 < i_1 < j_1 \leq j_2$ ,

- (3a)  $i_1 < j_1 \leq i_2 < j_2$ , (3b)  $i_2 < j_2 \leq i_1 < j_1$ ,  
(4)  $i_2 = i_1 < j_1 = j_2$ .

For the case 4), by the LLT the sum (52) restricted to the configurations (4) is less than  $\sum_{0 \leq i_1 < j_1 < n} 1_{\underline{r}=\underline{s}} \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}) \leq \sum \frac{C}{j_1 - i_1} \leq Cn \text{Log } n$ .

(1a) and (1b), (2a) and (2b), (3a) and (3b) are respectively the same up to the exchange of indices 1 and 2. To bound (52), it suffices to consider the subsums corresponding to 1a), 2a), 3a).

**1a)** ( $i_1 \leq i_2 < j_1 < j_2$ ) Setting  $m_1 = i_1$ ,  $m_2 = i_2 - i_1$ ,  $m_3 = j_1 - i_2$ ,  $m_4 = j_2 - j_1$ ,  $m_5 = n - j_2$ , we have:  $Z_{j_1} - Z_{i_1} = \tau^{m_1} Z_{m_2+m_3}$ ,  $Z_{j_2} - Z_{i_2} = \tau^{m_1+m_2} Z_{m_3+m_4}$  with  $m_1, m_2 \geq 0, m_3, m_4 > 0$ ; hence:

$$\begin{aligned}
& \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}, Z_{j_2} - Z_{i_2} = \underline{s}) = \mathbb{P}(Z_{m_2+m_3} = \underline{r}, \tau^{m_2} Z_{m_3+m_4} = \underline{s}) \\
& = \sum_{\underline{\ell}} [\mathbb{P}(Z_{m_2} = \underline{\ell}) \mathbb{P}(Z_{m_3} = \underline{r} - \underline{\ell}) \mathbb{P}(Z_{m_4} = \underline{s} - \underline{r} + \underline{\ell})] \\
(53) \quad & = \int e^{-2\pi i(\underline{r}, \underline{u}) + (\underline{s} - \underline{r}, \underline{v})} \sum_{\underline{\ell}} e^{-2\pi i(\underline{\ell}, \underline{t} - \underline{u} + \underline{v})} \Psi(\underline{t})^{m_2} \Psi(\underline{u})^{m_3} \Psi(\underline{v})^{m_4} d\underline{t} d\underline{u} d\underline{v}.
\end{aligned}$$

The last equation can be shown by approximating the probability vector  $(p_j)_{j \in J}$  by probability vectors with finite support and using the continuity of  $\Psi$ .

**2a)** ( $i_1 < i_2 < j_2 \leq j_1$ ) Setting  $m_1 = i_1$ ,  $m_2 = i_2 - i_1$ ,  $m_3 = j_2 - i_2$ ,  $m_4 = j_1 - j_2$ ,  $m_5 = n - j_1$ , we have:  $Z_{j_1} - Z_{i_1} = \tau^{m_1} Z_{m_2+m_3+m_4}$ ,  $Z_{j_2} - Z_{i_2} = \tau^{m_1+m_2} Z_{m_3}$ , with  $m_1, m_4 \geq 0, m_2, m_3 > 0$ .

Since  $\mathbb{P}(Z_{m_2} + \tau^{m_2+m_3} Z_{m_4} = \underline{q}) = \sum_{\underline{\ell}} \mathbb{P}(Z_{m_2} = \underline{\ell}, \tau^{m_2+m_3} Z_{m_4} = \underline{q} - \underline{\ell}) = \sum_{\underline{\ell}} \mathbb{P}(Z_{m_2} = \underline{\ell}) \mathbb{P}(\tau^{m_2+m_3} Z_{m_4} = \underline{q} - \underline{\ell}) = \sum_{\underline{\ell}} \mathbb{P}(Z_{m_2} = \underline{\ell}) \mathbb{P}(\tau^{m_2} Z_{m_4} = \underline{q} - \underline{\ell}) = \mathbb{P}(Z_{m_2+m_4} = \underline{q})$ , we have:

$$\begin{aligned}
& \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}, Z_{j_2} - Z_{i_2} = \underline{s}) = \mathbb{P}(Z_{m_2+m_3+m_4} = \underline{r}, \tau^{m_2} Z_{m_3} = \underline{s}) \\
& = \mathbb{P}(Z_{m_2} + \tau^{m_2+m_3} Z_{m_4} = \underline{r} - \underline{s}, \tau^{m_2} Z_{m_3} = \underline{s}) \\
& = \mathbb{P}(Z_{m_2} + \tau^{m_2+m_3} Z_{m_4} = \underline{r} - \underline{s}) \mathbb{P}(Z_{m_3} = \underline{s}) = \mathbb{P}(Z_{m_2+m_4} = \underline{r} - \underline{s}) \mathbb{P}(Z_{m_3} = \underline{s}).
\end{aligned}$$

