Qualitative Properties of Solutions for the Noisy Integrate & Fire model in Computational Neuroscience

José Carrillo, Benoît Perthame, Delphine Salort, Didier Smets

Qualitative Properties of Solutions for the Noisy Integrate & Fire model in Computational Neuroscience

José Antonio Carrillo * Benoît Perthame†‡§ Delphine Salort¶‖** Didier Smets†‡

October 31, 2014

Abstract

The Noisy Integrate-and-Fire equation is a standard non-linear Fokker-Planck Equation used to describe the activity of a homogeneous neural network characterized by its connectivity b (each neuron connected to all others through synaptic weights); $b > 0$ describes excitatory networks and $b < 0$ inhibitory networks. In the excitatory case, it was proved that, once the proportion of neurons that are close to their action potential V_F is too high, solutions cannot exist for all times. In this paper, we show a priori uniform bounds in time on the firing rate to discard the scenario of blow-up, and, for small connectivity, we prove qualitative properties on the long time behavior of solutions. The methods are based on the one hand on relative entropy and Poincaré inequalities leading to L^2 estimates and on the other hand, on the notion of 'universal super-solution' and parabolic regularizing effects to obtain L^∞ bounds.

Key words Integrate and fire; Neural networks; Fokker-Planck equations; Global existence
Mathematics Subject Classification 35K60; 35Q84; 82C32; 92B20

1 Introduction

Large networks composed by individual neuron activity models have been proposed in the computational neuroscience literature [1, 22] to capture coherent structures such as synchronisation and stationary distributions with applications in decision making in behavioral neurosciences for instance [20, 9]. Most of these models start from stochastic differential equations for each of the neurons coupled through their spiking times. These ideas go back to the early works of [15] modeling neurons as electrical circuits for the evolution of the action potential. These spikes produced, when the action potential surpasses a certain threshold value V_F, induce electrical discharges in the neuronal network inhibiting or exciting other neurons depending on their connectivity rate and strength. Several assumptions have been proposed in the random distribution of the spiking times of neuronal network,

*Department of Mathematics, Imperial College London, London SW7 2AZ. email: carrillo@imperial.ac.uk
†Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France. Emails : benoit.perthame@upmc.fr, smets@ann.jussieu.fr
‡CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France
§INRIA-Paris-Rocquencourt, EPC MAMBA, Domaine de Voluceau, BP105, 78153 Le Chesnay Cedex, France
¶Sorbone Universités, UPMC, UMR 7238, Laboratoire de Biologie Computationnelle et Quantitative. Email: delphine.salort@upmc.fr
‖CNRS, UMR 7238, Laboratoire de Biologie Computationnelle et Quantitative, Paris, France
**Institut Jacques Monod, UMR 7592, Univ Paris Diderot, Sorbonne Paris Cité

but the diffusion approximation has been used in many different applications [4, 2, 3, 16, 21]. This approximation consists in assuming that all the neurons in the network follow an average stochastic differential equation of the form

$$\tau dV = -(V - V_L)\Delta t + \mu_C \, dt + \sigma_C \, dB_t \tag{1.1}$$

where $V_L < V_F$ is the leak potential and τ is the typical relaxation time for a neuron. The final parameters μ_C and σ_C take into account the interaction with the rest of the network and are of the form $\mu_C = bN(t)$ and $\sigma_C^2 = 2a$ in the absence of external inputs, with $a > 0$ and b constants in its simplest form. The parameter b represents the connectivity parameter being positive for excitatory networks and negative for inhibitory networks. Here, $N(t)$ takes into account the average number of spikes per unit of time within the network usually referred as the firing rate of the network.

This basic model is called an integrate and fire model since the action potential is integrated in time following the SDE (1.1) upto the time in which the action potential surpasses V_F. At that time, the neuron spikes and it is assumed that it relaxes instantaneously to the reset potential V_R. Typically the values of the action potentials involved are such that $V_L < V_R < V_F$. As usual in SDEs, we can at least formally write the evolution equation for the probability density of neurons having an action potential v at time $t > 0$ leading to a Fokker-Planck like equation that we called the noisy Integrate and Fire (NIF) equation:

$$\frac{\partial p}{\partial t}(v,t) + \frac{\partial}{\partial v}\left[\big(-v + bN(t)\big)p(v,t)\right] - a\frac{\partial^2 p}{\partial v^2}(v,t) = N(t)\delta(v - V_R), \qquad v \le V_F\,. \tag{1.2}$$

Here, we have assumed that without loss of generality $V_L = 0 < V_R < V_F$. With this choice of units, physiological values for the action potentials are $V_R = 10mV$ and $V_F = 20mV$ with typical time units in miliseconds, where τ was absorbed in the time units. The initial and boundary conditions associated to such model are

$$p(V_F, t) = 0, \qquad p(-\infty, t) = 0, \qquad p(v, 0) = p^0(v) \ge 0\,. \tag{1.3}$$

since neurons are not allowed at the threshold potential V_F. Finally, the right-hand side of (1.2) incorporates the percentage of neurons that surpasses the threshold value at time t and thus are reset to potential V_R. In fact, in order to have an evolution of a probability density, we need to verify that solutions to (1.2) satisfy

$$\int_{-\infty}^{V_F} p(v,t)\, dv = \int_{-\infty}^{V_F} p^0(v)\, dv = 1\,. \tag{1.4}$$

It is straightforward to check that relation (1.4) together with the boundary conditions (1.3) defines the firing rate of the network as the flux of probability at V_F. This gives

$$N(t) := -a\frac{\partial p}{\partial v}(V_F, t) \ge 0\,. \tag{1.5}$$

More complicated models incorporating the evolution for the voltage conductance have also been proposed , see for instance [8, 6, 19].

In this work, we plan to advance the analysis of properties of the solutions to the problem (1.2)-(1.3)-(1.5). The main difficulty of analyzing the behavior of solutions to (1.2)-(1.3)-(1.5) lies in the nonlinearity due to the poinwise value of the derivative appearing in the drift of the equation in voltage variable. These difficulties in analyzing the behaviour of solutions of (1.2) are already present in the stationary states given by

$$\begin{cases} \frac{\partial}{\partial v}\left[(-v+bN_\infty)p_\infty(v)\right] - a\frac{\partial^2 p_\infty}{\partial v^2}(v) = N_\infty \delta(v-V_R), \qquad v \le V_F, \\ p_\infty(V_F)=0, \qquad N_\infty = -a\frac{\partial p_\infty}{\partial v}(V_F), \qquad \int_{-\infty}^{V_F} p_\infty(v)dv = 1. \end{cases} \tag{1.6}$$

The existence of stationary states for this problem was analysed in [5, Theorem 3.1] where it is proved that for inhibitory networks, $b < 0$, and $b \ge 0$ small enough, there is a unique stationary state while the non-uniqueness of stationary states and the nonexistence can happen for larger values of the connectivity parameter $b > 0$. Moreover, the authors in [5] were able to show that no matter how small $b > 0$ is, there are initial data very concentrated on the firing voltage V_F such that the firing rate diverges in finite time. This blow-up phenomenon makes the dynamics for excitatory networks interesting since it is not clear how to understand the coexistence of steady states and the blow-up.

Well-posedness of classical solutions have been obtained locally in time for $b > 0$ and global in time for $b \le 0$ in [10]. The extension criteria for classical solutions is the pointwise control of the firing rate $N(t)$. The solutions exist as long as $N(t)$ remains bounded. The existence of solution and its derivation from stochastic processes have also been studied in [12]. However, no information about the long-time asymptotics of the firing rate is obtained and the local stability near stationary states was not discussed neither in the excitatory or inhibitory case. Let us mention that, even if global in time solutions was proved for the inhibitory case, the inhibitory dynamic is also complex to deal, in particular for strong inhibitory connections, because, the bigger $|b|$ is, the stronger the nonlinearity is. In this work, we give some answers of these above problems.

The rest of the paper is organized as follows. In section 2, we prove local asymptotic stability for stationary states via entropy inequality, for small connectivity ($|b|$ small). In the third section, we first prove, for the inhibitory case, uniform L^2 bounds for the firing rate in time which are independent of $b \le 0$ (and so of the strength of the nonlinearity) and the initial data. In the excitatory case, given an initial data, we show uniform bounds on N and the solution in L^2 as soon $b \ge 0$ is small enough. These uniform in time bounds lead also to the local asymptotic stability result in Section 2 with weaker conditions on the initial data. The techniques here make extensive use of the general entropy method (GRE) as in [17, 18, 5]. Section 4 is devoted, given $b \le 0$ and a bounded initial data, to find uniform L^∞ bounds both for the density and for the firing rate of classical solutions of the problem (1.2)-(1.3)-(1.5). We make use of maximum principle tools adapted to the case of parabolic equations with drift together with a change of variables already used in [10] to reduce the problem locally to standard parabolic regularity theory.

2 Local Asymptotic Stability of stationary states

As mentioned earlier, solutions of the Equation (1.2) can blow-up as proved in [5]. In our first result, we prove that if the strength of interconnections $|b|$ is small enough, then by choosing an initial data close enough to the unique stationary state defined by the Equation (1.6), exponential convergence of the solution to the stationary state holds. Let us recall that the existence and uniqueness of solution for the stationary problem for $b \le 0$ or small enough $b > 0$ was shown in [5, Theorem 3.1]. These apriori bounds can also be used to get global existence results of weak solutions, direction that we will not pursue here.

