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► **To cite this version:**

Daniele Di Pietro, Alexandre Ern. Equilibrated tractions for the Hybrid High-Order method. 2014. hal-01079026v1

HAL Id: hal-01079026

<https://hal.science/hal-01079026v1>

Preprint submitted on 31 Oct 2014 (v1), last revised 1 Dec 2014 (v2)

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Equilibrated tractions for the Hybrid High-Order method

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Abstract

We show how to recover equilibrated face tractions for the hybrid high-order method for linear elasticity recently introduced in [1], and prove that these tractions are optimally convergent.

Résumé

Tractions équilibrées pour la méthode hybride d'ordre élevé. Nous montrons comment obtenir des tractions de face équilibrées pour la méthode hybride d'ordre élevé pour l'élasticité linéaire récemment introduite dans [1] et prouvons que ces tractions convergent de manière optimale.

1. Introduction

In this work, we devise a reformulation the Hybrid High-Order method for linear elasticity introduced in [1] that identifies its local equilibrium (or conservation) properties and, at the same time, shows how balanced (or conservative) face tractions can be constructed by element-by-element post-processing. This is an important complement to the original analysis, as local equilibrium is an essential property in practice.

Our focus is on the linear elasticity problem described hereafter. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a bounded connected polygonal or polyhedral domain. For $X \subset \bar{\Omega}$, we denote by $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ respectively the standard inner product and norm of $L^2(X)$, and a similar notation is used for $L^2(X)^d$ and $L^2(X)^{d \times d}$. For a given external load $\mathbf{f} \in L^2(\Omega)^d$, we consider the following problem: Find $\mathbf{u} \in H_0^1(\Omega)^d$ such that

$$2\mu(\nabla_s \mathbf{u}, \nabla_s \mathbf{v})_\Omega + \lambda(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega. \quad (1)$$

with $\mu > 0$ and $\lambda \geq 0$ real numbers representing the scalar Lamé coefficients and ∇_s denoting the symmetric gradient operator. More general boundary conditions can be considered. The material is organized as follows: in Section 2 we outline the original formulation of the HHO method; in Section 3 we derive the local equilibrium formulation based on a new local displacement reconstruction.

2. The Hybrid High-Order method

We consider admissible mesh sequences in the sense of [2, Sect. 1.4]. Each mesh \mathcal{T}_h in the sequence is a finite collection $\{T\}$ of nonempty, disjoint, open, polytopic elements such that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} \bar{T}$ and $h = \max_{T \in \mathcal{T}_h} h_T$ (with h_T the diameter of T), and there is a matching simplicial submesh of \mathcal{T}_h with locally equivalent mesh size and which is shape-regular in the usual sense. For all $T \in \mathcal{T}_h$, the faces of T are collected in the set \mathcal{F}_T . Additionally, interfaces are collected in the set \mathcal{F}_h^i and boundary faces in \mathcal{F}_h^b . The diameter of a face $F \in \mathcal{F}_h$ is denoted by h_F . For the sake of brevity, we abbreviate $a \lesssim b$ the inequality $a \leq Cb$ for positive real numbers a and b and a generic constant C which can depend on the mesh regularity, on μ , d , and the polynomial degree, but is independent of h and λ . We also introduce the notation $a \simeq b$ for the uniform equivalence $a \lesssim b \lesssim a$.

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Let a polynomial degree $k \geq 1$ be fixed. The local and global spaces of degrees of freedom (DOFs) are

$$\mathbf{U}_T^k := \mathbb{P}_d^k(T)^d \times \left\{ \times_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F)^d \right\} \quad \forall T \in \mathcal{T}_h, \quad \mathbf{U}_h^k := \left\{ \times_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T)^d \right\} \times \left\{ \times_{F \in \mathcal{F}_h} \mathbb{P}_{d-1}^k(F)^d \right\}. \quad (2)$$

