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The L^1 gradient flow of a generalized scale invariant Willmore energy for radially non increasing functions.

François DAYRENS

October 30, 2014

Abstract

We use the minimizing movement theory to study the gradient flow associated with a non-regular relaxation of a geometric functional derived from the Willmore energy. Thanks to the coarea formula, one can define a Willmore energy on regular functions of $L^1(\mathbb{R}^d)$. This functional is extended to every L^1 function by taking its lower semi-continuous envelope. We study the flow generated by this relaxed energy for radially non-increasing functions, i.e. functions with balls as level sets. In the first part of the paper, we prove a coarea formula for the relaxed energy of such functions. Then we show that the flow consists on an erosion of the initial data. The erosion speed is given by a first order ordinary equation.

1 Introduction and settings.

This paper is devoted to De Giorgi's minimizing movement solutions of a generalized scale invariant Willmore flow. These solutions are obtained as limits of discrete generalized Willmore flows built upon the minimization in $L^1(\mathbb{R}^d)$ of a relaxation of the functional

$$\mathcal{F}(u) = \int_{\mathbb{R}^d} |\nabla u| \left| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right|^{d-1} dx + \frac{1}{2\tau} \|u - u_n\|_{L^1}^2 \quad (1)$$

where u_n is updated at each discrete time step and $u_0 \in L^1$ is an initial data (see below for more details on minimizing movement solutions).

The first term in \mathcal{F} is an instance of the generalized p -Willmore energy

$$W_p(u) = \int_{\mathbb{R}^d} |\nabla u| \left| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right|^p dx$$

which arose in image processing and 2D or 3D shape completion [4] [12] [19] [32].

Using the coarea formula (see Theorem 1.7), one can decompose $W_p(u)$ on every level sets:

$$W_p(u) = \int_{-\infty}^{+\infty} \left(\int_{\partial\{u \geq t\}} |\vec{H}|^p d\mathcal{H}^{d-1} \right) dt$$

with \vec{H} the mean curvature vector on the boundary of the level set $\{u \geq t\}$ and \mathcal{H}^{d-1} the $(d-1)$ -Hausdorff measure.

This leads us to consider the p -Willmore energy of a hypersurface M of \mathbb{R}^d :

$$W_p(M) = \int_M |H|^p dA$$

where H is its scalar mean curvature and dA its area measure. As one can see, for $p = d-1$, $W_{d-1}(\lambda M) = W_{d-1}(M)$ for any $\lambda > 0$ therefore we call \mathcal{F} in (1) scale invariant. We refer to [5] to see how direct method of calculus of variations can be used on that kind of energies.

For $p = 2$ and $d = 3$, W_2 is the traditional conformal Willmore energy of a surface in \mathbb{R}^3 and its classical smooth flow is given by

$$\frac{d}{dt}M_t = \Delta_{M_t}H_t + 2H_t(H_t^2 - K_t)$$

with Δ_{M_t} the Laplace-Beltrami operator on M_t , H_t its mean curvature and K_t its Gauss curvature. We refer to [37], [28, 29] and [19] for topics concerning this regular flow.

For $p = 2$ and $d = 2$, W_2 is the famous Bernoulli-Euler *elastica*:

$$\int \kappa^2 ds$$

where κ is the curvature of a regular curve in the plane. The flow of a family of curves (γ_t) is given by the equation

$$\frac{d}{dt}\gamma_t = \left(\kappa_t'' + \frac{1}{2}\kappa_t^3 \right) \vec{n}_t$$

with \vec{n}_t the unit normal vector of γ_t . The elastica theory is a part of mechanics of solid materials initially developed by Euler and Bernoulli and devoted to the study of the deflection of thin beams, we refer to [36] for a historical overview. Elastica has many applications in shape completion and inpainting, we refer to [13] and [15] for an account on image topics. We can also cite [12] for a functional lifting approach on approximating elastica for image processing. Reader will find in [17] links between elastica and amodal completion methods based on human visual perception.