Theorem 2.1. *Assume that $|b|$ is small enough, then there is a constant $\mu > 0$ such that if the initial data satisfies*

$$\int_{-\infty}^{V_F} p_\infty \left(\frac{p^0 - p_\infty}{p_\infty}\right)^2 (v) dv \leq \frac{1}{2|b|},$$

then, for all $t \geq 0$

$$\int_{-\infty}^{V_F} p_\infty \left(\frac{p - p_\infty}{p_\infty}\right)^2 (v,t) dv \leq e^{-\mu t} \int_{-\infty}^{V_F} p_\infty \left(\frac{p^0 - p_\infty}{p_\infty}\right)^2 (v) dv.$$

The method is a direct application of relative entropy methods and Poincaré inequalities combined with a control of boundary terms discovered in the same context in [7]. We first recall some facts. The relative entropy differential inequality, generalized from [5] for the case $b = 0$, relates the quantities

$$h(v,t) := \frac{p(v,t)}{p_\infty(v)}, \qquad \text{and} \qquad \nu(t) := \frac{N(t)}{N_\infty}. \tag{2.1}$$

The computation for classical solutions gives, for any smooth convex function $G : \mathbb{R}^+ \longrightarrow \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{V_F} p_\infty(v) G\big(h(v,t)\big) dv = &- a \int_{-\infty}^{V_F} p_\infty(v)\; G''\big(h(v,t)\big)\; \left[\frac{\partial h}{\partial v}\right]^2 (v,t)\, dv \\ &- N_\infty \left[G\big(\nu(t)\big) - G\big(h(V_R,t)\big) - \big(\nu(t) - h(V_R,t)\big) G'\big(h(V_R,t)\big)\right] \\ &+ b(N - N_\infty) \int_{-\infty}^{V_F} \frac{\partial p_\infty}{\partial v}(v) \left[G\big(h(v,t)\big) - h(v,t) G'\big(h(v,t)\big)\right] dv. \end{aligned} \tag{2.2}$$

Another tool that we wil use is the following Poincaré inequality which can be found in [5, Appendix]; there is a constant $\gamma > 0$ depending on b such that for any function $h(v)$ satisfying $\int_{-\infty}^{V_F} p_\infty(v) h(v) dv = 1$, we have

$$\gamma \int_{-\infty}^{V_F} p_\infty(v)\; \big(h(v) - 1\big)^2 dv \leq \int_{-\infty}^{V_F} p_\infty(v)\; \left[\frac{\partial h}{\partial v}\right]^2 (v)\, dv. \tag{2.3}$$

Notice that even if the Poincaré inequality stated in [5, Proposition 4.3] was only for $b = 0$, its proof in [5, Appendix] is valid for b small enough since it only uses the behavior or p_∞ at $v = V_F$ and $v = -\infty$. In fact, p_∞ is implicitly given by the formula

$$p_\infty(v) = \frac{N_\infty}{a} e^{-\frac{(v - bN_\infty)^2}{2a}} \int_{\max(v,V_R)}^{V_F} e^{\frac{(w - bN_\infty)^2}{2a}} dw\,, \tag{2.4}$$

and thus, the behavior of $p_\infty(v) \simeq V_F - v$ at $v = V_F$ and $p_\infty(v) \simeq e^{-\frac{v^2}{2a}}$ at $v = -\infty$ does not depend on b. Therefore, the proof in [5, Appendix] applies equally well for b small enough, leading to (2.3).

Also we will use both notations ∂_v and $\frac{\partial}{\partial v}$ to denote the partial derivative in v. After this preliminary material, we can give the proof of the asymptotic behavior.

Proof of Theorem 2.1. We choose $G(h) = (h-1)^2$ and manipulate the three terms in the right hand side of (2.2). The first term is negative and gives the main control when combined with the Poincaré inequality.

We begin our arguments with the second term which is also negative by the convexity property. But we can also use it to provide us with a control of $\nu(t)$. We claim that for some constant $c_0 > 0$ we have

$$
\begin{aligned}
-N_\infty \left[G(\nu(t)) - G(h(V_R,t)) - \big(\nu(t) - h(V_R,t)\big)G'(h(V_R,t))\right]& \\
\le -c_0 G(\nu(t)) + \frac{a}{2}\int_{-\infty}^{V_F} p_\infty(v)\left[\frac{\partial h}{\partial v}\right]^2 (v,t)\,dv.& \qquad (2.5)
\end{aligned}
$$

Indeed, for the function $G(\cdot)$ at hand, we have, for all $0 < \varepsilon < 1/2$,

$$
\begin{aligned}
G(\nu(t)) - G(h(V_R,t)) - \big(\nu(t) - h(V_R,t)\big)G'(h(V_R,t)) &= \big(\nu(t) - h(V_R,t)\big)^2 \\
&\ge \varepsilon(\nu(t)-1)^2 - 2\varepsilon(h(V_R,t)-1)^2.
\end{aligned}
$$

Now, using the Sobolev injection of $L^\infty(I)$ in $H^1(I)$ for a sufficiently small neighborhood I of V_R where p_∞ is bounded below due to its expression in (2.4), and the Poincaré inequality (2.3), we obtain that there exists a constant C such that

$$
\big|h(V_R,t) - 1\big|^2 \le C\int_{-\infty}^{V_F} p_\infty(v)\ \left[\frac{\partial h}{\partial v}\right]^2 (v,t)\,dv
$$

which proves Estimate (2.5), choosing $\varepsilon < 1/2$ such that $2CN_\infty\varepsilon \le a/2$.

We now control the last term in the right hand side of (2.2) which is of cubic nature because it comes from the nonlinerity of the NIF equation. We first observe that $G(h) - hG'(h) = 1 - h^2$, and thus

$$
\int_{-\infty}^{V_F} \partial_v p_\infty(v)\left[G\big(h(v,t)\big) - h(v,t)G'\big(h(v,t)\big)\right]dv = 2\int_{-\infty}^{V_F} p_\infty\left[\partial_v h(v,t)\,(h(v,t)-1) + \partial_v h(v,t)\right]dv\,,
$$

where we take into account the Dirichlet boundary condition on p in (1.3). The interest of forcing the term $h(v,t)-1$ in the above equality can be seen when we handle the full cubic term. We can estimate it with two quantities under control as follows. Using the Cauchy-Schwarz inequality, we obtain that

$$
\begin{aligned}
&|b||N(t) - N_\infty|\int_{-\infty}^{V_F} p_\infty\left|\partial_v\big(h(v,t)\big)\ (h(v,t)-1)\right|dv \\
&\qquad\le \frac{|b|}{a}N_\infty^2(\nu(t)-1)^2 + a|b|\int_{-\infty}^{V_F} p_\infty\,(\partial_v h(v,t))^2\,dv\ \int_{-\infty}^{V_F} p_\infty\,(h(v,t)-1)^2\,dv.
\end{aligned}
$$

For the other term, we write that

$$
|b||N - N_\infty|\int_{-\infty}^{V_F} p_\infty\,|\partial_v h(v,t)|\,dv \le \frac{2b^2}{a}N_\infty^2(\nu(t)-1)^2 + \frac{a}{2}\int_{-\infty}^{V_F} p_\infty(v)\,(\partial_v h(v,t))^2\,dv.
$$

We can now conclude the proof of the Theorem 2.1. We include the previous estimates in the entropy equality (2.2) and obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v)\ \big(h(v,t)-1\big)^2 dv &\le -c_0(\nu(t)-1)^2 + C(b^2+|b|)(\nu(t)-1)^2 \\
&- a\int_{-\infty}^{V_F} p_\infty(v)\ \left[\frac{\partial}{\partial v}h(v,t)\right]^2 dv\left(1 - |b|\int_{-\infty}^{V_F} p_\infty(v)\ (h(v,t)-1)^2\ dv\right). \qquad (2.6)
\end{aligned}
$$

We now take b such that $C(b^2+|b|) \leq c_0$ and the initial data such that

$$|b| \int_{-\infty}^{V_F} p_\infty(v) \quad (h(v,t)-1)^2 \, dv < 1/2,$$

then $\int_{-\infty}^{V_F} p_\infty(v)\big(h(v,t)-1\big)^2 dv$ decreases for all times. Next the Poincaré inequality (2.3) together with (2.6) gives the exponential rate of convergence. $\square$

3 Improved uniform estimates in L^2 and of long term convergence.

3.1 Uniform estimates of N on L^2.

The aim of this part is to prove L^2 bounds on the global activity $N(t)$ for general initial data. In the inhibitory case, we show that for large enough time intervals, these uniforms in time bounds can be chosen independently of the choice of the initial data and of the interconnection strength $b \leq 0$, see Theorem 3.1 below. This is one more indication toward the understanding of the long time asymptotics in the inhibitory case. We will see in next section that uniform in time pointwise bounds can also be obtained under more stringent conditions on the initial data. We remind the reader that in the inhibitory case, there is only a unique stationary solution for each value of $b < 0$, see [5, Theorem 3.1]. However, for $b < 0$ with large $|b|$, we are unable yet to show that this unique stationary state attracts the global dynamics of the problem (see Theorem 3.5 for the proof of convergence of stationary state for small $|b|$, $b \leq 0$ with very low condition on the initial data).

In the case of excitatory networks, for any $b > 0$, it was proved in [5, Theorem 2.2], that if the initial data is enough concentrated near from V_F, blow-up occurs. Then the results obtained in Theorem 3.1 in the inhibitory case can not be extended to the excitatory case. However, we prove uniform in time bounds of N in L^2 under the constraint that the initial data, closed to V_F, is small enough with respect to $b > 0$. Other approaches to control the firing rate in the excitatory case can be seen in [12]. Let us remind the reader that for $b > 0$ small enough, it was shown in [5, Theorem 3.1] the existence of stationary states, for b within a range determined by V_F, V_R, and a.

