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**To cite this version:**

Andreas Baudisch, Amador Martin-Pizarro, Martin Ziegler. A MODEL THEORETIC STUDY OF RIGHT-ANGLED BUILDINGS. 2014. hal-01079813v3

**HAL Id: hal-01079813**

**https://hal.science/hal-01079813v3**

Submitted on 2 Dec 2014 (v3), last revised 12 Aug 2015 (v4)

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# A MODEL THEORETIC STUDY OF RIGHT-ANGLED BUILDINGS

ANDREAS BAUDISCH, AMADOR MARTIN-PIZARRO AND MARTIN ZIEGLER

Abstract. We study the model theory of right-angled buildings with infinite residues. For every Coxeter graph we obtain a complete theory with a natural axiomatisation, which is $\omega$-stable and equational. Furthermore, we provide sharp lower and upper bounds for its degree of ampleness, computed exclusively in terms of the associated Coxeter graph. This generalises and provides an alternative treatment of the free pseudospace.

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## 1. Introduction

A *Coxeter group* $(W, \Gamma)$ consists of a group $W$ with a fixed set $\Gamma$ of generators and defining relations $(\gamma \cdot \delta)^{m_{\gamma,\delta}} = 1$, where $m_{\gamma,\gamma} = 1$ and $m_{\gamma,\delta} = m_{\delta,\gamma}$, for $\gamma \neq \delta$, is either $\infty$ or an integer larger than 1. We will exclusively consider finitely generated Coxeter groups, with $\Gamma$ finite. A *word* $w$ is a finite sequence on the generators from $\Gamma$, and $w$ is *reduced* if its length is minimal with respect to all words representing the same element of $W$. A *chamber system* $(X, W, \Gamma)$ for the Coxeter group $(W, \Gamma)$ is a set $X$, equipped with a family of equivalence relations $(\sim_\gamma, \gamma \in \Gamma)$. If $w = \gamma_1 \cdots \gamma_n$ is a reduced word, a *reduced path of type* $w$ from $x$ to $y$ in $X$ is a sequence $x = x_0, \ldots, x_n = y$ such that $x_{i-1}$ and $x_i$ are $\sim_{\gamma_i}$-related and different for every $1 \leq i \leq n$. A chamber system $(X, W, \Gamma)$ is a *building* if each $\sim_\gamma$-class contains at least two elements and such that, for every pair $x$ and $y$ in $X$,

*Date*: November 20, 2014.
1991 *Mathematics Subject Classification.* Primary 03C45; Secondary 51E24.
*Key words and phrases.* Model Theory, ampleness, Coxeter, buildings.
The second author conducted research with support of programme ANR-09-BLAN-0047 Modig, ANR-13-BS01-0006 Valcomo as well as an Alexander von Humboldt-Stiftung Forschungsstipendium für erfahrene Wissenschaftler 3.3-FRA/1137360 STP.

there exists a element $g \in W$ such that there is a reduced path of type $w$ from $x$ to $y$ if and only if the word $w$ represents $g$. It follows that $g$ is uniquely determined by $x$ and $y$, and that the reduced path connecting $x$ and $y$ is uniquely determined by its type $w$. We refer the reader to [6] for a pleasant introduction to buildings.

The Coxeter group $(W, \Gamma)$ is *right-angled* if for every $\gamma \neq \delta$, the value $m_{\gamma,\delta}$ is either 2 or $\infty$. So $W$ is determined by its *Coxeter diagram*: a graph with vertex set $\Gamma$ such that $\gamma$ and $\delta$ have an edge connecting them, which we denote by $R(\gamma, \delta)$, if $m_{\gamma,\delta} = \infty$. In an abuse of notation, we will denote this graph by $\Gamma$ as well. The elements of $\Gamma$ will be referred to as *colours* or *levels*. Note that, for involutions $\gamma$ and $\delta$, the relation $(\gamma \cdot \delta)^2 = 1$ means that $\gamma$ and $\delta$ commute. We call a word $w'$ a *permutation* of $w$ if it can be obtained from $w$ by a sequence of swaps of commuting consecutive generators. In right-angled Coxeter groups, a word $w$ is reduced if and only if no permutation of $w$ has the form $w_1 \cdot \gamma \cdot \gamma \cdot w_2$, for some generator $\gamma$. Every element of $W$ is represented by a unique reduced word, up to permutation. In a building $(X, W, \Gamma)$, the the class $x/\sim_\gamma$ is the $\gamma$-*residue* of $x$. A right-angled Coxeter group admits a unique (up to isomorphism) countable building $\mathrm{B}_0(\Gamma)$ with infinite residues [7, Proposition 5.1], which we call *rich*.

A right-angled building can also be described in terms of an incidence geometry or, as we will refer to, a *coloured graph*. The vertices of colour $\gamma$ are equivalence classes of $\sim^\gamma$, the transitive closure of all $\sim_{\gamma'}$, for $\gamma' \neq \gamma$. Two vertices are connected by an edge if the corresponding classes intersect. The coloured graph associated to the rich building given by the diagram

$[0, n]$

is, as noticed by Tent [15], the prime model of the theory of the *free $n$-dimensional pseudospace*. A model of a theory is prime if it elementarily embeds into every model. The $n$-dimensional pseudospace, also considered by the authors in [2], witnessed the strictness of the ample hierarchy for $\omega$-stable theories. We are indebted to Tent for pointing out the connection between the free pseudospace and Tits buildings, which was the starting point of the present work.

Recall that a structure $M$ is *of finite Morley rank* if it is equipped with a (natural-valued) dimension function on its non-empty definable sets, called *Morley rank*, characterised by the following principle: If $X \subset M^n$ is definable, then $\mathrm{RM}(X) \geq n + 1$ if and only if $X$ contains an infinite family of pairwise disjoint definable sets $Y_i$ with $\mathrm{RM}(Y_i) \geq n$. If $M$ carries a group structure, this notion of dimension is well-behaved. For example, given a definable fibration $S \subset X \times Y \to Y$, the subset consisting of those $y$ in $Y$ such that the fibre over $y$ has dimension $k$ is definable for every $k$ in $N$. If all fibres have constant Morley rank $k$, then $\mathrm{RM}(X) = \mathrm{RM}(Y) + k$. For an algebraically closed field with no additional structure, definable sets are exactly the Zariski constructible ones, and Morley rank coincides with the Zariski dimension. The Morley rank of a type over a set $A$ is the minimum of the Morley rank of the formulae it contains. If $A$ is an $\omega$-saturated model, this notion agrees with the Cantor-Bendixon rank on the space of types with the Stone topology.

Motivated by a famous conjecture about the structure of *strongly minimal* sets (that is, irreducible definable sets of rank 1), the Algebraicity Conjecture states that every simple group of finite Morley rank can be seen as an algebraic group over an algebraically closed field, which is itself interpretable in the mere group structure.

Though the general conjecture on strongly minimal sets was proven to be false [9], work on the Algebraicity Conjecture, which remains open, has become a fruitful research area, combining ideas from model theory as well as the classification of simple finite groups. Groups of finite Morley rank form a particular class of $\omega$-stable theories, for which Morley rank induces a combinatorial (ordinal-valued) dimension. If an $\omega$-stable theory does not interpret a certain incidence configuration present in euclidean space, then it interprets neither infinite fields nor specific possible counterexamples to the Algebraicity Conjecture, called *bad groups* [12]. The notion of $n$-ampleness [13, 4] for a theory generalises the incidence configuration given in euclidean $(n+1)$-space by flags of affine subspaces of increasing dimension, from a single point to a hyperplane. Ampleness introduces thus a geometrical hierarchy, according to which algebraically closed fields are $n$-ample for every $n$. The first two levels of this hierarchy suffice to describe the structure of definable groups [10, 12]: they are virtually abelian, if the $\omega$-stable group is not 1-ample, and virtually nilpotent if the group has finite Morley rank and is not 2-ample. However, little is known from 2 onwards. In particular, whether the ample hierarchy was strict remained long unknown. Evans conjectured that his example could be accordingly modified to illustrate the strictness. Extending the construction in [3], where a 2-ample theory was produced which interprets no infinite group (and thus, no infinite field is interpretable), the aforementioned free $n$-dimensional pseudospace was constructed [2, 15] for every $n$, whose theory is $\omega$-stable and $n$-ample yet not $(n+1)$-ample. In particular, the $n$-dimensional pseudospace is a graph with $n+1$ many colours, labelled from 0 to $n$ such that the induced subgraph on consecutive colours is an infinite pseudoplane, whose theory was known to be 1-ample but not 2-ample.

In this article, we will provide an alternative approach to the above construction, which incorporates as well the rich buildings of every right-angled Coxeter group. As explained in Section 2, a right-angled building $B$ can be recovered from its associated colored graph $M$: The elements of $B$ correspond to the *flags* of $M$, coloured subgraphs of $M$ isomorphic to $\Gamma$. This article deals with the model theory of $\mathrm{M}_0(\Gamma)$, the coloured graph associated to the rich building $\mathrm{B}_0(\Gamma)$ given by $\Gamma$. Though models of the theory of $\mathrm{M}_0(\Gamma)$ need not arise from buildings, given flags $F$ and $G$ in a model, a notion of a *reduced path* between $F$ and $G$ can be defined, whose corresponding word consists of letters which are non-empty connected subsets of $\Gamma$. We show that the coloured graphs associated to rich buildings are *simply connected*: any two reduced paths between two given flags have the same word, up to permutation. A combinatorial study of the reduction of a path between two flags allows us to show the following crucial result (*cf.* Lemma 3.26 and Corollary 3.27):

**Theorem A.** *Simple connectedness is an elementary property.*

Let $\mathrm{PS}_\Gamma$ denotes the collection of sentences stating the following two elementary properties: Simple connectedness and that, given any flag $G$ and a colour $\gamma$, there are infinitely many flags which differ from $G$ only at the vertex of colour $\gamma$. The following theorem (*cf.* Theorem 4.12 and Corollaries 4.14 and 6.13) yields that $\mathrm{PS}_\Gamma$ axiomatises the complete theory of $\mathrm{M}_0(\Gamma)$:

**Theorem B.** *The theory* $\mathrm{PS}_\Gamma$ *is complete and* $\omega$*-stable of rank* $\omega^{K-1}$*, where* $K$ *is the cardinality of a connected component of* $\Gamma$ *of largest size. The coloured graph*

$\mathrm{M}_0(\Gamma)$, *associated to the countable rich building* $\mathrm{B}_0(\Gamma)$, *is the unique prime model of* $\mathrm{PS}_\Gamma$.

Morley rank defines a notion of independence, which agrees in the $\omega$-stable case with non-forking, as introduced by Shelah. A remarkable feature of non-forking independence, which rules out the existence of infinite definable groups, is *total triviality*: whenever we consider a base set of parameters $D$, given tuples $a$, $b$ and $c$ such that $a$ is independent both from $b$ and from $c$ over $D$, then it is independent from $b, c$ over $D$. Recall that a canonical basis of a type $p$ over a model is some set, fixed pointwise by exactly those automorphisms $\alpha$ of a sufficiently saturated model $N$ fixing the global non-forking extension $\mathbf{p}$ of $p$ over $N$. If $\mathrm{Cb}(p)$ exists, then it is unique, up to interdefinability, though generally canonical bases only exist as *imaginary* elements in the expansion $T^{\mathrm{eq}}$ of an $\omega$-stable theory $T$. Within the wider class of stable theories, there is a distinguished subclass consisting of the *equational* ones, where each definable set in every cartesian product $N^n$ of a model $N$ is a boolean combination of instances of $n$-equations $\varphi(x, y)$, that is, the tuple $x$ has length $n$ and the family of finite intersections of instances $\varphi(x, a)$ has the descending chain condition. Whilst all known examples of stable theories arising naturally in nature are equational, the only stable non-equational theory constructed so far [8] is an expansion of the free 2-dimensional pseudospace. We obtain the following (*cf.* Corollaries 7.25 and 7.28 and Proposition 7.26):

**Theorem C.** *The theory* $\mathrm{PS}_\Gamma$ *is equational, totally trivial with weak elimination of imaginaries: every type over a model has a canonical basis consisting of real elements.*

To conclude, we provide lower and upper bounds on the ample degree of the theory $\mathrm{PS}_\Gamma$, which can be deduced in terms of the underlying Coxeter graph $\Gamma$. Set $r$ to be the minimal valency of the non-isolated points of $\Gamma$ and $n$ the largest integer such that the graph $[0, n]$, as before, embeds as a full subgraph of $\Gamma$. We deduce (*cf.* Theorem 8.6):

**Theorem D.** *The theory* $\mathrm{PS}_\Gamma$ *is* $n$*-ample but not* $(|\Gamma| - r + 1)$*-ample*

These bounds are sharp and attained by $[0, n]$, by the circular graph on $n+2$ points or by the extremal case of the complete graph $\mathrm{K}_N$ on $N \geq 2$ elements, whose theory is 1-ample but not 2-ample.

## 2. Buildings and geometries

**Definition 2.1.** A graph consists of set of vertices together with a symmetric and irreflexive binary relation. Two vertices $a$ and $b$ are *adjacent* if the pair $(a, b)$ lies in the relation.

Given a finite graph $\Gamma$, its associated *right-angled Coxeter group* $(W, \Gamma)$ consists of the group $W$ generated by the elements of $\Gamma$ with defining relations:

$$\begin{aligned} &\gamma^2 = 1 && \text{for all } \gamma \text{ in } \Gamma \\ &\gamma\delta = \delta\gamma && \text{if } \gamma \text{ and } \delta \text{ are not adjacent.} \end{aligned}$$

As a convention, no element $\gamma$ *commutes* with itself.

**From now on, all Coxeter groups are right-angled.**

Fix a Coxeter group $(W, \Gamma)$. A word $v = \gamma_1 \cdots \gamma_n$ in the generators is *reduced* if there is no pair $i \neq j$ such that $\gamma_i$ equals $\gamma_j$ and commutes (i.e. is not adjacent) with every letter occurring between $\gamma_i$ and $\gamma_j$. Two words are *equivalent* if they represent the same element of $W$. Clearly, every word is equivalent to a reduced one. A reduced word $w$ *commutes* with $\gamma \in \Gamma$ if every element of $w$ does.

A word $v'$ is a *permutation* of $v$ if it can be obtained from $v$ by a sequence of commutations on pairs of commuting generators. A permutation of a reduced word is again reduced.

The following is easy to see.

**Lemma 2.2.** *Two reduced words $u$ and $v$ are equivalent if and only if $u$ is a permutation of $v$.*

For $s$ a subset of $\Gamma$, let $\langle s \rangle$ denote the subgroup of $W$ generated by $s$.

**Corollary 2.3.** *Given two subsets $s$ and $t$ of $\Gamma$,*

$$\langle s \cap t \rangle = \langle s \rangle \cap \langle t \rangle.$$

**Definition 2.4.** A *chamber system* $(X, W, \Gamma)$ for the Coxeter group $(W, \Gamma)$ consists of a set $X$ equipped with a family of equivalence relations $\sim_\gamma$ for each $\gamma \in \Gamma$. Given a word $w = \gamma_1 \cdots \gamma_n$, a *path of type $w$* from $x$ to $y$ in $X$ is a sequence $x = x_0, \ldots, x_n = y$ such that $x_{i-1}$ and $x_i$ are different and $\sim_{\gamma_i}$-related for every $1 \leq i \leq n$. A path of type $w$ is *reduced* if $w$ is.

A chamber system $(X, W, \Gamma)$ is a *building* if each $\sim_\gamma$-class contains at least two elements, and such that, for every pair $x$ and $y$ in $X$, there exists a element $g \in W$ witnessing that there is a reduced path of type $w$ from $x$ to $y$ if and only if the word $w$ represents $g$.

We will refer to a chamber system $(X, W, \Gamma)$ uniquely by the underlying set $X$ if the corresponding Coxeter group $(W, \Gamma)$ is clear. The following two lemmas can be easily shown.

**Lemma 2.5.** *In a building $X$, a reduced path of type $w$ connecting $x$ and $y$ is uniquely determined by $w$, $x$ and $y$.*

We will denote the existence of a path of type $w$ connecting $x$ to $y$ by $x \xrightarrow{w} y$. In particular, we have that $x \xrightarrow{\gamma} y$ if and only if $x \neq y$ are $\sim_\gamma$-related.

**Lemma 2.6.** *A chamber system $X$ is a building if and only if the following four conditions hold:*

*(a) Every $\sim_\gamma$-class has at least two elements.*
*(b) Every two elements of $X$ are connected by a path.*
*(c) Given two commuting generators $\gamma$ and $\delta$, if the elements $x$ and $y$ are connected by a path of type $\gamma\delta$, then $x$ and $y$ are also connected by a path of type $\delta\gamma$.*
*(d) There is no non-trivial closed reduced path.*

A chamber system satisfying conditions $(b)$ and $(c)$ is called *strongly connected*. A strongly connected chamber system is a *quasi-building* if it satisfies condition $(d)$. Lemma 2.5 holds for quasi-buildings, as well.

**Remark 2.7.** Given elements $b \xrightarrow{w} a$ in a quasi-building $A$ and $\lambda \in \Gamma$ commuting with $w$, if $a \sim_\lambda a^* \in A$, then there is a unique $b^*$ in $A$ with $b \sim_\lambda b^* \xrightarrow{w} a^*$.

*Proof.* Iterating 2.6 ($c$), the reduced path $b \xrightarrow{w} a \sim_\lambda a^*$ yields some element $b^*$ such that $b \sim_\lambda b^* \xrightarrow{w} a^*$. Furthermore, this element is uniquely determined by $b$, $a^*$ and the reduced word $\lambda \cdot w$, by Lemma 2.5. □

We will now produce certain *simple* extensions of a given quasi-building $A$. Fix some $\lambda$ in $\Gamma$ and an equivalence class $a/{\sim_\lambda}$ in $A$. We will extend $A$ to a quasi-building $A(a^*)$ containing a new element $a^*$ in $a/{\sim_\lambda}$. Let $B$ be the set of elements $x$ in $A$ which are connected to $a$ by a reduced path of type $w$, where $w$ and $\lambda$ commute. In particular, the generator $\lambda$ does not occur in $w$. Furthermore, if $b$ and $c$ in $B$ are $\sim_\gamma$-related, then $\lambda$ and $\gamma$ commute.

For every $b \in B$, introduce a new element $b^*$ and extend the chamber structure of $A$ to

$$A(a^*) = A \cup \{b^*\}_{b \in B}$$

by setting

$$b^* \sim_\gamma c^* \;\Leftrightarrow\; b \sim_\gamma c$$

for all $b$, $c$ in $B$, and

$$b^* \sim_\gamma a' \;\Leftrightarrow\; \lambda = \gamma \text{ and } b \sim_\lambda a',$$

for all $b \in B$ and $a' \in A$.

In particular, if $b$ is in $B$ and $b \xrightarrow{w} a$, then we obtain a reduced path $b^* \xrightarrow{w} a^*$.

**Lemma 2.8.** *The chamber system $A(a^*)$ is a quasi-building.*

*Proof.* Properties ($b$) and ($c$) of Lemma 2.6 can be easily shown. For example, suppose that $\delta$ and $\gamma_1$ commute, suppose $b^* \sim_{\gamma_1} d^* \sim_\delta c^*$. Then $b \sim_{\gamma_1} d \sim_\delta c$, so there is some $d'$ in $A$ such that $b \sim_\delta d' \sim_{\gamma_1} c'$, whence $b^* \sim_\delta (d')^* \sim_{\gamma_1} c^*$. The other cases are treated in a similar fashion. For property ($d$), since $A$ is strongly connected, a reduced path in $A(a^*)$ is either fully contained in $A$ or fully contained in $A(a^*) \setminus A$. The chamber system $A(a^*) \setminus A$ does not contain any non-trivial closed reduced path since it is isomorphic to $B$.

□

**Corollary 2.9.** *For every Coxeter group $(W, \Gamma)$, there is a countable building in which all $\sim_\gamma$-equivalence classes are infinite.*

We will now show that every subset of a quasi-building has a strongly connected hull, which is attained by a sequence of simple extensions.

**Proposition 2.10.** *Given a strongly connected subset $A$ of a quasi-building $X$ and $a^*$ in $X \setminus A$ with $a^* \sim_\lambda a \in A$, the smallest strongly connected subset of $X$ containing $A \cup \{a^*\}$ is isomorphic to $A(a^*)$.*

*Proof.* Observe that $A$ is a quasi-building, since $X$ is. Let $B \subset A$ be as in the construction of $A(a^*)$, that is, the set of elements $b \in A$, with $b \xrightarrow{w} a$ and $w$ commutes with $\lambda$.

Every $b$ in $B$ yields a unique $b^*$ in $X$ such that $a^* \xrightarrow{w} b^* \xrightarrow{\lambda} b$, by Remark 2.7. Clearly $b^*$ does not depend on the choice of $w$. By symmetry, the element $b$ is determined by $b^*$. Thus, the elements $b^*$ are pairwise distinct. Note that none of the $b^*$'s belong to $A$, since $a \xrightarrow{\gamma} a^* \xrightarrow{w} b^*$ and $A$ is strongly connected.

Therefore $A(a^*)$ may be identified with a subset of $X$, which is contained in every strongly connected extension of $A \cup \{a^*\}$. We need only show that the chamber structure of $A \cup \{a^*\}$ agrees with the structure induced by $X$. Given distinct

elements $b$ and $c$ in $B$ with $b \sim_\delta c$, we need only show that $b^* \sim_\delta c^*$, since the converse follows by replacing $a$, $b$ and $c$ with $a^*$, $b^*$ and $c^*$.

The set $A$ is strongly connected, so there are reduced words $u$ and $v$ such that $a \xrightarrow{u} b$ and $a \xrightarrow{v} c$ in $A$.

**Claim.** *Let $a$, $b$ and $c$ be distinct elements of a quasi-building $X$ with $b \sim_\delta c$. If $a \xrightarrow{u} b$ and $a \xrightarrow{v} c$ for reduced words $u$ and $v$. Then there are three possibilities:*

*(1) $u \cdot \delta$ is reduced.*
*(2) $v \cdot \delta$ is reduced.*
*(3) There exists a reduced word $w \cdot \delta$ and an element $a'$ such that $a \xrightarrow{w} a'$, $a' \xrightarrow{\delta} b$ and $a' \xrightarrow{\delta} c$.*

*The third case implies that both $u$ and $v$ are equivalent to $w \cdot \delta$*

*Proof of the claim:* If $u \cdot \delta$ is not reduced, up to permutation, we may assume that $u = w \cdot \delta$. Choose $a'$ in $X$ with $a \xrightarrow{w} a' \xrightarrow{\delta} b$. Thus $a' \sim_\delta c$. Either $a' = c$, so $w$ is equivalent to $v$, which gives case (2), or $a' \xrightarrow{\delta} c$, which gives case (3).
*End of the proof of the claim.*

If we apply the previous claim to our situation, we obtain three possibilities:

(1) The word $u \cdot \delta$ is reduced. Up to permutation, we have that $v = u \cdot \delta$. In particular, the elements $\gamma$ and $\delta$ commute. Since $b \sim_\delta c \sim_\gamma c^*$, there is some $c'$ in $X$ with $b \sim_\delta c' \sim_\gamma c$. Note that, since $a^* \xrightarrow{v} c^*$ and $a^* \xrightarrow{u} b^* \xrightarrow{\delta} c'$, we have that $a^* \xrightarrow{v} c'$ as well. Thus $c^* = c'$, by Remark 2.7, so $c^* \sim_\delta b^*$.
(2) The word $v \cdot \delta$ is reduced, which is treated similar to the first case.
(3) For some reduced word $w \cdot \delta$, there is an $a'$ such that $a \xrightarrow{w} a'$, $a' \xrightarrow{\delta} b$ and $a' \xrightarrow{\delta} c$. Since $A$ is strongly connected, the element $a'$ lies in $A$. By the first case, we have that $(a')^* \sim_\delta b^*$ and $(a')^* \sim_\delta c^*$, whence $b^* \sim_\delta c^*$.