A generic collection of DOFs from \mathbf{U}_h^k is denoted by $\mathbf{v}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h})$, and, for a given $T \in \mathcal{T}_h$, $\mathbf{v}_T = (\mathbf{v}_T, (\mathbf{v}_F)_{F \in \mathcal{F}_T}) \in \mathbf{U}_T^k$ indicates its restriction to \mathbf{U}_T^k . For all $T \in \mathcal{T}_h$, we define a high-order local displacement reconstruction operator $\mathbf{p}_T^k : \mathbf{U}_T^k \rightarrow \mathbb{P}_d^{k+1}(T)^d$ by solving the following (well-posed) pure traction problem: For a given $\mathbf{v}_T \in \mathbf{U}_T^k$, $\mathbf{p}_T^k \mathbf{v}_T$ is such that

$$(\nabla_s \mathbf{p}_T^k \mathbf{v}_T, \nabla_s \mathbf{w})_T = (\nabla_s \mathbf{v}_T, \nabla_s \mathbf{w})_T + \sum_{F \in \mathcal{F}_T} (\mathbf{v}_F - \mathbf{v}_T, \nabla \mathbf{w} \mathbf{n}_{TF})_F \quad \forall \mathbf{w} \in \mathbb{P}_d^{k+1}(T)^d, \quad (3)$$

and the rigid-body motion components of $\mathbf{p}_T^k \mathbf{v}_T$ are prescribed so that $\int_T \mathbf{p}_T^k \mathbf{v}_T = \int_T \mathbf{v}_T$ and $\int_T \nabla_{ss}(\mathbf{p}_T^k \mathbf{v}_T) = \sum_{F \in \mathcal{F}_T} \int_F \frac{1}{2}(\mathbf{n}_{TF} \otimes \mathbf{v}_F - \mathbf{v}_F \otimes \mathbf{n}_{TF})$ where ∇_{ss} is the skew-symmetric gradient operator. Additionally, we define the divergence reconstruction $D_T^k : \mathbf{U}_T^k \rightarrow \mathbb{P}_d^k(T)$ such that, for a given $\mathbf{v}_T \in \mathbf{U}_T^k$,

$$(D_T^k \mathbf{v}_T, q)_T = (\nabla \cdot \mathbf{v}_T, q)_T + \sum_{F \in \mathcal{F}_T} (\mathbf{v}_F - \mathbf{v}_T, q \mathbf{n}_{TF})_F \quad \forall q \in \mathbb{P}_d^k(T). \quad (4)$$

We introduce the local bilinear form $a_T : \mathbf{U}_T^k \times \mathbf{U}_T^k \rightarrow \mathbb{R}$ such that

$$a_T(\mathbf{w}_T, \mathbf{v}_T) := 2\mu \{(\nabla_s \mathbf{p}_T^k \mathbf{w}_T, \nabla_s \mathbf{p}_T^k \mathbf{v}_T)_T + s_T(\mathbf{w}_T, \mathbf{v}_T)\} + \lambda(D_T^k \mathbf{w}_T, D_T^k \mathbf{v}_T)_T, \quad (5)$$

where the stabilizing bilinear form $s_T : \mathbf{U}_T^k \times \mathbf{U}_T^k \rightarrow \mathbb{R}$ is such that

$$s_T(\mathbf{w}_T, \mathbf{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1}(\pi_F^k(\mathbf{P}_T^k \mathbf{w}_T - \mathbf{w}_F), \pi_F^k(\mathbf{P}_T^k \mathbf{v}_T - \mathbf{v}_F))_F, \quad (6)$$

and the second displacement reconstruction $\mathbf{P}_T^k : \mathbf{U}_T^k \rightarrow \mathbb{P}_d^{k+1}(T)$ is such that, for all $\mathbf{v}_T \in \mathbf{U}_T^k$, $\mathbf{P}_T^k \mathbf{v}_T := \mathbf{v}_T + (\mathbf{p}_T^k \mathbf{v}_T - \pi_T^k \mathbf{p}_T^k)$. Let $\mathbf{I}_T^k : H^1(T)^d \rightarrow \mathbf{U}_T^k$ be the reduction map such that, for all $T \in \mathcal{T}_h$ and all $\mathbf{v} \in H^1(T)^d$, $\mathbf{I}_T^k \mathbf{v} = (\pi_T^k \mathbf{v}, (\pi_F^k \mathbf{v})_{F \in \mathcal{F}_T})$. The potential reconstruction \mathbf{p}_T^k and the bilinear form s_T are conceived so that they satisfy the following two key properties:

(i) *Stability*. For all $\mathbf{v}_T \in \mathbf{U}_T^k$,

$$\|\nabla_s \mathbf{p}_T^k \mathbf{v}_T\|_T^2 + s_T(\mathbf{v}_T, \mathbf{v}_T) \simeq \|\nabla_s \mathbf{v}_T\|_{L^2(T)^{d \times d}}^2 + j_T(\mathbf{v}_T, \mathbf{v}_T), \quad (7)$$

with bilinear form $j_T : \mathbf{U}_T^k \times \mathbf{U}_T^k \rightarrow \mathbb{R}$ such that $j_T(\mathbf{w}_T, \mathbf{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1}(\mathbf{w}_T - \mathbf{w}_F, \mathbf{v}_T - \mathbf{v}_F)_F$.

(ii) *Approximation*. For all $\mathbf{v} \in H^{k+2}(T)^d$,

$$\{\|\nabla_s(\mathbf{v} - \mathbf{p}_T^k \mathbf{I}_T^k \mathbf{v})\|_T^2 + s_T(\mathbf{I}_T^k \mathbf{v}, \mathbf{I}_T^k \mathbf{v})\}^{1/2} \lesssim h_T^{k+1} \|\mathbf{v}\|_{H^{k+2}(T)^d}. \quad (8)$$

We observe that, unlike s_T , the bilinear form j_T only satisfies $j_T(\mathbf{I}_T^k \mathbf{v}, \mathbf{I}_T^k \mathbf{v}) \lesssim h^k \|\mathbf{v}\|_{H^{k+1}(T)^d}$. The discrete problem reads: Find $\mathbf{u}_h \in \mathbf{U}_{h,0}^k := \{\mathbf{u}_h \in \mathbf{U}_h^k \mid \mathbf{u}_F \equiv \mathbf{0} \quad \forall F \in \mathcal{F}_h^b\}$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\mathbf{w}_T, \mathbf{v}_T) = \sum_{T \in \mathcal{T}_h} (f, \mathbf{v}_T)_T \quad \forall \mathbf{v}_h \in \mathbf{U}_{h,0}^k. \quad (9)$$

The following convergence result was proved in [1]:

Theorem 1 (Energy error estimate). *Let $\mathbf{u} \in H_0^1(\Omega)^d$ and $\mathbf{u}_h \in \mathbf{U}_{h,0}^k$ denote the unique solutions to (1) and (9), respectively, and assume $\mathbf{u} \in H^{k+2}(\Omega)^d$ and $\nabla \cdot \mathbf{u} \in H^{k+1}(\Omega)$. Then, letting $\hat{\mathbf{u}}_h \in \mathbf{U}_{h,0}^k$ be such that $\hat{\mathbf{u}}_T := \mathbf{I}_T^k \mathbf{u}$ for all $T \in \mathcal{T}_h$, the following holds (with $\|\mathbf{v}_T\|_{a,T}^2 = a_T(\mathbf{v}_T, \mathbf{v}_T)$ for all $\mathbf{v}_T \in \mathbf{U}_T^k$):*

$$\sum_{T \in \mathcal{T}_h} \|\mathbf{u}_T - \hat{\mathbf{u}}_T\|_{a,T}^2 \lesssim h^{2(k+1)} (\|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^{k+1}(\Omega)})^2. \quad (10)$$

Moreover, assuming elliptic regularity, $\sum_{T \in \mathcal{T}_h} \|\mathbf{u} - \mathbf{p}_T^k \mathbf{u}_T\|_{L^2(T)^d}^2 \lesssim h^{2(k+2)} (\|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^{k+1}(\Omega)})^2$.