Hence, in case 2a), we get:

$$(54) \quad \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}, Z_{j_2} - Z_{i_2} = \underline{s}) = \mathbb{P}(Z_{m_2+m_4} = \underline{r} - \underline{s}) \mathbb{P}(Z_{m_3} = \underline{s}).$$

**3a)** ( $i_1 < j_1 \leq i_2 < j_2$ ) The events  $Z_{j_1} - Z_{i_1} = \underline{r}$  and  $Z_{j_2} - Z_{i_2} = \underline{s}$  are independent.

Following the method of [27], now we estimate  $\text{Var}(V_{n,\underline{p}}) = \mathbb{E}(V_{n,\underline{p}}^2) - (\mathbb{E}V_{n,\underline{p}})^2$ . Recall that  $V_{n,\underline{p}} = \sum_{0 \leq i,j < n} 1_{Z_j - Z_i = \underline{p}} = \sum_{0 \leq i < j < n} (1_{Z_j - Z_i = \underline{p}} + 1_{Z_j - Z_i = -\underline{p}}) + n 1_{\underline{p}=\underline{0}}$ . We have

$$\begin{aligned} V_{n,\underline{p}}^2 &= \sum_{0 \leq i_1 < j_1 < n; 0 \leq i_2 < j_2 < n} (1_{Z_{j_1} - Z_{i_1} = \underline{p}} + 1_{Z_{j_1} - Z_{i_1} = -\underline{p}}) (1_{Z_{j_2} - Z_{i_2} = \underline{p}} + 1_{Z_{j_2} - Z_{i_2} = -\underline{p}}) \\ &\quad + 2n 1_{\underline{p}=\underline{0}} \sum_{0 \leq i < j < n} (1_{Z_j - Z_i = \underline{p}} + 1_{Z_j - Z_i = -\underline{p}}) + n^2 1_{\underline{p}=\underline{0}}. \end{aligned}$$

The last term gives 0 in the computation of the variance, so that it suffices to bound

$$\begin{aligned} &\sum_{0 \leq i_1 < j_1 < n; 0 \leq i_2 < j_2 < n} \mathbb{E}[(1_{Z_{j_1} - Z_{i_1} = \underline{p}} + 1_{Z_{j_1} - Z_{i_1} = -\underline{p}}) (1_{Z_{j_2} - Z_{i_2} = \underline{p}} + 1_{Z_{j_2} - Z_{i_2} = -\underline{p}})] \\ &- \sum_{0 \leq i_1 < j_1 < n; 0 \leq i_2 < j_2 < n} [\mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{p}) + \mathbb{P}(Z_{j_1} - Z_{i_1} = -\underline{p})] [\mathbb{P}(Z_{j_2} - Z_{i_2} = \underline{p}) + \mathbb{P}(Z_{j_2} - Z_{i_2} = -\underline{p})], \end{aligned}$$

i.e., the sum over the finite set  $(\underline{r}, \underline{s}) \in \{\pm \underline{p}\}$  of the sums

$$\sum_{0 \leq i_1 < j_1 < n; 0 \leq i_2 < j_2 < n} [\mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}, Z_{j_2} - Z_{i_2} = \underline{s}) - \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}) \mathbb{P}(Z_{j_2} - Z_{i_2} = \underline{s})].$$

B) By the previous analysis, we are reduced to cases (1a), (2a), (3a). For (3a), by independence, we get 0. As  $\mathbb{E}(V_{n,\underline{p}}^2) - (\mathbb{E}V_{n,\underline{p}})^2 \geq 0$ , for the bound of (1a), we can neglect the corresponding centering terms, since they are subtracted and non negative.

**(1a)** For  $\underline{r}, \underline{s} \in \mathbb{Z}^2$ , let  $a_{\underline{r}, \underline{s}}(n) := \sum_{0 \leq i_1 \leq i_2 < j_1 < j_2 < n} \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}, Z_{j_2} - Z_{i_2} = \underline{s})$ ,  $a(\underline{p}, n) := \sum_{\underline{r}, \underline{s} \in \{\pm \underline{p}\}} a_{\underline{r}, \underline{s}}(n)$ .