If $b^* \sim_\delta a'$ for some $b$ in $B$ and $a'$ in $A$, then the path $b \sim_\lambda b^* \sim_\delta a'$ cannot be reduced, since $b^*$ does not lie in $A$. Thus $\delta = \lambda$ and $b^* \sim_\lambda a'$.

□

Corollary 2.9 and Proposition 2.10 yield immediately the following result:

**Corollary 2.11.** *(*cf. [7, Proposition 5.1]*) Every Coxeter group $(W, \Gamma)$ has, up to isomorphism, a unique countable building $\mathrm{B}_0(\Gamma)$, in which all $\sim_\gamma$-equivalence classes are infinite.*

We will study the model theory of buildings using the following expansion of the natural language:

**Definition 2.12.** Let $X$ be a chamber system for $(W, \Gamma)$ and $s$ a subset of $\Gamma$. By $\sim_\emptyset$, we denote the diagonal in $X \times X$. Otherwise, for $\emptyset \neq s \subset \Gamma$, the relation $\sim_s$ is the transitive closure of all $\sim_\gamma$, with $\gamma \in s \subset \Gamma$. The $\sim_s$-class of an element is called its *$s$-residue*. In particular, its *$\gamma$-residue* is its $\sim_\gamma$-class, which is often called *$\gamma$-panel* in the literature.

For $\gamma$ in $\Gamma$, set $\sim^\gamma = \sim_{\Gamma \setminus \{\gamma\}}$. The chamber system $(X, \sim^\gamma)_{\gamma \in \Gamma}$ is called the associated *dual chamber system* of $X$.

It is easy to see that $x \sim_s y$ if and only if $x \xrightarrow{w} y$ for some $w \in \langle s \rangle$. If $X$ is a quasi-building, the word $w$ is uniquely determined as an element of $W$, so Corollary

2.3 implies

$$\sim_{s_1 \cap s_2} = \sim_{s_1} \cap \sim_{s_2},$$

and particularly

$$(\dagger) \qquad \sim_\gamma = \bigcap_{\beta \neq \gamma} \sim^\beta .$$

Thus, for a quasi-building $X$, the chamber system $(X, \sim_\gamma)_{\gamma \in \Gamma}$ is definable in its associated dual chamber system $(X, \sim^\gamma)_{\gamma \in \Gamma}$. Clearly, the latter is only definable if countable disjunctions are allowed.
The aim of this article is to study the complete theory of $\mathrm{B}^0(\Gamma)$, the associated dual chamber system of $\mathrm{B}_0(\Gamma)$, which was defined in Corollary 2.11.

**Lemma 2.13.** *A quasi-building $X$ has the following properties, which are elementary in the above expanded language:*

*(1) Given $x$ and $y$ in $X$ with $x \sim^\gamma y$ for all $\gamma \in \Gamma$, then $x = y$.*
*(2) If $(x_\gamma)_{\gamma \in \Gamma}$ is a* coherent *sequence in $X$, i.e. whenever $\gamma$ and $\delta$ are adjacent, there exists some $y_{\gamma,\delta}$ in $X$ with $x_\gamma \sim^\gamma y_{\gamma,\delta} \sim^\delta x_\delta$, then there exists an element $z \in X$ with $z \sim^\gamma x_\gamma$ for all $\gamma \in \Gamma$.*

*Proof.* Property (1) clearly follows from condition (†).

In order to show Property (2), note that any singleton is a quasi-building and has property (2). By Proposition 2.10, we need only show now that, if $A$ is a quasi-building with Property (2), then so does $A(a^*)$.

Let $(x_\gamma)_{\gamma \in \Gamma}$ be a coherent sequence in $A(a^*)$, where $a^* \sim_\lambda a \in A$. Each $x_\gamma$ lies either in $A$ or is $\sim_\lambda$ connected to some element in $A$. Since only the $\sim^\gamma$-class of $x_\gamma$ matters, we may assume that all $x_\gamma$ belong to $A$, for $\gamma \neq \lambda$. If $x_\lambda$ is in $A$, then the result follows, since Property (2) holds in $A$. Otherwise, if $x_\lambda$ does not belong to $A$, we have that $x_\lambda \xrightarrow{w} a^*$, for a reduced word $w$ which commutes with $\lambda$. Thus $x_\lambda \sim^\lambda a^*$ and we may assume that $x_\lambda = a^*$.

By assumption, for every $\gamma$ adjacent with $\lambda$, there is some $y_{\lambda,\gamma}$ in $A(a^*)$ such that $a^* \sim^\lambda y_{\lambda,\gamma} \sim^\gamma x_\gamma$. The element $y_{\lambda,\gamma}$ cannot lie in $A$, for otherwise the reduced path $a \sim_\lambda a^* \sim^\lambda y_{\lambda,\gamma}$ implies that so is $a^*$. Thus $y_{\lambda,\gamma}$ is of the form $b^*_{\lambda,\gamma}$ for some $b_{\lambda,\gamma} \in A$.

It follows that $a \sim^\gamma b_{\lambda,\gamma} \sim^\gamma x_\gamma$. Replacing the element $x_\lambda$ in the sequence $(x_\gamma)$ by $a$, yields a new sequence contained in $A$ and coherent. Thus, we find an element $c$ in $A$ such that $c \sim^\lambda a$ and $c \sim^\gamma x_\gamma$ for $\gamma \neq \lambda$. Observe that $c^* \sim^\lambda a^*$ and $c^* \sim^\gamma x_\gamma$, so $c^*$ is the desired element.

□

**Definition 2.14.** A chamber system $(X, \sim^\gamma)_{\gamma \in \Gamma}$ is a *dual quasi-building* if it has properties (1) and (2) from Lemma 2.13.

The following remark can easily be verified.

**Remark 2.15.** A chamber system $(X, \sim^\gamma)_{\gamma \in \Gamma}$ is a dual quasi-building if and only if for every coherent sequence $(x_\gamma)_{\gamma \in \Gamma}$ there is a unique $z$ with $z \sim^\gamma x_\gamma$ for all $\gamma \in \Gamma$.

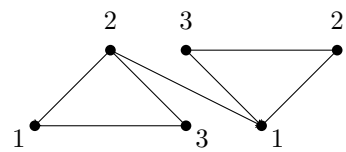
**Definition 2.16.** A $\Gamma$-*graph* $M$ is a coloured graph with colours $\mathcal{A}_\gamma(M)$ for $\gamma$ in $\Gamma$, and no edges between elements of $\mathcal{A}_\gamma(M)$ and $\mathcal{A}_\delta(M)$ if $\gamma$ and $\delta$ are not adjacent.

A *flag* $F$ of the $\Gamma$-graph $M$ is a subgraph $F = \{f_\gamma\}_{\gamma\in\Gamma}$, where each $f_\gamma$ lies in $\mathcal{A}_\gamma(M)$, such that the map $\gamma \mapsto f_\gamma$ induces a graph isomorphism between $\Gamma$ and $F$.

The $\Gamma$-graph $M$ is a $\Gamma$-*space* if the following two additional properties are satisfied:

(1) Every vertex belongs to a flag of $M$.
(2) Any two adjacent vertices in $M$ can be expanded to a flag of $M$.

In particular, if $\Gamma$ is the complete graph $\mathbb{K}_3$, then the following $\mathbb{K}_3$-graph is not a $\mathbb{K}_3$-space:




**Theorem 2.17.** *The class of dual quasi-buildings for* $(W, \Gamma)$ *and the class of* $\Gamma$-*spaces are bi-interpretable.*

*Proof.* In a chamber system $(X, \sim^\gamma)_{\gamma\in\Gamma}$, we interpret a $\Gamma$-graph $\mathcal{M}(X)$ as follows: for every $\gamma \in \Gamma$, the colour $\mathcal{A}_\gamma$ consists of the $\gamma$-residues of elements in $X$, put in a disjoint way; two elements $u$ and $v$ of $X$ are identified in $\mathcal{A}_\gamma$ if $u \sim^\gamma v$. For the graph structure on $\mathcal{M}(X)$, we impose that two elements $u$ and $v$ are adjacent if $u \in \mathcal{A}_\gamma$ and $v \in A_\delta$, the elements $\gamma$ and $\delta$ are adjacent and there is some $z \in X$ with $z \sim^\gamma u$ and $z \sim^\delta v$.

Since every $x \in X$ gives rise to the flag $\phi(x) = \{x/\sim^\gamma \mid \gamma \in \Gamma\}$ of $\mathcal{M}(X)$, it is straight-forward to see that $\mathcal{M}(X)$ is a $\Gamma$-space.

Given a $\Gamma$-graph $M$, we define a chamber system $(\mathcal{X}(M), \sim^\gamma)_{\gamma\in\Gamma}$, whose underlying set is the collection of flags of $M$. Two flags $F$ and $G$ are $\sim^\gamma$-related if their $\gamma$-vertices agree.

We will first show that $\mathcal{X}(M)$ is a dual quasi-building. We need only check Property (2). Assume that $(F_\gamma)$ is a coherent system of flags. Let $f_\gamma$ denote the $\gamma$-vertex of $F_\gamma$. If $\gamma$ and $\delta$ are adjacent, there is a flag $G$ such that $F_\gamma \sim^\gamma G \sim^\delta F_\delta$. In particular, the flag $G$ contains an edge between $f_\gamma$ and $f_\delta$. So $H = (f_\gamma)$ is a flag of $M$ and $H \sim^\gamma F_\gamma$ for all $\gamma \in \Gamma$.

For a chamber system $(X, \sim^\gamma)_{\gamma\in\Gamma}$, the correspondence $x \mapsto \phi(x)$ defines a map $\phi : X \to \mathcal{X}((\mathcal{M}(X))$. It is easy to see that $\phi(x) \sim^\gamma \phi(y)$ if and only if $x \sim^\gamma y$. If $X$ is a dual quasi-building, Property (1) implies that $\phi$ is injective and Property (2) that $\phi$ is surjective. Thus $X$ and $\mathcal{X}((\mathcal{M}(X))$ are definably isomorphic.

Given a $\Gamma$-graph $M$, the correspondence $a : \mathcal{M}(\mathcal{X}(M)) \to M$ which associates to each class $F/\sim^\gamma$ the $\gamma$-vertex of $F$ is a bijection between $\mathcal{M}(\mathcal{X}(M))$ and the collection of vertices of $M$ which belong to a flag of $M$. For adjacent $\gamma$ and $\delta$, there is an edge between $F/\sim^\gamma$ and $G/\sim^\delta$ if and only if $a(F/\sim^\gamma)$ and $a(G/\sim^\delta)$ belong to a common flag of $M$. This shows that $a$ is an definable isomorphism if $M$ is a $\Gamma$-space.

To conclude, observe that $a \circ \phi$ is the identity map on the dual quasi-building $(X, \sim^\gamma)_{\gamma\in\Gamma}$. □

In order to describe the model-theoretical properties of the dual quasi-building $\mathrm{B}^0(\Gamma)$ (defined on page 8), we will therefore consider the first-order theory of

$$\mathrm{M}_0(\Gamma) = \mathcal{M}(\mathrm{B}^0(\Gamma)),$$

its associated $\Gamma$-space.

## 3. Simply connected $\Gamma$-spaces

Recall that by a Coxeter group we mean a right-angled finitely generated Coxeter group. From now on, fix a Coxeter group $(W, \Gamma)$, which we will denote simply by $W$, with underlying Coxeter graph $\Gamma$.

**Notation.** For subsets of of the graph $\Gamma$, *connected* means path-connected. A *letter* is a non-empty connected subset of $\Gamma$. Characters such as $s$ and $t$ will exclusively refer to letters. A *word* $u$ is a finite sequence of letters.

Every generator $\gamma$ in $\Gamma$ defines the letter $\{\gamma\}$. In this way, every word in the generators can be considered as a word in the above sense.

**Definition 3.1.** Two letters $s$ and $t$ *commute* if $s \cup t$ is not a letter, i.e. if the elements of $s$ commute with all elements of $t$. In particular, no letter commutes with itself. Two words commute if their letters respectively do. A word is *commuting* if it consists of pairwise commuting letters. A *permutation* of a word is obtained by repeatedly permuting adjacent commuting letters. Two words $u$ and $v$ are *equivalent*, denoted by $u \approx v$, if one can be permuted into the other.

The following is easy to see.

**Remark 3.2.** A commuting word $w = s_1 \cdots s_n$ is determined up to equivalence by its *support*

$$|w| = s_1 \cup \cdots \cup s_n,$$

where the $s_i$'s are the connected components of $|w|$.

We will often write $w$ instead of $|w|$ if $w$ is a commuting word.

Throughout this section, we will work inside some ambient $\Gamma$-space.

**Definition 3.3.** A *weak flag path* $P$ from $F$ to $G$ is a finite sequence $F = F_0, \ldots, F_n = G$ such that the colours where $F_{i+1}$ and $F_i$ differ form a letter $s_{i+1}$. To such a path, we associate the word $u = s_1 \cdots s_n$ and denote this by $F \underset{u}{\to} G$.

In the light of Theorem 2.17, we transfer to $\Gamma$-spaces Definition 2.12 and say that two flags $F$ and $G$ are *$A$-equivalent*,

$$F \sim_A G,$$

if the set of colours where $F$ and $G$ differ is contained in $A \subset \Gamma$. Similarly as in [2, Lemma 6.3], by decomposing any subset of $\Gamma$ as a disjoint union of its connected components, we obtain the following.

**Lemma 3.4.** *Two flags $F$ and $G$ are $A$-equivalent if and only if they can be connected by a weak flag path whose word consists of letters contained in $A$. In particular, setting $A = \Gamma$, any two flags can be connected by a weak flag path.*

*Proof.* For $F = \{f_\gamma \mid \gamma \in \Gamma\}$ and $G = \{g_\gamma \mid \gamma \in \Gamma\}$, let $s_1, \ldots, s_n$ be the connected components of $\{\gamma \in \Gamma \mid f_\gamma \neq g_\gamma\}$. Set

$$F_i = \{f_\gamma \mid \gamma \notin s_1 \cup \ldots \cup s_i\} \cup \{g_\gamma \mid \gamma \in s_1 \cup \ldots \cup s_i\}.$$

Then $F_0, \ldots, F_n$ is a weak path which connects $F$ and $G$, and it has word $s_1 \cdots s_n$. □

The proof yields the following two corollaries.

**Corollary 3.5.** *Given flags $F$ and $G$, there exists a commuting word $u$, unique up to equivalence, such that $F \underset{u}{\to} G$. For every permutation $u'$ of $u$, there is a unique weak flag path from $F$ to $G$ with word $u'$.*

**Corollary 3.6.** *In a path $F \underset{u}{\to} H \underset{v}{\to} G$ with commuting words $u$ and $v$, the flag $H$ is uniquely determined by $F$, $G$, $u$ and $v$.*

To each relation $F \underset{s}{\to} G$ associate the natural bijection $\mathcal{A}_s(F) \to \mathcal{A}_s(G)$. It is easy to see, that a weak flag path $P : F_0 \underset{s_0}{\longrightarrow} F_1 \ldots F_{n-1} \underset{s_{n-1}}{\longrightarrow} F_n$ is completely determined by $F_0$ (likewise by $F_n$) and the sequence of associated maps. If $s_1 \cdots s_n$ is commuting, this sequence of maps is equivalent to the collection $\mathcal{A}_{s_i}(F_0) \to \mathcal{A}_{s_i}(F_n)$, which gives an alternative proof of 3.5. More generally, the following lemma holds.

**Lemma 3.7** (Permutation of a path)**.** *Given a weak flag path $P : F_0 \underset{u}{\to} F_n$ and a permutation $u'$ of $u$, there is a unique weak flag path $P' : F_0 \underset{u'}{\longrightarrow} F_n$ such that the associated map of each letter in $P$ is the same as the associated map of the corresponding occurrence of that letter in $P'$.*

Such a path $P'$ is a *permutation* of $P$.

**Definition 3.8.**

- A *splitting* of a letter $s$ is a (possibly trivial) word, whose letters are properly contained in $s$. Given words $u$ and $v$, we say that $u \prec v$ if $u$ is equivalent to a word obtained from $v$ by replacing at least one occurrence of a letter in $v$ by a splitting. We write $u \preceq v$ if either $u \prec v$ or $u \approx v$.
- Whenever $F \underset{s}{\to} G$ and there is no weak flag path from $F$ to $G$ whose word is a splitting of $s$, write $F \xrightarrow{s} G$. A *flag path* from $F$ to $G$ with word $u = s_1 \cdots s_n$, denoted by $F \xrightarrow{u} G$, is a weak flag path $F = F_0, \ldots, F_n = G$ such that $F_i \xrightarrow{s_{i+1}} F_{i+1}$ for $i = 0, \ldots, n-1$.

It is easy to see that the relation $\prec$ is transitive, irreflexive and well-founded (*cf.* [2, Lemma 5.26].) A permutation of a flag path is again a flag path. If $F \underset{s}{\to} G$, whether $F \xrightarrow{s} G$ depends on the ambient $\Gamma$-space.

**Lemma 3.9.** *If $F \underset{u}{\to} G$, then $F \xrightarrow{v} G$ for some $v \preceq u$.*

*Proof.* Suppose $F \underset{u}{\to} G$. If this is not a flag path, it contains a step $F' \underset{s}{\to} G'$ which can be replaced by $F' \underset{w}{\to} G'$, where $w$ is a splitting of $G$. This yields a weak

flag path $F \underset{u'}{\longrightarrow} G$ with $u' \prec u$. Since $\prec$ is well-founded, this procedure stops with a flag path $F \xrightarrow{v} G$ for some word $v \preceq u$. □

**Definition 3.10.**

- A word $v = s_1 \cdots s_n$ is *reduced* if there is no pair $i \neq j$ such that $s_i \subset s_j$ and $s_i$ commutes with all letters in $v$ between $s_i$ and $s_j$.
- A flag path is *reduced* if its associated word is.
- The reduced word $v$ is a *reduct* of $u$ if it can obtained from $u$ by the following rules

  COMMUTATION: Permute consecutive commuting letters.

  ABSORPTION: If $s$ is contained in $t$, replace a subword $s \cdot t$ (or $t \cdot s$) by $t$.

  SPLITTING: Replace a subword $s \cdot s$ by a splitting of $s$.

  We will denote this by $u \xrightarrow{*} v$ (*cf.* [2, Definition 5.24]). Clearly $u \xrightarrow{*} v$ implies $v \preceq u$.

It is easy to see that a word $u$ is reduced if and only if any permutation of $u$ is. Similarly, a path $P$ is reduced if and only if any permutation of $P$ is.

Consider indexes $i \neq j$ in a word $v = s_1 \cdots s_n$ such that $s_i \subset s_j$ and $s_i$ commutes with all letters in $v$ between $s_i$ and $s_j$. Using Commutation and Absorption, we can delete the occurrence of the letter $s_i$. If $s_i = s_j$, we may also replace $s_j$ by a splitting of $s_j$. We call such an operation a *generalised* Absorption or Splitting. It is easy to see that every reduct of a word can be obtained by a sequence of generalised Absorptions and Splittings, followed by a permutation.

**Lemma 3.11.** *If $F \xrightarrow{u} G$, then $F \xrightarrow{v} G$ for some reduced $v$ with $u \xrightarrow{*} v$.*

*Proof.* If the path $F \xrightarrow{u} G$ is not reduced, possibly after permutation, we may assume that it contains a subpath $F' \xrightarrow{s} H' \xrightarrow{t} G'$, where $s \subset t$ (or $t \subset s$). One of the following reduction steps now applies:

PROPER ABSORPTION: If $s \subsetneq t$, remove $H'$ since $F' \xrightarrow{t} G'$, for otherwise, there would be a a splitting $x$ of $t$ such that $F' \underset{x}{\rightarrow} G'$, which implies $H' \underset{s \cdot x}{\longrightarrow} G'$, contradicting $H' \xrightarrow{t} G'$.

ABSORPTION/SPLITTING: If $s = t$, note that $F' \sim_s G'$. Lemmata 3.4 and 3.9 yield:

ABSORPTION: $F' \xrightarrow{s} G'$, or

SPLITTING: $F' \xrightarrow{x} G'$ for some splitting $x$ of $s$.

Therefore, the flag $H'$ can be removed.

Note that both Absorption and Splitting yield words which are $\prec$-smaller than $u$. Thus, the process must eventually stop. □

**Remark 3.12.** We will see in Remark 4.15 that, for every reduction $u \xrightarrow{*} v$, there is a flag path of word $u$ in a suitable $\Gamma$-space which can be reduced to a path with word $v$ by the above procedure.

**Corollary 3.13.** *Any two flags can be connected by a reduced path.*

The following property of the ambient space will ensure that all reduced paths between two given flags have equivalent words (*cf.* Proposition 3.19).

**Definition 3.14.** A $\Gamma$-space $M$ is *simply connected* if there are no non-trivial closed reduced flag paths.

**Lemma 3.15.** *The $\Gamma$-space $M$ is simply connected if and only if the word of any closed flag path can be reduced to the trivial word* 1.

*Proof.* Suppose the condition on the right holds. Given a closed reduced flag path with word $u$, since $u \xrightarrow{*} 1$, then $u = 1$, as $u$ is already reduced. For the other direction, given a closed path $P$ with word $u$, apply Lemma 3.11 to obtain a closed reduced path whose word $v$ is a reduct of $u$. If $M$ is simply connected, the word $v$ must be 1, thus $u \xrightarrow{*} 1$. □

**Theorem 3.16.** *The $\Gamma$-space $\mathrm{M}_0(\Gamma)$ is simply connected.*

*Proof.* First of all, observe that $F \underset{\gamma}{\rightarrow} G$ implies $F \xrightarrow{\gamma} G$ for all singletons $\gamma \in \Gamma$. Therefore, given two different elements $x$ and $y$ in $\mathrm{B}^0(\Gamma)$, we have that $x \sim_\gamma y$ if and only if $\phi(x) \xrightarrow{\gamma} \phi(y)$. This observation, together with the fact that $\mathrm{B}_0(\Gamma)$ is a building, yields the desired conclusion once we show that, if $\phi(x) \xrightarrow{s} \phi(y)$ in $\mathrm{M}_0(\Gamma)$, then $s$ is a singleton.