Our main results are based on the idea of estimating the localized relative entropy of the solution $p(v,t)$ toward a given stationary state p^1_∞ for an excitatory case $b_1 > 0$. They can be summarized as follows.

Theorem 3.1. *Let $b_1 > 0$ such that there exists a stationary state of Equation* (1.2) *and let p^1_∞ be the corresponding stationary state. Let V_M with $V_F > V_M > V_R$ and define*

$$S(b_1, V_M) := \int_{V_M}^{V_F} \frac{\big(p^0\big)^2}{p^1_\infty} dv$$

with an initial data chosen such that $S(b_1, V_M) < +\infty$. Then:

i) There exists a constant C independent of $S(b_1, V_M)$ and there exists a time $T > 0$ depending only on V_M and $S(b_1, V_M)$, such that for all intervals $I \subset (T, +\infty)$ and for all $b \leq 0$,

$$\int_I N(t)^2 dt \leq C(1 + |I|) \tag{3.1}$$

holds.

ii) Assume $b > 0$ is small enough depending on $S(b_1, V_M)$ and V_M, then there exists a constant C such that for all intervals $I \subset \mathbb{R}^+$,

$$\int_I N(t)^2 dt \leq C(1 + |I|) \tag{3.2}$$

holds.

Remark 3.2. *The proof of Theorem* 3.1 *only involves the dynamics of the probability density solution of Equation* (1.2) *closed to V_F, that*

$$\left\| \int_{-\infty}^{V_F} p(t, v) dv \right\|_{L^\infty(\mathbb{R}^+)} < +\infty$$

and the existence of stationary states of Equation (1.2) *for some $b_1 > 0$ strictly decreasing in the neighborhood of V_F. Hence, the proof makes use of only a small part of the structure of Equation* (1.2)*, and therefore may be applied in more general settings.*

Remark 3.3. *The proof is based on an entropy inequality in the spirit of* (2.2) *with however a main difference: the solution of Equation* (1.2) *is not compared to a stationary state associated to Equation* (1.2) *with the same b, but with a stationary state p^1_∞ of Equation* (1.2) *associated to a given $b_1 > 0$ fixed independently of b.*

As mentioned earlier these results are based on the localization of a relative entropy toward a fixed steady state for a given $b_1 > 0$. In order to estimate these localized estimates for solutions to (1.2), we will compute the evolution of weighted quantities localized around V_F of the form

$$\int_{-\infty}^{V_F} W(v, t)\gamma(v) dv \qquad \text{with } W(v, t) = \frac{(p(v, t))^2}{p^1_\infty(v)} = p^1_\infty(v)(H(v, t))^2 , \tag{3.3}$$

where the notation $H := p/p^1_\infty$ is used and with a localization weight $\gamma(v)$ to be chosen. Actually, in both cases we need to localize near V_F, and thus the weight $\gamma \in \mathcal{C}^\infty(-\infty, V_F)$ is such that the support of $\gamma \subset (\alpha, V_F]$ with $\alpha > V_R > 0$. Remember we can assume V_R positive without loss of generality. Let us define the function γ explicitly. For $\alpha \in (-\infty, V_F)$, let us fix the notation $\beta = (V_F - \alpha)^2$. Then, we define our cutoff function γ as follows

$$\gamma(v) = e^{\frac{-1}{\beta - (V_F - v)^2}} \text{ if } v \geq \alpha, \quad \gamma(v) = 0 \text{ else.} \tag{3.4}$$

The following technical Lemma on γ holds.

Lemma 3.4. *Let $V_F > 0$ and let $\alpha \in (-\infty, V_F)$. Then γ given by* (3.4) *is a positive increasing function on (α, V_F) and the following properties holds:*

i)

$$\lim_{v \to V_F} \frac{\gamma'}{\gamma}(v) = 0.$$

ii) There exists a constant C such that on (α, V_F), we have

$$\gamma'^2 + \gamma''^2 + \gamma'''^2 \leq C\gamma. \tag{3.5}$$

iii) There exists a constant $\alpha < \delta < V_F$ such that

$$\gamma''(\delta) = 0 \text{ and } \gamma''(v) \leq 0 \text{ on } (\delta, V_F). \tag{3.6}$$

Proof of Lemma 3.4. A direct computation leads to

$$\gamma'(v) = \frac{2(V_F - v))}{\beta - (V_F - v)^2}\gamma(v)\,,$$

and so γ is increasing and

$$\lim_{v \to V_F} \frac{\gamma'}{\gamma}(v) = 0.$$

Moreover, we have

$$\gamma'(v)^2 = \gamma^2(v)\left(\frac{2(V_F - v))}{\beta - (V_F - v)^2}\right)^2 \text{ with } \lim_{v\to\alpha} \gamma\left(\frac{2(V_F - v))}{\beta - (V_F - v)^2}\right)^2 = 0.$$

We deduce that there exists a constant C such that $\gamma'^2 \leq C\gamma$. Computing $\gamma''(v)$, we obtain

$$\gamma''(v) = \left(\frac{4(V_F - v)^2}{(\beta - (V_F - v)^2)^4} - \frac{2}{(\beta - (V_F - v)^2)^2} - \frac{8(V_F - v)^2}{(\beta - (V_F - v)^2)^3}\right)\gamma.$$

We deduce the last point of Lemma 3.4 since the signs of the second derivative are different at α and V_F, and that there exists a constant C such that $\gamma''^2 \leq C\gamma$. A similar tedious but trivial argument computing $\gamma'''(v)$ leads to find that there exists a constant $C > 0$ such that $(\gamma''')^2 \leq C\gamma$ which ends the proof of Lemma 3.4. □

3.1.1 Proof of Theorem 3.1 in the inhibitory case.

The proof of Theorem 3.1 in the inhibitory case can be reduced to prove that for all $\alpha < V_F$ close enough to V_F, there exist three positive constants C_1, C_2, and C_3 independent of $b \leq 0$ such that

$$\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leq -C_1\frac{N(t)^2}{N_\infty^1} + C_2 + (bN(t) - C_3)\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv. \tag{3.7}$$

Indeed, let us assume that estimate (3.7) holds and let us prove Theorem 3.1. Let us define

$$I(t) := \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.$$

By assumption we have $I(0) < +\infty$. Since $b \leq 0$, (3.7) implies that $I'(t) \leq C_2 - C_3 I(t)$, and thus, assuming t large enough depending on the initial value $I(0)$, we deduce

$$\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leq 2\frac{C_2}{C_3}\,,$$

which implies Theorem 3.1 by integrating (3.7) on any interval I.

Let us now prove estimate (3.7). By using (1.2) and (1.6) for $b_1 > 0$, it is straightforward to check that the function W satisfies

$$\partial_t W + \frac{\partial}{\partial v}\big[(-v + bN(t))W\big] - a\frac{\partial^2}{\partial^2 v}W + 2ap^1_\infty\left(\frac{\partial H}{\partial v}\right)^2 + \left(bN(t) - b_1 N^1_\infty\right) H^2 \frac{\partial p^1_\infty}{\partial v} = N^1_\infty \delta(v - V_R)\left(\frac{2N}{N^1_\infty} - H\right) H\,, \qquad (3.8)$$

where we have used the notation (3.3). Multiplying equation (3.8) by the weight $\gamma \in \mathcal{C}^\infty(-\infty, V_F)$ and taking into account that the support of $\gamma \subset (\alpha, V_F]$ with $\alpha > V_R > 0$, we deduce after integration in v that

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv = &\int_{-\infty}^{V_F} (-v + bN(t))W(v,t)\gamma'(v)dv + a\frac{\partial W}{\partial v}(V_F,t)\gamma(V_F) - aW(V_F,t)\gamma'(V_F)\\ &- 2a\int_{-\infty}^{V_F} p^1_\infty(v)\left[\frac{\partial H}{\partial v}(v,t)\right]^2 \gamma(v)dv + a\int_{-\infty}^{V_F} W(v,t)\gamma"(v)dv\\ &- \left(bN(t) - b_1 N^1_\infty\right)\int_{-\infty}^{V_F} \frac{\partial p^1_\infty}{\partial v}(v)H(v,t)^2\gamma(v)dv\,. \end{aligned} \qquad (3.9)$$

At this stage, we use the structure of p^1_∞ closed to V_F due to its almost explicit expression given in [5, Section 3]. We observe that if α is close enough to V_F, then $\partial_v p^1_\infty \le -p^1_\infty$ on the support of γ. Indeed, p^1_∞ is $\mathcal{C}^\infty$ closed to V_F with $p^1_\infty(V_F) = 0$ and $\partial_v p^1_\infty(V_F) < 0$. This implies that, as soon $b \le 0$ and α close enough to V_F, the non linear term can be controlled by

$$-\left(bN(t) - b_1 N^1_\infty\right)\int_{-\infty}^{V_F} \frac{\partial p^1_\infty}{\partial v}(v)H^2(v,t)\gamma(v)dv \le \left(bN(t) - b_1 N^1_\infty\right)\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv. \qquad (3.10)$$