Setting  $m_1 = i_1, m_2 = i_2 - i_1, m_3 = j_1 - i_2, m_4 = j_2 - j_1, m_5 = n - j_2$  (so that  $(m_1, \dots, m_5)$  runs in the set  $\tilde{M}_n^{(5)}$  of Lemma 6.4), by (53) we have for the generating function  $A_{\underline{r}, \underline{s}}(\lambda) := \sum_{n \geq 0} \lambda^n a_{\underline{r}, \underline{s}}(n)$ ,  $0 \leq \lambda < 1$ :

$$A_{\underline{r}, \underline{s}}(\lambda) = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} e^{-2\pi i(\langle \underline{r}, \underline{t} \rangle + \langle \underline{s}, \underline{u} \rangle)} \sum_n \lambda^n \sum_{(m_1, \dots, m_5) \in \tilde{M}_n^{(5)}} \Psi(\underline{t})^{m_2} \Psi(\underline{t} + \underline{u})^{m_3} \Psi(\underline{u})^{m_4} d\underline{t} d\underline{u}.$$

Using  $\sum_{\underline{r}, \underline{s} \in \{\pm \underline{p}\}} e^{-2\pi i(\langle \underline{r}, \underline{t} \rangle + \langle \underline{s}, \underline{u} \rangle)} = \cos 2\pi \langle \underline{p}, \underline{t} \rangle \cos 2\pi \langle \underline{p}, \underline{u} \rangle$ , for the generating function  $A_{\underline{p}}(\lambda) := \sum_{n \geq 0} \lambda^n a(\underline{p}, n)$ , we have by (51) with  $\alpha = \Psi(\underline{t}), \beta = \Psi(\underline{u}), \gamma = \Psi(\underline{t} + \underline{u})$ :

$$A_{\underline{p}}(\lambda) = 4 \frac{\lambda^2}{(1 - \lambda)^2} \int \int \frac{\cos 2\pi \langle \underline{p}, \underline{t} \rangle \cos 2\pi \langle \underline{p}, \underline{u} \rangle \Psi(\underline{u}) \Psi(\underline{t} + \underline{u})}{(1 - \lambda \Psi(\underline{t})) (1 - \lambda \Psi(\underline{u})) (1 - \lambda \Psi(\underline{t} + \underline{u}))} d\underline{t} d\underline{u}.$$

For an aperiodic r.w., the bound obtained for  $A_{\underline{p}}(\lambda)$  in [27] for  $\underline{p} = \underline{0}$  is valid. Indeed the bounds for the domains  $T_{i,\delta}$  and  $E_\delta$  hold: for  $T_{i,\delta}$  this is clear; for  $E_\delta$  this is because  $(1 - \Psi(\underline{t})) (1 - \Psi(\underline{u})) (1 - \Psi(\underline{t} + \underline{u}))$  does not vanish on  $E_\delta$ . (See [27] p. 227 for the notation for the domains.) The main contribution comes from a small neighborhood  $U_\delta$  of diameter  $\delta > 0$  of  $\underline{0}$  and the factor  $\cos 2\pi \langle \underline{p}, \underline{t} \rangle \cos 2\pi \langle \underline{p}, \underline{u} \rangle$  plays no role.



It follows that  $A_{\underline{p}}(\lambda) \leq \frac{C}{(1-\lambda)^3}$ . Since  $a(\underline{p}, n)$  increases with  $n$ , we obtain  $a(\underline{p}, n) = O(n^2)$  by using the following elementary ‘‘Tauberian’’ argument:

Let  $(u_k)$  be a non decreasing sequence of non negative numbers. If there is  $C$  such that  $\sum_{k=0}^{+\infty} \lambda^k u_k \leq \frac{C}{(1-\lambda)^3}$ ,  $\forall \lambda \in [0, 1[$ , then  $u_n \leq C'n^2$ ,  $\forall n \geq 1$ . Indeed we have:

$$\lambda^n u_n = (1-\lambda) \left( \sum_{k=n}^{+\infty} \lambda^k \right) u_n \leq (1-\lambda) \sum_{k=n}^{+\infty} \lambda^k u_k \leq (1-\lambda) \sum_{k=0}^{+\infty} \lambda^k u_k \leq \frac{C}{(1-\lambda)^2}.$$

For  $\lambda = 1 - \frac{1}{n}$  in the previous inequality, we get:  $u_n \leq Cn^2 (1 - \frac{1}{n})^{-n} \leq C'n^2$ .