If $\phi(x) \xrightarrow{s} \phi(y)$, then $\phi(x) \sim_s \phi(y)$, which means that for every $\delta \notin s$, the $\delta$-vertex of $\phi(x)$ equals the $\delta$-vertex of $\phi(y)$, that is $x \sim^\delta y$ for $\delta \notin s$, or equivalently $x \xrightarrow{w} y$, where the letters in $w$ are elements of $s$. Hence $\phi(x) \xrightarrow{w} \phi(y)$. So $w = s$ and $s$ is a singleton. □

An interesting feature of simply connected $\Gamma$-spaces is that the word of a reduced flag path connecting two given flags is unique, up to equivalence. For the proof of the following proposition we need a definition and a lemma.

**Definition 3.17.** The letter $t$ is (properly) *left-absorbed* by the word $s_1 \cdots s_n$, resp. *right-absorbed*, if and only if $t$ is (properly) contained in some $s_i$ and commutes with $s_1 \cdots s_{i-1}$, resp. with $s_{i+1} \cdots s_n$. A word $u$ is *left-absorbed*, resp. *right-absorbed*, by $v$ if each letter in $u$ is.

A word is reduced if and only if it cannot be written as $u \cdot t \cdot v$, where $t$ is either right-absorbed by $u$ or left-absorbed by $v$. The word $u = s_1 \cdots s_n$ left-absorbs $t$ if and only if $u^{-1}$ right-absorbs $t$, where $u^{-1} = s_n \cdots s_1$.

**Lemma 3.18.** *Let $u \cdot s$ be reduced and $x$ be a splitting of $s$. Every reduct of $u \cdot x$ is equivalent to $u \cdot x_1$ for some $x_1 \preceq x$.*

*Proof.* Since $u \cdot s$ is reduced, a generalised Absorption or Splitting for $u \cdot x$ cannot happen for a pair $s_i \subset s_j$, where $s_i$ is contained in $u$. So letters contained in $u$ will never be deleted. □

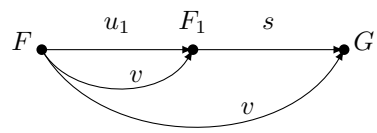
**Proposition 3.19.** *If the flags $F$ and $G$ in a simply connected $\Gamma$-space $M$ are connected by reduced flag paths with respective words $u$ and $v$, then $u \approx v$.*

*Proof.* We prove it by induction on the $\prec$-ranks of $u$ and $v$. If $F = G$, the claim is equivalent to simple connectedness. We may therefore assume that $F \neq G$. Since $u \cdot v^{-1}$ belongs to a closed non-trivial flag path, it cannot be a reduced word. So assume that $u = u_1 \cdot s$ and $s$ is right-absorbed by $v$. Splitting the first path accordingly,

$$F \xrightarrow{u_1} F_1 \xrightarrow{s} G,$$

we distinguish two cases:

(1) The letter $s$ is properly right-absorbed by $v$. Then $v$ is the only reduct of $v \cdot s$ and therefore $F \xrightarrow{v} G \xrightarrow{s} F_1$ gives $F \xrightarrow{v} F_1$.

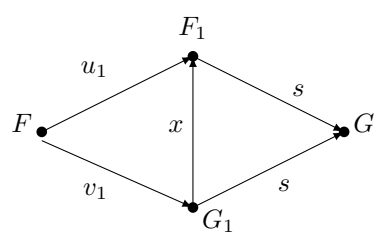



Since $u_1 \prec u$, induction yields that $u_1 \approx v$. In particular, the letter $s$ is right-absorbed by $u_1$, contradicting that $u$ is reduced.

(2) After a permutation $v$ has the form $v_1 \cdot s$. We split the second path as

$$F \xrightarrow{v_1} G_1 \xrightarrow{s} G.$$

We have then either $G_1 \xrightarrow{s} F_1$ or $G_1 \xrightarrow{x} F_1$ for a reduced splitting $x$ of $s$. If $G_1 \xrightarrow{s} F_1$, then $F \xrightarrow{v} F_1$, which contradicts as before that $F \xrightarrow{u_1} F$. So $G_1 \xrightarrow{x} F_1$.




By Lemma 3.18 the path $F \xrightarrow{v_1} G_1 \xrightarrow{x} F_1$ reduces to a path $F \xrightarrow{v_1 \cdot x_1} F_1$ with $x_1 \preceq x$. Since $v_1 \cdot x_1 \prec v$, induction yields that $v_1 \cdot x_1 \approx u_1$. So $v_1 \cdot x_1 \cdot s \approx u$ is reduced, which is only possible if $x_1 = 1$. Whence $v_1 \approx u_1$ and therefore $v \approx u$. □

**Definition 3.20.** Given $F$ and $G$ flags in a simply connected $\Gamma$-space $A$, we say that the word $u$ *connects* $F$ *and* $G$, if $u$ is the word of a reduced path from $F$ to $G$. Since $u$ is uniquely determined up to equivalence we denote it by $\mathrm{w}_A(F, G)$

In order to show that simple connectedness is an elementary property, we will first give a general description of a reduction of a flag path.

For $i \neq j$, a pair of letters $s_i$, $s_j$, occurring in $v$ is called *reduced* if either $s_i$ and $s_j$ are not comparable, or if none of the two letters commutes with all letters between $s_i$ and $s_j$. The word $v$ is reduced if and only if all pairs of letters occurring

in $v$ are. Two disjoint subwords $w_1$ and $w_2$ of a common (possibly not reduced) word $v$ are *reduced* if all pairs of letters $s$ and $t$, where $s$ occurs in $w_1$ and $t$ occurs in $w_2$, are reduced in $v$. By a sequence of generalised Absorptions and Splittings applied to letters in $w_1$ and $w_2$, replace $w_1$, $w_2$ by a pair $w_1^*$, $w_2^*$, which is reduced in the resulting word $v^*$. We call such a process a *reduction* of $w_1$, $w_2$ in $v$. If $v$ is the word of a flag path, we call a corresponding transformation of the path also a reduction of $w_1$, $w_2$.

**Lemma 3.21** (Reduction Lemma)**.** *Let $w_1 \cdot w \cdot w_2$ be the (possibly non-reduced) word of a flag path, where both $w_1$ and $w_2$ are reduced. Then there are words $u_1, v_1, u_2, v_2$ and a reduction of $w_1$, $w_2$ in the path with resulting word $w_1^* \cdot w \cdot w_2^*$, such that:*

- *$w_1 \approx u_1 \cdot v_1$ and $v_2 \cdot u_2 \approx w_2$,*
- *$w_1^* = u_1 \cdot x_1$ for some $x_1 \preceq v_1$ and $x_2 \cdot u_2 = w_2^*$ for some $x_2 \preceq v_2$,*
- *$v_1$ and $v_2$ commute with $w$,*
- *$|v_1|$ and $|v_2|$ are contained in $|w_1| \cap |w_2|$.*

*Proof.* Since both $w_1$ and $w_2$ are reduced, we may assume that the words $w_1^*$ and $w_2^*$ are obtained by a sequence of generalised Absorptions and Splitting, each one involving a letter in the first word and a letter in the last word, where after every step we apply a reduction on both the first and the last word. It is enough to show by induction that, at every intermediate step of the reduction, the word $w_1' \cdot w \cdot w_2'$ satisfies the conclusion of the lemma. Start by setting $u_j = w_j$ and $v_j = x_j$ the empty word, for $j = 1, 2$. Assume that $u_1, v_1, x_1, v_2, u_2, x_2$ witness this at the $i^{\text{th}}$ step. In particular $w_1' = u_1 \cdot x_1$ and $x_2 \cdot u_2 = w_2'$, where $x_i \preceq v_i$ for $i = 1, 2$.

We will treat the case of a generalised Splitting and leave to the reader the easier case of a generalised Absorption. Suppose hence that $w_1'' \cdot w \cdot w_2''$ is obtained from $w_1' \cdot w \cdot w_2'$ by a generalised Splitting followed by a reduction of the first and last word. Then there is a letter $s$ occurring both in $w_1'$ and $w_2'$, which commutes with $w$ as well as with the letters of $w_1'$ on its right (resp. the letters of $w_2'$ on its left). Suppose furthermore that the word $w_1''$ is obtained from $w_1'$ by deleting $s$. Note that $w_1''$ is reduced.

The word $w_2''$ is obtained from $w_2'$ by replacing $s$ by a splitting $y$ of $s$ followed by a further reduction. By Lemma 3.18, we may assume that $w_2''$ is reduced and obtained, up to a permutation, by replacing $s$ with some word $y_2 \preceq y \prec s$. If $s$ occurred in $u_2$, then set $u_2'$ the word obtained by removing $s$ from $u_2$ as well as $v_2' = v_2 \cdot s$ and $x_2' = x_2 \cdot y_2$. If $s$ occurred in $x_2$, then replace $s$ by $y_2$ and leave $v_2$ and $u_2$ unchanged.

Likewise, modify the words $u_1$, $v_1$ and $x_1$ accordingly. □

**Definition 3.22.** Given a letter $s$ and a natural number $n$, the reduced word $w$ *satisfies* $E(s, n)$ if $|w| \subset s$ and no permutation of $w$ is a product of $n$ many words $u_i$, with $|u_i| \subsetneq s$.

The properties $E(s, n)$ get stronger as $n$ increases. In particular, the word $w$ satisfies $E(s, 0)$ if and only if $|w| \subset s$ and $w \neq 1$. Similarly, the word $w$ satisfies $E(s, 1)$ if and only if $|w| = s$.

**Corollary 3.23.** *Let $u = s_1 \cdots s_n$ be a reduced word and $w_1, \ldots, w_n$ reduced words with $|w_k| = s_k$. Consider two indices $i < j$ and a reduction $w_i^*, w_j^*$ of the pair $w_i, w_j$ in $w = w_i \cdots w_j$ as in the Reduction Lemma 3.21. Then the following holds*

(1) *If $w_i$ satisfies $E(s_i, m)$ for some $m > 0$, then $w_i^*$ satisfies $E(s_i, m-1)$, similarly for $w_j$ and $w_j^*$.*
(2) *Assume that $w_i$ and $w_j$ satisfy $E(s_i, 2)$ and $E(s_j, 2)$, respectively. If a pair $w_{i'}, w_{j'}$ is already reduced in $w_1 \cdots w_n$, then the corresponding pair remains reduced in $w^* = w_1 \cdots w_i^* \cdots w_j^* \cdots w_n$.*

*Proof.* Choose $u_i, v_i, x_i, u_j, v_j, x_j$ as in the Reduction Lemma. Since $|v_i|$ is contained in $s_i \cap s_j$ and $v_i$ commutes with $s_{i+1} \cdots s_{j-1}$, then $|v_i|$ is a proper subset of $s_i$, for otherwise the pair $s_i, s_j$ would not be reduced in $w$. As $w_i \approx u_i \cdot v_i$, if $w_i$ has property $E(s_i, m)$, then $u_i$ has property $E(s_i, m-1)$, and so does $w_i^*$.

Assume that $w_i$ and $w_j$ satisfy $E(s_i, 2)$ and $E(s_j, 2)$, respectively. Therefore $w_i^*$ and $w_j^*$ have property $E(s_i, 1)$ and $E(s_j, 1)$, respectively, and we have $|w_i^*| = s_i$ and $|w_j^*| = s_j$. Suppose that $w_{i'}, w_{j'}$ is reduced in $w$. Since the words $w_i^*$ and $w_j^*$ commute with the same letters as $w_i$ and $w_j$, respectively, the pair $w_{i'}, w_{j'}$ remains reduced in $w^*$ if $\{i', j'\}$ and $\{i, j\}$ are disjoint. By symmetry it suffices to consider the following three other cases:

Case $i' < j' = i$. We have to show that the pair $w_{i'}, u_i \cdot x_i$ is reduced in $w^*$ if the pair $w_{i'}, u_i \cdot v_i$ is reduced in $w$. This follows easily from $|x_i| \subset s_i = |u_i|$.

Case $i = i' < j' < j$. Here we have to show that the pair $u_i \cdot x_i, w_{j'}$ is reduced in $w^*$ if the pair $u_i \cdot v_i, w_{j'}$ is reduced in $w$. This follows easily from the fact that $v_i$ and $x_i$ commute with $w_{j'}$.

Case $i = i' < j < j'$. Again we have to show that the pair $u_i \cdot x_i, w_{j'}$ is reduced in $w^*$ if the pair $u_i \cdot v_i, w_{j'}$ is reduced in $w$. This follows easily from $|x_i|, |v_i|, |v_j| \subset |u_j|$. □

**Proposition 3.24.** *Let $u = s_1 \cdots s_n$ be a reduced word. Given reduced flag paths $F_{i-1} \xrightarrow{w_i} F_i$ such that each $w_i$ has property $E(s_i, n)$, then the path*

$$F_0 \xrightarrow{w_1} F_1 \ldots F_{n-1} \xrightarrow{w_n} F_n$$

*has a reduction of length $\geq n$.*

*Proof.* Choose any enumeration of all pairs $i < j$ of indices between 1 and $n$ and apply the Reduction Lemma to each pair in order. Observe that $k$ occurs in at most $n-1$ reductions. At every step of the reduction, the resulting words satisfy $E(s_k, 2)$, so the resulting path

$$F_0 \xrightarrow{w_1^*} F_1^* \ldots F_{n-1}^* \xrightarrow{w_n^*} F_n,$$

is reduced, by Corollary 3.23 (2). None of the words $w_i^*$ is trivial, by Corollary 3.23 (1).

□

Together with Remark 4.15, the previous proposition shall imply the following (a priori) stronger form.

**Remark 3.25.** Let $u = s_1 \cdots s_n$ be a reduced word and $w_1, \ldots, w_n$ reduced words with property $E(s_i, n)$. Then every reduct of $w_1 \cdots w_n$ has length at least $n$.

**Lemma 3.26.** *For each number $n$ and letter $s$, the property*

"There is a reduced path $F \xrightarrow{w} G$, where $w$ satisfies $E(s, n)$"

*is an elementary property of $F$ and $G$,*

*Proof.* By Lemmata 3.4, 3.9 and 3.11, there is, for each proper subset $A$ of $s$, a reduced flag path $F \xrightarrow{u} G$ with $|u| \subset A$ if and only if $F$ and $G$ agree on all colours outside $A$. □

We can now conclude that simple connectedness is elementary.

**Corollary 3.27.** *A $\Gamma$-space $M$ is simply connected if and only if for all non-trivial reduced $u = s_1 \cdots s_n$ there is no sequence $F_0 \xrightarrow{w_1} F_1 \dots F_{n-1} \xrightarrow{w_n} F_n = F_0$ with reduced words $w_i$ which satisfy $E(s_i, n)$.*

*Proof.* Suppose $M$ satisfies the right condition. Any reduced flag path

$$F_0 \xrightarrow{s_1} F_1 \xrightarrow{s_2} \dots \xrightarrow{s_n} F_n$$

clearly satisfies the hypothesis for every $1 \leq i \leq n$, for otherwise, by Lemma 3.4, there would be a flag path from $F_{i-1}$ to $F_i$ whose associated word is a splitting of $s_i$. Thus $F_0 \neq F_n$, so $M$ is simply connected.

Suppose now that $M$ is simply connected and consider a weak path $F_0 \xrightarrow{w_1} F_1 \dots F_{n-1} \xrightarrow{w_n} F_n$ as in the statement. Proposition 3.24 yields that this path has a reduction of length at least $n$. Since $M$ is simply connected, we conclude that $F_0 \neq F_n$. □

Simple connectedness allows us to generalise [2, Remark 4.9], which will be needed for the proof of Proposition 4.7.

**Lemma 3.28.** *Given two adjacent colours $\gamma$ and $\delta$, if $M$ is a simply connected $\Gamma$-space, the subgraph $\mathcal{A}_{\gamma,\delta}(M) = \mathcal{A}_\gamma(M) \cup \mathcal{A}_\delta(M)$ has no non-trivial circles.*

*Proof.* Since any edge in $\mathcal{A}_{\gamma,\delta}$ lies within a flag in $M$, by property (2) of Definition 2.16, a path with no repetitions in $\mathcal{A}_{\gamma,\delta}$ induces a (possibly non-reduced) flag path of the form

$$F_0 \xrightarrow{w_1} F_1 \xrightarrow{w_2} \dots F_n,$$

where the reduced words $w_1, \dots, w_n$ have the following properties:

(a) $\delta \in |w_{2k+1}| \subset \Gamma \setminus \{\gamma\}$,
(b) $\gamma \in |w_{2k}| \subset \Gamma \setminus \{\delta\}$.

By repeatedly applying Lemma 3.21 to each pair $w_i, w_j$ for $i \neq j$, it is easy to see that the above conditions remain in the reduct $w_1^* \cdots w_n^*$ of the word $w_1 \cdot w_n$. In particular, the word $w_1^* \cdots w_n^*$ is not trivial and thus $F_0 \neq F_n$. Hence, the original path in $\mathcal{A}_{\gamma,\delta}$ was not closed.

□

## 4. The Theory $\mathrm{PS}_\Gamma$

**Definition 4.1.** The theory $\mathrm{PS}_\Gamma$ is a collection of sentences stating that the underlying $\Gamma$-space satisfies the following properties:

(1) simple connectedness,
(2) for any colour $\gamma$ in $\Gamma$, the $\sim_\gamma$-class of any flag $G$ is infinite.

Axiom (1) is a first-order property, by Corollary 3.27. Clearly, so is Axiom (2). The $\Gamma$-space $\mathrm{M}_0(\Gamma)$, as defined on page 10, is a model of $\mathrm{PS}_\Gamma$ by Theorem 3.16, so $\mathrm{PS}_\Gamma$ is consistent.

The rest of this section is devoted to proving the completeness of $\mathrm{PS}_\Gamma$.

We first generalise [2, Definition 4.3].

**Definition 4.2.** Fix some letter $s$, and let $F$ be a flag in a $\Gamma$-graph $A$. Create a new flag $F^* = \{b^*_\gamma\}_{\gamma\in\Gamma}$ which agrees with $F$ on the colours of $\Gamma \setminus s$ but $b^*_\gamma \notin A$ for $\gamma \in s$. We define a $\Gamma$-graph

$$A(F^*)$$

with vertices $A \cup F^*$ and edges those of $A$ and of $F^*$. A $\Gamma$-graph $B \supset A$ is a *simple extension* of $A$ of *type* $s, F$ if it is $A$-isomorphic to $A(F^*)$.

Note that $F^* \underset{s}{\to} F$ by construction.

**Remark 4.3.** If $A$ is a $\Gamma$-space, then so is $A(F^*)$.

These simple extensions generalise those simple extensions defined after Remark 2.7, as the following easy remark shows.

**Remark 4.4.** Denote $\partial s = s \cup \{\delta \in \Gamma \mid \text{adjacent to some } \gamma \in s\}$ as the set of all $\gamma$ in $\Gamma$ not commuting with $s$. There is a 1-to-1-correspondence between the new flags $H^*$ of $A(F^*)$ and the flags $H$ of $A$ with $H \sim_{\Gamma\setminus\partial s} F$ given by

$$H^* \sim_{\Gamma\setminus\partial s} F^* \text{ and } H^* \sim_s H.$$

In order to prove that the theory $\mathrm{PS}_\Gamma$ is complete, we will need the appropriate interpretation of a strongly connected subset in this context.

**Definition 4.5.** A non-empty subgraph $D$ of a $\Gamma$-space $M$ is *nice* if it satisfies the following conditions:

- Any point $a$ in $D$ lies in a flag in $D$.
- Given flags $F$ and $G$ in $D$ and a letter $s$, if $F \xrightarrow{s} G$ in $D$, then $F \xrightarrow{s} G$ in $M$.

Any nice set is the union of all the flags contained in it. Niceness is a transitive property. Proposition 3.19 yields that a non-empty subset $D$ of a simply connected $\Gamma$-space $M$ is nice if and only if the following hold:

- Any point $a$ in $D$ lies in a flag in $D$.
- Given flags $F$ and $G$ in $D$ and a reduced word $u$, if $F \xrightarrow{u} G$ in $M$, then there exists such a path in $D$ with the same word.

In particular, if $D$ is nice in $M$, then $D$ is simply connected whenever $M$ is.

**Remark 4.6.** (1) The $\Gamma$-space $A$ is nice in $A(F^*)$.
(2) A nice subset of a $\Gamma$-space is itself a $\Gamma$-space.

*Proof.* For (1), given flags $G$ and $H$ in $A$ with $G \xrightarrow{t} H$ in $A$, suppose there is a splitting $x = t_1 \cdots t_n$ of $t$ such that

$$G = F_0 \underset{t_1}{\longrightarrow} F_1 \cdots F_{n-1} \underset{t_n}{\longrightarrow} F_n = H$$

in $A(F^*)$. We replace each $F_i$ by a flag $F'_i$ in $A$ as follows: If $F_i$ belongs to $A$, set $F'_i = F_i$. Otherwise, by Remark 4.4, the flag $F_i$ has the form $H^*_i$ and we set $F'_i = H_i$. Note that $F'_{i-1} \sim_{t_i} F'_i$, so $G$ and $H$ can be connected in $A$ be a weak flag path whose word is $\preceq$-smaller than $t_1 \cdots t_n$, contradicting $G \xrightarrow{t} H$.
In order to show (2), consider two elements $a$ and $b$ in a nice subset $A$ of $M$, connected by a flag $G$ in $M$. Choose flags $F$ and $H$ in $A$ containing $a$ and $b$ respectively. Let $\gamma$ be the colour of $a$ and $\delta$ the colour of $b$. Since $F \sim_{\Gamma\setminus\{\gamma\}} G \sim_{\Gamma\setminus\{\delta\}} H$, there is a reduced path $F \xrightarrow{u} G' \xrightarrow{v} H$ in $M$ such that $\gamma$ does not

occur in $u$ and $\delta$ does not occur in $v$. By niceness, we may therefore assume that $G'$ belongs to $A$. Clearly $G'$ contains $a$ and $b$. $\square$

We can now prove the analogue version of [2, Lemma 4.21].

**Proposition 4.7.** *Given a flag $F$ in a nice subset $A$ of a simply connected $\Gamma$-space $M$, and a flag $F^*$ in $M$ which is $s$-equivalent to $F$ for some letter $s$, the following are equivalent:*

*(a) The $\Gamma$-graph $A \cup F^*$ is a simple extension of $A$ and is nice in $M$.*
*(b) Whenever $G$ is a flag in $A$ and $x$ a splitting of $s$, then $F^* \underset{x}{\not\to} G$ in $M$.*

*Proof.* $(a) \to (b)$: Set $B = A \cup F^*$. If

$$F^* \underset{x}{\to} G$$

in $M$, then we cannot have that $F^* \xrightarrow{s} G$ in $B$, for $B$ is nice. Thus, there is a splitting $x'$ of $s$ such that $F^* \xrightarrow{x'} G$ in $B$. All flags in $B$ which are $s$-equivalent to $F$ are either $F^*$ itself or contained in $A$, by Remark 4.4, so $F^* \sim_t G'$ for some $G'$ in $A$, where $t \subsetneq s$ is the first letter of $x'$. This is impossible, for no vertex of $F^*$ with colour in $s$ lies in $A$.
$(b) \to (a)$: Since $F$ lies in $A$, the hypothesis implies that $F^* \xrightarrow{s} F$. We will first show that, for any flag $G$ in $A$, if $F^* \xrightarrow{u} G$ in $M$, then $s \preceq u$. We may assume that $u$ is reduced. Since $A$ is nice, there is a reduced path connecting $F \xrightarrow{v} G$ in $A$, which remains reduced in $M$, again by niceness of $A$. The word $u$ is thus a reduct of $s \cdot v$. If in the reduction splitting ever occurs, it produces a flag in $A$ which connects to $F^*$ by a splitting of $s$, contradicting the assumption. Thus, the reduction involves only commutation and possibly absorption of $s$ by $u$. Hence $s \preceq u$.