As we can see with both flows above, the smooth Willmore flow is a geometric flow defined by a fourth order equation. A celebrated example of geometric flow is the mean curvature flow, which is defined by a second order equation derived from the area of a hypersurface. The mean curvature flow received a lot of attention. For instance, in [26], Huisken studied the regular flow and singularity appearance. To deal with singularities, the flow is extended in [11] to varifolds, a weak generalization of surfaces. The so called level set approach to mean curvature flow is introduced in [16] and [21, 22, 23, 24], see [25] for a good overview on surface evolution using level set methods. Finally, [14] and [31] use a minimizing movement approach to study this flow and the second paper even provides an algorithm to compute it.

Among many generalizations of the area of a surface, let us focus on the theory developed in the framework of *BV* functions, i.e. functions with bounded variation. In this setting, the area of the boundary of a set E coincides with the total variation of its characteristic functions:

$$P(E) = TV(1_E)$$

where TV is the total variation (see [20] and [38]) and P the perimeter of E (or the area of ∂E). The coarea formula for BV functions states that

$$TV(u) = \int_{-\infty}^{+\infty} P(\{u \geq t\}) dt.$$

It is natural to wonder whether the gradient flow of TV can be recovered from the individual gradient flows of the perimeter of each level set. Authors in [6] show that the total variation flow is different and, when starting with $u_0 = 1_E$, it follows

$$u_t = (1 - c_E t)_+ 1_E$$

with c_E a constant depending only on E . In contrast, for the individual flow of level set, [31] proved that the minimizing movement flow for P corresponds to the mean curvature flow for regular surfaces. Besides, TV is a first order functional, similar to our second order functional \mathcal{F} , used in [35] for image denoising problems.

Returning to our fourth order flow in L^1 defined by the minimization of (1), we will prove a very similar evolution for radially non increasing functions in the weaker setting of De Giorgi's minimizing movement theory [1] [18] [31]. We start with a non negative, bounded, compactly supported and radially non increasing initial data $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$. That means there exists a function $r : [0, a] \rightarrow \mathbb{R}^+$ such that

$$\{u_0 \geq t\} = B(0, r(t)).$$

We prove in this paper that the minimizing movement flow is an erosion of the initial data. This erosion is described by an ordinary differential equation involving only the radius function r .

The plan of the paper is as follow: in section 2, we prove a coarea formula for Willmore energy. That allows us to show in section 3 that a minimizer of (1) is radially non increasing. We iterate to construct a minimizing sequence and then, we describe the minimizing movement. The flow we find is very close to the total variation flow in [6]; that is an erosion of the initial data. We prove in Theorem 4.1 that the minimizing movement $t \mapsto u_t$ exists and it is described through a function $\lambda : [0, +\infty[\rightarrow \mathbb{R}^+$ (see Figure 1):

$$u_t(x) = \min(u_0(x), \lambda(t)).$$

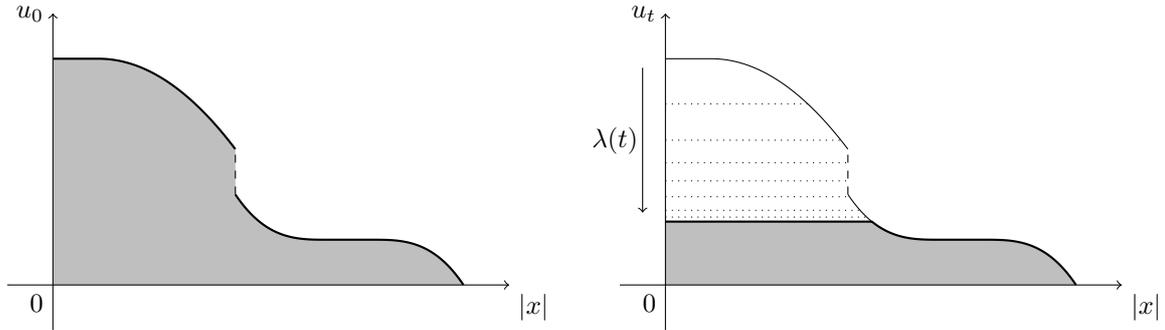


Figure 1: Minimizing movement of radially non-increasing function for Willmore energy.