**(2a)** Let  $b_{\underline{r}, \underline{s}}(n)$  be the sum

$$\sum_{0 \leq i_1 < i_2 < j_2 \leq j_1 < n} [\mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}, Z_{j_2} - Z_{i_2} = \underline{s}) - \mathbb{P}(Z_{j_1} - Z_{i_1} = \underline{r}) \mathbb{P}(Z_{j_2} - Z_{i_2} = \underline{s})]$$

and  $b(\underline{p}, n) := \sum_{\underline{r}, \underline{s} \in \{\pm p\}} b_{\underline{r}, \underline{s}}(n)$ .

Putting  $m_1 = i_1$ ,  $m_2 = i_2 - i_1$ ,  $m_3 = j_2 - i_2$ ,  $m_4 = j_1 - j_2$ , we have by (54):

$$b(\underline{p}, n) = \sum_{\underline{r}, \underline{s} \in \{\pm p\}} \sum_{\substack{m_1+m_2+m_3+m_4 < n \\ m_1, m_4 \geq 0, m_2, m_3 \geq 1}} \mathbb{P}(Z_{m_3} = \underline{s}) [\mathbb{P}(Z_{m_2+m_4} = \underline{r} - \underline{s}) - \mathbb{P}(Z_{m_2+m_3+m_4} = \underline{r})].$$

We will use Lemma 6.3 to bound  $b(\underline{p}, n)$ .

If  $m_3 \underline{\ell}_1 = \underline{s} \pmod{D}$ , then either  $(m_2 + m_4) \underline{\ell}_1 = \underline{r} - \underline{s} \pmod{D}$  and  $(m_2 + m_4 + m_3) \underline{\ell}_1 = \underline{r} \pmod{D}$ , or  $(m_2 + m_4) \underline{\ell}_1 \neq \underline{r} - \underline{s} \pmod{D}$  and  $(m_2 + m_4 + m_3) \underline{\ell}_1 \neq \underline{r} \pmod{D}$ . In the latter case, the corresponding probabilities are 0. Moreover,  $\mathbb{P}(Z_{m_3} = \underline{s}) = 0$  if  $m_3 \underline{\ell}_1 \neq \underline{s} \pmod{D}$ . Therefore the sum reduces to those indices such that  $m_3 \underline{\ell}_1 = \underline{s} \pmod{D}$ ,  $(m_2 + m_4) \underline{\ell}_1 = \underline{r} - \underline{s} \pmod{D}$  and  $(m_2 + m_4 + m_3) \underline{\ell}_1 = \underline{r} \pmod{D}$  and the bound (49) applies for the non zero terms. We obtain:

$$b(\underline{p}, n) \leq 4 \sum_{\substack{m_1+m_2+m_3+m_4 < n \\ m_1, m_4 \geq 0, m_2, m_3 \geq 1}} \left[ \frac{1}{m_3} \frac{m_3}{(m_2 + m_4)(m_2 + m_3 + m_4)} + \frac{1}{m_3 (m_2 + m_4 + m_3)^{\frac{3}{2}}} \right].$$

The sum of the first term restricted to  $\{m_i \geq 1, m_1 + m_2 + m_3 + m_4 \leq n\}$  reads:

$$\begin{aligned} & \sum_{k=1}^n \sum_{m_1+m_2+m_3+m_4=k} \frac{1}{(m_2 + m_4)(m_2 + m_4 + m_3)} = \sum_{k=1}^n \sum_{\ell=1}^k \sum_{m_2+m_4=\ell, m_1+m_3=k-\ell} \frac{1}{\ell(\ell + m_3)} \\ & \leq \sum_{k=1}^n \sum_{\ell=1}^k \sum_{m_1+m_3=k-\ell} \ell \cdot \frac{1}{\ell(\ell + m_3)} \leq \sum_{k=1}^n \sum_{\ell=1}^k \left( \frac{1}{\ell} + \dots + \frac{1}{k} \right) = \frac{1}{2} n(n+1). \end{aligned}$$

Likewise, we have:

$$\begin{aligned} & \sum_{k=1}^n \sum_{m_1+m_2+m_3+m_4=k} \frac{1}{m_3(m_2+m_4+m_3)^{\frac{3}{2}}} \leq n \sum_{k=1}^n \sum_{m_2+m_3+m_4=k} \frac{1}{m_3 k^{\frac{3}{2}}} \\ & \leq n \sum_{k=1}^n \frac{1}{k^{3/2}} \sum_{m_3=1}^k \frac{k}{m_3} \leq n \sum_{k=1}^n \frac{\text{Log}k}{\sqrt{k}} \leq C n^{3/2} \text{Log}n. \end{aligned}$$

Therefore, we obtain  $b(n, \underline{p}) = O(n^2)$ .