In particular, for any flag $G$ in $A$ with $F^* \xrightarrow{u} G$, there exists a letter $t$ in $u$ containing $s$. Actually, by Lemma 3.9 it suffices to assume $F^* \underset{u}{\to} G$.

In order to show that $B = A \cup F^*$ is a simple extension of $A$, we need to show that there is no new edge consisting of an element $b$ in $\mathcal{A}_\gamma(B) \setminus A$ and some $a$ in $\mathcal{A}_\delta(A)$. Otherwise, there exists a flag $F'$ in $M$ passing through $a$ and $b$. Take a flag $G$ in $A$ containing $a$.

Note that $\gamma$ is in $s$. Suppose first that $\delta$ lies in $s$ as well. Since $F^* \sim_{\Gamma\setminus\{\gamma\}} F' \sim_{\Gamma\setminus\{\delta\}} G$, we obtain reduced words $u$ and $v$ such that $\gamma$ does not occur in $u$, the colour $\delta$ does not occur in $v$ and

$$F^* \xrightarrow{u} F' \xrightarrow{v} G.$$

The reduction $F^* \xrightarrow{w} G$ in $M$ satisfies that every letter in $w$ does not contain either $\gamma$ or $\delta$, contradicting the previous discussion.

If $\delta$ does not lie in $s$, then, with the choice of flags as before, we obtain the following path

$$F \xrightarrow{s} F^* \xrightarrow{u} F' \xrightarrow{v} G$$

As before, since $A$ is nice, this implies that $F \xrightarrow{w} G$ in $A$, where each letter in $w$ avoids either $\gamma$ or $\delta$. This induces a path in $\mathcal{A}_{\gamma,\delta}(A)$ between $a'$ and $a$, where $a'$ is the $\delta$-vertex of $F$. which, together with the connection $a - b - a'$, yields a nont-trivial circle, contradicting Lemma 3.28.

Let us now show that $B$ is nice in $M$. Given $G_1 \xrightarrow{t} G_2$ flags in $B$, we distinguish the following cases:

- Both flags lie in $A$. Then $G_1 \xrightarrow{t} G_2$ also in $A$, and thus in $M$, since $A$ is nice.
- Both flags lie in $B \setminus A$. By Remark 4.4, we have $G_1 \sim_{\Gamma\setminus\partial s} G_2$ and whence $t \subset \Gamma \setminus \partial s$. Thus we find $H_1$ and $H_2$ in $A$ such that $H_1 \sim_{\Gamma\setminus\partial s} H_2$ and $G_i \sim_s H_i$ for $i = 1, 2$. This implies $H_1 \xrightarrow{t} H_2$ in $B$ and also in $A$. Therefore $H_1 \xrightarrow{t} H_2$ in $M$, for $A$ is nice, which implies that $G_1 \xrightarrow{t} G_2$ in $M$ as well.
- Exactly one flag, say $G_1$, lies in $B \setminus A$. Again by Remark 4.4, we have that $F^* \xrightarrow{w} G_1$ for a word $w$ which commutes with $s$ and a flag $H_1$ in $A$ with $G_1 \underset{s}{\rightarrow} H_1$. Since $F^* \underset{w\cdot t}{\longrightarrow} G_2$, some letter of $w \cdot t$ must contain $s$, so $s \subset t$. If $s = t$, we have $G_1 \xrightarrow{t} G_2$ in $M$ for the same reason. Otherwise, if $s \subsetneq t$, then $H_1 \xrightarrow{t} G_2$ in $B$. Since $A$ is nice, we have $H_1 \xrightarrow{t} G_2$ in $M$, which implies $G_1 \xrightarrow{t} G_2$ in $M$.

□

In particular, setting $s = \{\gamma\}$, for $\gamma$ in $\Gamma$, we deduce the following result.

**Corollary 4.8.** *Given a flag $G$ in a nice subset $A$ of a simply connected $\Gamma$-space $M$ and $\gamma$ in $\Gamma$, if the flag $F$ is $\{\gamma\}$-equivalent to $G$ and the $\gamma$-vertex of $F$ does not lie in $A$, then the set $B = A \cup F$ is nice and a simple extension of $A$ of type $\{\gamma\}, G$.*

The next proposition shows that a simply connected space is the increasing union of simple extensions of nice subsets. sets (*cf.* [2, Theorem 4.22]).

**Proposition 4.9.** *Given a nice subset $A$ of a simply connected $\Gamma$-space $M$ and $b$ in $M$, there exists a nice subset $B$ containing $b$ which can be obtained from $A$ by a finite number of simple extensions.*

*Proof.* Given a flag $F$ in $M$ containing $b$, choose a reduced path

$$F = F_0 \xrightarrow{s_1} F_1 \dots \xrightarrow{s_n} F_n$$

connecting $F$ with a flag $F_n$ in $A$ such that the word $u = s_1 \cdots s_n$ is $\prec$-minimal. We prove the claim by $\prec$-induction on $u$. If $u = 1$, there is nothing to show. Otherwise, minimality of $u$ implies that there is no path which connect $F_{n-1}$ to a flag in $A$ whose word is a splitting of $s_n$. It follows from Proposition 4.7 that $A' = A \cup F_{n-1}$ is a simple extension of $A$ of type $s_n, F_n$, so $A'$ is nice in $M$. Now $F$ can be connected to some flag in $A'$ by a reduced word path, whose word $u'$ is $\prec$-minimal such, with $u' \preceq s_1 \cdots s_{n-1} \prec u$. By induction, the element $b$ is contained in a nice set $B$ which can be obtained from $A'$ (and thus, from $A$) by a finite number of simple extensions. □

**Lemma 4.10.** *Let $s$ be a letter in $\Gamma$ which is not a singleton. There are $\gamma$ and $\gamma'$ in $s$ distinct such that both $s \setminus \{\gamma\}$ and $s \setminus \{\gamma'\}$ are connected.*

*Proof.* By repeatedly removing edges, it is enough to prove it for *a spanning tree* $s$, that is, whenever we remove an edge between two points in $s$, the resulting graph is no longer connected. In particular, such an $s$ contains no cycles. The assertion now follows, since any non-trivial tree has at least two extremal points. □

**Corollary 4.11.** *Given a letter $s$ and a flag $G$ contained in some finite nice subset $A$ of an $\omega$-saturated model $M$ of $\mathrm{PS}_\Gamma$, then $M$ contains a simple extension of $A$ of type $s, G$ which is nice in $M$.*

*Proof.* For $s = \{\gamma\}$, pick any flag $G^*$ which is $\gamma$-equivalent to $G$ and its $\gamma$-vertex does not lie in (the finite set) $A$, by property (2). The set $A \cup G^*$ is nice in $M$ and a simple of $A$ type $s, G$, by Corollary 4.8.

Suppose now that $|s| \geq 2$. By Proposition 4.7 and saturation, it is enough to produce, for every $n$, a flag $G_n$ with $G_n \underset{s}{\to} G$ and $G_n \underset{x}{\not\to} G'$, whenever $G'$ is a flag in $A$ and $x$ a splitting of $s$ of length at most $n$.

By Lemma 4.10, find two subletters $s_0$ and $s_1$ of $s$ of cardinality $|s| - 1$ and not commuting with each other. Since $A$ contains only finitely many flags, simple connectedness yields an upper bound $N$ for the length of the word of any reduced flag path between any two flags in $A$. Set $A_0 = A$ and $G_0 = G$. By induction on $|s|$, there is a sequence of pairs $\{(G_i, A_i)\}_{i \leq N+n}$ such that $G_i$ is a flag in the (finite) nice set $A_i$ and $A_{i+1} = A_i \cup G_{i+1}$ is a simple extension of type $s_{f(i)}, G_i$, where $f(i)$ in $\{0, 1\}$ is the residue of $i$ modulo 2. The flag path

$$G_{N+n} \xrightarrow{s_{f(N+n-1)}} G_{N+n-1} \dots G_1 \xrightarrow{s_0} G_0$$

is reduced, since $s_0$ and $s_1$ do not commute.

Clearly $G_{N+n} \underset{s}{\to} G_0$. If $G_{N+n} \underset{x}{\to} G'$ in $A$ for some splitting $x$ of $s$ of length at most $n$, reduction would yield a reduced path $G \xrightarrow{u} G'$, where $u$ is also a splitting of $s$. By niceness of $A$, we may assume that this path lies in $A$, so $u$ has length at most $N$. Proposition 3.19 implies that $x \cdot u \xrightarrow{*} s_{f(N+n-1)} \cdots s_0$. However, at every step of the reduction, the number of letters of size exactly $|s| - 1$ is bounded by $N + n$, contradicting our choice of $s_{f(N+n-1)} \cdots s_0$. □

We can now conclude that the theory $\mathrm{PS}_\Gamma$ is complete and that the type of a nice set is determined by its quantifier-free type.

**Theorem 4.12.** *Any two $\omega$-saturated models of $\mathrm{PS}_\Gamma$ have the back-and-forth property with respect to the collection of partial isomorphisms between finite nice substructures. In particular, any partial isomorphism $f : A \to A'$ between two finite nice subsets of two models of $\mathrm{PS}_\Gamma$ is elementary. The theory $\mathrm{PS}_\Gamma$ is complete.*

*Proof.* Let $M$ and $M'$ be two $\omega$-saturated models and let $f : A \to A'$ be a partial isomorphism between two finite nice substructures. Given $b$ in $M$, Proposition 4.9 yields a nice subset $B$ containing $A \cup \{b\}$ such that $A \leq B$ in finitely many steps. By $\omega$-saturation of $M'$ and Corollary 4.11 (finitely many times), we obtain a nice subset $B'$ of $M'$ containing $A'$ such that $f$ extends to isomorphism between $B$ and $B'$.

Since any model $M$ is nice in any elementary extension, replacing the models by appropriate saturated extensions, we produce a back-and forth system. Completeness of $\mathrm{PS}_\Gamma$ then follows, since any two flags have the same quantifier-free type. □

**Corollary 4.13.** *The type of a nice set $A$ is determined by its quantifier-free type.*

*Proof.* For finite sets, this follows from Theorem 4.12. For infinite nice sets, note that they are direct unions of finite nice subsets. □

**Corollary 4.14.** *The theory* $\mathrm{PS}_\Gamma$ *is* $\omega$*-stable and the model* $\mathrm{M}_0(\Gamma)$ *is the unique (up to isomorphism) countable prime model.*

*Proof.* It suffices to count 1-types over a countable subset of $A$, which we may assume nice inside a given saturated model. Every simple extension of $A$ is uniquely determined, up to $A$-isomorphism, by its type $s, G$, by Corollary 4.13. Therefore, if $A$ is countable, there are, up to $A$-isomorphism, only countably many simple extensions.

Now, every 1-type is realised in a finite tower of simple extensions over $A$, by Proposition 4.9, so there are only countably many types.

Given $\gamma$ in $\Gamma$ and a flag $G$ in a nice subset $A$, the type over $A$ of the simple extension $B = A \cup F$ of type $\{\gamma\}, G$ is determined by its quantifier-free type, which amounts to saying, by Corollary 4.8, that $F$ and $G$ are $\gamma$-equivalent but that the $\gamma$-vertex of $F$ does not lie in $A$. Therefore, if $A$ is finite, the type of $B$ over $A$ is isolated. Thus, the countable model $\mathrm{M}_0(\Gamma)$ is constructible and hence is the prime model. □

Uniqueness of prime models and Theorem 2.17 yield the uniqueness result in Corollary 2.11.

**Remark 4.15.** Let $M$ be an $\omega$-saturated model of $\mathrm{PS}_\Gamma$. For every word $v$, there is a path $F \xrightarrow{v} G$ in $M$. Whenever a (possibly non-reduced) word $u$ can be reduced to $v$, then there is a flag path from $F$ to $G$ with word $u$.

## 5. Non-splitting reductions

In order to describe the geometrical complexity of $\mathrm{PS}_\Gamma$, we will need several auxiliary results on the combinatorics of reduction of words when no splitting occurs, generalising some of the results of [2]. For the sake of self-containment, we will provide, whenever possible, different proofs.

**Definition 5.1.** A letter $s$ is a *beginning* (resp. *end*) of the word $u$ if $u \approx s \cdot v$ (resp. $u \approx v \cdot s$). The *initial segment* (resp. *final segment*) of $u$ is the commuting subword whose letters are beginnings (resp. ends) of $u$.

By abuse of the language, we say that the word $u$ is an *initial subword* of $v$ if $u$ is an initial subword (in the proper sense) of some permutation of $v$. Likewise for *final subword*. The initial segment of $v$ is the largest commuting initial subword of $v$. Inductively on the sum of their lengths, it is easy to see that any two words $u$ and $v$ have a *largest common initial subword*, resp. a *largest common final subword*, none of which need be commutative.

Common initial subwords can be removed, as seen easily.

**Lemma 5.2.** *If* $u \cdot v' \approx u \cdot v''$*, then* $v' \approx v''$*.*

**Lemma 5.3.** *Let* $a$ *be a final subword of* $c \cdot d$ *such that every end of* $a$ *commutes with* $d$*. Then* $a$ *and* $d$ *commute and* $a$ *is a final subword of* $c$*.*

*Proof.* Proceed by induction on the length of $a$. If $a$ is the trivial word, there is nothing to show. Otherwise, write $a = s \cdot a_1$. By induction, the subword $a_1$ commutes with $d$ and is a final subword of $c \approx c' \cdot a_1$. Since $c \cdot d \approx c' \cdot d \cdot a_1$, Lemma 5.2 implies that $s$ is an end of $c' \cdot d$.

If $s$ occurs in $d$, then $s$ and $a_1$ commute, so $s$ is an end of $a$ and hence it must commute with $d$, by hypothesis, which is a contradiction. Therefore, the word $s$

must occur in $c'$, so in particular it must commute with $d$. Therefore, the word $a$ commutes with $d$ and is a final subword of $c$, as desired. □

**Definition 5.4.** A *non-splitting reduction* of the word $u$, denoted by $u \to v$, is a reduced word $v$ obtained from $u$ by a reduction process, where only COMMUTATION and ABSORPTION occur (*cf.* Definition 3.10).

By induction on the length of the reduction $u \to v$, the following can be easily shown.

**Lemma 5.5.** *If $v$ is a non-splitting reduction of $u$, then every $u' \preceq u$ has a non-splitting reduction $v' \preceq v$.*

**Corollary 5.6** (*cf.* [2, Corollary 5.3])**.** *Every word admits exactly one non-splitting reduction, up to permutation.*

*Proof.* Assume that $v_1$ and $v$ are two non-splitting reducts of $u$. In particular $v_1 \preceq u$, so $v_1 \to v'$ for some $v' \preceq v$, by Lemma 5.5. Thus $v'$ must be equivalent to $v_1$, and hence $v_1 \preceq v$. Similarly, we obtain $v \preceq v_1$, so $v_1 \approx v$. □

**Notation.** Given reduced words $u$ and $v$, we denote by $[u \cdot v]$ the *non-splitting reduct* of $u \cdot v$, which is defined up to permutation.

**Corollary 5.7** (*cf.* [2, Lemma 5.29])**.** *Given reduced words $u$, $v$ and $x$ with $x \preceq v$, then $[u \cdot x] \preceq [u \cdot v]$.*

**Corollary 5.8.** *If $u$ and $v$ are reduced words, then $u \preceq [u \cdot v]$.* □

**Proposition 5.9.** *(Symmetric Decomposition Lemma,* cf. [2, Corollary 5.23]*) Given two reduced words $u$ and $v$, there are unique decompositions (up to permutation):*

$$u = u_1 \cdot u' \cdot w \qquad\qquad w \cdot v' \cdot v_1 = v,$$

*such that:*

*(a) $w$ is a commuting word,*
*(b) $u'$ is properly left-absorbed by $v_1$,*
*(c) $v'$ is properly right-absorbed by $u_1$,*
*(d) $u'$, $w$ and $v'$ pairwise commute,*
*(e) $u_1 \cdot w \cdot v_1$ is reduced.*

*Furthermore,*

$$[u \cdot v] = u_1 \cdot w \cdot v_1.$$

*Proof.* We show the existence of such a decomposition by induction on the sum of the lengths of $u$ and $v$. If $u \cdot v$ is already reduced, then set $u_1 = u$ and $v_1 = v$, and define $v'$, $u'$ and $w$ to be the trivial word.

Otherwise, up to permutation and changing the roles of $u$ and $v$ , we may assume that $u = \tilde{u} \cdot s$, where $s$ is left-absorbed by $v$. In particular $[s \cdot v] = v$. By induction, we find words $\tilde{u}_1$, $\tilde{u}'$, $\tilde{w}$, $\tilde{v}'$ and $\tilde{v}_1$ with the desired properties such that

$$\tilde{u} = u_1 \cdot \tilde{u}' \cdot \tilde{w} \qquad\qquad \tilde{w} \cdot v' \cdot \tilde{v}_1 = v.$$

Observe that $s$ cannot be left-absorbed by $\tilde{w}$ nor by $\tilde{v}'$, for $u$ is reduced. Thus $s$ commutes with $\tilde{w} \cdot \tilde{v}'$ and is absorbed by $\tilde{v}_1$. There are two possibilities:

(1) The letter $s$ is properly absorbed by $\tilde{v}_1$. Set $u' = \tilde{u}' \cdot s$, $w = \tilde{w}$ and $v_1 = \tilde{v}_1$.

(2) Up to permutation, the letter $s$ is a beginning of $\tilde{v}_1 = s \cdot v_1^*$. Set $u' = \tilde{u}'$, $w = \tilde{w} \cdot s$ and $v_1 = v_1^*$.

Let us finish by showing the uniqueness of the above decomposition. Note first that $[u \cdot v] = u_1 \cdot w \cdot v_1$, since $u \cdot v \to u_1 \cdot w \cdot v_1$. The word $w$ is exactly the intersection of the final segments of $u$ and $v^{-1}$, since $u_1 \cdot w \cdot v_1$ is reduced. Furthermore, the word $u_1 \cdot w$ is the largest common initial subword of $u$ and $[u, v]$, for otherwise, there exists a letter $s$ with $v_1 = s \cdot \tilde{v}_1$ and $u' = s \cdot \tilde{u}'$, contradicting that $u'$ is properly left-absorbed by $v_1$. Similarly, the word $w \cdot v_1$ is the largest common final subword of $v$ and $[u, v]$, providing the desired result. □

**Corollary 5.10** (*cf.* [2, Corollary 5.14])**.** *Let $u$ and $v$ be reduced words. Then $u$ is left-absorbed by $v$ if and only if $[u \cdot v] = v$.*

*Proof.* Note that $[u \cdot v] = v$ if and only if $u_1 \cdot w \cdot v_1 \approx v' \cdot w \cdot v_1$. Lemma 5.2 implies $u_1 \approx v'$. So $u_1$ is properly right-absorbed by itself, which can only happen if $u_1 = 1$. Thus $u \approx u' \cdot w$ is left-absorbed by $v$. □

**Corollary 5.11** (*cf.* [2, Corollary 5.22])**.** *A reduced word is commuting if and only if it left-absorbs itself.*

*Proof.* Suppose first that we have two reduced words $u$ and $v$, such that $u$ right-absorbs $v$ and $v$ left-absorbs $u$. The proof of 5.10 yields both $u_1 = v' = 1 = v_1 = u'$, so $u = v = w$ is commuting. The statement now follows easily, since Corollary 5.10 yields that if $u$ left-absorbs itself, then it also right-absorbs itself. □

**Corollary 5.12.** *If the reduced word $u$ is left-absorbed by the reduced word $v$, we can write (up to permutation) :*

$$u = u' \cdot w \qquad\qquad w \cdot v_1 = v,$$

*such that*

*(1) $w$ is a commuting word,*
*(2) $u'$ is properly left-absorbed by $v_1$,*
*(3) $u'$ and $w$ commute.*

□

**Corollary 5.13.** *Given reduced words $u$, $v$ and $x$ such that $[u \cdot v] = u \cdot x$, then $x$ is a final subword of $v$. In particular, the word $x$ commutes with any word commuting with $v$.*

*Proof.* Decompose

$$u = u_1 \cdot u' \cdot w \qquad\qquad w \cdot v' \cdot v_1 = v,$$

as in Proposition 5.9. Hence $u_1 \cdot w \cdot v_1 = [u \cdot v] = u_1 \cdot w \cdot u' \cdot x$, so $v_1 \approx u' \cdot x$, by Lemma 5.2. Thus $u' = 1$ and $x$ is a final subword of $v$, as desired. □

**Lemma 5.14.** *If the reduced word $u$ is left-absorbed by $v_1 \cdot v_2$, then $u \approx u_1 \cdot u_2$, where each $u_i$ is left-absorbed by $v_i$, and $u_2$ commutes with $v_1$ (and therefore with $u_1$).*

*Proof.* If $u$ is empty, there is nothing to show. Otherwise, write $u = u' \cdot s$. By induction on the length of $u$, obtain a decomposition $u' \approx u_1' \cdot u_2'$, such that $u_i'$ is absorbed by $v_i$ and $u_2'$ commutes with $v_1$. We distinguish two cases: If $s$ is absorbed by $v_1$, then $u_2'$ commutes with $s$, so decompose $u \approx (u_1' \cdot s) \cdot u_2'$. Otherwise, the letter $s$ commutes with $v_1$ and is left-absorbed by $v_2$. Set $u \approx u_1 \cdot (u_2 \cdot s)$. □

**Corollary 5.15.** *Given a reduced word $u \approx u_1 \cdot \tilde{u}$, where $\tilde{u}$ denotes the final segment of $u$, then $[u \cdot u^{-1}] = u_1 \cdot \tilde{u} \cdot {u_1}^{-1}$.*

*Proof.* It suffices to show that the word $u_1 \cdot \tilde{u} \cdot {u_1}^{-1}$ is reduced. Otherwise, there is an end $s$ of $u_1$ which commutes with $\tilde{u}$. In that case, the letter $s$ is an end of $u$ and hence it occurs in $\tilde{u}$, which is a contradiction. □

For the proof of the next lemma, we will require the following notation: Recall (*cf.* Remark 4.4) that $\Gamma \setminus \partial s$ is the set of those colours commuting with $s$. For a letter $t$, denote by $\mathrm{C}_t(s)$ the commuting word with support $t \setminus \partial s$ (*cf.* Remark 3.2). For a word $v = t_1 \cdots t_n$, set

$$\mathrm{C}_v(s) = \mathrm{C}_{t_1}(s) \cdots \mathrm{C}_{t_n}(s).$$

**Lemma 5.16.** *(Division Lemma) Given reduced words $u \preceq v$, there exists a reduced word $w$, unique up to permutation, such that for every reduced word $x$,*

$$[x \cdot u] \preceq v \Leftrightarrow x \preceq w.$$

We denote it by $v/u$. Since $v/u \preceq v/u$, Corollary 5.8 implies that $v/u \preceq v$. Furthermore, since $1 \preceq v/u$, the condition $u \preceq v$ is necessary for the existence of $v/u$.