Moreover, Theorem 4.2 states that λ is a solution of the ordinary differential equation

$$\begin{cases} \lambda'(t) &= -\frac{\omega_d}{\alpha_d^2} \frac{1}{r_0(\lambda(t))^{2d}} \\ \lambda(0) &= \sup u_0, \end{cases} \quad (2)$$

where $\frac{\omega_d}{\alpha_d^2}$ is simply a dimensional constant. ODE (2) contains one question we cannot answer for now: what happens if the initial condition $\lambda(0)$ satisfies $r(\lambda(0)) = 0$? Implicitly, ODE (2) says the erosion starts with a non finite speed.

1.1 Minimizing movements.

We now recall the notion of minimizing movement. The idea is to define the gradient flow for a possibly non regular functional. Let H be a Hilbert space and consider

$$f : H \rightarrow \overline{\mathbb{R}}.$$

Minimizing movements can be defined in Banach spaces but are easier to understand and to handle in Hilbert spaces, refer to [2], [3] and [18] for a good account on this theory. In this paper we will build a minimizing movement in a self contained way. Recall first the basics of this theory.

If f is regular (at least C^1) then its (minimizing) gradient flow is defined by the ordinary differential equation :

$$\begin{cases} u'(t) &= -\nabla f(u(t)) \\ u(0) &= x. \end{cases} \quad (3)$$

In order to extend this notion to a non regular functional, we can use the implicit Euler scheme for solving this ordinary differential equation: for small time parameter $\tau > 0$:

$$\frac{u_{n+1} - u_n}{\tau} = -\nabla f(u_{n+1}).$$

Write

$$F(y) = f(y) + \frac{1}{2\tau} \|y - u_n\|^2$$

and the equation becomes

$$\nabla F(u_{n+1}) = 0.$$

Thus, we transform ODE (3) into the following minimization problem: given a starting point $x \in H$, find a sequence (u_n) satisfying $u_0 = x$ and

$$u_{n+1} \in \operatorname{argmin}_{v \in H} \left(f(v) + \frac{1}{2\tau} \|v - u_n\|^2 \right).$$

Moreover, it yields the same solutions for regular functions as shown by Theorem 1.4.

Definition 1.1 (Minimizing sequence). *Let $\tau > 0$ and $x_0 \in \operatorname{dom}(f) = \{x \in H \mid -\infty < f(x) < +\infty\}$, the minimizing sequence $(x_\tau(n))$ of f starting from x_0 is defined by induction : $x_\tau(0) = x_0$ and $x_\tau(n+1)$ is a minimizer of*

$$y \mapsto f(y) + \frac{1}{2\tau} \|y - x_\tau(n)\|^2.$$

From that sequence, we can define a piecewise constant path

$$u_\tau(t) = x_\tau(\lceil t/\tau \rceil)$$

with $\lceil \cdot \rceil$ the integer part.

Definition 1.2 (Minimizing movement). *We say that $u : [0, +\infty[\rightarrow H$ is a minimizing movement for f starting from x_0 if there exists a minimizing sequence such that u is a uniform limit of u_τ when $\tau \rightarrow 0$ (up to a subsequence $\tau_i \rightarrow 0$).*

Remark 1.3. If the starting point x_0 is a local finite minimizer of f , then $u(t) = x_0$ is the only minimizing movement starting from x_0 . Indeed (adding a constant we can suppose that this minimum is positive) in a ball $B(x_0, \delta)$ we have $0 < f(x_0) \leq f(x)$. Then we obtain

$$f(x_0) < f(x) + \frac{1}{2\tau} \|x - x_0\|^2 \quad \text{on } B(x_0, \delta)$$

with equality only when $x = x_0$.

For τ small enough $\tau < \frac{\delta^2}{2f(x_0)}$, we have

$$f(x_0) < f(x) + \frac{1}{2\tau} \|x - x_0\|^2 \quad \text{on } H$$

with the same equality case.

Thus the minimizing sequence is constant, so be the minimizing movement.