The previous estimations imply  $\text{Var}(V_{n,\underline{p}}) = O(n^2)$ . By (26), for  $\underline{p} \in L(W)$ ,  $\lim_n V_{n,\underline{p}}/\mathbb{E}V_{n,\underline{p}} = 1$  a.e. follows as in [27] and therefore  $\lim_n V_{n,\underline{p}}/V_{n,\underline{0}} = 1$  a.e.  $\square$

## 7. Appendix II: mixing, moments and cumulants

For the sake of completeness, we recall in this appendix some general results on mixing of all orders, moments and cumulants (see [25] and the references given therein). Implicitly we assume existence of moments of all orders when they are used.

For a real random variable  $Y$  (or for a probability distribution on  $\mathbb{R}$ ), the cumulants (or semi-invariants) can be formally defined as the coefficients  $c^{(r)}(Y)$  of the cumulant generating function  $t \rightarrow \ln \mathbb{E}(e^{tY}) = \sum_{r=0}^{\infty} c^{(r)}(Y) \frac{t^r}{r!}$ , i.e.,  $c^{(r)}(Y) = \frac{\partial^r}{\partial t^r} \ln \mathbb{E}(e^{tY})|_{t=0}$ . Similarly the joint cumulant of a random vector  $(X_1, \dots, X_r)$  is defined by

$$(55) \quad c(X_1, \dots, X_r) = \frac{\partial^r}{\partial t_1 \dots \partial t_r} \ln \mathbb{E}(e^{\sum_{j=1}^r t_j X_j})|_{t_1=\dots=t_r=0}.$$

This definition can be given as well for a finite measure  $\nu$  on  $\mathbb{R}^r$ . Its cumulant is noted  $c_\nu(x_1, \dots, x_r)$ . The joint cumulant of  $(Y, \dots, Y)$  ( $r$  copies of  $Y$ ) is  $c^{(r)}(Y)$ .

For any subset  $I = \{i_1, \dots, i_p\} \subset J_r := \{1, \dots, r\}$ , we put

$$m(I) = m(i_1, \dots, i_p) := \mathbb{E}(X_{i_1} \cdots X_{i_p}), \quad s(I) = s(i_1, \dots, i_p) := c(X_{i_1}, \dots, X_{i_p}).$$

The cumulants of a process  $(X_j)_{j \in \mathcal{J}}$ , where  $\mathcal{J}$  is a set of indexes, is the family

$$\{c(X_{i_1}, \dots, X_{i_r}), (i_1, \dots, i_r) \in \mathcal{J}^r, r \geq 1\}.$$

The following formulas link moments and cumulants and vice-versa:

$$(56) \quad c(X_1, \dots, X_r) = s(J_r) = \sum_{\mathcal{P}} (-1)^{p-1} (p-1)! m(I_1) \cdots m(I_p),$$

$$(57) \quad \mathbb{E}(X_1 \cdots X_r) = m(J_r) = \sum_{\mathcal{P}} s(I_1) \cdots s(I_p),$$

where in both formulas,  $\mathcal{P} = \{I_1, I_2, \dots, I_p\}$  runs through the set of partitions of  $J_r = \{1, \dots, r\}$  into  $p \leq r$  nonempty intervals, with  $p$  varying from 1 to  $r$ .