*Proof.* Note that we may assume that $u$ consists of a single letter: given $u = u_1 \cdot u_2$, suppose that the statement holds for both $u_1$ and $u_2$. Since $u \preceq v$, we have $u_2 \preceq v$, so $v/u_2$ exists. Now $u_1 \cdot u_2 \preceq v$ implies $u_1 \preceq v/u_2$. Thus, set $v/u = (v/u_2)/u_1$, which exists. Note that

$$[x \cdot u] = [[x \cdot u_1] \cdot u_2] \preceq v \Leftrightarrow [x \cdot u_1] \preceq v/u_2 \Leftrightarrow x \preceq v/u,$$

as desired. Therefore, assume $u = s \preceq v$.

We can write $v = v_1 \cdot t \cdot v_2$, such that $s \subset t$ and no letter in $v_2$ contains $s$. Set $w = [v_1 \cdot t \cdot \mathrm{C}_{v_2}(s)]$. No letter in $v_1 \cdot t$ is left-absorbed during the non-splitting reduction $v_1 \cdot t \cdot \mathrm{C}_{v_2}(s) \to w$, for $v$ is reduced.

Clearly $[w \cdot s] = w \preceq v$. Given $x \preceq w$ reduced, Corollary 5.7 implies that $[x \cdot s] \preceq [w \cdot s] \preceq v$, which gives one implication. For the other, assume that $[x \cdot s] \preceq v$. We distinguish two cases : if $s$ is absorbed by $x$, then $x = [x \cdot s] \preceq v = v_1 \cdot t \cdot v_2$, so we may decompose $x \approx x_1 \cdot x_2$, where $x_1 \preceq v_1 \cdot t \cdot$ and $x_2 \preceq v_2$. Since $x_2$ does not right-absorb $s$, Lemma 5.14 implies that $x_2$ commutes with $s$, which is right-absorbed by $x_1$. This implies $x_2 \preceq \mathrm{C}_{v_2}(s)$ and thus $x \preceq w$.

If $s$ is not absorbed by $x$, then Proposition 5.9 yields a decomposition $x \approx x_1 \cdot x'$, where $x'$ is properly absorbed by $s$ and $[x \cdot s] = x_1 \cdot s$. The word $x_1 \cdot s$ right-absorbs $s$, so $x_1 \cdot s \preceq w$ by the previous discussion. Since $x' \prec s$, we conclude that $x \approx x_1 \cdot x' \preceq w$, as desired. □

**Definition 5.17.** According to Lemma 5.16, we denote by $\mathcal{S}_{\mathrm{R}}(u)$ the largest reduced word, unique up to permutation, such that for every reduced word $x$,

$$[u \cdot x] \preceq u \Leftrightarrow x \preceq \mathcal{S}_{\mathrm{R}}(u).$$

**Corollary 5.18.** *Given a reduced word $u$, the word $\mathcal{S}_{\mathrm{R}}(u) \preceq u$ is commutative. A reduced word $x$ is right-absorbed by $u$ if and only if $x \preceq \mathcal{S}_{\mathrm{R}}(u)$.*

*In particular, if $\tilde{u}$ denotes the final segment of $u$, then $\tilde{u} \preceq \mathcal{S}_{\mathrm{R}}(u)$.*

*Proof.* Note that $x$ is right-absorbed by $u$ if and only if $[u \cdot x] = u$. Since $u \preceq [u \cdot x]$, by Corollary 5.8, this is equivalent to $[u \cdot x] \preceq u$, that is $x \preceq \mathcal{S}_{\mathrm{R}}(u)$.

In particular, the word $\mathcal{S}_{\mathrm{R}}(u)$ is right-absorbed by $u$. Thus $[u \cdot [\mathcal{S}_{\mathrm{R}}(u) \cdot \mathcal{S}_{\mathrm{R}}(u)]] = [[u \cdot \mathcal{S}_{\mathrm{R}}(u)] \cdot \mathcal{S}_{\mathrm{R}}(u)] = [u \cdot \mathcal{S}_{\mathrm{R}}(u)] = u$, so $\mathcal{S}_{\mathrm{R}}(u) \preceq [\mathcal{S}_{\mathrm{R}}(u) \cdot \mathcal{S}_{\mathrm{R}}(u)] \preceq \mathcal{S}_{\mathrm{R}}(u)$, which implies that $\mathcal{S}_{\mathrm{R}}(u)$ is commutative, by Corollary 5.11. □

## 6. Forking

We work inside a sufficiently saturated model $M$ of the theory $\mathrm{PS}_\Gamma$, which is $\omega$-stable, by Corollary 4.14. In particular, Morley rank induces an ordinal-valued dimension function on types. Given subsets $A$, $B$ and $C$ of $M$, we say that $A$ is *independent from $B$ over $C$* or $\mathrm{tp}(A/B)$ *does not fork over $C$*, denoted by

$$A \underset{C}{\downarrow} B,$$

if $\mathrm{RM}(\mathrm{tp}(A/BC)) = \mathrm{RM}(\mathrm{tp}(A/C))$. Among many other properties, this notion of independence is invariant under automorphisms of $M$, symmetric and transitive (*cf.* [16]).

In a similar fashion as in [2, Section 7], we will describe non-forking over nice sets and canonical bases in $\mathrm{PS}_\Gamma$. We will also show that this theory, which has weak elimination of imaginaries, has trivial forking and furthermore is totally trivial, as defined in [5].

Recall the terminology introduced in Definition 3.20: A word $u$ connects $F$ to $G$ if it is the word of a reduced path from $F$ to $G$. This word is unique, up to permutation, and denoted by $\mathrm{w}(F, G)$.

The following result describes the type of a flag over a nice set and will help us to determine the nature of non-forking.

**Proposition 6.1.** *Given a flag $F$ and a reduced path with word $u$ which connects $F$ to a flag $G$ lying in a nice set $D$, the following are equivalent:*

*(a) For any flag $G'$ in $D$,*

$$\mathrm{w}(F, G') = [u \cdot \mathrm{w}(G, G')],$$

*that is, the word connecting $F$ to $G'$ is equivalent to $[u \cdot v]$, the non-splitting reduct of $u \cdot v$, where $v$ is the reduced word connecting $G$ to $G'$.*

*(b) The word $u$ is the $\preceq$-smallest word connecting $F$ to some flag in $D$.*

*(c) The word $u$ is $\preceq$-minimal among words connecting $F$ to some flag in $D$.*

This generalisation of [2, Proposition 7.2 ] has essentially the same proof.

*Proof.* (a)→(b) follows from Corollary 5.8. The implication (b)→(c) is trivial.

For (c)→(a), let $v$ be the word of a reduced path from $G$ to some flag $G'$ in $D$. By niceness, we may assume that the path is fully contained in $D$. Choose a decomposition $u = u_1 \cdot u' \cdot w$ and $w \cdot v' \cdot v_1 = v$, as in Proposition 5.9, with corresponding paths

$$F \xrightarrow{u_1 \cdot u'} F_1 \xrightarrow{w} G \xrightarrow{w} G_1 \xrightarrow{v' \cdot v_1} G',$$

where $G_1$ is some flag in $D$. The word $b$ connecting $F_1$ to $G_1$ is a reduct of $w \cdot w$, so $b \preceq w$. The word $c$ connecting $F$ with $G_1$ is hence a reduct of $u_1 \cdot u' \cdot b$, so

$$c \preceq u_1 \cdot u' \cdot b \preceq u_1 \cdot u' \cdot w \preceq u.$$

Minimality of $u$ implies that $c \approx u$, so $b \approx w$. Hence, the reduction of the path $F \to G'$ has word $u_1 \cdot w \cdot v_1 = [u \cdot v]$. □

**Definition 6.2.** A *base-point* of the flag $F$ over the nice set $D$ is a flag $G$ in $D$ such that any of the conditions of Proposition 6.1 hold.

Recall Definition 5.17 of $\mathcal{S}_{\mathrm{R}}(u)$. We easily conclude the following.

**Corollary 6.3.** *Let $G$ be a base-point of $F$ over the nice set $D$ and $u$ the word which connects $F$ to $G$. Let $v$ be a reduced word connecting $G$ to some flag $G_1$ in $D$. The flag $G_1$ is a base-point of $F$ over $D$ if and only if $v$ is right-absorbed by $u$ if and only if $u = [u \cdot v]$ if and only if $v \preceq \mathcal{S}_{\mathrm{R}}(u)$.*

**Lemma 6.4.** *(*cf. [2, Lemma 7.4]*) Let $G$ be a base-point of $F$ over the nice set $D$ and denote by $P$ a reduced path $F = F_0 \cdots F_n = G$ with word $u$. Then $D \cup P$ is nice and the extension $F_0 \cdots F_n/D$ is uniquely determined, up to isomorphism, by $G$ and $u$.*

*Proof.* If $u$ is trivial, there is nothing to show. Otherwise, write $u = u' \cdot s$. Minimality of $u$ yields that no splitting of $s$ can connect $F_{n-1}$ to a flag in $D$. Thus Proposition 4.7 implies that the set $D' = D \cup F_{n-1}$ is nice and a simple extension of $D$ of type $s, G$.

We will now show that $F_{n-1}$ is a base-point of $F$ over $D'$, by Proposition 6.1 (c). Otherwise, there is a reduced word $v \prec u'$ which connects $F$ to a flag $H'$ in $D'$. By minimality of $u$, the flag $H'$ cannot lie in $D$. By Remark 4.4, there exists some flag $H$ in $D$ such that $H' \xrightarrow{s} H$. This gives a reduced path from $F$ to $H$ whose word is some reduction of $v \cdot s$, contradicting the minimality of $u$.

By induction, if $P'$ denotes the subpath $F = F_0 \cdots F_{n-1}$, then the set $D' \cup P'$ is nice and the extension $F_0 \cdots F_{n-1}/D'$ is uniquely determined, up to isomorphism, by $G'$ and $u'$. Therefore, the extension $F_0 \cdots F_n/D$ is uniquely determined, up to isomorphism, by $G$ and $u$. □

**Corollary 6.5.** *Given a reduced word $u$ and a flag $G$ in a nice set $D$, there is a flag $F$ such that $u$ connects $F$ to $G$, which is a base-point of $F$ over $D$. The type of $F$ over $D$ is uniquely determined by $G$ and $u$.*

Recall that the word $\mathrm{w}(F, G)$ is determined only up to a permutation. So the $\mathrm{tp}(F/D)$ depends only on the equivalence class of $u$.

*Proof.* Observe that, if such a flag $F$ exists and $P$ denotes the reduced path from $F$ to $G$ with word $u$, Lemma 6.4 and Corollary 4.13 imply that the type of $P$ over $D$ is uniquely determined by $G$ and $u$.

Thus, we need only show existence of such a flag $F$, by induction on the length of $u$. If $u = 1$, there is nothing to do. Otherwise, write $u = s \cdot u'$ and choose a flag $F'$ connecting to $G$ by $u'$ such that $G$ is a base-point of $F'$ over $D$. Let $P'$ denote the reduced path $F' \xrightarrow{u'} G$. By Lemma 6.4, the set $D' = D \cup P'$ is nice. Corollary 4.11 yields a simple extension $D' \cup F$ of type $s, F'$. Proposition 4.7 implies that $F'$ is base-point of $F$ over $D'$. We need only show that $G$ is a base-point of $F$. Hence, let $G'$ be an arbitrary flag in $D \subset D'$. We have

$$\mathrm{w}(F, G') = [s \cdot \mathrm{w}(F', G')] = [s \cdot [u' \cdot \mathrm{w}(G, G')]] = [u \cdot \mathrm{w}(G, G')],$$

as desired. □

If we denote the type of $F$ over $G$, resp. over $D$, by

$$\mathrm{p}_u(G) \text{ resp. } \mathrm{p}_u(G)|D,$$

we conclude the following.

**Proposition 6.6.** *Let $G$ be a flag in the nice set $D$ and $u$ a reduced word, then $\mathrm{p}_u(G)|D$ is the unique non-forking extension of $\mathrm{p}_u(G)$ to $D$.*

*Proof.* We may replace $D$ by a sufficiently saturated elementary substructure containing it, which is again nice in $M$. Observe that $\mathrm{p}_u(G)|D$ is *invariant over* $G$, that is, every automorphism of $D$ fixing the flag $G$ fixes $\mathrm{p}_u(G)|D$ as a collection of formulae. Thus, the type $\mathrm{p}_u(G)|D$ does not divide over $G$, so $\mathrm{p}_u(G)|D$ is a non-forking extension of $\mathrm{p}_u(G)$. Uniqueness now follows since the collection of non-forking extensions of $\mathrm{p}_u(G)$ to $D$ are conjugate under the automorphisms of $D$ fixing $G$. □

**Corollary 6.7.** *(*cf. [2, Lemmata 7.4 and 7.6]*) Given a flag $F$ and a nice set $D$, the flag $G$ in $D$ is a base-point of $F$ over $D$ if and only if $F \mathop{\smile\!\!\!\!\!\!\mid}_G D$.*

Recall that a type $p$ is *stationary* if it has a unique non-forking extension to a global type $\mathbf{p}$ over $M$. Types over elementary substructures are always stationary in a stable theory. The *canonical base* $\mathrm{Cb}(p)$ of a stationary type $p$ is some set, fixed pointwise by exactly those automorphisms $\alpha$ of $M$ fixing the global non-forking extension $\mathbf{p}$ of $p$. If $\mathrm{Cb}(p)$ exists, then it is unique, up to interdefinability, and $\mathbf{p}$ is the unique non-forking extension to $M$ of its restriction to $\mathrm{Cb}(p)$. Canonical bases exist as *imaginary* elements in the expansion $T^{\mathrm{eq}}$ of an $\omega$-stable theory $T$.

As in Remark 3.2, given a reduced word $u$, we do not distinguish between the word $\mathcal{S}_{\mathrm{R}}(u)$ and its support. For a flag $G$, the class of $G$ modulo $\mathcal{S}_{\mathrm{R}}(u)$ can be identified with a subset of the real sort, namely with the set of vertices of $G$ whose colours do not belong to $\mathcal{S}_{\mathrm{R}}(u)$. To simplify the notation, we will denote this class by $G/\mathcal{S}_{\mathrm{R}}(u)$.

**Corollary 6.8.** *The class $G/\mathcal{S}_{\mathrm{R}}(u)$ is a canonical base of $\mathrm{p}_u(G)$.*

*Proof.* Two types $\mathrm{p}_u(G)$ and $\mathrm{p}_u(G_1)$ have a common global non-forking extension if and only if $\mathrm{p}_u(G)|D = \mathrm{p}_u(G_1)|D$ for some nice set $D$ which contains $G$ and $G_1$. This is equivalent, by Corollary 6.3, to $\mathrm{w}(G, G_1) \preceq \mathcal{S}_{\mathrm{R}}(u)$, which is equivalent to $G_1 \sim_{\mathcal{S}_{\mathrm{R}}(u)} G$. □

**Definition 6.9.** Given reduced words $u$ and $v$, we say that $u$ is a *proper left-divisor* of $v$ if $u \not\approx v$ and there is a reduced word $w$ such that $[u \cdot w] = v$.

If $u$ is a proper left-divisor of $v$, it follows that $u \prec v$. In particular, being a proper left-divisor is a well-founded relation. Let $\mathrm{R}_{\mathrm{div}}$ be its foundation rank, and likewise let $\mathrm{R}_{\prec}$ denote the foundation rank of reduced words with respect to $\prec$.

The foundation rank on types associated to the forking relation is called *Lascar rank*, denote by U-rank. In particular, the type $p$ over the set $A$ has Lascar rank at least $\alpha + 1$ if there is an extension $q \supset p$ over some superset $B \supset A$ which forks over $A$ and $\mathrm{U}(q) \geq \alpha$.

The following result can be proved exactly as [2, Theorem 7.10 and Lemma 7.11]

**Lemma 6.10.** *For every flag $G$ and every reduced word $u$,*

$$\mathrm{U}(\mathrm{p}_u(G)) = \mathrm{R}_{\mathrm{div}}(u) \leq \mathrm{RM}(\mathrm{p}_u(G)) \leq \mathrm{R}_{\prec}(u).$$

In general $\mathrm{R}_{\mathrm{div}}(u)$, $\mathrm{RM}(\mathrm{p}_u(G))$ and $\mathrm{R}_{\prec}(u)$ need not agree (*cf.* [2, Remark 7.14]). They coincide however in the following special case, the proof of which is a straightforward modification of the proof of [2, Lemma 7.12], together with Lemma 4.10.

**Lemma 6.11.** *(*cf. [2, Corollary 7.13]*) For every reduced word $u = s_1 \cdots s_n$, with $|s_i| \geq |s_{i+1}|$ for $i = 1, \ldots, n-1$,*

$$\mathrm{R}_{\mathrm{div}}(u) = \mathrm{R}_{\prec}(u) = \omega^{|s_1|-1} + \cdots + \omega^{|s_n|-1}.$$

**Remark 6.12.** The Morley rank of the type $\mathrm{p}_u(G)$ can be easily computed thanks to the following observation: The rank of $\mathrm{p}_u(G)$ is strictly larger than $\alpha$ if and only if either

(1) the word $u$ has a proper left-divisor $v$ such that $\mathrm{p}_v(G)$ has at least rank $\alpha$,

or

(2) the type $\mathrm{p}_u(G)$ is an accumulation point of a family of types $\mathrm{p}_v(G)$, each of rank at least $\alpha$.

**Corollary 6.13.** *The theory $\mathrm{PS}_\Gamma$ is $\omega$-stable of Morley rank $\omega^{K-1}$, where $K$ is the cardinality of a connected component of $\Gamma$ of largest size.*

*Proof.* Decompose $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$ into its connected components. Similarly as in Corollary 8.4, it is easy to see that each restriction $M_i = \mathcal{A}_{\Gamma_i}(M)$ is a model of $\mathrm{PS}_{\Gamma_i}$. The structure $M$ can be considered as the disjoint union of the structures $M_i$'s, so the Morley rank of $M$ is the maximum of the Morley ranks of the structures $M_i$. We may therefore assume that $\Gamma$ is connected.

Given any vertex $a$, choose a flag $F$ containing $a$ as well as a flag $G$ independent from $F$ over $\emptyset$. If $\mathrm{p}_u(G)$ is the type of $F$ over $G$, then the word $u$ must be equal to $\Gamma$, since the canonical base $G/\mathcal{S}_{\mathrm{R}}(u)$ is algebraic over the empty set. By Lemma 6.11, we have

$$U(F) = \mathrm{RM}(F) = \mathrm{R}_{\prec}(\Gamma) = \omega^{K-1}.$$

By Lascar inequalities, we have that $\mathrm{U}(F/a) + \mathrm{U}(a) \leq \mathrm{U}(F) = \omega^{K-1}$, so $\mathrm{U}(a) = \omega^{K-1}$, since $\mathrm{U}(a) > 0$. Since

$$\omega^{K-1} = \mathrm{U}(a) \leq \mathrm{RM}(a) \leq \mathrm{RM}(F) = \omega^{K-1},$$

we have equality, as desired. □

Two types $p$ and $q$, possibly over different sets of parameters, are *non-orthogonal*, if there is a common extension $C$ of both sets of parameters, and two realisations $a$ and $b$ of the corresponding non-forking extensions of $p$ and $q$ to $C$ such that $a$ forks with $b$ over $C$. As in [2, Theorem 7.15], we conclude the following.

**Remark 6.14.** Every type over a nice set $D$ is non-orthogonal to some $\mathrm{p}_s(G)|D$, where $G$ lies in $D$.

Given a reduced flag path $P : F \xrightarrow{u} G$, we will conclude this section by describing the flags one can obtain from the collection of vertices of the flags occurring in $P$, as well as describing how the flags in $P$ can vary (or *wobble*), whilst the endpoints are fixed.

**Lemma 6.15.** *(*cf. [2, Lemma 6.18]*) Let $A$ be the set of vertices of the flags occurring in a reduced flag path $P$. Then any flag in $A$ occurs in some permutation of $P$.*

*Proof.* Write $P : F \xrightarrow{u} G$ and note first that $G$ is the base-point of $F$ over $G$. If $u = 1$, then $F = G$ is the only flag in $A$, so there is nothing to prove. Otherwise, write $u = s \cdot v$ and decompose the path $P$ as $F \xrightarrow{s} H \xrightarrow{v} G$. If we denote by $B$ the set of vertices of the flags occurring in the reduced path $H \xrightarrow{v} G$, Lemma 6.4 yields that $B$ is nice. By induction and Proposition 4.7, the nice set $B \cup F$ is a simple extension of $B$ of type $s, H$.

Given any flag $K$ in $A$, we distinguish two cases: if $K$ lies in $B$, by induction $K$ occurs in some permutation of $H \xrightarrow{v} G$, which induces a permutation of $P$. Otherwise, by Remark 4.4, there exist a reduced word $w$ commuting with $s$ and a flag $K_1$ in $B$ such that $F \xrightarrow{s} H \xrightarrow{w} K_1$ and $F \xrightarrow{w} K \xrightarrow{s} K_1$. By Corollary 3.6, we may assume that the second path is a permutation of the first. By induction, the flag $K_1$ belongs to a permutation $H \xrightarrow{w} K_1 \xrightarrow{v_2} G$ of $H \xrightarrow{v} G$. So $F \xrightarrow{w} K \xrightarrow{s} K_1 \xrightarrow{v_2} G$ is a permutation of $P$. □

The following generalises Remark 4.4 and Lemma 6.15.

**Lemma 6.16.** *Let $G$ be a base-point of $F$ over the nice set $D$ and $P$ be a reduced path connecting $F$ to $G$. For every flag $K'$ in the nice set $D \cup P$, there are flags $K$ occurring in some permutation of $P$ and $G'$ in $D$, such that $w = \mathrm{w}(K, G)$ commutes with $v = \mathrm{w}(G, G')$ and $K \xrightarrow{v} K' \xrightarrow{w} G'$.*

*Proof.* If $P$ is trivial, set $K = G$ and $G' = K'$. Otherwise, decompose $P$ into $F \xrightarrow{s} F' \xrightarrow{u'} G$ and set $P' : F' \xrightarrow{u'} G$. Note that $G$ is also a base-point of $F'$ over $D$. If $K'$ is contained in $D \cup P'$, find, by induction on the length of $P$, a flag $K$ occurring in some permutation of $P'$ and $G'$ in $D$, as desired. Otherwise, Remark 4.4 implies the existence of a flag $K_1'$ in $D \cup P'$ such that $w' = \mathrm{w}(F', K_1')$ commutes with $s$ and $F \xrightarrow{w'} K' \xrightarrow{s} K_1'$. By induction, we find a flag $K_1$ occurring in a permutation of $P'$ and a flag $G'$ in $D$ such that $w_1 = \mathrm{w}(G, G')$ commutes with $v_1 = \mathrm{w}(K_1, G)$ and $K_1 \xrightarrow{w_1} K_1' \xrightarrow{v_1} G'$.