Moreover, since γ is increasing, $b \le 0$, and $\alpha > V_R > 0$, we get

$$\int_{-\infty}^{V_F} \left((-v + bN(t))W\right)\gamma'(v)dv \le 0.$$

Finally, the boundary conditions in (1.3)-(1.5) and (1.6) together with L'Hopital rule, implies that

$$-a\frac{\partial W}{\partial v}(V_F,t) = \frac{N(t)^2}{N^1_\infty} \qquad \text{and} \qquad W(V_F,t) = 0.$$

Collecting all the last three estimates, using that γ is nonnegative and increasing and that $\gamma''(v) \le 0$ for $v \in (\delta, V_F)$ as proven in Lemma 3.4, we deduce from (3.9) that

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \le &-\frac{N(t)^2}{N^1_\infty}\gamma(V_F)\\ &- 2a\int_{-\infty}^{V_F} p^1_\infty(v)\left[\frac{\partial H}{\partial v}(v,t)\right]^2 \gamma(v)dv + a\int_{-\infty}^{\delta} W(v,t)\gamma"(v)dv\\ &+ \left(bN(t) - b_1 N^1_\infty\right)\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv. \end{aligned} \qquad (3.11)$$

Hence, to prove (3.7), and so Theorem 3.1 part *i)*, it remains to prove the following claim: there exists a positive constant C such that for all $\varepsilon > 0$ small enough,

$$\left|\int_{-\infty}^{\delta} W(v,t)\gamma"(v)dv\right| \leq \varepsilon C\left(\int_{-\infty}^{V_F}\left(\frac{\partial H}{\partial v}(v,t)\right)^2 p^1_\infty(v)\gamma(v)dv + \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv\right) + \frac{C}{\varepsilon} \tag{3.12}$$

holds. Indeed, assuming (3.12), taking α close enough to V_F such that estimate (3.10) holds, and taking ε small enough, we find estimate (3.7) from (3.11), which proves Theorem 3.1 part *i)*.

We are now reduce to show (3.12). The main difficulty of the proof of this claim is that

$$\lim_{v\to\alpha}\frac{\gamma"}{\gamma} = +\infty$$

and so, the choice of the structure of γ in Lemma 3.4 here is crucial. To control the right-hand side of (3.12), we write

$$\int_{-\infty}^{\delta} W(v,t)\gamma''(v)dv = \int_{-\infty}^{\delta} H(v,t)\frac{\partial}{\partial v}\left(\int_{-\infty}^{v} p(t,w)dw\right)\gamma''(v)dv.$$

After integration by parts, recalling that $\gamma''(\delta) = 0$ due to Lemma 3.4, we find

$$\begin{aligned}\int_{-\infty}^{\delta} W(v,t)\gamma''(v)dv = &-\int_{-\infty}^{\delta}\frac{\partial H}{\partial v}(v,t)\left(\int_{-\infty}^{v} p(t,w)dw\right)\gamma''(v)dv\\ &-\int_{-\infty}^{\delta} H(v,t)\left(\int_{-\infty}^{v} p(t,w)dw\right)\gamma'''(v)dv\,,\end{aligned}$$

and so, using that

$$\int_{-\infty}^{V_F} p(v,t)dt = 1, \forall t \geq 0\,,$$

we obtain

$$\left|\int_{-\infty}^{\delta} W(v,t)\gamma''(v)dv\right| \leq \int_{-\infty}^{\delta}\left|\frac{\partial H}{\partial v}(v,t)\right||\gamma''(v)|dv + \int_{-\infty}^{\delta} H(v,t)|\gamma'''(v)|dv\,.$$

Using that on (α,δ), there exists a constant $C > 0$ such that $p^1_\infty(v) \geq C > 0$, estimate (3.5) of Lemma 3.4, and the inequality $cd \leq \varepsilon c^2 + \frac{1}{\varepsilon}d^2$, we finally conclude

$$\left|\int_{-\infty}^{\delta} W(v,t)\gamma"(v)dv\right| \leq \varepsilon C\left(\int_{-\infty}^{\delta}\left(\frac{\partial H}{\partial v}(v,t)\right)^2 p^1_\infty(v)\gamma(v)dv + \int_{-\infty}^{\delta} W(v,t)\gamma(v)dv\right) + \frac{C}{\varepsilon}$$

leading to (3.12) by the positivity of the integrands. □

3.1.2 Proof of Theorem 3.1 in the excitatory case.

The proof of Theorem 3.1 in the excitatory case can be reduced to prove that there exist four positive constants C_1, C_2, C_3, and C_4 depending only on γ defined in (3.4), and p^1_∞ such that for all $t \geq 0$,

$$\frac{dI}{dt} \leq N(t)^2\left(C_1 b^2 - \frac{\gamma(V_F)}{N^1_\infty} + b^2 C_2 I(t)\right) + C_3 - C_4 I(t)\,. \tag{3.13}$$

Here, as in the previous subsection, we use the notation

$$I(t) := \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.$$

Indeed, assuming estimate (3.13) and giving $I_0 < +\infty$, define

$$M := \max\left(I_0, \frac{C_3}{C_4}\right).$$

Let us chose $b > 0$ small enough such that

$$C_1 b^2 - \frac{\gamma(V_F)}{N_\infty^1} + b^2 C_2 M < 0.$$

Then, in this case, for all $t \geq 0$, we have

$$I(t) = \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \leq M\,,$$

which implies Theorem 3.1 in the excitatory case.

Let us now prove estimate (3.13). Coming back to Equation (3.9), since $b > 0$ we can proceed as in the previous case to have sign control on the terms

$$\int_{-\infty}^{V_F} -vW(v,t)\gamma'(v)dv \leq 0 \tag{3.14}$$

and

$$b_1 N_\infty^1 \int_{-\infty}^{V_F} \gamma(v)\frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^2 dv \leq -b_1 N_\infty^1 \int_{-\infty}^{V_F} \gamma(v)W(v,t)dv\,, \tag{3.15}$$

for α close enough to V_F. The term with $\gamma''(v)$ in (3.9) is treated analogously to the previous subsection. Hence, when $b > 0$, we have to deal differently with the terms

$$bN(t)\int_{-\infty}^{V_F} W(v,t)\gamma'(v)dv \geq 0 \quad \text{ and } \quad -bN(t)\int_{-\infty}^{V_F} \gamma(v)\frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^2 dv \geq 0.$$

We now make the following claim: let γ defined as in (3.4). Choosing α close enough to V_F, there exists a constant C such that

$$\begin{aligned} bN(t)\int_{-\infty}^{V_F} W(v,t)\gamma'(v)dv \leq{}& \varepsilon\, C\left(\int_{-\infty}^{V_F} \gamma(v)p_\infty^1(v)\left|\frac{\partial H}{\partial v}(v,t)\right|^2 dv + \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv\right) \\ &+ \frac{C(bN(t))^2}{\varepsilon}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv + \frac{C(1+(bN(t))^2)}{\varepsilon} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} -bN(t)\int_{-\infty}^{V_F} \gamma(v)\frac{\partial p_\infty^1}{\partial v}(v)H(v,t)^2 dv \leq{}& \varepsilon\, C\left(\int_{-\infty}^{V_F} \gamma(v)p_\infty^1(v)\left|\frac{\partial H}{\partial v}(v,t)\right|^2 dv + \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv\right) \\ &+ \frac{C(1+(bN(t))^2)}{\varepsilon} + \frac{C(bN(t))^2}{\varepsilon}\int_{-\infty}^{V_F} W(v,t)\gamma(v)dv\,, \end{aligned} \tag{3.17}$$

for all $\varepsilon > 0$.

The proof of (3.13), and thus the proof of Theorem 3.1 in the excitatory case, is obtained from (3.9) taking into account (3.12), (3.14), (3.15), (3.16), and (3.17).

We are then reduced to show the claims (3.16) and (3.17). Let us first prove estimate (3.16) which follows the idea of the proof of (3.12). The difference is that in our present case $\gamma' \geq 0$ on (α, V_F), so that we have to split γ' in two pieces, close and far from V_F. More precisely, using estimate (3.6) of Lemma 3.4, we obtain that there exists $\alpha < \omega < V_F$ such that $\gamma' \leq \gamma$ on (ω, V_F). We introduce two nonnegative functions $\gamma_1, \gamma_2 \in \mathcal{C}^\infty(-\infty, V_F)$ with $\gamma_1 + \gamma_2 = 1$ and such that, for $\varepsilon_1 > 0$ small enough:

- γ_1 is decreasing on $(-\infty, V_F)$ with $\gamma_1 = 0$ on $(\omega + \varepsilon_1, V_F)$.
- γ_2 is an increasing function with $\gamma_2 = 0$ on $(-\infty, \omega)$.