There are lot of results explaining why the minimizing movement corresponds to the classical gradient flow in smooth case, for instance (see [2, esempio 1.1]):

Theorem 1.4. *Consider a C^2 function $f : H \rightarrow \mathbb{R}$. If f is Lipschitz or $\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$ then for all starting point x_0 , f admits a unique minimizing movement starting from x_0 given by*

$$\begin{cases} u'(t) &= -\nabla f(u(t)) \\ u(0) &= x_0. \end{cases}$$

1.2 Scale invariant Willmore energy.

We introduce the Willmore energy to which we will apply the minimizing movement principles. The Willmore energy is especially known for hypersurfaces : consider M a C^2 hypersurface of \mathbb{R}^d with $d \geq 2$ and its principal curvatures $\kappa_1, \dots, \kappa_{d-1}$, then the (scalar) mean curvature on every point p on M is given by

$$H(p) = \frac{1}{d-1} \sum_{i=1}^{d-1} \kappa_i(p).$$

We refer to [9] for generalities on differential geometry. We can now define the Willmore energy of the hypersurface M .

Definition 1.5. Let M be a C^2 hypersurface of \mathbb{R}^d and H its mean curvature, the scale invariant Willmore energy of M is given by

$$W(M) = \int_M |H|^{d-1} d\mathcal{H}^{d-1},$$

with \mathcal{H}^{d-1} the $d-1$ Hausdorff measure on \mathbb{R}^d or equivalently the area measure on M .

The Willmore energy can be also defined with other exponents than $d-1$ but in that case it is no more invariant under scaling transformations. The extension to functions of the Willmore energy is straightforward.

Definition 1.6. Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be a C^2 function, its scale invariant Willmore energy is given by

$$W_f(u) = \frac{1}{(d-1)^{d-1}} \int_{\mathbb{R}^d} |\nabla u| \left| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right|^{d-1} dx.$$

With the convention $|\nabla u| \left| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right|^{d-1} = 0$ if $|\nabla u| = 0$.

The link between the two definitions can be done using the coarea formula. We recall that formula (see [20]).

Theorem 1.7 (General coarea formula). *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a lipschitz continuous map with $m \leq n$ then for every measurable function $g \in L^1(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^m} \left(\int_{f^{-1}(y)} g(x) d\mathcal{H}^{n-m}(x) \right) d\mathcal{H}^m(y) = \int_{\mathbb{R}^n} g(x) J_f(x) d\mathcal{H}^n(x).$$

J_f is the generalized jacobian of f .

Applying this formula to the function

$$g(x) = 1_{\{x \mid J_f(x)=0\}}$$

one obtains the following theorem (similar to Sard's lemma)

Corollary 1.8. *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a lipschitz continuous map with $m \leq n$ then for \mathcal{H}^m -almost every $y \in \mathbb{R}^m$*

$$\mathcal{H}^{n-m}(\{x \in \mathbb{R}^n \mid f(x) = y, J_f(x) = 0\}) = 0.$$

For $m = 1$ we have $J_f(x) = |\nabla f(x)|$ therefore this corollary implies that for almost every $t \in \mathbb{R}$: $\{x \mid f(x) = t\}$ is a hypersurface up to a \mathcal{H}^{n-1} -negligible set.

Consider a hypersurface given by the boundary of a level set of a regular function u i.e. for a t -level set:

$$\{u \geq t\} = \{x \in \mathbb{R}^d \mid u(x) \geq t\}.$$

Then the mean curvature vector at every point of the boundary of the level set is given by

$$\vec{H} = \frac{-1}{d-1} \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) \vec{n}$$

where \vec{n} is the normal vector

$$\vec{n} = \frac{\nabla u}{|\nabla u|}.$$

The mean curvature is simply the scalar:

$$H = \frac{-1}{d-1} \operatorname{div} \frac{\nabla u}{|\nabla u|}.$$

By the previous corollary, almost every boundary of level set of u are regular hypersurfaces and then their Willmore energies can be computed. Summing over all of these energies and using the general coarea formula one will obtain

Proposition 1.9 (coarea formula). *For any C^2 function $u : \mathbb{R}^d \rightarrow \mathbb{R}$, we have*

$$W_f(u) = \int_{-\infty}^{+\infty} W(\partial\{u \geq t\}) dt.$$

1.3 Functional context.

We introduce the general context of this paper. A way to un-regularize the Willmore energy adding good functional analysis properties is to consider a relaxation of W .