3. Local equilibrium formulation

The difficulty in devising an equivalent local equilibrium formulation for problem (9) comes from the stabilization term s_T , which introduces a non-trivial coupling of interface DOFs inside each element. In this section, we introduce post-processed discrete displacement and stress reconstructions that allow us to circumvent this difficulty. For a given element $T \in \mathcal{T}_h$, define the following bilinear form on $\underline{\mathbf{U}}_T^k$:

$$\tilde{a}_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := 2\mu \{(\nabla_s \mathbf{p}_T^k \underline{\mathbf{w}}_T, \nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T)_T + j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T)\} + \lambda(D_T^k \underline{\mathbf{w}}_T, D_T^k \underline{\mathbf{v}}_T)_T, \quad (11)$$

where the only difference with respect to the bilinear form a_T defined by (5) is that we have stabilized using j_T . Denoting by $\|\cdot\|_{\tilde{a},T}$ the local seminorm induced by \tilde{a}_T on $\underline{\mathbf{U}}_T^k$, one can prove that for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$,

$$\|\underline{\mathbf{v}}_T\|_{\tilde{a},T} \simeq \|\underline{\mathbf{v}}_T\|_{a,T}. \quad (12)$$

We next define the isomorphism $\underline{\mathbf{c}}_T^k : \underline{\mathbf{U}}_T^k \rightarrow \underline{\mathbf{U}}_T^k$ such that

$$\tilde{a}_T(\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) = a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) + (2\mu)j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \quad \forall \underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k, \quad (13)$$

and rigid-body motion components prescribed as above. We also introduce the stress reconstruction $\underline{\mathbf{S}}_T^k : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{P}_d^k(T)^{d \times d}$ such that

$$\underline{\mathbf{S}}_T^k := (2\mu \nabla_s \mathbf{p}_T^k + \lambda \mathbf{I}_d D_T^k) \circ \underline{\mathbf{c}}_T^k. \quad (14)$$

Lemma 2 (Equilibrium formulation). *The bilinear form a_T defined by (5) is such that, for all $\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$,*

$$a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) = (\underline{\mathbf{S}}_T^k \underline{\mathbf{w}}_T, \nabla_s \underline{\mathbf{v}}_T)_T + \sum_{F \in \mathcal{F}_T} (\boldsymbol{\tau}_{TF}(\underline{\mathbf{w}}_T), \underline{\mathbf{v}}_F - \underline{\mathbf{v}}_T)_F, \quad (15)$$

with interface traction $\boldsymbol{\tau}_{TF} : \underline{\mathbf{U}}_T^k \rightarrow \mathbb{P}_{d-1}^k(F)^d$ such that

$$\boldsymbol{\tau}_{TF}(\underline{\mathbf{w}}_T) = \underline{\mathbf{S}}_T^k \underline{\mathbf{w}}_T \cdot \mathbf{n}_{TF} + h_F^{-1} [((\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T)_F - \underline{\mathbf{w}}_F) - ((\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T)_T - \underline{\mathbf{w}}_T)]. \quad (16)$$

Proof. Let $\tilde{\underline{\mathbf{w}}}_T := \underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T$. We have, using the definitions (13) of $\underline{\mathbf{c}}_T^k$ and (11) of the bilinear form \tilde{a}_T ,

$$\begin{aligned} a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) &= \tilde{a}_T(\tilde{\underline{\mathbf{w}}}_T, \underline{\mathbf{v}}_T) - (2\mu)j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \\ &= 2\mu \{(\nabla_s \mathbf{p}_T^k \tilde{\underline{\mathbf{w}}}_T, \nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T)_T + j_T(\tilde{\underline{\mathbf{w}}}_T - \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T)\} + \lambda(D_T^k \tilde{\underline{\mathbf{w}}}_T, D_T^k \underline{\mathbf{v}}_T)_T \\ &= (\underline{\mathbf{S}}_T^k \underline{\mathbf{w}}_T, \nabla_s \underline{\mathbf{v}}_T)_T + \sum_{F \in \mathcal{F}_T} (\underline{\mathbf{S}}_T^k \underline{\mathbf{w}}_T \cdot \mathbf{n}_{TF}, \underline{\mathbf{v}}_F - \underline{\mathbf{v}}_T)_F + (2\mu)j_T(\tilde{\underline{\mathbf{w}}}_T - \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T), \end{aligned}$$