With this partition of unity, we obtain that

$$\int_{-\infty}^{V_F} W(v,t)\gamma' dv = \int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_1 dv + \int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_2 dv.$$

As $\gamma_2 = 0$ on $(-\infty, \omega)$, we have

$$\int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_2(v) dv \leq \int_{-\infty}^{V_F} W(v,t)\gamma(v) dv$$

using that $\gamma' \leq \gamma$ on (ω, V_F), and so for all $\varepsilon > 0$,

$$bN(t)\int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_2(v) dv \leq \left(\varepsilon + \frac{b^2 N^2(t)}{\varepsilon}\right)\int_{-\infty}^{V_F} W(v,t)\gamma(v) dv.$$

Let us now control the term

$$\int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_1(v) dv.$$

Following the proof of (3.12), we find that

$$\int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_1(v) dv = \int_{-\infty}^{V_F} H(v,t)\frac{\partial}{\partial v}\left(\int_{-\infty}^{v} p(w,t) dw\right)\gamma'(v)\gamma_1(v) dv.$$

After integration by parts, we obtain that

$$\int_{-\infty}^{V_F} W(v,t)\gamma'(v)\gamma_1(v) dv \leq \int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}(v,t)\right|\gamma'(v)\gamma_1(v) dv + \int_{-\infty}^{V_F} H(v,t)|(\gamma'\gamma_1)'(v)| dv\,.$$

To control

$$bN(t)\int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}(v,t)\right|\gamma'(v)\gamma_1(v) dv,$$

we apply inequality $cd \leq \varepsilon c^2 + \frac{1}{\varepsilon}d^2$ with $c = \partial_v H \gamma' \gamma_1$ and $d = bN(t)$ to obtain that for all $\varepsilon > 0$,

$$bN(t)\int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}(v,t)\right|\gamma'(v)\gamma_1(v) dv \leq \varepsilon \int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}(v,t)\right|^2 (\gamma'(v)\gamma_1(v))^2 dv + \frac{1}{\varepsilon}(bN(t))^2(V_F - \alpha).$$

Using Lemma 3.4, we obtain that there exists a constant C such that $(\gamma'\gamma_1)^2 \leq C\gamma$. Moreover, as $p^1_\infty(v)$ is a decreasing function on (α, V_F) if α close enough to V_F, we have $p^1_\infty(v) \geq p_\infty(\omega+\varepsilon_1) > 0$ on the support of γ_1. We deduce that for all $\varepsilon > 0$,

$$bN(t)\int_{-\infty}^{V_F}\left|\frac{\partial H}{\partial v}(v,t)\right|\gamma'(v)\gamma_1(v)dv \leq \frac{C\varepsilon}{p^1_\infty(\omega+\varepsilon_1)}\int_{-\infty}^{V_F}p^1_\infty(v)\left|\frac{\partial H}{\partial v}(v,t)\right|^2\gamma(v)dv + \frac{(bN(t))^2(V_F-\alpha)}{\varepsilon}\,.$$

We proceed similarly with the term

$$bN(t)\int_{-\infty}^{V_F}H(v,t)|(\gamma'\gamma_1)'(v)|dv,$$

to show that

$$bN(t)\int_{-\infty}^{V_F}H(v,t)|(\gamma'\gamma_1)'(v)|dv \leq \frac{C\varepsilon}{p^1_\infty(\omega+\varepsilon_1)}\int_{-\infty}^{V_F}W(v,t)\gamma(v)dv + \frac{(bN(t))^2(V_F-\alpha)}{\varepsilon}\,.$$

which finally proves estimate (3.16).

Let us now prove estimate (3.17). Integration by parts implies that

$$\int_{-\infty}^{V_F}\gamma(v)\frac{\partial p^1_\infty}{\partial v}(v)H(v,t)^2dv = -\int_{-\infty}^{V_F}\gamma'(v)W(v,t)dv - 2\int_{-\infty}^{V_F}\frac{\partial H}{\partial v}(v,t)H(v,t)\gamma(v)p^1_\infty(v)dv.$$

The term

$$bN(t)\int_{-\infty}^{V_F}\gamma'(v)W(v,t)dv$$

is estimated using (3.16). To control the term

$$J(t) := bN(t)\int_{-\infty}^{V_F}\frac{\partial H}{\partial v}(v,t)H(v,t)\gamma(v)p^1_\infty(v)dv,$$

we use the inequality $cd \leq \varepsilon c^2 + \frac{1}{\varepsilon}d^2$ with $c = |\partial_v H|$ and $d = 2bN(t)H$. We deduce that there exists a constant C such that for all $\varepsilon > 0$

$$J(t) \leq C\varepsilon\int_{-\infty}^{V_F}\left(\frac{\partial H}{\partial v}(v,t)\right)^2 p^1_\infty(v)\gamma(v)dv + \frac{(CbN(t))^2}{\varepsilon}\int_{-\infty}^{V_F}\gamma(v)W(v,t)dv\,,$$

which ends the proof of estimate (3.17). As a consequence, we conclude the proof of the claims, the estimate (3.13), and Theorem 3.1 part *ii)*. □

3.2 Application of Theorem 3.1 to convergence toward stationary states.

In this part, we prove, using Theorem 3.1, that the conclusion of Theorem 2.1 still holds when we relax the conditions on the initial data.

Theorem 3.5. *Given p_∞ a stationary state solution to* (1.6) *with associated coupling constant b. Let $V_M < V_F$ and assume that*

$$S_{b,V_M} := \int_{V_M}^{V_F}p_\infty\left(\frac{p^0-p_\infty}{p_\infty}\right)^2(v)dv < +\infty.$$

i) There exist a positive constant C, depending only on S_{b,V_M} and V_M, and $\mu > 0$ such that

$$\int_{-\infty}^{V_F} p_\infty \left(\frac{p-p_\infty}{p_\infty}\right)^2 (v,t)dv \leq e^{-\nu t} \int_{-\infty}^{V_F} p_\infty \left(\frac{p^0-p_\infty}{p_\infty}\right)^2 (v)dv\,,$$

for all $0 < b \leq C$ and $t \geq 1$.

ii) There exist a positive constant C, independent of S_{b,V_M}, and $T > 0$ such that

$$\int_{-\infty}^{V_F} p_\infty \left(\frac{p-p_\infty}{p_\infty}\right)^2 (v,t)dv \leq e^{-\nu(t-T)} \int_{-\infty}^{V_F} p_\infty \left(\frac{p-p_\infty}{p_\infty}\right)^2 (v,T)dv\,,$$

for all $t \geq T+1$ and for all $0 < -b \leq C$.

Proof of Theorem 3.5. The proof of this Theorem is close to the proof of Theorem 2.1. Let us follow the notations (2.1) of Theorem 2.1 and recall that $G(h) = (h-1)^2$. Our starting point is the identity (2.2). By integrating by parts in the last term of (2.2), we obtain

$$-b(N(t)-N_\infty)\int_{-\infty}^{V_F} \frac{\partial p_\infty}{\partial v}(v)h^2(v,t)dv = 2b(N(t)-N_\infty)\int_{-\infty}^{V_F} p_\infty(v)h(v,t)\frac{\partial h}{\partial v}(v,t)dv\,.$$

Using the inequality $cd \leq \varepsilon c^2 + \frac{1}{\varepsilon}d^2$ with $c = \sqrt{p_\infty}\partial_v h$, $d = 2b(N(t)-N_\infty)\sqrt{p_\infty}h$, and $\varepsilon = a$ in Equation (2.2), we find that

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v)G\left(h(v,t)\right)dv \leq &- N_\infty\left[G\left(\nu(t)\right) - G\left(h(V_R,t)\right) - \left(\nu(t)-h(V_R,t)\right)G'\left(h(V_R,t)\right)\right]\\ &- a\int_{-\infty}^{V_F} p_\infty(v)\,\left(\frac{\partial h}{\partial v}(v,t)\right)^2 dv\\ &+ \frac{4|b(N(t)-N_\infty)|^2}{a}\int_{-\infty}^{V_F} p_\infty(v)\left(h(v,t)-1\right)^2 dv + \frac{4|b(N-N_\infty)(t)|^2}{a}\,.\end{aligned}$$

Applying Poincaré inequality (2.3), we deduce

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v)G\left(h(v,t)\right)dv \leq &- N_\infty\left[G\left(\nu(t)\right) - G\left(h(V_R,t)\right) - \left(\nu(t)-h(V_R,t)\right)G'\left(h(V_R,t)\right)\right]\\ &+ \left(-\nu a + \frac{4|b(N-N_\infty)|^2}{a}\right)\int_{-\infty}^{V_F} p_\infty(v)G\left(h(v,t)\right)dv\\ &+ \frac{4|b(N-N_\infty)|^2}{a}\,.\end{aligned}$$

Using the proof of (2.5), it is easy to check that there exists $C > 0$ such that for all $0 < \varepsilon < \frac{1}{2}$, we have

$$\begin{aligned}&G\left(\nu(t)\right) - G\left(h(V_R,t)\right) - \big(\nu(t)-h(V_R,t)\big)G'\big(h(V_R,t)\big) \geq\\ &\qquad \varepsilon(\nu(t)-1)^2 - 2\varepsilon C\int_{-\infty}^{V_F} p_\infty(v)\,\left(\frac{\partial h}{\partial v}(v,t)\right)^2 dv\,.\end{aligned}$$

Poincaré inequality (2.3) again implies that

$$\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v)G\left(h(v,t)\right)dv \leq \left(-\nu a + \varepsilon C + \frac{4|b(N-N_\infty)|^2}{a}\right)\int_{-\infty}^{V_F} p_\infty(v)G\left(h(v,t)\right)dv$$
$$+\frac{(N-N_\infty)^2}{N_\infty}\left(\frac{4b^2N_\infty}{a}-\varepsilon\right).$$

Then, taking ε small enough and $|b|$ small enough such that

$$\frac{4b^2N_\infty}{a}-\varepsilon \leq 0$$

we finally conclude that

$$\frac{d}{dt}\int_{-\infty}^{V_F} p_\infty(v)G\left(h(v,t)\right)dv \leq \left(-\nu a + \varepsilon C + \frac{4|b(N-N_\infty)|^2}{a}\right)\int_{-\infty}^{V_F} p_\infty(v)G\left(h(v,t)\right)dv\,.$$