Definition 1.10 (functional Willmore energy). *For $u \in L^1$, its Willmore energy is :*

$$W(u) = \begin{cases} W_f(u) & \text{if } u \text{ is } C_c^2 \\ +\infty & \text{otherwise} \end{cases}$$

where C_c^2 is the class of C^2 functions with compact support. We let \overline{W} be the lower semi continuous envelop of W , i.e. \overline{W} is the relaxed energy :

$$\overline{W}(u) = \inf \left\{ \liminf_{n \rightarrow +\infty} W(u_n) \mid u_n \rightarrow u \text{ in } L^1 \right\}.$$

Like for functions, we define a relaxed energy for subset of \mathbb{R}^d .

Definition 1.11 (set convergence). *Let (E_n) be a sequence of \mathbb{R}^d subsets. We say that (E_n) converges to some subset E in L^1 if the characteristic function sequence (1_{E_n}) converges to 1_E in $L^1(\mathbb{R}^d)$.*

Remark 1.12. Recall the symmetric difference between two sets :

$$E, E' \subset \mathbb{R}^d : E \Delta E' = (E \setminus E') \cup (E' \setminus E).$$

(E_n) converges to a set E in L^1 iff $|E_n \Delta E| \rightarrow 0$, with $|A|$ the Lebesgue measure on \mathbb{R}^d of the subset A .

Definition 1.13 (geometric Willmore energy). *Let E be a subset of \mathbb{R}^d . We say E is regular if $|E| > 0$ and its border ∂E belongs to C^2 i.e. it is locally the graph of a C^2 function. We identify the Willmore energy of E with the energy of its border.*

This notion can be extended to every measurable set by relaxing it with the L^1 topology. For a measurable set E , we define :

$$\overline{W}(E) = \inf \left\{ \liminf_{n \rightarrow +\infty} W(E_n) \mid E_n \rightarrow E \text{ in } L^1 \text{ and } E_n \text{ compact} \right\}.$$

By convention $W(\emptyset) = 0$ and then $\overline{W}(E) = 0$ for every negligible set E . We have the following recent result on the Willmore functional, it confirms that W and \overline{W} coincide on regular functions and on regular sets, see [30, Theorem 4.4].

Theorem 1.14. *If $u \in C_c^2(\mathbb{R}^d)$ then $\overline{W}(u) = W(u)$. In the same way for a compact regular subset E , $\overline{W}(E) = W(E)$.*

The relaxation of the Willmore energy has been the purpose of several papers, we refer to [7], [8], [33] and [34] for interesting readings.

1.4 Study objectives.

Now we can explain our main goal and how we will reach it. Consider an initial data $u_0 \in L^1(\mathbb{R}^d)$, we investigate the convergence of a minimizing sequence of the energy \overline{W} in L^1 to a map $t \mapsto u_t \in L^1(\mathbb{R}^d)$. The general problem is far too hard for now and we focus on some particular initial data : the class of radially non-increasing functions.

Definition 1.15 (radially non-increasing function). *$u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a radially non-increasing function if there exists a function $r : \mathbb{R} \rightarrow \mathbb{R}^+$ such that, for almost every $t \in \mathbb{R}$*

$$\{u \geq t\} = \overline{B}(0, r(t))$$

up to a Lebesgue negligible set.

r is called the radius function of u .

As we work with almost everywhere defined functions, it is convenient to consider only purely radial functions $u(x) = u(|x|)$ and $B(0, r(t)) = \{u \geq t\}$ with B representing either the closed or open ball, depending on u and t . All arguments hereafter are valid whenever the sets $B(0, r(t))$ and $\{u \geq t\}$ coincide up to a negligible set (with respect to the Lebesgue measure of \mathbb{R}^d). For the sake of simplicity, we will no longer precise this subtlety.