Let $P_1 : K_1 \xrightarrow{v_1} G$ be the part of a permutation of $P'$ which connects $K_1$ to $G$. Lemma 6.4 implies that the set $D_1 = D \cup P_1$ is nice. Furthermore, the flag $K_1$ is a base-point of $F'$ over $D_1$. Set $u_1 = \mathrm{w}(F', K_1)$. Since $F' \xrightarrow{u_1} K_1 \xrightarrow{w_1} K_1'$, we conclude that $w' = [u_1 \cdot w_1]$. In particular, the letter $s$ must commute with both $u_1$ and $w_1$. Since $F \xrightarrow{s} F' \xrightarrow{u_1} K_1$, there exists thus a unique flag $K$ such that $F \xrightarrow{u_1} K \xrightarrow{s} K_1$. Clearly, the flag $K$ occurs in a permutation of $P$ with word $u_1 \cdot s \cdot v_1$ and $\mathrm{w}(K, G) = s \cdot v_1 = \mathrm{w}(K', G')$. We need only show that $\mathrm{w}(K, K')$ commutes with $s \cdot v_1$. Since $K \xrightarrow{u_1^{-1}} F \xrightarrow{w'} K'$, the word $\mathrm{w}(K, K')$ is some reduction of $u_1^{-1} \cdot w'$ (possibly with splitting), so $\mathrm{w}(K, K')$ commutes with $s$, since both $u_1$ and $w$ do. The flag path $K \xrightarrow{s} K_1 \xrightarrow{w_1} K_1' \xrightarrow{s} K'$ must reduce to one with word $\mathrm{w}(K, K')$, which commutes with $s$, so the word

$$s \cdot w_1 \cdot s \approx w_1 \cdot s \cdot s$$

must reduce to $w_1 = \mathrm{w}(K, K')$, which commutes with $s \cdot v_1$, as desired. □

**Corollary 6.17.** *Let $A$ be the nice set consisting of the vertices of the flags occurring in a reduced flag path $P : F \xrightarrow{u} G$. Every flag $K$ in $A$ is uniquely determined by $\mathrm{w}(K, G)$. Thus, the only automorphism of $A$ fixing one of the endpoints of $P$ is the identity.*

*Proof.* Given a flag $K$ in $A$, we may decompose $P$ as $F \xrightarrow{v} K \xrightarrow{w} G$, where $w = \mathrm{w}(K, G)$. If $K'$ is another flag in $A$ with $w = \mathrm{w}(K', G)$, then $K'$ occurs as well in a permutation $P'$ of $P : F \xrightarrow{v'} K \xrightarrow{w'} G$. Since $w \approx w'$, Lemma 5.2 implies that $v' \approx v$. We may therefore permute $P$ in order to decompose it as $F \xrightarrow{v} K' \xrightarrow{w} G$. This implies that $K' = K$. □

**Definition 6.18.** The *wobbling* of a reduced product $u \cdot v$ is $\mathrm{Wob}(u, v) = \mathcal{S}_{\mathrm{R}}(u) \cap \mathcal{S}_{\mathrm{R}}(v^{-1})$, that is, the collection of those $\gamma$ in $\Gamma$ which are both right-absorbed by $u$ and left-absorbed by $v$.

Note that $\mathrm{Wob}(u, v)$ cannot be equal to $|u|$ nor to $|v|$, for $u \cdot v$ is reduced.

**Lemma 6.19.** *(Wobbling Lemma* cf. [2, Lemma 7.4]*) Given two reduced paths between the flags $F$ and $G$ with the same word $u = s_1 \cdots s_i \cdots s_n$,*

$$\begin{tikzcd}
 & H_1 \arrow[r] & \cdots \arrow[r] & H_{n-1} \arrow[dr, "s_n"] & \\
F \arrow[ur, "s_1"] \arrow[dr, "s_1"'] & & & & G, \\
 & H'_1 \arrow[r] & \cdots \arrow[r] & H'_{n-1} \arrow[ur, "s_n"'] &
\end{tikzcd}$$

*then $H_i$ and $H'_i$ are $\mathrm{Wob}(s_1 \cdots s_i, s_{i+1} \cdots s_n)$-equivalent for every $i$ in $\{1, \ldots, n-1\}$.*

*In particular, the tuple $H_i/\mathrm{Wob}(s_1 \cdots s_i, s_{i+1} \cdots s_n)$ enumerating the vertices of $H_i$ with colours in $\Gamma \setminus \mathrm{Wob}(s_1 \cdots s_i, s_{i+1} \cdots s_n)$ lies in $\mathrm{dcl}(F, G)$.*

*Proof.* Given two different flag paths as in the above picture, we prove the statement by induction on the index $i < n$. For $i = 1$, let $w_1$ be the reduced word connecting $H_1$ to $H'_1$. Since $H_1 \xrightarrow{s_1} F \xrightarrow{s_1} H'_1$, it follows that $w_1 \preceq s_1$ and $w_1$ is right absorbed by $s_1$. If $w_1 = s_1$, it contradicts Proposition 3.19, since $H_1 \xrightarrow{s_2 \cdots s_n} G$. Therefore, the word $w_1$ is a proper splitting of $s_1$. Furthermore, since $w_1 \cdot s_2 \cdots s_n \xrightarrow{*} s_2 \cdots s_n$, no letter from $s_2 \cdots s_n$ can be absorbed during the reduction, for the word $u$ is reduced. Corollary 5.10 implies that $w_1$ is completely absorbed by $s_2 \cdots s_n$, so $|w_1| \subset \mathrm{Wob}(s_1, s_2 \cdots s_n)$, as desired.

Let now $H_i \xrightarrow{w_i} H'_i$, resp. $H_{i+1} \xrightarrow{w_{i+1}} H'_{i+1}$. By induction, the word $w_i$ has support in $\mathrm{Wob}(s_1 \cdots s_i, s_{i+1} \cdots s_n)$. Lemma 5.14 yields that $w_i \approx w_i^1 \cdot w_i^2$, where $w_i^1$ is left-absorbed by $s_{i+1}$ and $w_i^2$ commutes with $s_{i+1}$ in order to be left-absorbed by $s_{i+2} \cdots s_n$. Thus $w_i^1 \prec s_{i+1}$. Since

$$s_{i+1} \cdot w_i \cdot s_{i+1} \to w_i^2 \cdot s_{i+1} \cdot s_{i+1}$$

reduces to $w_{i+1}$, we conclude that $w_{i+1} \approx w_i^2 \cdot x$, where $x \preceq s_{i+1}$. Note that $x \prec s_{i+1}$, since $H_{i+1} \xrightarrow{s_{i+2} \cdots s_n} G$, so $x$ is a splitting of $s_{i+1}$ which must be absorbed by $s_{i+2} \cdots s_n$ and so is $w_{i+1}$. The word $w_i^2$ is right-absorbed by $s_1 \cdots s_i$, by induction, and commutes with $s_{i+1}$. Since $x$ is a proper splitting of $s_{i+1}$, we have that and $s_1 \cdots s_{i+1} \cdot w_i^2 \cdot x \to s_1 \cdots s_{i+1}$, so $w_{i+1}$ is right-absorbed by $s_1 \cdots s_{i+1}$, as desired. □

**Lemma 6.20.** *(Base-point Lemma* cf. [2, Lemma 7.18]*) Given a flag $H$ with base-point $G$ in the nice set $A$, let $H \xrightarrow{v} G$ be a reduced flag path connecting $H$ to $G$. If $H/W$ lies in $\mathrm{acl}(A)$ for some subset $W \subset \Gamma$, then $|v| \subset W$.*

*Proof.* Choose a flag $H' \models \mathrm{tp}(H/\mathrm{acl}(A))$ with $H \mathop{\smile\!\!\!\!\!\!\mid}_A H'$. Note that $G$ is a base-point in $A$ for $H'$ as well. Thus Proposition 6.6 and transitivity of non-forking imply that $H \mathop{\smile\!\!\!\!\!\!\mid}_G H'$. Furthermore, the flags $H$ and $H'$ are $W$-equivalent, so the reduced word $w$ connecting $H$ to $H'$ has support in $W$. By Proposition 6.1, the word $w$ is the non-splitting reduct of $v \cdot v^{-1}$ and equals $v_1 \cdot \tilde{v} \cdot v_1^{-1}$, where $\tilde{v}$ is the final segment of $v \approx v_1 \cdot \tilde{v}_1$, by Corollary 5.15. Thus, the set $|v|$ is contained in $|w| \subset W$, as desired. □

**Corollary 6.21.** *Nice sets are algebraically closed.*

*Proof.* Let $b$ be algebraic over the nice set $A$. Suppose that $b$ has colour $\gamma$ and choose some flag $F$ containing it. Pick some base-point $G$ for $F$ over $A$ and let $P : F \xrightarrow{u} G$ be a reduced flag path connecting $F$ to $G$. By assumption, the set $F/(\Gamma \setminus \{\gamma\}) = \{b\}$ lies in $\mathrm{acl}(A)$, so $|u| \subset \Gamma \setminus \{\gamma\}$, by Lemma 6.20, that is, the element $b$ equals the $\gamma$-vertex of $G$, which lies in $A$. □

## 7. Equationality

In this section, we will show that the theory $\mathrm{PS}_\Gamma$ is equational.

**Definition 7.1.** A parameter-free formula $\varphi(x, y)$, where the tuple $x$ has length $n$, is an *$n$-equation* if the family of finite intersections of instances $\varphi(x, a)$ (where $a$ belongs to a sufficiently saturated model $N$) has the descending chain condition (DCC).

A complete theory $T$ is *$n$-equational* if every definable set in $N^n$ is a Boolean combination of instances of $n$-equations. A theory is *equational* if it is $n$-equational for every $n$ in $\mathbb{N}$.

Equational theories are stable [14]. Equationality is preserved under naming parameters and bi-interpretability [11]. However, it remains open whether equationality follows from 1-equationality. The rest of this section is devoted to showing that the theory $\mathrm{PS}_\Gamma$ is equational. As in the previous section, let $M$ denote a sufficiently saturated model of $\mathrm{PS}_\Gamma$, inside of which we work.

**Definition 7.2.** Given a word $u = s_1 \cdots s_n$ and a flag $G$ in $M$, let $\mathrm{P}_u(X, G)$ be the formula stating the existence of a sequence $X = F_0, \ldots, F_n = G$ of flags with $F_{i-1} \sim_{s_i} F_i$ for $1 \leq i \leq n$.

It follows from Lemmata 3.4, 3.7 and 3.9 that $M \models \mathrm{P}_u(F, G)$ if and only if there exists some reduction $u' \preceq u$ with $F \xrightarrow{u'} G$. Thus, if $w \preceq u$, then the sentence $\forall X \forall Y\, (\mathrm{P}_w(X, Y) \to \mathrm{P}_u(X, Y))$ holds in all $\Gamma$-spaces. The converse is true, for models of $\mathrm{PS}_\Gamma$ when the word $u$ is reduced, as shown below.

**Remark 7.3.** Let $w$ be a reduced word and $w_1, \ldots, w_n$ be arbitrary words. Then

$$\mathrm{PS}_\Gamma \models \forall X \forall Y \left( \mathrm{P}_w(X, Y) \to \bigvee_{i=1}^{n} \mathrm{P}_{w_i}(X, Y) \right)$$

if and only if $w \preceq w_i$ for some $i$.

*Proof.* We need only prove right-to-left. By Corollary 6.5, there are flags $F$ and $G$ in our saturated model $M$ such that $F \xrightarrow{w} G$. Thus $\mathrm{P}_{w_i}(F, G)$ for some $i$, so $F \xrightarrow{w'} G$ for some reduced $w' \preceq w_i$. Proposition 3.19 implies that $w \approx w'$, so $w \preceq w_i$. □

**Lemma 7.4.** *Given two arbitrary words $u$ and $v$, there is a finite collection of reduced words $w_1, \dots, w_n$ such that, for any reduced word $w$,*

$$w \preceq u, v \iff \bigvee_{i=1}^{n} w \preceq w_i.$$

*In particular, the words $w_i \preceq u, v$ for all $i$.*

*Proof.* In $M$, the formula $\mathrm{P}_u(F,G) \wedge \mathrm{P}_v(F,G)$ implies that $\mathrm{P}_w(F,G)$ for some reduced word $w \preceq u, v$. By compactness, there is a finite set $w_1, \dots, w_n$ of reduced words satisfying that $w_i \preceq u, v$ for $i = 1, \dots, n$ and

$$\mathrm{PS}_\Gamma \models \forall X \forall Y \Big( \mathrm{P}_u(X,Y) \wedge \mathrm{P}_v(X,Y) \to \bigvee_{i=1}^{n} \mathrm{P}_{w_i}(X,Y) \Big).$$

Thus, if $w \preceq u, v$, the formula $\mathrm{P}_w(X,Y)$ implies $\bigvee_{i=1}^{n} \mathrm{P}_{w_i}(X,Y)$. Whence $w \preceq w_i$ for some $i$, by Remark 7.3. □

**Lemma 7.5.** *Given flags $G_1$ and $G_2$ in $M$, and reduced words $u_1$ and $u_2$ such that none of the formulae $\mathrm{P}_{u_1}(X,G_1)$ and $\mathrm{P}_{u_2}(X,G_2)$ implies the other, if $P$ denotes a reduced path from $G_1$ to $G_2$, then the conjunction $\mathrm{P}_{u_1}(X,G_1) \wedge \mathrm{P}_{u_2}(X,G_2)$ is equivalent to*

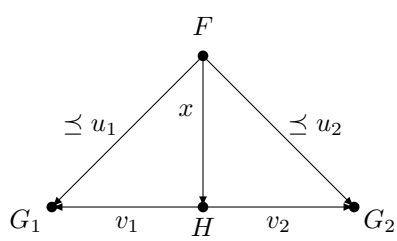
$$\bigvee_{i=1}^{n} \mathrm{P}_{w_i}(X,H_i)$$

*for some flags $H_1, \dots, H_n$ occurring in some permutation of $P$ and reduced words $w_1, \dots, w_n \prec u_1, u_2$.*

If $\mathrm{P}_{u_1}(X,G_1)$ and $\mathrm{P}_{u_2}(X,G_2)$ are disjoint, set $n = 0$. If $G_1 = G_2$, this is the content of Lemma 7.4.

*Proof.* Choose any realisation $F \models \mathrm{P}_{u_1}(X,G_1) \wedge \mathrm{P}_{u_2}(X,G_2)$ and a basepoint $H$ of $F$ over the nice set determined by the reduced path $P$. The flag $H$ occurs in some permutation of $P$, by Lemma 6.15.

Set $x = \mathrm{w}(F,H)$ and $v_i = \mathrm{w}(H,G_i)$ for $i = 1,2$. Observe that $\mathrm{w}(F,G_i) \preceq u_i$ for $i = 1,2$. Proposition 6.1 implies that $[x \cdot v_i] = \mathrm{w}(F,G_i) \preceq u_i$, so $x \preceq u_i/v_i$ for $i = 1,2$, by Lemma 5.16. We obtain the following diagram:




In particular, a flag $F$ realises $\mathrm{P}_{u_1}(X,G_1) \wedge \mathrm{P}_{u_2}(X,G_2)$ if and only if there is some flag $H$ occurring in some permutation of $P$ with

$$\mathrm{P}_{u_1/v_1}(F,H) \wedge \mathrm{P}_{u_2/v_2}(F,H).$$

Lemma 7.4 applied to $u_1/v_1$ and $u_2/v_2$ yields reduced words $w_1, \dots, w_n$ describing the above intersection. We need only show that $w_j \prec u_1, u_2$ for $j = 1, \dots, n$. Clearly $w_j \preceq u_i/v_i \preceq u_i$. Suppose however that $w_j \approx u_1$ for some $j = 1, \dots, n$,

so $u_1 \preceq u_i/v_i$ and thus $u_1 \preceq [u_1 \cdot v_1] \preceq u_1$. Hence $\mathrm{P}_{u_1}(X, G_1) = \mathrm{P}_{u_1}(X, H)$. Also $u_1 \preceq [u_1 \cdot v_2] \preceq u_2$, so $\mathrm{P}_{u_1}(X, G_1) = \mathrm{P}_{u_1}(X, H) \subset \mathrm{P}_{u_2}(X, H) \subset \mathrm{P}_{u_2}(X, G_2)$, contracting our hypothesis. □

Since the relation $\prec$ is well-founded, we conclude the following.

**Corollary 7.6.** *The formulae* $\mathrm{P}_u(X, Y)$ *are equations.*

By bi-interpretatibility, in order to conclude that the theory $\mathrm{PS}_\Gamma$ is equational, we need only show that every formula whose free variables enumerate flags is a Boolean combination of formulae $\mathrm{P}_u(X, G)$. For that, we will first introduce the notion of nice envelopes.

A colour-preserving graph homomorphism $f : A \to B$ between two $\Gamma$-spaces induces a homomorphism $\chi(f) : \chi(A) \to \chi(B)$ between the chamber systems of flags of $A$ and $B$. It is easy to see that this defines an isomorphism between the category of $\Gamma$-spaces and the category of dual quasi-buildings. (*cf.* Theorem 2.17)

**Definition 7.7.** Suppose that both $A$ and $B$ are simply connected. Given $X \subset \chi(A)$ and $Y \subset \chi(B)$, the map $\phi : X \to Y$ is *contracting* if and only if for any two flags $F$ and $G$ in $X$,

$$\mathrm{w}_B(\phi(F), \phi(G)) \preceq \mathrm{w}_A(F, G).$$

We say that $\phi$ is an isometry if

$$\mathrm{w}_B(\phi(F), \phi(G)) = \mathrm{w}_A(F, G).$$

The following is easy to see:

**Lemma 7.8.** *If $A$ and $B$ are simply connected, a map $\chi(A) \to \chi(B)$ is contracting if and only if it is an homomorphism of chamber systems.*

**Definition 7.9.** If $A \subset B$ are $\Gamma$-spaces, a *specialisation* from $B$ to $A$ is a homomorphism $f : B \to A$ such that $f\restriction_A = \mathrm{Id}_A$.

**Remark 7.10.** If $B = A \cup F$ is a simple extension of $A$ of type $s, G$, then the map $f$ fixing all elements of $A$ which sends the $\gamma$-vertex of $F$ to the $\gamma$-vertex of $G$, for $\gamma$ in $s$, is a specialisation.

**Lemma 7.11.** *Given $\Gamma$-spaces $A \subset B$, with $B$ simply connected, the subspace $A$ is nice in $B$ if and only if $B$ can be specialised to $A$.*

*Proof.* Suppose that $B$ can be specialised to $A$, and let $P : F \xrightarrow{u} G$ be a reduced path in $B$ connecting two flags $F$ and $G$ in $A$. The specialisation maps $P$ to a connecting path $P'$ in $A$ between $F$ and $G$ with word $u' \preceq u$, so $A$ is nice in $B$.

If $A$ is nice in $B$, observe that $A$ specialises to itself. Choose then a maximal specialisation $f : C \to A$, with $C \subset B$ nice in $B$. If $C \neq B$, Proposition 4.9 yields a proper simple extension $C'$ of $C$, which is again nice in $B$. By the remark 7.10, there is a specialisation $C' \to C$. Composing it with $f$ contradicts the maximality of $f$. □

Given two nice subsets $N_1$ and $N_2$ of our fixed saturated model $M$, an *isomorphism* means a bijection $f : N_1 \to N_2$ such that both $f$ and $f^{-1}$ are homomorphisms of $\Gamma$-spaces. If $N_1$ and $N_2$ have a common subset $A$ , we say that they are *$A$-isomorphic*, denoted by $N_1 \simeq_A N_2$, if there is an isomorphism between $N_1$ and $N_2$ fixing all elements in $A$.

**Definition 7.12.** Let $N \subset M$ be nice and $A$ be some subset of $N$.

(1) We say that $N$ is a *nice hull* of $A$ if every nice subset of $M$ containing $A$ has a nice subset $N'$ which is $A$-isomorphic to $N$.
(2) The $\Gamma$-space $N$ is *incompressible* over $A$ if every $A$-homomorphism $f : N \to N$ is an automorphism of $N$.
(3) The nice subset $N$ is *strongly incompressible* over $A$ if every $A$-homomorphism $f : N \to M$ induces an isomorphism of $N$ with a nice subset of $M$.

We will see in Proposition 7.18 that, if $N$ is incompressible over $A$, then the only $A$-endomorphism of $N$ is the identity.

Lemma 7.11 implies the following easy observation.

**Remark 7.13.** If $N$ is incompressible over $A$, then it contains no proper nice subset $A \subset N' \subsetneq N$. Likewise if $N$ is strongly incompressible over $A$.

**Lemma 7.14.**

*(1) If the nice set $N$ is strongly incompressible over $A$, then it is incompressible and a nice hull of $A$.*
*(2) If $N$ is a nice hull of $A$ and $N'$ is incompressible over $A$, then $N \simeq_A N'$.*

*Proof.* For (1), let $N$ be strongly incompressible over $A$. To show that $N$ is incompressible over $A$, consider an $A$-homomorphism $f : N \to N$. This must induce an $A$-isomorphism with a nice subset $N_1$ of $N$ containing $A$, so $N_1 = N$ by Remark 7.13.

Let us now show that $N$ is a nice hull. Given any nice subset $N'$ of $M$ containing $A$, choose a specialisation $f : M \to N'$, by Lemma 7.11. The map $f\restriction_N$ must then induce an $A$-isomorphism of $N$ with a nice subset of $N'$, as desired.

Suppose now $N$ and $N'$ are as stated in (2). Since $N$ is a nice hull, there is some nice subset $N''$ of $N'$ which is $A$-isomorphic to $N$. We conclude that $N'' = N'$ by the Remark 7.13. □

We now have all the necessary ingredients to conclude the following result.

**Theorem 7.15.** *Every subset $A$ of $M$ has a unique, up to $A$-isomorphism, strongly incompressible extension. If $A$ is finite, so is this extension.*

*Proof.* We give the proof for $A$ finite, and leave the general case to reader.

Proceed by induction over $|A|$. If $A$ is empty, then any flag is strongly incompressible over $\emptyset$. Otherwise, write $A = A_0 \cup \{a\}$ and choose, by induction, a finite strongly incompressible extension $N_0$ of $A_0$. Among all flags passing through $a$, choose one, say $F$, with $\preceq$-minimal word $u = \mathrm{w}_M(F, G)$, where $G$ is some base-point of $F$ over $N_0$. Assume furthermore that $u$ is $\preceq$-minimal among all possible $A_0$-copies of $N_0$. Let $P : F \xrightarrow{u} G$ be a reduced flag path connecting $F$ to $G$. Set $N = N_0 \cup P$, which is a finite nice subset of $M$, by Lemma 6.4. In order to show that $N$ is strongly incompressible over $A$, consider an $A$-homomorphism $f : N \to M$. By induction, the map $f\restriction_{N_0}$ induces an $A_0$-isomorphism between $N_0$ and the nice set $f(N_0)$, so $f(N_0)$ is also strongly incompressible over $A_0$. The map $f$ is contracting, by Lemma 7.8, so $\mathrm{w}(f(F), f(G)) \preceq u$. Since $a$ is contained in $f(F)$, minimality of $u$ implies that $\mathrm{w}(f(F), f(G)) = u$, thus $f(G)$ is a base-point of $f(F)$ over $f(N_0)$. Therefore, the set $f(P)$ determines a reduced path from $f(F)$ to $f(G)$. Hence, the set $f(N)$ is nice, by Lemma 6.4, and $f$ is an $A$-isomorphism, as desired. □

Together with Lemma 7.14, we conclude:

**Corollary 7.16.** *Every set $A$ has nice hull $\mathrm{N}(A)$, which is incompressible and unique, up to $A$-isomorphism. If $A$ is finite, then so is $\mathrm{N}(A)$.*

Thus, the three notions in Definition 7.12 coincide.