Applying estimate (3.1) of Theorem 3.1 when $b>0$ and estimate (3.2) of Theorem 3.1 when $b<0$, we obtain that there exists a constant $C>0$ as in Theorem 3.5 and there exists $T\geq 0$ and $\mu>0$ such that for all $t\in(T+1,+\infty)$,

$$\int_T^t\left(-\nu a+\varepsilon C+\frac{2|b(N-N_\infty)|^2}{a}\right)(s)ds \leq -\mu(t-T)\,.$$

We remind that T could be arbitrary in the $b>0$ case. This concludes the proof of Theorem 3.5. □

4 Uniform L^∞ bounds for the inhibitory case

So far, we have obtained controls for the solutions in L^2 spaces. Another method, based on the comparison principle adapted to the nonlinear drift, gives time-uniform controls in L^∞. We present this method in several subsections leading to the following main result:

Theorem 4.1. *For $b<0$ and $p^0\in L^1_+\cap L^\infty(-\infty,V_F)\cap C^1((-\infty,V_F])$ with $p^0(V_F)=0$, given a classical solution p to* (1.2)-(1.5)*, it satisfies $\sup\limits_{t\geq 0}\|p(t)\|_\infty<\infty$ and $\sup\limits_{t\geq 0}N(t)<\infty$.*

By classical solutions of (1.2), we mean a function $p(v,t)$ which is for all $t>0$ C^1 in time and C^2 in the voltage variable $v\in(-\infty,V_R)\cup(V_R,V_F]$, it satisfies the Dirichlet boundary conditions in v at $-\infty$ and at V_F, and such that the first derivatives from left and right exist at $v=V_R$ and from the left at $v=V_F$ defining the firing rate in (1.5). Moreover, $p(v,t)$ satisfies (1.2) in the classical sense for $v\in(-\infty,V_R)\cup(V_R,V_F)$ and $t>0$, and $p(v,t)$ satisfies (1.2)-(1.3) in the distributional sense.

Notice that the last condition is equivalent to say that $p(v,t)$ satisfies (1.2) in the classical sense for $v\in(-\infty,V_R)\cup(V_R,V_F)$ and $t>0$, and that $\frac{\partial p}{\partial v}(v,t)$ has a jump discontinuity at $v=V_R$ of value $-N(t)/a$. We point out that classical solutions were shown to exist under the conditions $p^0\in L^1_+\cap C^1(-\infty,V_F)$ in [10]. They used a suitable change of variables translating (1.2)-(1.5) to a free boundary problem, see subsection 4.2, together with an integral equation for the firing rate solved by fixed point arguments.

Moreover, the authors show that this solution can be extended in a unique way as soon as the firing rate $N(t)$ is bounded. They also prove that solutions exist globally in time for $b<0$ by a contradiction argument on this extension criteria using the firing rate. Here, we will show the global in time existence of a unique classical solution by showing apriori L^∞ bounds on the solution and its firing rate $N(t)$.

Let us finally comment that, as it will cleared later on, the regularity assumptions on the initial data can be improved to $p^0 \in L^1_+ \cap L^\infty(-\infty, V_F)$ such that

$$\limsup_{v \to V_F} \frac{p^0(v)}{V_F - v} < \infty \,.$$

In the rest of this section, we always assume that all the functions of the voltage variable under consideration are non-negative, vanish at $v = V_F$ and at $v = -\infty$, and are smooth separately on $(-\infty, V_R]$ and $[V_R, V_F]$. By smooth, we precisely mean that solutions are at least $C^2((-\infty, V_R)) \cap C^1((-\infty, V_R])$ and $C^2((V_R, V_F)) \cap C^1([V_R, V_F])$. The notation $C^1(J)$ with J a closed interval means that the one-sided derivatives exist and are continuous up to the boundary of the interval. If functions are time dependent, then they are assumed to be at least C^1 for $t > 0$. In other words, that the functions are in the same functional framework as the classical solutions of the problem (1.2)-(1.5) constructed in [10].

4.1 Comparison principle and super-solutions

Definition 4.2. *Let $b < 0$, $V_0 \in [-\infty, V_F)$ and $T > 0$. We say that $\bar{p}$ is a universal (classical) super-solution to* (1.2) *on $[V_0, V_F] \times [0, T]$ if*

$$\frac{\partial \bar{p}}{\partial t}(v,t) - \frac{\partial}{\partial v}\big(v\, \bar{p}(v,t)\big) - a\frac{\partial^2 \bar{p}}{\partial v^2}(v,t) \geq \bar{N}(t)\delta(v - V_R) \tag{4.1}$$

on $(V_0, V_F) \times (0, T)$ in the distributional sense and on $((V_0, V_F)\backslash\{V_R\}) \times (0, T)$ in the classical sense, where $\bar{N}(t) := -a\frac{\partial \bar{p}}{\partial v}(V_F, t) \geq 0$ and

$$\bar{p}(\cdot, t) \text{ is non-increasing on } [V_0, V_F] \quad \forall t \in [0, T].$$

Notice that, as for classical solutions, the condition that the super-solution $\bar{p}$ satisfies (4.1) is equivalent to say that $\bar{p}$ satisfies (4.1) on $((V_0, V_F)\backslash\{V_R\}) \times (0, T)$ in the classical sense, and that $\frac{\partial \bar{p}}{\partial v}(v, t)$ has a decreasing jump discontinuity at $v = V_R$ of at least $\bar{N}(t)/a$ length.

Lemma 4.3. *Let $V_0 \in (-\infty, V_F)$, $b \leq 0$, and $T > 0$. Let $\bar{p}$ be a universal classical super-solution and p be a classical solution to* (1.2)-(1.5) *on $[V_0, V_F] \times [0, T]$, and assume that*

$$\bar{p}(v, 0) \geq p(v, 0) \quad \forall v \in [V_0, V_F] \qquad \text{and that} \qquad \bar{p}(V_0, t) \geq p(V_0, t) \quad \forall t \in [0, T].$$

Then $\bar{p} \geq p$ on $[V_0, V_F] \times [0, T]$ and if $\bar{p}(\cdot, 0) - p(\cdot, 0)$ is not identically zero then $\bar{p} > p$ in $(V_0, V_F) \times (0, T]$.

Proof. We treat the case $V_0 < V_R$, the other one being simpler. Since $N(t) \leq \bar{N}(t)$ whenever $p(\cdot, t) \leq \bar{p}(\cdot, t)$, as long as the latter holds we may write

$$\frac{\partial \bar{p}}{\partial t}(v,t) - \frac{\partial}{\partial v}\left[\big(v - bN(t)\big)\bar{p}(v,t)\right] - a\frac{\partial^2 \bar{p}}{\partial v^2}(v,t) \geq \bar{N}(t)\delta(v - V_R) + bN(t)\frac{\partial \bar{p}}{\partial v}(v,t) \geq N(t)\delta(v - V_R),$$

where we have used the fact that $\bar{p}$ is a universal super-solution, that $b \leq 0$ and that $\frac{\partial \bar{p}}{\partial v}(v, t)$ by assumption. Therefore if we set $w := \bar{p} - p$ then as long as $w(\cdot, t) \geq 0$ we obtain

$$\frac{\partial w}{\partial t}(v,t) - \frac{\partial}{\partial v}\left[\big(v - bN(t)\big)w(v,t)\right] - a\frac{\partial^2 w}{\partial v^2}(v,t) \geq 0$$

and it follows from the weak and strong forms of the maximum principle for linear parabolic equations that $w(\cdot,t)\geq 0$ holds for all $t\in[0,T]$ and then that for $t\in(0,T]$ that $w(\cdot,t)>0$ on (V_0,V_F) unless w is identically zero. $\square$

We shall consider two classes of universal super-solutions. For the first one, we choose $V_0=-\infty$ and we explicitly define

$$P(v,t)=\begin{cases}\exp(t) & \text{for } \ v\leq V_R,\\ \exp(t)\,\frac{V_F-v}{V_F-V_R} & \text{for } \ V_R\leq v\leq V_F.\end{cases} \tag{4.2}$$

Lemma 4.4. *Given arbitrary $\alpha>0$ and $b\leq 0$, the function αP in* (4.2) *is a universal super-solution to* (1.2) *on $[-\infty,V_F]\times\mathbb{R}^+$.*

Proof. This function P is clearly non-increasing. By linearity, it is enough to check the inequality for $\alpha=1$. Indeed, for $v\leq V_R$ since P is independent of v, expanding the equation it is enough to check that $\frac{\partial P}{\partial t}-P\geq 0$, which actually holds with an equality. At $v=V_R$ the jump in the derivative exactly matches the derivative at V_F. Finally, for $v\geq V_R$ we have

$$\frac{d}{dt}\big(\exp(t)\big)\frac{V_F-v}{V_F-V_R}+\exp(t)\frac{v}{V_F-V_R}-\exp(t)\frac{V_F-v}{V_F-V_R}=\exp(t)\frac{v}{V_F-V_R}\geq 0.$$

$\square$

Corollary 4.5. *For $b\leq 0$ and $p^0\in L^1_+\cap L^\infty(-\infty,V_F)\cap C^1((-\infty,V_F])$ with $p^0(V_F)=0$, classical solutions to* (1.2)-(1.5) *are globally defined in time with $p\in L^\infty\big((-\infty,V_F)\times(0,T)\big)$ and $N\in L^\infty(0,T)$ for all $T>0$.*

Proof. Unique classical solutions to (1.2)-(1.3) are constructed in [10] in a maximal time interval T^* by fixed point arguments, where T^* is characterized as

$$T^*=\sup\{t>0\,:\,N(t)<\infty\}\,.$$

Now, it suffices to consider a sufficiently large constant α so that $p^0(v)\leq\alpha P(v,0)$ for all $v\in(-\infty,V_F)$ that is possible under the assumptions on p^0. To check this, just realize that due to the continuity and derivability from the left assumptions at $v=V_F$, there is $M>0$ such that

$$\limsup_{v\to V_F}\frac{p^0(v)}{V_F-v}\leq M$$

and thus, we can choose α large enough so that $M<\frac{\alpha}{V_F-V_R}$ and $p^0(v)\leq\alpha P(v,0)$ for all $v\in(V_F-\epsilon_0,V_F)$ for some $\epsilon_0>0$. Taking α even larger, it is easy to show that $p^0(v)\leq\alpha P(v,0)$ for all $v\in(-\infty,V_F)$ using that p^0 is bounded in $(-\infty,V_F)$.