Here is the main direction of the paper. Starting from a radially non-increasing initial data u_0 , the study of the minimizing movement requires the minimization of the functional

$$\mathcal{F}(u) = \overline{W}(u) + \frac{1}{2\tau} \|u - u_0\|_{L^1}^2.$$

As u_0 is invariant under rotation of \mathbb{R}^d , it is natural to think that a minimizer is also invariant. It is true and the proof is driven by the following ideas. Consider any possibly non radial candidate u , look at its level sets, replace them by balls with same volume. Construct the function \tilde{u} having these balls as level sets. Using the coarea formula, one can compute the Willmore energy of u and \tilde{u} with their level sets. If balls are minimizers of the geometric Willmore energy, we deduce an inequality between Willmore energies of the level sets. This inequality is preserved by coarea formula and then

$$\overline{W}(\tilde{u}) \leq \overline{W}(u).$$

By taking balls as level sets, we get closer from the level sets of the initial condition u_0 . By preserving the volume of the level sets, a classical lemma allows us to compute the L^1 norm of $u - u_0$ and $\tilde{u} - u_0$ using the volumes of (almost) every level sets. That means that \tilde{u} is closer from u_0 than u is, and then

$$\mathcal{F}(\tilde{u}) \leq \mathcal{F}(u).$$

In fact, some arguments are not accurate in generality, we have to be more precise. The first gap is that Willmore minimizers are not the spheres but the planes. We have to add "compact hypersurface" to have spheres as minimizers. As a consequence, our level sets have to be compact. The second difficulty is that the relaxed Willmore energy does not satisfy the coarea formula, in the general sense. This is the subject of the next section and we need to suppose that the functions are non-negative. These two restrictions lead us to consider only non-negative and compactly supported functions and this cannot be removed in order to make these ideas work.

After that, our minimizers are described only through their radius functions r . Studying them becomes easier and it is the subject of the third part. We demonstrate that minimizers are in fact among truncations of the initial radius function r_0 . Estimations on the best beheaded function allows us to iterate the process in order to find a minimizing sequence. This sequence of functions consists on multiple truncations of the initial data until vanishing. Then we prove that these truncations converge to an erosion of the initial data, see Figure 1. Some properties are finally obtained about the erosion speed.

Remark 1.16. In this study, the fidelity term in \mathcal{F} of the minimizing movement is given by a L^1 norm:

$$\|u - u_0\|_{L^1}.$$

Using it with indicator functions $u = 1_E$ and $u_0 = 1_{E_0}$, one finds

$$d_{L^1}(E, E_0) = |E \Delta E_0|.$$

This natural distance d_{L^1} is actually not a good one for the mean curvature flow. Indeed, minimizing movement techniques lead to a pinning phenomenon (see [10, chapter 8]). The mean curvature flow can be stationary for non minimal hypersurfaces (minimal for area). For instance, the minimizing sequence for a circle in the plane is constant for τ small enough therefore the minimizing movement flow of this circle is stationary. To avoid this phenomenon, one rather uses the distance

$$\delta(E, E_0) = \int_{E \Delta E_0} |d_{E_0}(x)| dx$$

with d_E the classical distance or the signed distance to the set E (see [14] and [31]).

In our case, we keep the distance d_{L^1} for two reasons. The first one is that there is no easy coarea

formula equivalent to Lemme 2.3 giving a suitable distance between two functions in terms of the δ -distance between their level sets. The second one is that we work with radial functions i.e. functions with balls as level sets. The (geometric) scale invariant Willmore energy is minimal for every ball (no matter the radius). Whatever the fidelity term we use on level sets, a minimizer will always be the initial ball. Thus the flow of each level set is stationary.

One point of this paper is that even if the level set Willmore flow is stationary, the function flow is non trivial.

Notations

$ \cdot $	either absolute value of a real, modulus of a vector or Lebesgue measure on \mathbb{R}^d
α_d	volume of the unit ball of \mathbb{R}^d
ω_d	area of the unit sphere of \mathbb{R}^d
$A\Delta B$	symmetric difference between the sets A and B
\mathcal{H}^k	k -dimensional Hausdorff measure

2 Coarea formula for Willmore energy.

In this section, we will discuss the coarea formula for Willmore energy (non necessarily scale invariant). Let $p \in]0, +\infty[$: we extend Definitions 1.5 and 1.6 to curvature with power p with the same conventions.