Now let be given a random field of real random variables  $(X_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$  and a summable weight  $R$  from  $\mathbb{Z}^d$  to  $\mathbb{R}^+$ . For  $Y := \sum_{\underline{\ell} \in \mathbb{Z}^d} R(\underline{\ell}) X_{\underline{\ell}}$  we obtain from the definition (55):

$$(58) \quad c^{(r)}(Y) = c(Y, \dots, Y) = \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, \dots, X_{\underline{\ell}_r}) R(\underline{\ell}_1) \cdots R(\underline{\ell}_r).$$

### Limiting distribution and cumulants

For our purpose, we state in terms of cumulants a particular case of a theorem of M. Fréchet and J. Shohat, generalizing classical results of A. Markov. Using the formulas linking moments and cumulants, a special case of their “generalized statement of the second limit-theorem” can be expressed as follows:

**Theorem 7.1.** [13] *Let  $(Z^n, n \geq 1)$  be a sequence of centered real r.v. such that*

$$(59) \quad \lim_n c^{(2)}(Z^n) = \sigma^2, \quad \lim_n c^{(r)}(Z^n) = 0, \quad \forall r \geq 3,$$

*then  $(Z^n)$  tends in distribution to  $\mathcal{N}(0, \sigma^2)$ . (If  $\sigma = 0$ , then the limit is  $\delta_0$ ).*

**Theorem 7.2.** (cf. Theorem 7 in [24]) *Let  $(X_{\underline{k}})_{\underline{k} \in \mathbb{Z}^d}$  be a random process and  $(R_n)_{n \geq 1}$  a summation sequence on  $\mathbb{Z}^d$ . Let  $(Y^n)$  be defined by  $Y_n = \sum_{\underline{\ell}} R_n(\underline{\ell}) X_{\underline{\ell}}$ ,  $n \geq 1$ . If*

$$(60) \quad \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, \dots, X_{\underline{\ell}_r}) R_n(\underline{\ell}_1) \cdots R_n(\underline{\ell}_r) = o(\|Y^n\|_2^r), \quad \forall r \geq 3,$$

*then  $\frac{Y^n}{\|Y^n\|_2}$  tends in distribution to  $\mathcal{N}(0, 1)$  when  $n$  tends to  $\infty$ .*

*Proof.* Let  $\beta_n := \|Y^n\|_2 = \|\sum_{\underline{\ell}} R_n(\underline{\ell}) X_{\underline{\ell}}\|_2$  and  $Z^n = \beta_n^{-1} Y^n$ .

Using (58), we have  $c^{(r)}(Z^n) = \beta_n^{-r} \sum_{(\underline{\ell}_1, \dots, \underline{\ell}_r) \in (\mathbb{Z}^d)^r} c(X_{\underline{\ell}_1}, \dots, X_{\underline{\ell}_r}) R_n(\underline{\ell}_1) \cdots R_n(\underline{\ell}_r)$ . The theorem follows then from the assumption (60) by Theorem 7.1 applied to  $(Z^n)$ .  $\square$

**Definition 7.3.** *A measure preserving  $\mathbb{N}^d$  (or  $\mathbb{Z}^d$ )-action  $T : \underline{n} \rightarrow T^{\underline{n}}$  on a probability space  $(X, \mathcal{A}, \mu)$  is  $r$ -mixing,  $r > 1$ , if for all sets  $B_1, \dots, B_r \in \mathcal{A}$*

$$\lim_{\min_{1 \leq \ell < \ell' \leq r} \|\underline{n}_\ell - \underline{n}_{\ell'}\| \rightarrow \infty} \mu\left(\bigcap_{\ell=1}^r T^{-\underline{n}_\ell} B_\ell\right) = \prod_{\ell=1}^r \mu(B_\ell).$$

**Notation 7.4.** For  $f$  in the space  $L_0^\infty(X)$  of measurable essentially bounded functions on  $(X, \mu)$  with  $\int f d\mu = 0$ , we apply the definition of moments and cumulants to  $(T^{\underline{n}_1} f, \dots, T^{\underline{n}_r} f)$  and put

$$(61) \quad m_f(\underline{n}_1, \dots, \underline{n}_r) = \int_X T^{\underline{n}_1} f \cdots T^{\underline{n}_r} f d\mu, \quad s_f(\underline{n}_1, \dots, \underline{n}_r) := c(T^{\underline{n}_1} f, \dots, T^{\underline{n}_r} f).$$

To use the property of mixing of all orders, we need the following lemma.