Now, we can use Lemmas 4.3 and 4.4 to deduce that $p(v,t)\leq\alpha P(v,t)$ for all $v\in(-\infty,V_F)$ and $t\geq 0$, and thus

$$N(t)\leq\exp(t)\frac{a}{V_F-V_R}$$

for all $t>0$. This shows that $T^*=\infty$ and the stated uniform time dependent bounds. $\square$

The second class of universal super-solutions is time independent but is only defined for $v \geq V_0 := 0$. We consider the solutions Q_1 and Q_2 to the problems

$$\begin{cases} -aQ_1' - vQ_1 = a & \text{on } (V_R, V_F), \qquad Q_1(V_F) = 0, \\ -aQ_2' - vQ_2 = 0 & \text{on } (0, V_R), \qquad Q_2(V_R) = Q_1(V_R), \end{cases} \tag{4.3}$$

and we define Q equal to Q_1 on $[V_R, V_F]$ and to Q_2 on $[0, V_R]$. Notice that as mentioned in the introduction, we can assume without loss of generality that $0 < V_R < V_L$

Lemma 4.6. *The function Q is non negative and strictly monotone decreasing on $[0, V_F]$. Moreover, given any $\beta > 0$ the function βQ is a universal classical super-solution to equation* (1.2) *on $[0, V_F] \times \mathbb{R}^+$ with $\bar{N}(t) = a\beta$.*

Proof. The positivity and monotonicity properties are straightforward. Since $vQ_1 = vQ_2$ at $v = V_R$, and since $Q_1(V_F) = 0$, we have

$$-a\frac{\partial(\beta Q_1)}{\partial v}(V_R) + a\frac{\partial(\beta Q_2)}{\partial v}(V_R) = a\beta = -a\frac{\partial(\alpha Q_1)}{\partial v}(V_F),$$

and in particular $\bar{N}(t) \equiv \bar{N} = a\beta$. This together with the definition of Q in (4.3), implies that

$$\frac{\partial}{\partial v}\left(-a\frac{\partial(\beta Q)}{\partial v} - v\beta Q\right) = \bar{N}\delta(v = V_R).$$

The conclusion follows. □

4.2 Change of variable

The next step is to reduce equation (1.2), away from the singularity at $v = V_R$ and with $N(t)$ considered as a data, to the linear heat equation on a time varying domain. This allows us to later use standard regularizing effects. This kind of change of variables was at the basis of the existence result in [10] since the whole problem (1.2)-(1.5) is equivalent to a free boundary problem for the heat equation with a Dirac Delta source in a moving point. This connection to a Stefan-like free boundary problem clarifies the fact that for $b < 0$ we have global existence of solutions while for $b > 0$ there is blow-up for any value of b for suitably chosen initial data. We refer to [14, 13, 10] for further discussions on this connection to free boundary problems.

The construction of the change of variables is well-known in kinetic theory, since it is the one making equivalent the heat equation and the linear Fokker-Planck equation, see [11] for more references and nonlinear versions of it, together with the method of characteristics to eliminate a linear drift term, we only include the details here for the sake of completeness and to do it adapted to our purposes. Since we want to use it in the regions where the classical solutions are smooth, we will avoid writing the right-hand side in the new variables, we refer to [10] for this. On a formal level we first write

$$q(y, \tau) := f(\tau)p(h(y, \tau), g(\tau))$$

for some arbitrary smooth functions f, g and h. Direct computations lead to the equality

$$\begin{aligned} &\frac{1}{f(\tau)g'(\tau)}\left[\partial_\tau q - a\partial_{yy}q\right](y, \tau) \\ &\quad = \left[\partial_t p - a\frac{(\partial_y h(y, \tau))^2}{g'(\tau)}\partial_{vv}p + \frac{f'(\tau)}{f(\tau)g'(\tau)}p + \frac{\partial_\tau h(y, \tau) - a\partial_{yy}h(y, \tau)}{g'(\tau)}\partial_v p\right](h(y, \tau), g(\tau)). \end{aligned} \tag{4.4}$$

To annihilate the right-hand side of (4.4), in view of (1.2), we therefore impose the conditions

$$(\partial_y h(y,\tau))^2 = g'(\tau), \qquad \frac{\partial_\tau h(y,\tau)}{g'(\tau)} = -h(y,\tau) + bN(g(\tau)), \qquad \text{and} \qquad f'(\tau) = -f(\tau)g'(\tau). \tag{4.5}$$

Notice that the first equality actually implies that $\partial_{yy}h$ vanishes. In terms of $t = g(\tau)$, the second equation in (4.5) translates into

$$\frac{d}{dt}h(y, g^{-1}(t)) = -h(y, g^{-1}(t)) + bN(t),$$

whose solution is given by the variation of constants formula (choosing a reference time $t_0 \geq 0$)

$$h(y,\tau) = h(y, g^{-1}(t_0)) \exp(-(t-t_0)) + \int_{t_0}^{t} bN(s) \exp(-(t-s))\, ds.$$

In particular,

$$(\partial_y h(y,\tau))^2 = (\partial_y h(y, g^{-1}(t_0)))^2 \exp(-2(t-t_0)).$$

In view of the invariance of the heat equation by translation and by the scaling $(y,\tau) \to (\lambda y, \lambda^2 \tau)$, and since $\partial_y h(y, g^{-1}(t_0))$ does not depend on y, there is no loss of generality in requiring that $h(y, g^{-1}(t_0)) = y$ and that $g^{-1}(t_0) = \frac{1}{2}$, so that the first equation in (4.5) is met if and only if

$$g(\tau) = t_0 + \frac{1}{2}\log(2\tau),$$

and then the third one if and only if

$$f(\tau) = C\tau^{-\frac{1}{2}}$$

for some arbitrary constant C which only reflects the linearity of the equation. Choosing it conveniently we finally write

$$q(y,\tau) = \exp(-(t-t_0))\, p\Big(\exp(-(t-t_0))y + \int_{t_0}^{t} bN(s)\exp(-(t-s))\, ds, t\Big), \tag{4.6}$$

where $t = g(\tau) = t_0 + \frac{1}{2}\log(2\tau)$, or equivalently

$$\tau = \frac{1}{2}\exp(2(t-t_0)). \tag{4.7}$$

Summarizing, we have

Lemma 4.7. *Let p be a classical solution to (1.2)-(1.5) and $0 \leq t_0 \leq T$. Then the function q defined by (4.6) is a classical solution to the heat equation*

$$\partial_\tau q - a\partial_{yy} q = 0$$

on the set Ω_{t_0} of (τ, y) in $\mathbb{R}^2$ such that $\frac{1}{2}\exp(-2t_0) \leq \tau \leq \frac{1}{2}\exp(2(T-t_0))$,

$$y < \sqrt{2\tau}V_F - \int_0^{\frac{1}{2}\log(2\tau)} bN(s+t_0)\exp(s)\, ds, \qquad \textit{and} \qquad y \neq \sqrt{2\tau}V_R - \int_0^{\frac{1}{2}\log(2\tau)} bN(s+t_0)\exp(s)\, ds.$$

Remark 4.8. *Let us finally point out, althought not needed for our purposes, that q also satisfies*

$$\partial_\tau q - a\partial_{yy} q = M(\tau)\delta\left(v - \sqrt{2\tau}V_R + \int_0^{\frac{1}{2}\log(2\tau)} bN(s+t_0)\exp(s)\,ds\right)$$

in the distributional sense for (τ, y) in $\mathbb{R}^2$ such that $\frac{1}{2}\exp(-2t_0) < \tau < \frac{1}{2}\exp(2(T-t_0))$ and

$$y < \sqrt{2\tau}V_F - \int_0^{\frac{1}{2}\log(2\tau)} bN(s+t_0)\exp(s)\,ds\,,$$

where

$$M(\tau) = -a\frac{\partial q}{\partial y}\left(\sqrt{2\tau}V_F - \int_0^{\frac{1}{2}\log(2\tau)} bN(s+t_0)\exp(s)\,ds, \tau\right).$$

This is the Stefan-like free boundary problem that was used for the well-posedness theory in [10].

4.3 A uniform bound

Let $p^0 \in L^1_+ \cap L^\infty(-\infty, V_F) \cap C^1((-\infty, V_F])$ with $p^0(V_F) = 0$ be an initial datum for (1.2)-(1.5). Then, there exists α^0 sufficiently large (depending on p^0) so that

$$\alpha^0 P(\cdot, 0) > p^0 \qquad \text{on} \qquad (-\infty, V_F)$$

as it was done in the proof of Corollary 4.5. Let then β^0 be sufficiently large so that

$$\beta^0 Q > \alpha^0 P(\cdot, 1) \qquad \text{on} \qquad [0, V_F).$$

We extend the function Q to the interval $(-\infty, 0)$ being constant equal to its value at zero. Since $P(\cdot, 1)$ is also constant on that interval, we have

$$\beta^0 Q > \alpha^0 P(\cdot, 1) \qquad \text{on} \qquad (-\infty, V_F). \tag{4.8}$$

We are going to show the following lemma which obviously shows Theorem 4.1.