**Corollary 7.17.** *The algebraic closure $\mathrm{acl}(A)$ of a finite set $A$ is finite and contained in $\mathrm{N}(A)$.*

*Proof.* Let $\mathrm{N}(A)$ be the nice envelope of a finite set $A$. Nice sets are algebraically closed, by Corollary 6.21. Thus, the set $\mathrm{acl}(A)$ is contained in $\mathrm{N}(A)$, which is finite. □

**Proposition 7.18.** *The nice envelope $\mathrm{N}(A)$ is rigid over $A$, that is, its only automorphism fixing $A$ pointwise is the identity.*

*Proof.* Again, we leave the case of infinite $A$ to the reader and assume $A$ finite.

If $A$ is empty, recall that $\mathrm{N}(\emptyset)$ consists of a single flag, so the result is obvious. By the proof of Theorem 7.15, if $A = A_0 \cup \{a\}$, then $\mathrm{N}(A) = N_0 \cup P$, where $N_0$ is a nice hull of $A_0$ and $P$ is a reduced flag path connecting a flag $F$ containing $a$ to its basepoint $G$ over $N_0$. Furthermore, the word $\mathrm{w}(F, G)$ is $\preceq$-minimal among all words $\mathrm{w}(F, G')$, where $G'$ has the same type as $G$ over $A_0$. If $\gamma$ denotes the colour of $a$, then $u$ has a unique beginning $s$, which contains $\gamma$.

By Remark 7.10, choose a specialisation $\phi : N_0 \cup P \to N_0$, collapsing the whole path $P$ onto $G$. Let now $f$ be some automorphism of $\mathrm{N}(A)$ fixing $A$. The map $(\phi \circ f)\restriction_{N_0}$, which is an $A_0$-automorphism, by strong incompressibility of $N_0$, must then be the identity, by induction on $|A|$. Thus $f$ is the identity on $N_0 \setminus G$.

Since $f$ is an automorphism, we have that $\mathrm{w}(f(F), f(G)) = \mathrm{w}(F, G) = u$. As the flag $f(F)$ lies in $N_0 \cup P$, Lemma 6.16 yields some flag $K$ with $P : F \xrightarrow{u_1} K \xrightarrow{u_2} G$ where $\mathrm{w}(K, f(F))$ and $u_2$ commute. If $P'$ denotes the subpath $K \xrightarrow{u_2} G$, the flag $K$ is a base-point of $F$ over $N_0 \cup P'$. Thus, the word $\mathrm{w}(F, f(F))$ equals $[u_1 \cdot \mathrm{w}(K, f(F))]$. Since $a$ is contained in $f(F)$, the flags $F$ and $f(F)$ are $(\Gamma \setminus \{\gamma\})$-equivalent, so $\gamma$ does not occur in $\mathrm{w}(F, f(F))$. If $u_1$ is not trivial, then $s$ must be its beginning, for $u$ has only $s$ as a beginning, and hence, either $s$ or a larger letter containing it must occur in $\mathrm{w}(F, f(F))$. Thus, we conclude that $u_1 = 1$ and the word $\mathrm{w}(F, f(F))$ commutes with $u$.

As flag $f(G)$ lies in $N_0 \cup P$, by Lemma 6.16 there is a flag $K_1$ with $P : F \xrightarrow{v_1} K \xrightarrow{v_2} G$ such that $\mathrm{w}(K, f(G))$ and $v_2$ commute. We have the following permutations:

$$v_1 \cdot v_2 \cdot \mathrm{w}(F, f(F)) \approx \mathrm{w}(F, f(F)) \cdot u \approx \mathrm{w}(F, f(F)) \cdot \mathrm{w}(f(F), f(G)) \approx [v_1 \cdot \mathrm{w}(K, f(G))].$$

Corollary 5.13 implies that $v_2 \cdot \mathrm{w}(F, f(F))$ is a final subword of $\mathrm{w}(K, f(G))$, which commutes with $v_2$. We conclude that $v_2 = 1$ and hence $f(G)$ lies in $N_0$. The map $f$ maps $N_0$ to itself, so it is the identity on $N_0$. Since $f$ induces a permutation of $P$, Corollary 6.17 implies that $f$ is the identity on $N$. □

**Corollary 7.19.** *The algebraic closure $\mathrm{acl}(A)$ of $A$ is rigid over $A$, so it equals $\mathrm{dcl}(A)$.*

*Proof.* Note that $\mathrm{acl}(A)$ is contained in $\mathrm{N}(A)$, by Corollary 7.17, so $\mathrm{N}(A) = \mathrm{N}(\mathrm{acl}(A))$. Every $A$-automorphism of $\mathrm{acl}(A)$ extends to an automorphism of $\mathrm{N}(A)$, which must then be the identity, by Proposition 7.18. □

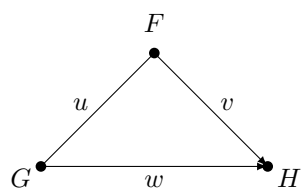
**Proposition 7.20.** *All types are stationary.*

This needs no longer hold if we consider types over subsets of $M^{\mathrm{eq}}$.

*Proof.* We need only prove the statement for 1-types, and thus it suffices to prove it for types of a single flag. Let $p$ be the type of a flag $F$ over the parameter set $A$ and consider $q$ a global non-forking extension of $p$ to $M$. Since $q = \mathrm{p}_u(G)|M$ for some flag $G$, its canonical parameter is $B = G/\mathcal{S}_{\mathrm{R}}(u)$, by Corollary 6.8, which is interdefinable with a set of real elements.

The theory is stable, so the number of global non-forking extension of $p$ is bounded, that is, the set $B$ is algebraic over $A$ and hence definable over $A$, by Corollary 7.19. The type $p$ is thus stationary. □

We will show that the theory $\mathrm{PS}_\Gamma$ is equational, by proving that the type of finitely many flags is determined by the collection of words connecting each pair of flags. For two flags this follows from Lemma 6.4: A reduced path $P : F \xrightarrow{u} G$ determines a nice subset, whose type is determined by $u$. For three flags $F$, $G$ and $H$ as below:




we need the following Proposition.

**Proposition 7.21.** *(1) Given $u$, $v$ and $w$ reduced words with $u \cdot v \xrightarrow{*} w$, there is a decomposition:*

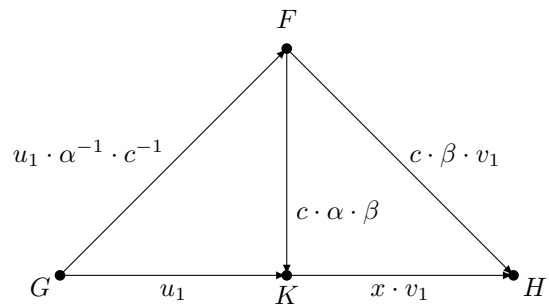
$$\begin{aligned} u &\approx u_1 \cdot \alpha^{-1} \cdot c^{-1}, \\ v &\approx c \cdot \beta \cdot v_1, \\ w &\approx u_1 \cdot x \cdot v_1, \end{aligned}$$

*such that $\alpha$, $\beta$ and $x$ pairwise commute, the word $x$ is properly left-absorbed by $c$, the word $\alpha$ is properly left-absorbed by $v_1$, and $\beta$ is right-absorbed by $u_1$. The words $u_1$, $v_1$, $c$, $x$, $\alpha$ and $\beta$ are unique up to permutation.*

*(2) Assume further that $F$, $G$ and $H$ are flags such that $\mathrm{w}(G, F) = u$, $\mathrm{w}(F, H) = v$ and $\mathrm{w}(G, H) = w$. Then there is a reduced path $P : G \xrightarrow{w} H$, and a base-point $K$ of $F$ over $P$ such that:*

$$\begin{aligned} \mathrm{w}(G, K) &= u_1 \\ \mathrm{w}(K, H) &= x \cdot v_1, \\ \mathrm{w}(F, K) &= c \cdot \alpha \cdot \beta \end{aligned}$$

Since $x$ is properly left-absorbed by $c$, the product $c \cdot c^{-1}$ can be reduced to $x$. Thus, the existence of a decomposition as in (1) implies $u \cdot v \xrightarrow{*} w$. By Lemma 6.15, we may assume in (2) that the flag $K$ occurs in the path $P$.




*Proof.* Let us first prove the existence of such a decomposition. By Remark 4.15, find flags $F$, $G$ and $H$ such that $G \xrightarrow{u} F \xrightarrow{v} H$ and $G \xrightarrow{w} H$. Let $P$ be a reduced flag path connecting $G$ to $H$. By Lemma 6.15, we may choose a base-point $K$ of $F$ over $P$ which occurs in the path $P$. Set $\mathrm{w}(G,K) = w_1$, $\mathrm{w}(K,H) = w_2$, and $\mathrm{w}(F,K) = y$. Assume that $y$ is $\preceq$-minimal among all choices of $P$, which implies

(‡) No end of $y$ is contained in $\mathrm{Wob}(w_1, w_2)$.

Indeed, if $y = y' \cdot s$, with $s \subset \mathrm{Wob}(w_1, w_2)$, decompose $F \xrightarrow{y'} K' \xrightarrow{s} K$ for some flag $K'$. Observe that $K$ is also a base-point for $K'$ over the nice set determined by $P$, so by Proposition 6.1 the reduction of $s \cdot w_1^{-1}$, resp. $s \cdot w_2$, is non-splitting and equals $w_1^{-1}$, resp. $w_2$. Replacing $K$ by $K'$, we obtain a permutation $P'$ of $P$, so that $F$ connects to $P'$ with word $y'$, contradicting the minimality of $y$.

Proposition 6.1 implies that $[y \cdot w_1^{-1}] = u^{-1}$ and $[y \cdot w_2] = v$. By Proposition 5.9, up to permutations of $w_1$ and $u$, write:

$$\begin{aligned} w_1 &= u_1 \cdot x', \\ y &\approx c_1^{-1} \cdot \beta, \\ u &= u_1 \cdot c_1 \end{aligned}$$

where $x'$ and $\beta$ commute, the word $x'$ is properly left-absorbed by $c_1$ and $\beta$ is right-absorbed by $u_1$. Let $K'$ be a flag in $P$ such that $G \xrightarrow{u_1} K' \xrightarrow{x'} K$. Since $x'$ is right-absorbed by $y$, the flag $K'$ is also a base-point of $F$ over $P$ by Corollary 6.3. Replacing $K'$ by $K$, we may assume that $x' = 1$.

Likewise, write

$$\begin{aligned} w_2 &= x \cdot v_1, \\ y &\approx c_2 \cdot \alpha, \\ v &= c_2 \cdot v_1 \end{aligned}$$

where $x$ and $\alpha$ commute, the word $x$ is properly right-absorbed by $c_2$ and $\alpha$ is left-absorbed by $v_1$.

Note that $y = c_2 \cdot \alpha \approx c_1^{-1} \cdot \beta$. However, no end of $\alpha$ can be an end of $\beta$, by (‡). Thus, every end of $\alpha$ commutes with $\beta$ and Lemma 5.3 yields that $\alpha$ commutes with $\beta$ and is a final subword of $c_1^{-1}$. Likewise for $\beta$ and $c_2$. After possible permutations

of $c_1$ and $c_2$, write:

$$c_1^{-1} = c \cdot \alpha,$$
$$c_2 = c \cdot \beta.$$

Let us now show that $\beta$ and $x$ commute. Otherwise, write $x = x_1 \cdot s \cdot x_2$, where $x_1$ and $\beta$ commutes, but $\beta$ and $s$ do not. Since $x$ is right-absorbed by $c_2 = c \cdot \beta$, the letter $s$ must be absorbed by $\beta$, and therefore right-absorbed by $u_1$. Since $s$ commutes with $x_1$, the word $u_1 \cdot x \cdot v_1$ is not reduced, which is a contradiction.

We have now

$$\begin{aligned} u &= u_1 \cdot \alpha^{-1} \cdot c^{-1}, \\ v &= c \cdot \beta \cdot v_1, \\ w &= w_1 \cdot w_2 = u_1 \cdot x \cdot v_1, \\ y &= c \cdot \alpha \cdot \beta. \end{aligned}$$

The only property left to show is that $\alpha$ is properly left-absorbed by $v_1$. Otherwise, apply Corollary 5.12 to produce, up to permutation, the following decompositions:

$$\alpha = \alpha' \cdot \omega \qquad\qquad \omega \cdot v_2 = v_1,$$

where $\omega$ is a commuting word, the word $\alpha'$ is properly left-absorbed by $v_2$, and $\alpha'$ and $\omega$ commute. Since $w_2 \approx \omega \cdot x \cdot v_2$, there is a flag $K'$ in some permutation of $P$ such that $K \xrightarrow{\omega} K' \xrightarrow{x \cdot v_2} H$. Now, the word $\omega$ is right-absorbed by $y \approx c \cdot \beta \cdot \alpha' \cdot \omega$, so the flag $K'$ is also a base-point of $F$ over $P$, by Corollary 6.3. Replacing $K$ by $K'$, and substituting:

$$\begin{aligned} u_1 &\rightsquigarrow u_1 \cdot \omega \\ v_1 &\rightsquigarrow v_2 \\ \alpha &\rightsquigarrow \alpha' \\ \beta &\rightsquigarrow \beta \cdot \omega \end{aligned}$$

gives that the new words $u_1$, $v_1$, $c$, $x$, $\alpha$, $\beta$ and the new base-point $K$ have the desired properties.

For uniqueness, let $u_1 \cdot a$ be the largest common initial subword of $u = u_1 \cdot \alpha^{-1} \cdot c^{-1}$ and $w = u_1 \cdot x \cdot v_1$ (*cf.* the discussion after Definition 5.1). Since $\alpha$ and $x$ commute and each one is properly absorbed by $v_1$, resp. $c$, the word $a$ must commutes with $\alpha$ and $x$, and is an initial subword of both $c^{-1} \approx a \cdot c'$ and $v_1 \approx a \cdot v_1'$. In particular, the words $\alpha \cdot c'$ and $x \cdot v_1'$ are uniquely determined, up to a permutation, for $u_1 \cdot a$ is.

Observe that $c'$ is the largest common final subword of $\alpha \cdot c'$ and $v^{-1} = {v_1'}^{-1} \cdot a^{-1} \cdot \beta^{-1} \cdot a \cdot c'$, for otherwise, the largest common final subword would then contain a letter $s$ from $\alpha$, which is an end of ${v_1'}^{-1} \cdot a^{-1} \cdot \beta^{-1} \cdot a$, contradicting that $\alpha$ is properly left-absorbed by $v_1'$. Therefore, the words $\alpha$ and $a^{-1} \cdot \beta \cdot a \cdot v_1'$ are uniquely determined.

Since $x$ and $a^{-1} \cdot \beta \cdot a$ commute, the word $v_1'$ is the largest common final subword of $x \cdot v_1'$ and $a^{-1} \cdot \beta \cdot a \cdot v_1'$, so $x$ and $a^{-1} \cdot \beta \cdot a$ are uniquely determined. The result now follows by applying the following auxiliary result to the words $u_1 \cdot a$ and $a^{-1} \cdot \beta \cdot a$, in order to determine $u_1$, $a$ and $\beta$, as desired.

**Claim.** *Given reduced words $e \approx u \cdot a$ and $f \approx a^{-1} \cdot b \cdot a$, where $u$ right-absorbs $b$, then $a$, $u$ and $b$ are uniquely determined by $e$ and $f$.*

*Proof of the claim:* We proceed by induction on the length of $f$. Clearly, if $f = 1$, then $b = a = 1$ and thus $e = u$.

Otherwise, note that $e$ right-absorbs $f$ if and only if $a = 1$, in which case $e = u$ and $f = b$. Therefore, if $e$ does not right-absorb $f$, write $a = a' \cdot s$. In particular, the word $e$ has an end which is simultaneously a beginning and an end of $f$.

We show first that the only possible ends $t$ of $e$ which are both a beginning and an end of $f$ are exactly the ends of $a$. If not, the end $t$ must be an end of $u$ which commutes with $a$. Likewise $b \approx t \cdot b' \cdot t$, which contradicts that $u$ right-absorbs $b$, since $b'$ and $t$ do not commute.

Removing $s$ yields the reduced words $u \cdot a'$ and $a'^{-1} \cdot b \cdot a'$, and the result now follows by induction.
*End of the proof of the claim.*

□

**Corollary 7.22.** *The type* $\mathrm{tp}(F, G, H)$ *of three flags* $F$*,* $G$ *and* $H$ *is uniquely determined by* $\mathrm{w}(F, G)$*,* $\mathrm{w}(G, H)$ *and* $\mathrm{w}(F, H)$*.*

*Proof.* Proposition 7.21 yields a reduced path $P : G \xrightarrow{w_1} K \xrightarrow{w_2} H$ from $G$ to $H$, where $K$ is a base-point of $F$ over the nice set determined by $P$, such that $w_1$, $w_2$ and $\mathrm{w}(F, K)$ are uniquely determined, up to permutation, by the words $\mathrm{w}(F, G)$, $\mathrm{w}(G, H)$ and $\mathrm{w}(F, H)$. By Corollary 6.5, the type of $F$ over $P$ is uniquely determined by $\mathrm{w}(F, K)$ and $K$. Note that $H$ is a base-point of $G$ over the nice set $H$, so Lemma 6.4 yields that the type of $P = G \cdots K \cdots H$ is determined by the word $w_1 \cdot w_2$. Thus, the type of $GKH$ is determined by the equivalence classes of $w_1$ and $w_2$, as desired. □

To extend this result to arbitrary sets of flag, we need the following lemma. Given a $\Gamma$-space $A$, recall that $\chi(A)$ denotes the chamber system of the flags in $A$ (*cf.* the discussion before Definition 7.7).

**Lemma 7.23.** *A non-empty collection* $X$ *of flags equals* $\chi(A)$ *for some nice subset* $A$ *if and only if, whenever* $F$ *and* $G$ *in* $X$ *are connected by a reduced word* $u$ *in* $M$*, then there is path in* $X$ *with word* $u$ *connecting* $F$ *to* $G$*.*

If $X = \chi(A)$, the remark after Definition 4.5 implies that $A$ is the union of the flags in $X$.

*Proof.* One direction is an equivalent definition of niceness, so we need only show that, if $X$ satisfies the right-hand condition, then $X = \chi(A)$ for some nice subset $A$. Set $A$ the collection of all vertices of flags in $X$. In order to show that $X = \chi(A)$ and that $A$ is nice, it suffices to show that any flag in $A$ is a flag from $X$, that is, we need only show, by induction on $|S|$, that given a collection of elements $S$ in $A$ lying in a flag $H$ in $M$, there is a flag in $X$ containing $S$. If $S$ is a singleton, there is nothing to prove. Otherwise, enumerate $S = \{a_1, \ldots, a_r\}$ and set $T \subset \Gamma$ the collection of colours of $a_1, \ldots, a_{r-1}$, and $\gamma$ the colour of $a_r$. Choose a flag $F$ in $X$ containing $a_r$ and, by induction, a flag $G$ in $X$ containing $a_1, \ldots, a_{r-1}$. Observe that

$$F \sim_{\Gamma \setminus \{\gamma\}} H \sim_{\Gamma \setminus T} G,$$

so there are reduced words $u$ and $v$, such that $\gamma$ does not occur in $u$, the support of $v$ is disjoint from $T$ and $F \xrightarrow{u} H \xrightarrow{v} G$. Reducing this path in $M$ gives a word

$w_1 \cdot w_2$, where $\gamma$ does not occur in $w_1$ and each letter in $w_2$ is disjoint from $T$. By assumption, there is a reduced path in $X$ of the form $F \xrightarrow{w_1} H' \xrightarrow{w_2} G$. Clearly, the path $H'$ lies in $X$ and contains the set $S$, as desired. □

We will now extend Corollary 7.22 to arbitrarily many flags.

**Theorem 7.24.** *The type* $\mathrm{tp}(F_1, \ldots, F_n)$ *of a sequence of flags is uniquely determined by* $\mathrm{w}(F_i, F_j)$ *for* $1 \leq i \neq j \leq n$.

*Proof.* Recall the notion of isometry, as in Definition 7.7. Given two collections of flags $X$ and $X'$, we will show that a surjective isometry $\phi : X \to X'$ induces an elementary map between nice subsets of $M$.

Suppose first that $X$ satisfies the conditions of Lemma 7.23. Then so does $X'$, and there are nice subsets $A$ and $A'$ such that $X = \chi(A)$ and $X' = \chi(A')$. In particular, the map $\phi$ induces a graph isomorphism $f : A \to A'$, for two flags $F$ and $G$ in $A$ are $\gamma$-equivalent if and only if $\mathrm{w}(F, G)$ does not contain $\gamma$. By Remark 4.6 (2), nice subsets are $\Gamma$-spaces, so $f$ is an elementary map, by Corollary 4.13.

We need only show that $\phi$ can be extended to some supersets of flags, which satisfy the conditions of 7.23. Given flags $F$ and $G$ in $X$, with respective images $F'$ and $G'$ in $X'$, if $\mathrm{w}(F, G) = u$, then choose reduced paths $P : F \xrightarrow{u} G$ and $P' : F' \xrightarrow{u'} G'$ with

$$X \underset{F,G}{\downarrow} P \quad \text{and} \quad X' \underset{F',G'}{\downarrow} P'.$$

Let $\psi$ be the isometry $P \to P'$ which maps $(F, G)$ to $(F', G')$. In order to show that $\phi \cup \psi$ is a well-defined isometry between $X \cup P$ and $X' \cup P'$, consider some flag $H$ in $X$. It suffices to show that $\phi \cup \psi$ induces an isometry $H \cup P \to \phi(H) \cup P'$. By Corollary 7.22, the map $\phi{\restriction}_{H,F,G}$ is elementary. So is $\psi$, by Lemma 6.4. As the type $\mathrm{tp}(H/F, G)$ is stationary, by Proposition 7.20, it follows that $\phi{\restriction}_{H,F,G} \cup \psi$ is an elementary map and thus an isometry.

Considering all possible pairs of flags $(F, G)$ in $X$, we obtain the desired superset satisfying 7.23. □

The above result together and Corollary 7.6 yield the following, by compactness.

**Corollary 7.25.** *The theory* $\mathrm{PS}_\Gamma$ *is equational.*

Equationality, or rather, Theorem 7.24 allows us to show total triviality and hence weak elimination of imaginaries.

**Proposition 7.26** (*cf.* [2, Lemma 7.22])**.** *The theory* $\mathrm{PS}_\Gamma$ *is totally trivial: over a set of parameters* $D$*, given tuples* $a$*,* $b$ *and* $c$ *such that* $a$ *is independent both from* $b$ *and from* $c$ *over* $D$*, then it is independent from* $b, c$ *over* $D$*.*

*Proof.* By taking a non-forking extension to a small submodel containing $D$, we may assume that $D$ is nice. Since every element is contained in a flag, it suffices to show total triviality when $a$, $b$ and $c$ enumerate three flags $F$, $H$ and $K$.