**Lemma 7.5.** *For every sequence  $(\underline{n}_1^k, \dots, \underline{n}_r^k)$  in  $(\mathbb{Z}^d)^r$ , there are a subsequence with possibly a permutation of indices (still written  $(\underline{n}_1^k, \dots, \underline{n}_r^k)$ ), an integer  $\kappa(r) \in [1, r]$ , a subdivision  $1 = r_1 < r_2 < \dots < r_{\kappa(r)-1} < r_{\kappa(r)} \leq r$  of  $\{1, \dots, r\}$  and a constant integral vector*

$\underline{a}_j$  such that

$$(62) \quad \lim_k \min_{1 \leq s \neq s' \leq \kappa(r)} \|\underline{n}_{r_s}^k - \underline{n}_{r_{s'}}^k\| = \infty,$$

$$(63) \quad \underline{n}_j^k = \underline{n}_{r_s}^k + \underline{a}_j, \text{ for } r_s < j < r_{s+1}, \text{ } s = 1, \dots, \kappa(r) - 1, \text{ and for } r_{\kappa(r)} < j \leq r.$$

If  $(\underline{n}_1^k, \dots, \underline{n}_r^k)$  satisfies  $\lim_k \max_{i \neq j} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$ , then  $\kappa(r) > 1$ .

*Proof.* Remark that if  $\sup_k \max_{i \neq j} \|\underline{n}_i^k - \underline{n}_j^k\| < \infty$ , then  $\kappa(r) = 1$  so that (62) is void and (63) is void for the indexes such that  $r_{s+1} = r_s + 1$ . The proof of the lemma is by induction. The result is clear for  $r = 2$ . Suppose that the subsequence for the sequence of  $(r - 1)$ -tuples  $(\underline{n}_1^k, \dots, \underline{n}_{r-1}^k)$  is built.

Let  $1 \leq r_1 < r_2 < \dots < r_{\kappa(r-1)} \leq r - 1$  be the corresponding subdivision of  $\{1, \dots, r - 1\}$ , as stated above for the sequence  $(\underline{n}_1^k, \dots, \underline{n}_{r-1}^k)$ . If  $(\underline{n}_1^k, \dots, \underline{n}_{r-1}^k)$  satisfies  $\lim_k \max_{1 < i < j \leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$ , then  $\kappa(r - 1) > 1$  by construction in the induction process.

Now consider  $(\underline{n}_1^k, \dots, \underline{n}_r^k)$ . If  $\lim_k \|\underline{n}_r^k - \underline{n}_i^k\| = +\infty$ , for  $i = 1, \dots, r - 1$ , then we take  $1 \leq r_1 < r_2 < \dots < r_{\kappa(r-1)} < r_{\kappa(r)} = r$  as new subdivision of  $\{1, \dots, r\}$ . If  $\liminf_k \|\underline{n}_r^k - \underline{n}_{i_s}^k\| < +\infty$  for some  $s \leq \kappa(r - 1)$ , then along a new subsequence (still denoted  $\underline{n}_r^k$ ) we have  $\underline{n}_r^k = \underline{n}_{i_s}^k + \underline{a}_r$ , where  $\underline{a}_r$  is a constant integral vector. After changing the labels, we insert  $n_r$  in the subdivision of  $\{1, \dots, r - 1\}$  and obtain the new subdivision of  $\{1, \dots, r\}$ .

For the last condition on  $\kappa$ , suppose that  $\lim_k \max_{1 < i < j \leq r} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$ . Then, if  $\liminf_k \max_{1 < i < j \leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| < +\infty$ , necessarily,  $\kappa(r) > 1$ . If, on the contrary, the sequence  $(\underline{n}_1^k, \dots, \underline{n}_{r-1}^k)$  satisfies  $\lim_k \max_{1 < i < j \leq r-1} \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$ , then  $\kappa(r - 1) > 1$ , so that  $\kappa(r) \geq \kappa(r - 1) > 1$ .  $\square$

**Lemma 7.6.** *If a  $\mathbb{Z}^d$ -dynamical system on  $(X, \mathcal{A}, \mu)$  is mixing of order  $r \geq 2$ , then, for any  $f \in L_0^\infty(X)$ ,*

$$\lim_{\max_{i \neq j} \|\underline{n}_i - \underline{n}_j\| \rightarrow \infty} s_f(\underline{n}_1, \dots, \underline{n}_r) = 0.$$

*Proof.* The notation  $s_f$  was introduced in (61). Suppose that the above convergence does not hold. Then there is  $\varepsilon > 0$  and a sequence of  $r$ -tuples  $(\underline{n}_1^k = \underline{0}, \dots, \underline{n}_r^k)$  such that  $|s_f(\underline{n}_1^k, \dots, \underline{n}_r^k)| \geq \varepsilon$  and  $\max_{i \neq j} \|\underline{n}_i^k - \underline{n}_j^k\| \rightarrow \infty$  (we use stationarity).