Lemma 4.9. *If β^0 is sufficiently large (depending now only on a, b, V_R, $\|p^0\|_\infty$, and the derivative from the left of p^0 at $v = V_F$), then $p(v,t) \le \beta^0 Q(v)$ for all $v \in (-\infty, V_F]$ and for all $t \ge 0$.*

Assume by contradiction that this is not the case and let t_0 be the first time for which there exists $v_0 \in (-\infty, V_F)$ such that $p(t_0, v_0) = \beta^0 Q(v_0)$. Since P is a universal super-solution to (1.2) which is decreasing in v and increasing in t, it follows from Lemma 4.3 (with the choice $V_0 = -\infty$) that $p(\cdot, t) \le \alpha^0 P(\cdot, t)$ for all $t \ge 0$. Therefore from (4.8), we deduce that $t_0 \ge 1$. Also, since $\beta^0 Q$ is a universal supersolution on $(0, V_F)$, we infer once more from Lemma 4.3 (with the choice $V_0 = 0$) that we can choose $v_0 \le 0$, and thus

$$p(v_0, t_0) = \beta^0 Q(v_0) = \beta^0 Q(0).$$

For $t \le t_0$, we have $p(\cdot, t) \le \beta^0 Q$ and in particular

$$N(t) = -a\partial_x p(V_F, t) \le -a\beta^0 Q'(V_F) = a\beta^0.$$

Let q be defined from p by the change of variable (4.6). In view of the previous estimate on $N(t)$ and since $b \le 0$, we have, for $\tau_0 := \frac{1}{2}\exp(-1) \le \tau \le \frac{1}{2}$, that is $t_0 - \frac{1}{2} \le t \le t_0$ due to (4.7),

$$\sqrt{2\tau} V_R + \int_{\frac{1}{2}\log(2\tau)}^{0} bN(s+t_0)\exp(s)\,ds \ge \sqrt{2\tau} V_R + \int_{\frac{1}{2}\log(2\tau)}^{0} ba\beta^0 \exp(s)\,ds = \sqrt{2\tau}(V_R - ba\beta^0) + ba\beta^0.$$

In particular, we obtain

$$\sqrt{2\tau} V_R + \int_{\frac{1}{2}\log(2\tau)}^{0} bN(s+t_0)\exp(s)\,ds \ge \frac{\sqrt{2\tau}}{2} V_R$$

for $\max(\frac{1}{2}\exp(-1), \tau_{\min}) \le \tau \le \frac{1}{2}$, where

$$\tau_{\min} = \frac{1}{2}\left(1 - \frac{V_R}{V_R - 2ba\beta^0}\right)^2.$$

Notice that

$$\frac{1}{2} - \tau_{\min} \ge \frac{1}{2}\frac{V_R}{V_R - 2ba\beta^0},$$

and as as consequence, we have the inclusion of the parabolic cylinders

$$\Lambda_{v_0,r} := [v_0 - r, v_0 + r] \times [\frac{1}{2} - \frac{r^2}{a}, \frac{1}{2}] \subseteq \Omega_{t_0},$$

where Ω_{t_0} is defined in Lemma 4.7, provided

$$r \le \frac{1}{2}\exp(-\frac{1}{2})V_R \qquad \text{and} \qquad \frac{r^2}{a} \le \min\left(\frac{1}{2}(1-\exp(-1)), \frac{1}{2}\frac{V_R}{V_R - 2ba\beta^0}\right). \tag{4.9}$$

We are now in position to make use of the regularizing effect of parabolic equations.

Lemma 4.10. *If q is a solution of the heat equation*

$$\partial_\tau q - a\partial_{yy} q = 0$$

on the cylinder $\Lambda_{v_0,r}$, then we have the estimate

$$|q(v_0, \frac{1}{2})| \le Kar^{-3}\|q\|_{L^1(\Lambda_{v_0,r})}$$

where $K > 0$ is a universal constant.

Proof. Using the scaling $(y,\tau) \mapsto \Big((y - v_0)/r, (\tau - \frac{1}{2})a/r^2\Big)$ one reduces the estimate to the case $a = 1$, $r = 1$, on the unit cylinder $[-1,1] \times [-1,0]$, which yields the constant K by standard parabolic regularization. □

Lemma 4.10 together with the property that $p(t,\cdot)$ is a probability density give

$$\beta^0 Q_2(0) = p(v_0, t_0) = q(v_0, \frac{1}{2}) \le Kar^{-3}\int_{\frac{1}{2}-\frac{r^2}{a}}^{\frac{1}{2}}\int_{v_0-r}^{v_0+r} q(y,\tau)\,dy\,d\tau \le \frac{K}{r} = O(\sqrt{\beta}^0)$$

due to (4.9). This is a contradiction if we choose β^0 large enough concluding the proof of both Lemma 4.9 and Theorem 4.1.

Acknowledgment: JAC acknowledges support from projects MTM2011-27739-C04-02, 2009-SGR-345 from Agència de Gestió d'Ajuts Universitaris i de Recerca-Generalitat de Catalunya, the Royal Society through a Wolfson Research Merit Award, and the Engineering and Physical Sciences Research Council (UK) grant number EP/K008404/1. BP and DS are supported by the french "ANR blanche" project Kibord: ANR-13-BS01-0004 funded by the French Ministry of Research.

References

[1] P. C. Bressloff, *Waves in neural media*, Lecture Notes on Mathematical Modelling in the Life Sciences, Springer, New York, 2014. From single neurons to neural fields.

[2] R. Brette and W. Gerstner, *Adaptive exponential integrate-and-fire model as an effective description of neural activity*, Journal of neurophysiology, 94 (2005), pp. 3637–3642.

[3] N. Brunel, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking networks*, J. Comp. Neurosci., 8 (2000), pp. 183–208.

[4] N. Brunel and V. Hakim, *Fast global oscillations in networks of integrate-and-fire neurons with long firing rates*, Neural Computation, 11 (1999), pp. 1621–1671.

[5] M. J. Cáceres, J. A. Carrillo, and B. Perthame, *Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states*, Journal of Mathematical Neuroscience, 1 (2011), p. 7.

[6] M. J. Cáceres, J. A. Carrillo, and L. Tao, *A numerical solver for a nonlinear fokker-planck equation representation of neuronal network dynamics*, J. Comp. Phys., 230 (2011), pp. 1084–1099.

[7] M. J. Cáceres and B. Perthame, *Beyond blow-up in excitatory integrate and fire neuronal networks: refractory period and spontaneous activity*, Journal of Theoretical Biology, 350 (2014), pp. 81–89.

[8] D. Cai, L. Tao, A. V. Rangan, and D. W. McLaughlin, *Kinetic theory for neuronal network dynamics*, Commun. Math. Sci., 4 (2006), pp. 97–127.

[9] J. A. Carrillo, S. Cordier, D. G., and M. S., *Complexity reduction of rate-equations models for two-choice decision-making*, PLoS ONE, 8 (2013), p. e80820.

[10] J. A. Carrillo, M. d. M. González, M. P. Gualdani, and M. E. Schonbek, *Classical solutions for a nonlinear fokker-planck equation arising in computational neuroscience*, Comm. in Partial Differential Equations, 38 (2013), pp. 385–409.

[11] J. A. Carrillo and G. Toscani, *Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J., 49 (2000), pp. 113–142.

[12] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *Global solvability of a networked integrate-and-fire model of McKean-Vlasov type*, 2012. arXiv 1211.0299.

[13] A. Fasano and M. Primicerio, *A critical case for the solvability of Stefan-like problems*, Math. Methods Appl. Sci., 5 (1983), pp. 84–96.

[14] A. Friedman, *Free boundary problems for parabolic equations. I. Melting of solids.*, J. Math. Mech., 8 (1959), pp. 499–517.

[15] L. Lapicque, *Recherches quantitatives sur l'excitation électrique des nerfs traitée comme une polarisation*, J. Physiol. Pathol. Gen, 9 (1907), pp. 620–635.

[16] M. Mattia and P. Del Giudice, *Population dynamics of interacting spiking neurons*, Phys. Rev. E, 66 (2002), p. 051917.

[17] P. Michel, S. Mischler, and B. Perthame, *General relative entropy inequality: an illustration on growth models*, J.Math.Pures Appl., 84 (2005), pp. 1235–1260.

[18] B. Perthame, *Transport equations in biology*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.

[19] B. Perthame and D. Salort, *On a voltage-conductance kinetic system for integrate & fire neural networks*, Kinet. Relat. Models, 6 (2013), pp. 841–864.

[20] R. Ratcliff and P. Smith, *A comparison of sequential sampling sampling models for two-choice reaction time*, Psychol Rev, 111 (2004), pp. 333–367.

[21] A. Renart, N. Brunel, and X.-J. Wang, *Mean-field theory of irregularly spiking neuronal populations and working memory in recurrent cortical networks*, in Computational Neuroscience: A comprehensive approach, J. Feng, ed., Chapman & Hall/CRC Mathematical Biology and Medicine Series, 2004.

[22] H. Tuckwell, *Introduction to Theoretical Neurobiology*, Cambridge Univ. Press, Cambridge, 1988.