Let $G$ be a base-point of $F$ over $D$. In particular, we have $F \downarrow_G H$ and $F \downarrow_G K$.

Choose another realisation $F'$ of $\mathrm{tp}(F/G)$ such that $F' \downarrow_G H, K$. Then $\mathrm{w}(F', G) = \mathrm{w}(F, G)$ and

$$\mathrm{w}(F', H) = [\mathrm{w}(F', G) \cdot \mathrm{w}(G, H)] = [\mathrm{w}(F, G) \cdot \mathrm{w}(G, H)] = \mathrm{w}(F, H),$$

since $F \downarrow_G H$. Likewise for $\mathrm{w}(F, K)$. Theorem 7.24 implies that $F$ and $F'$ have the same type over $G, H, K$, so $F \downarrow_G H, K$, as desired. □

Since $\mathrm{PS}_\Gamma$ is $\omega$-stable, we conclude by [5, Proposition 7] the following.

**Corollary 7.27.** *The theory* $\mathrm{PS}_\Gamma$ *is perfectly trivial, that is, given given any set of parameters* $D$ *and tuples* $a$*,* $b$ *and* $c$ *such that* $a$ *and* $b$ *are both independent over* $D$*, then so are they over* $D \cup \{c\}$*.*

An $\omega$-stable theory $T$ has *weak elimination of imaginaries* if the canonical base of every stationary type can be chosen to be a subset of the real sort. In this case, types over algebraically closed sets are always stationary.

**Corollary 7.28.** *The theory* $\mathrm{PS}_\Gamma$ *has weak elimination of imaginaries, and given any stationary type* $\mathrm{tp}(a_1, \ldots, a_n/B)$*, then*

$$\mathrm{Cb}(\mathrm{tp}(a_1 \ldots a_n/B)) = \mathrm{Cb}(\mathrm{tp}(a_1/B)) \cup \cdots \cup \mathrm{Cb}(\mathrm{tp}(a_n/B)).$$

*Proof.* Considering a small elementary substructure $N$, we may assume that the stationary type $p = \mathrm{tp}(a_1, \ldots, a_n/N)$. Suppose that the canonical base of every unary type over $N$ is interdefinable with a finite subset of the real sort. Thus, we may choose finite subsets $C_1, \ldots, C_n$ such that $\mathrm{Cb}(\mathrm{tp}(a_i/N))$ is interdefinable with $C_i$. Total triviality (Proposition 7.26) implies that $p$ does not fork over $C = C_1 \cup \ldots \cup C_n$. By Proposition 7.20, the restriction of $p$ to $B$ is stationary, so $C$ is interdefinable with $\mathrm{Cb}(p)$.

Therefore, we need only show that the canonical base of every unary $\mathrm{tp}(a/N)$ over the nice set $N$ is interdefinable with a finite subset of the real sort. Given a flag $F$ containing $a$ independent from $N$, the type $\mathrm{tp}(F/a)$ is stationary, so $\mathrm{Cb}(\mathrm{tp}(a/N))$ is interdefinable with $\mathrm{Cb}(\mathrm{tp}(F/N))$. Corollary 6.8 yields now the desired result. □

**Corollary 7.29.** *The canonical base of a type is algebraic over two independent realisations.*

*Proof.* Again by Proposition 7.26 and taking small elementary substructures, we need only consider a unary type $p$ over some nice set $A$. Let $a$ and $a'$ be two independent realisations of $p$. Choose a flag $F$ containing $a$ independent from $A$ over $a$. Since the type of $F$ over $a$ is stationary, it follows that $\mathrm{Cb}(a/A)$ is interdefinable with $\mathrm{Cb}(F/A)$. By automorphisms, we may find a flag $F' \equiv_A F$ containing $a'$ with

$$F' \underset{a'}{\downarrow} AF.$$

In particular, we have that

$$FF' \underset{a,a'}{\downarrow} A$$

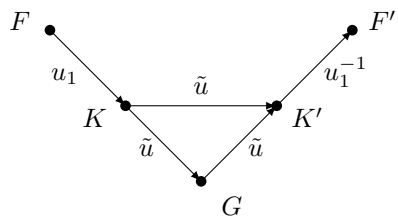
and

$$F \underset{A}{\downarrow} F',$$

thus, if $\mathrm{Cb}(F/A) \subset \mathrm{acl}(F, F') \cap A$, then $\mathrm{Cb}(a/A) = \mathrm{Cb}(F/A) \subset \mathrm{acl}(a, a')$, as desired.

Therefore, we need only prove the statement for the type of a flag $F$ over $A$. Let $F' \underset{A}{\downarrow} F$ and choose a base-point $G$ of $A$ over $A$. Suppose that $P : F \xrightarrow{u} G$ and let $\tilde{u}$ be the final segment of $u \approx u_1 \cdot \tilde{u}$. Then $w(F, F') = [u \cdot u^{-1}] = u_1 \cdot \tilde{u} \cdot {u_1}^{-1}$, by Corollary 5.15.

We obtain the following diagram:




Since $\mathrm{Wob}(u_1, \tilde{u} \cdot u_1^{-1}) \subset \mathcal{S}_{\mathrm{R}}(u)$, Lemma 6.19 implies that $K/\sim_{\mathcal{S}_{\mathrm{R}}(u)}$ lies in $\mathrm{acl}(F, F')$. Observe that $K \sim_{\tilde{u}} G$ and $\tilde{u} \preceq \mathcal{S}_{\mathrm{R}}(u)$, by Corollary 5.18. Thus, the canonical base $\mathrm{Cb}(F/A)$ is contained in $\mathrm{acl}(F, F')$ as well.

□

## 8. Ampleness

Recall the definition of $n$-ampleness [13, 4].

**Definition 8.1.** An $\omega$-stable theory $T$ is *$n$-ample* if, working inside a sufficiently saturated model, there are $n+1$ real tuples satisfying, possibly working over parameters, the following conditions:

(1) $\mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1}, a_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1})$ for every $0 \leq i < n$,
(2) $a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_i} a_0, \ldots, a_{i-1}$ for every $1 \leq i < n$,
(3) $a_n \not\mathop{\smile\!\!\!\!\!\!\mid} a_0$.

Note that $T$ is $n$-ample if and only if $T^{\mathrm{eq}}$ is [2, Corollary 2.4]. Furthermore, if $T$ is $n$-ample, it is $(n-1)$-ample. A theory is 1-based if and only if it is not 1-ample. It is CM-trivial if and only if is not 2-ample.

In order to find an upper bound for the ample degree of $\mathrm{PS}_{\Gamma}$, we will use the following result.

**Lemma 8.2.** *(*[2, Remark 2.5]*) If $T$ is $n$-ample, there are $n+1$ tuples $a_0, \ldots, a_n$ enumerating small elementary substructures of an ambient saturated model such that for every $0 \leq i < n-1$*

*(a)* $a_n \mathop{\smile\!\!\!\!\!\!\mid}_{a_{i-1}} a_i$.
*(b)* $\mathrm{acl}^{\mathrm{eq}}(a_i, a_{i+1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_i, a_n) = \mathrm{acl}^{\mathrm{eq}}(a_i)$.
*(c)* $a_n \not\mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_{i+1})} a_i$.

**Lemma 8.3.** *Let $X$ be -definable subset without parameters of a saturated model $N$ of an $\omega$-stable theory $T$. The* induced structure *on $X$ is the set of all relations on every cartesian power of $X$ which are definable in $M$ without parameters. If the theory of $X$ equipped with the induced structure is $n$-ample, then so is $T$.*

*Proof.* Let $a_0, \ldots, a_n$ witness that $X$ is $n$-ample for some $n$ in $\mathbb{N}$. Since $X$ is equipped with the full induced structure from $N$, properties (2) and (3) hold, when we consider these tuples in $N$. Furthermore, working in $N$, we have that

$$\mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1}, a_{i+1}) \cap X^{\mathrm{eq}} = \mathrm{acl}^{\mathrm{eq}}(a_0, \ldots, a_{i-1}) \cap X^{\mathrm{eq}},$$

where $X^{\mathrm{eq}}$ denotes those imaginary elements of $N^{\mathrm{eq}}$ having some representative in $X$. The following result together with (1) yield the desired result.

**Claim.** *If $A$ and $B$ are subsets of $X$, then*

$$\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}\left(\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B) \cap X^{\mathrm{eq}}\right).$$

*Proof of the claim:* Note that we may assume both $A$ and $B$ finite. If $e$ lies in $\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B)$, it is witnessed by some finite definable sets $\varphi(x, a)$, resp. $\psi(x, b)$, with $a$ in $A$, resp. $b$ in $B$. The canonical parameter $d$ of the definable set $E_a \cap F_b$, which contains $e$, belongs to $\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B) \cap X^{\mathrm{eq}}$.
*End of the proof of the claim.*

□

As in the previous sections, the $\Gamma$-space $M$ is a sufficiently saturated model of $\mathrm{PS}_\Gamma$.

A subgraph $Y \subset \Gamma$ is *full* if, whenever two vertices $x$ and $y$ in $Y$ are connected in $\Gamma$, then so are they in $Y$.

**Corollary 8.4.** *Let $\Gamma'$ be a full subgraph of $\Gamma$ and $F$ a fixed flag in $M$. Consider the $\Gamma'$-residuum of $F$*

$$X = \{F' \text{ flag in } M \mid F' \sim_{\Gamma'} F\}.$$

*The set $X$ is the collection of flags of some nice set $A$. The restriction $M' = \mathcal{A}_{\Gamma'}(A)$ to vertices with colours in $\Gamma'$ is a model of $\mathrm{PS}_{\Gamma'}$.*

*If $\mathrm{PS}_{\Gamma'}$ is $n$-ample, so is $\mathrm{PS}_\Gamma$.*

*Proof.* If two flags from $X$ are connected by a reduced flag-path $P$ with word $u$, the $|u|$ is a subset of $\Gamma'$, by simple connectedness. Thus, all flags in $P$ belong to $X$, so $X = \chi(A)$, for some nice subset $A$, by Lemma 7.23.

The restriction $G \mapsto G' = G\restriction_{\Gamma'}$ is a bijection between the flags in $A$ (i.e. the elements of $X$) and the flags of $M'$. Since $\Gamma'$ is full, the Coxeter group generated by $\Gamma'$ is a subgroup of $(W, \Gamma)$, so a reduced word on the letters of $\Gamma'$ remains so as a word in $\Gamma$. It follows that $M'$ is a simply connected $\Gamma'$-space. For every $\gamma \in \Gamma'$ and $G$ in $X$, every flag in $M$ which is $\gamma$-equivalent to $G$ belongs again to $X$. Thus $M'$ is a model of $\mathrm{PS}_{\Gamma'}$.

In order to show that, if $\mathrm{PS}_{\Gamma'}$ is $n$-ample, so is $\mathrm{PS}_\Gamma$, we need only show, by Lemma 8.3, that the induced structure on $M'$ (as an definable subset of $M$ with parameters in $F$) coincides with the structure of $M'$ as a model of $\mathrm{PS}_{\Gamma'}$. That is, that every definable relation on $M'$ which is definable in $M$ over $F$ is then $F$-definable in the $\Gamma'$-space $M'$, or equivalently, that a type in $M'$ over $F$ determines a unique type in $M$ over $F$. Assume therefore that the tuples $c_1$ and $c_2$ have the same type in $M'$ over $F'$. Choosing nice sets $D'_i$ in $M'$, for $i = 1, 2$ containing $F', c_i$, there is an elementary map $f' : D'_1 \to D'_2$ in $M'$, which maps $c_1$ to $c_2$ and is the identity on $F'$. The sets $D_i = D'_i \cup (F \setminus M')$ are clearly nice in $A$, and hence in $M$. The map $f'$ extends to an elementary map between $D_1$ and $D_2$, which is the identity on $F$. Thus the tuples $c_1$ and $c_2$ have the same type over $F$ in $M$, as desired. □

Recall (Definition 5.1) that a letter $s$ is an end of the word $u$ if $u \approx v \cdot s$. The final segment of $u$, denoted by $\tilde{u}$, is the commuting subword consisting of all ends of $u$. When a word $w$ is commuting, we will identify it with its support $|w|$.

**Lemma 8.5.** *([2, Lemma 8.2 and Proposition 8.3]) Consider nice sets $A$ and $B$ and a flag $F$ such that $F \mathop{\smile\!\!\!\!\!\!\mid}_B A$ and $\mathrm{acl}(AB) \cap \mathrm{acl}(AF) = \mathrm{acl}(A)$. Let $u = u_B$*

*(resp. $u_A$) be the minimal word connecting $F$ to a flag $G_B$ in $B$ (resp. $G_A$ in $A$). Consider the reduced word $v$ which connects $G_B$ to $G_A$ and the associated symmetric decomposition*

$$u = u_1 \cdot u' \cdot w \qquad\qquad w \cdot v' \cdot v_1 = v,$$

*as in Proposition 5.9. Then the word $w \cdot v_1$ is commuting. Furthermore, if*

$$F \underset{A\cap B}{\not\downarrow} A,$$

*then*

$$|v'| \nsubseteq \tilde{u} \subsetneq \tilde{u}_A.$$

*Proof.* By transitivity of non-forking, we have that $F \downarrow_{G_B} G_A$, so

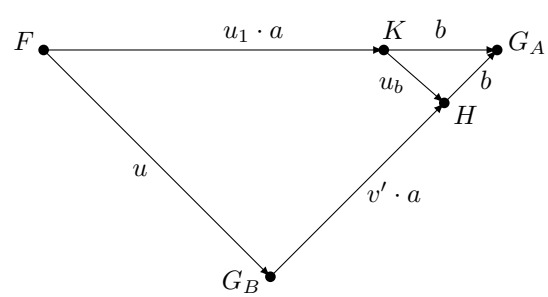
$$u_A = [u \cdot v] = u_1 \cdot w \cdot v_1.$$

We will first show that $w \cdot v_1$ is commuting. Considering its final segment, write $w \cdot v_1 \approx a \cdot b$ where $b$ is a commuting word. In order to show that $a$ is trivial, we need only show that $b$ and $a$ commute.

Now, the word $u' \cdot w$ is left absorbed by $w \cdot v_1$, so write $u' \cdot w \approx u_a \cdot u_b$, by Lemma 5.14, where $u_a$ is left-absorbed by $a$ and $u_b$ commutes with $a$ in order to be left-absorbed by $b$. Choose a flag path $P : G_B \xrightarrow{v' \cdot a} H \xrightarrow{b} G_A$, for some flag $H$. Choosing $P$ independent from $F$ over $G_B, G_A$, we may assume that $F \downarrow_{G_B} P$, so $G_B$ is a base-point of $F$ over the nice set determined by $P$. Therefore

$$\mathrm{w}(F, H) = [u \cdot v' \cdot a] = u_1 \cdot u' \cdot w \cdot v' \cdot a = u_1 \cdot u_b \cdot u_a \cdot v' \cdot a = u_1 \cdot a \cdot u_b.$$

Choose now a flag $K$ with $F \xrightarrow{u_1 \cdot a} K \xrightarrow{u_b} H$. Observe that $\mathrm{w}(K, G_A) \preceq [u_b \cdot b] = b$. On the other hand, the word $\mathrm{w}(F, G_A) = [u \cdot v] = u_1 \cdot w \cdot v_1 \approx u_1 \cdot a \cdot b$, so $\mathrm{w}(K, G_A) = b$.

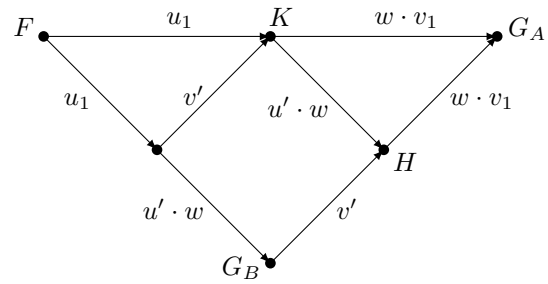



Set $\mathrm{C}(a)$ to be the set of colours commuting with $a$. Clearly $\mathrm{C}(a) \cap \mathcal{S}_{\mathrm{R}}(a) = \emptyset$ and both $\mathrm{Wob}(v' \cdot a, b)$ and $\mathrm{Wob}(u_1 \cdot a, b)$ are subsets of $S = \mathcal{S}_{\mathrm{R}}(a) \cup \mathrm{C}(a)$. Lemma 6.19 implies that $H/S$ lies in $\mathrm{acl}(AB)$ and $K/S$ in $\mathrm{acl}(AF)$. Since $u_b$ commutes with $a$ and $K \xrightarrow{u_b} H$, we have that

$$K/S = H/S \in \mathrm{acl}(AB) \cap \mathrm{acl}(AF) = \mathrm{acl}(A).$$

Lemma 6.20 implies that $b \subset S$. However, no letter of $b$ is contained in $\mathcal{S}_{\mathrm{R}}(a)$, since $a \cdot b$ is reduced and $b$ is commuting, so $b$ and $a$ commute, as desired.

Let us now show that $\tilde{u} \subset \tilde{u}_A$. Since $a = 1$ and $b = w \cdot v_1$, the previous diagram yields the following picture:




The final segment $\tilde{u} = \hat{u}_1 \cdot \tilde{u}' \cdot w$ for some commuting final subword $\hat{u}_1$ of $u_1$, which commutes with $u' \cdot w$. Since $v'$ is (properly) right-absorbed by $u_1$, we have $\mathcal{S}_{\mathrm{R}}(v') \subset \mathcal{S}_{\mathrm{R}}(u_1) \subset T = \hat{u}_1 \cup \mathrm{C}(\hat{u}_1)$, so

$$\mathrm{Wob}(v', w \cdot v_1) \subset \mathrm{Wob}(u_1, w \cdot v_1) \subset T.$$

Note that $|u' \cdot w| \subset T$ and as above, the flag $K/T$ lies in $\mathrm{acl}(A)$. Lemma 6.20 implies that $|w \cdot v_1| \subset T$. Since the word $\hat{u}_1 \cdot w \cdot v_1$ is reduced, no letter of the commuting word $w \cdot v_1$ is contained in $\hat{u}_1$. Therefore $w \cdot v_1$ and $\hat{u}_1$ commute, so

$$\tilde{u} = \hat{u}_1 \cdot w \cdot \tilde{u}' \subset \hat{u}_1 \cdot w \cdot v_1 = \tilde{u}_A.$$

To conclude, notice that if $|v'| \subset \mathcal{S}_{\mathrm{R}}(u_A)$, then $G_A/\mathcal{S}_{\mathrm{R}}(u_A) = G_B/\mathcal{S}_{\mathrm{R}}(u_A)$, which would imply $F \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} A$ by Corollary 6.8. Therefore, if $F \not\mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} A$, then neither $\tilde{u} = \tilde{u}_A$ nor $|v'| \subset \tilde{u}$: for the first case, if $\tilde{u} = \tilde{u}_A$, then $\tilde{u}' = v_1$ and hence $u' = v_1 = 1$, since $\tilde{u}'$ is properly absorbed by $v_1$. For the latter, if $|v'| \subset \tilde{u}$, then $|v'| \subset \tilde{u}_A \subset \mathcal{S}_{\mathrm{R}}(u_A)$. □

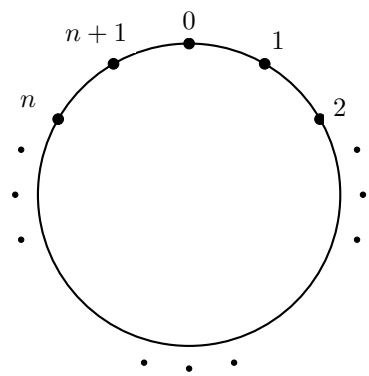
If the graph $\Gamma$ has no edges, the theory $\mathrm{PS}_\Gamma$ is the theory of an infinite set $M$ partitioned into $|\Gamma|$ many infinite sets $\mathcal{A}_\gamma$. This is a trivial theory of Morley rank 1 (and degree $|\Gamma|$) which is easily seen not to be 1-ample.

For a graph with at least one edge, we define its *minimal valency* as the minimum of the valencies of non-isolated vertices. In particular, it is at least 1.

**Theorem 8.6.** *Let $\Gamma$ be a graph with at least one edge. Let $r$ be its minimal valency and $n$ in $\mathbb{N}$ be maximal such that the graph $[0, n]$:*

*embeds as a full subgraph of $\Gamma$. Then the theory $\mathrm{PS}_\Gamma$ is $n$-ample but not $(|\Gamma|-r+1)$-ample.*

If $\Gamma$ contains a full subgraph isomorphic to $[0, n]$, for some $n \in \mathbb{N}$, then $n \leq |\Gamma|$ and $r \leq |\Gamma| - n$, since the graph $[0, n]$ has minimal valency 1. Thus $|\Gamma| - r + 1$ is always bigger than $n$, as expected. For the graph $[0, n]$, the theorem says that its associated theory is $n$-ample but not $(n + 1)$-ample (*cf.* [15, Theorem 3.3], [2, Theorem 8.4]), hence the bounds are best possible, similarly as for the graph consisting $0, \ldots, n + 1$ arranged in a circular way;




which has valency 2, so its theory is $n$-ample yet not $(n+1)$-ample.

In particular, the theory of $\Gamma$ is not 1-based if and only if $\Gamma$ contains at least one edge. The complete graph $\mathbb{K}_n$ has minimal valency $n-1$ and the theory $\mathrm{PS}_{\mathbb{K}_n}$ is CM-trivial for every $n$.

*Proof.* Suppose that $[0,n]$ embeds as a full subgraph in $\Gamma$. Fix some flag $F$ and consider the collection of flags $[0,n]$-equivalent to $F$. Corollary 8.4 and [2, Theorem 8.4] imply that $\mathrm{PS}_\Gamma$ is $n$-ample.

Suppose now that $\mathrm{PS}_\Gamma$ is $N$-ample for some natural number $N$, and let $a_0,\dots,a_N$ be enumerations of small models as in Lemma 8.2. Total triviality of $\mathrm{PS}_\Gamma$ implies that we may replace $a_N$ by a flag $F$. For $0 \le i \le N-1$, let $u^i$ be the reduced word connecting $F$ to a base-point $G_i$ in the nice set $a_i$. Lemma 8.5 applied to each triangle $(F, a_i, a_{i+1})$ implies that the final segment $\tilde{u}^{i+1}$ of $u^{i+1}$ is properly contained in the final segment $\tilde{u}^i$ of $u^i$. In particular $|\tilde{u}^1| \ge N-1$.

Let $v$ be the reduced word connecting $G_1$ to $G_0$. Lemma 8.5 implies the existence of a word $v'$ which is properly right-absorbed by $u^1$ and such that $|v'|$ is not contained in $\tilde{u}^1$. Let $\gamma$ be in $|v'| \setminus \tilde{u}^1$. Since $v'$ is absorbed by $u^1$, so is $\gamma$. Thus $\gamma$ commutes with $\tilde{u}^1$ and must be properly right-absorbed by $u^1$. Hence $\gamma$ is not isolated and has the valency at most $|\Gamma| - |\tilde{u}^1| - 1$. So $r \le |\Gamma| - N$, that is $N < |\Gamma| - r + 1$. □

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