By taking a subsequence (but keeping the same notation), we can assume that, for two fixed indexes  $i, j$ ,  $\lim_k \|\underline{n}_i^k - \underline{n}_j^k\| = \infty$ . By Lemma 7.5, there is a subdivision  $1 = r_1 < r_2 < \dots < r_{\kappa(r)-1} < r_{\kappa(r)} \leq r$  and constant integer vectors  $\underline{a}_j$  such that

$$(64) \quad \lim_k \min_{1 \leq s \neq s' \leq \kappa(r)} \|\underline{n}_{r_s}^k - \underline{n}_{r_{s'}}^k\| = \infty,$$

$$(65) \quad \underline{n}_j^k = \underline{n}_{r_s}^k + \underline{a}_j, \text{ for } r_s < j < r_{s+1}, \text{ } s = 1, \dots, \kappa(r) - 1, \text{ and for } r_{\kappa(r)} < j \leq r.$$

Let  $d\mu_k(x_1, \dots, x_r)$  denote the probability measure on  $\mathbb{R}^r$  defined by the distribution of the random vector  $(T^{\underline{n}_1^k} f(\cdot), \dots, T^{\underline{n}_r^k} f(\cdot))$ . We can extract a converging subsequence from the sequence  $(\mu_k)$ , as well as for the moments of order  $\leq r$ . Let  $\nu(x_1, \dots, x_r)$  (resp.  $\nu(x_{i_1}, \dots, x_{i_p})$ ) be the limit of  $\mu_k(x_1, \dots, x_r)$  (resp. of its marginal measures  $\mu_k(x_{i_1}, \dots, x_{i_p})$  for  $\{i_1, \dots, i_p\} \subset \{1, \dots, r\}$ ). Let  $\varphi_i, i = 1, \dots, r$ , be continuous functions with compact

support on  $\mathbb{R}$ . Mixing of order  $r$  and condition (64) imply

$$\begin{aligned} \nu(\varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_r) &= \lim_k \int_{\mathbb{R}^d} \varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_r d\mu_k = \lim_k \int \prod_{i=1}^r \varphi_i(f(T^{\underline{n}_i^k} x)) d\mu(x) \\ &= \lim_k \int \left[ \prod_{s=1}^{\kappa(r)-1} \prod_{r_s \leq j < r_{s+1}} \varphi_j(f(T^{\underline{n}_s^k + \underline{a}_j} x)) \right] \prod_{\kappa(r) \leq j \leq r} \varphi_j(f(T^{\underline{n}_{\kappa(r)}^k + \underline{a}_j} x)) d\mu(x) \\ &= \left[ \prod_{s=1}^{\kappa(r)-1} \left( \int \prod_{r_s \leq j < r_{s+1}} \varphi_j(f(T^{\underline{a}_j} x)) d\mu(x) \right) \right] \left[ \int \prod_{\kappa(r) \leq j \leq r} \varphi_j(f(T^{\underline{a}_j} x)) d\mu(x) \right]. \end{aligned}$$

Therefore  $\nu$  is the product of marginal measures corresponding to disjoint subsets: there are  $I_1 = \{i_1, \dots, i_p\}$ ,  $I_2 = \{i'_1, \dots, i'_{p'}\}$  two nonempty subsets of  $J_r = \{1, \dots, r\}$ , such that  $(I_1, I_2)$  is a partition of  $J_r$  and  $d\nu(x_1, \dots, x_r) = d\nu(x_{i_1}, \dots, x_{i_p}) \times d\nu(x_{i'_1}, \dots, x_{i'_{p'}})$ .

With  $\Phi(t_1, \dots, t_r) = \ln \int e^{\sum t_j x_j} d\nu(x_1, \dots, x_r)$  and the analogous formulas for  $\nu(x_{i_1}, \dots, x_{i_p})$  and  $\nu(x_{i'_1}, \dots, x_{i'_{p'}})$ , we get  $\Phi(t_1, \dots, t_r) = \Phi(t_{i_1}, \dots, t_{i_p}) + \Phi(t_{i'_1}, \dots, t_{i'_{p'}})$ ; hence  $c_\nu(x_1, \dots, x_r) = \frac{\partial^r}{\partial t_1 \dots \partial t_r} \Phi(t_1, \dots, t_r) |_{t_1 = \dots = t_r = 0} = 0$ , which contradicts  $\liminf_k |s_f(\underline{n}_1^k, \dots, \underline{n}_r^k)| > 0$ .  $\square$

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