where we have concluded observing that, using (3) with $\mathbf{w} = \mathbf{p}_T^k \tilde{\underline{\mathbf{w}}}_T$ and (4) with $q = D_T^k \tilde{\underline{\mathbf{w}}}_T$, and recalling the definition (14) of $\underline{\mathbf{S}}_T^k$, $(2\mu \nabla_s \mathbf{p}_T^k \tilde{\underline{\mathbf{w}}}_T, \nabla_s \mathbf{p}_T^k \underline{\mathbf{v}}_T)_T + \lambda(D_T^k \tilde{\underline{\mathbf{w}}}_T, D_T^k \underline{\mathbf{v}}_T)_T = (\underline{\mathbf{S}}_T^k \underline{\mathbf{w}}_T, \nabla_s \underline{\mathbf{v}}_T)_T + \sum_{F \in \mathcal{F}_T} (\underline{\mathbf{S}}_T^k \underline{\mathbf{w}}_T \cdot \mathbf{n}_{TF}, \underline{\mathbf{v}}_F - \underline{\mathbf{v}}_T)_F$. To obtain (15), it suffices to use the definition of j_T . \square

Lemma 3 (Equilibrated interface tractions). *Let $\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_{h,0}^k$ denote the unique solution to (9). Then, for all $F \in \mathcal{F}_h^i$ such that $F \subset \partial T_1 \cap \partial T_2$, the following holds:*

$$\boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2 F}(\underline{\mathbf{u}}_{T_2}) = \mathbf{0}. \quad (17)$$

Proof. Take as an ansatz collection of DOFs in (9) $\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T \in \mathcal{T}_h}, (\mathbf{v}_F)_{F \in \mathcal{F}_h}) \in \underline{\mathbf{U}}_{h,0}^k$ such that \mathbf{v}_F is arbitrary in $\mathbb{P}_{d-1}^k(F)^d$, whereas $\mathbf{v}_{F'} \equiv \mathbf{0}$ for all $F' \in \mathcal{F}_h \setminus \{F\}$ and $\mathbf{v}_T \equiv \mathbf{0}$ for all $T \in \mathcal{T}_h$. Then, using (15) with $\underline{\mathbf{w}}_T = \underline{\mathbf{u}}_T$ in (9), it is inferred that $a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) = (\boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2 F}(\underline{\mathbf{u}}_{T_2}), \mathbf{v}_F)_F = 0$, which proves the desired result since $\boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2 F}(\underline{\mathbf{u}}_{T_2}) \in \mathbb{P}_{d-1}^k(F)^d$. \square

To conclude, we show that the locally post-processed solution yields a new collection of DOFs that is an equally good approximation of the exact solution as is the discrete solution $\underline{\mathbf{u}}_h$. Consequently, the equilibrated face numerical tractions defined in (16) optimally converge to the exact tractions.

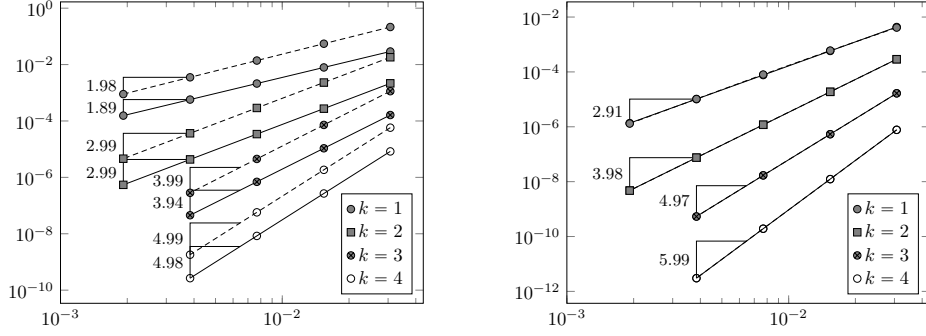


Figure 1: Convergence results in the energy-norm (left) and L^2 -norm (right) for the solution to (9) (solid lines) and its post-processing based on \mathbf{c}_T^k (dashed lines). The right panel shows that the post-processing has no sizable effect on element unknowns.

Proposition 4 (Convergence for $\mathbf{c}_T^k \mathbf{u}_T$). *Using the notation of Theorem 1, the following holds:*

$$\sum_{T \in \mathcal{T}_h} \|\mathbf{c}_T^k \mathbf{u}_T - \hat{\mathbf{u}}_T\|_{a,T}^2 \lesssim h^{2(k+1)} (\|\mathbf{u}\|_{H^{k+2}(\Omega)^d} + \lambda \|\nabla \cdot \mathbf{u}\|_{H^{k+1}(\Omega)})^2. \quad (18)$$

Proof. Let $T \in \mathcal{T}_h$. Recalling (13), we have

$$\begin{aligned} \tilde{a}_T(\mathbf{c}_T^k \mathbf{u}_T - \hat{\mathbf{u}}_T, \mathbf{v}_T) &= a_T(\mathbf{u}_T, \mathbf{v}_T) + (2\mu)j_T(\mathbf{u}_T, \mathbf{v}_T) - \tilde{a}_T(\hat{\mathbf{u}}_T, \mathbf{v}_T) \\ &= a_T(\mathbf{u}_T - \hat{\mathbf{u}}_T, \mathbf{v}_T) + (2\mu)s_T(\hat{\mathbf{u}}_T, \mathbf{v}_T) + (2\mu)j_T(\mathbf{u}_T - \hat{\mathbf{u}}_T, \mathbf{v}_T). \end{aligned}$$

Hence, using the Cauchy–Schwarz inequality followed by the stability property (7) and the norm equivalence (12) for \mathbf{v}_T ,

$$\begin{aligned} |\tilde{a}_T(\mathbf{c}_T^k \mathbf{u}_T - \hat{\mathbf{u}}_T, \mathbf{v}_T)| &\leq \left\{ \|\mathbf{u}_T - \hat{\mathbf{u}}_T\|_{a,T}^2 + (2\mu)s_T(\hat{\mathbf{u}}_T, \hat{\mathbf{u}}_T) + (2\mu)j_T(\mathbf{u}_T - \hat{\mathbf{u}}_T, \mathbf{u}_T - \hat{\mathbf{u}}_T) \right\}^{1/2} \|\mathbf{v}_T\|_{\tilde{a},T} \\ &\lesssim \left\{ \|\mathbf{u}_T - \hat{\mathbf{u}}_T\|_{a,T}^2 + (2\mu)s_T(\hat{\mathbf{u}}_T, \hat{\mathbf{u}}_T) \right\}^{1/2} \|\mathbf{v}_T\|_{\tilde{a},T}. \end{aligned}$$

Using (12) followed by the latter inequality, we infer that

$$\|\mathbf{c}_T^k \mathbf{u}_T - \hat{\mathbf{u}}_T\|_{a,T} \lesssim \|\mathbf{c}_T^k \mathbf{u}_T - \hat{\mathbf{u}}_T\|_{\tilde{a},T} = \sup_{\mathbf{v}_T \in \underline{\mathbf{u}}_T^k \setminus \{\mathbf{0}\}} \frac{\tilde{a}_T(\mathbf{c}_T^k \mathbf{u}_T - \hat{\mathbf{u}}_T, \mathbf{v}_T)}{\|\mathbf{v}_T\|_{\tilde{a},T}} \lesssim \left\{ \|\mathbf{u}_T - \hat{\mathbf{u}}_T\|_{a,T}^2 + (2\mu)s_T(\hat{\mathbf{u}}_T, \hat{\mathbf{u}}_T) \right\}^{1/2}.$$

The estimate (18) then follows squaring the above inequality, summing over $T \in \mathcal{T}_h$, and using (10) and (8), respectively, to bound the terms in the right-hand side. \square

To assess the estimate (18), we have numerically solved the pure displacement problem with exact solution $\mathbf{u} = (\sin(\pi x_1) \sin(\pi x_2) + 1/2x_1, \cos(\pi x_1) \cos(\pi x_2) + 1/2x_2)$ for $\mu = \lambda = 1$ on a h -refined sequence of triangular meshes. The corresponding convergence results are presented in Figure 1. In the left panel, we compare the quantities on the left-hand side of estimates (10) and (18). Although the order of convergence is the same, the original solution \mathbf{u}_h displays better accuracy in the energy-norm. This is essentially due to face unknowns, as confirmed in the right panel, where the square roots of the quantities $\sum_{T \in \mathcal{T}_h} \|\mathbf{u}_T - \hat{\mathbf{u}}_T\|_T^2$ and $\sum_{T \in \mathcal{T}_h} \|\mathbf{c}_T^k \mathbf{u}_T - \hat{\mathbf{u}}_T\|_T^2$ (both of which are discrete L^2 -norms of the error) are plotted.

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