Definition 2.1 (p -Willmore energies.). *Let u be a $C^2(\mathbb{R}^d, \mathbb{R})$ function and E be a C^2 subset of \mathbb{R}^d :*

$$W_p(u) = \frac{1}{(d-1)^p} \int_{\mathbb{R}^d} |\nabla u| \left| \operatorname{div} \frac{\nabla u}{|\nabla u|} \right|^p dx$$

and

$$W_p(E) = \int_{\partial E} |H|^{p-1} d\mathcal{H}^{d-1}.$$

We define also the relaxation of these energies for every function $u \in L^1(\mathbb{R}^d)$ and every measurable subset E :

$$\overline{W}_p(u) = \inf \left(\liminf_{n \rightarrow +\infty} W_p(u_n) \mid u_n \rightarrow u \text{ in } L^1 \right)$$

and

$$\overline{W}_p(E) = \inf \left(\liminf_{n \rightarrow +\infty} W_p(E_n) \mid E_n \rightarrow E \text{ in } L^1 \right).$$

Notice that $W_{d-1} = W$ with the previous notation 1.10.

The purpose here is to prove a coarea type formula for \overline{W}_p :

$$\int_{-\infty}^{+\infty} \overline{W}_p(\{u \geq t\}) dt = \overline{W}_p(u).$$

The validity of such a formula has been studied in [34]. In our special case of radially non-increasing functions, we will prove that it is true.

We will first state three lemmas, proofs are given in the appendix. The first one is a coarea inequality proved in [4, Theorem 4 and Remark 2].

Lemma 2.2 (coarea inequality). *Let $u \in L^1(\Omega)$, we have*

$$\int_{-\infty}^{+\infty} \overline{W}_p(\{u \geq t\}) dt \leq \overline{W}_p(u).$$

The second one is a classical lemma based on Cavalieri formula.

Lemma 2.3 (L^1 norm). For measurable functions $u, v : \mathbb{R}^d \mapsto \mathbb{R}$ we have

$$\int_{\mathbb{R}^d} |u - v| dx = \int_{-\infty}^{+\infty} |\{u \geq t\} \Delta \{v \geq t\}| dt$$

with $\{u \geq t\}$ the t -height level set of u .

The last one is only for technical issue.

Lemma 2.4 (Reattachment). Let $a, b, \alpha, \beta \in \mathbb{R}$ such that $a, b < 0$. There exists a decreasing function $f \in C^2([0, 1])$ such that

$$\begin{aligned} f(0) &= 1 & f(1) &= 0 \\ f'(0) &= a & f'(1) &= b \\ f''(0) &= \alpha & f''(1) &= \beta. \end{aligned}$$

Now we state and prove the main result of this section.

Theorem 2.5. Let $v : \mathbb{R}^d \rightarrow \mathbb{R}$ be a radially non-increasing function which is non-negative, bounded and compactly supported. Then

$$\int_{-\infty}^{+\infty} \overline{W}_p(\{v \geq t\}) dt = \overline{W}_p(v).$$

For such function v , we write $r : \mathbb{R} \rightarrow \mathbb{R}^+$ the radius function defined by the relation

$$\{v \geq t\} = B(0, r(t)).$$

Notice that $r(0) < +\infty$ is the radius of the support of v , $r(t) > 0$ for all t in $[0, \sup v[$ and $r(t) = 0$ for $t > \sup v$. We have then

$$\begin{aligned} \overline{W}_p(\{v \geq t\}) &= \omega_d r(t)^{d-1-p}, \\ \overline{W}_p(v) &= \omega_d \int_0^{\sup v} r(t)^{d-1-p} dt \end{aligned}$$

with ω_d the area of the unit sphere of \mathbb{R}^d .

Proof. Step 1: let r be the previous radius function, we write $b = \sup v < +\infty$ and $R = r(0) + 1$ (the "+1" will be important for strictly decrease). r is a non increasing function looking like on Figure 2.

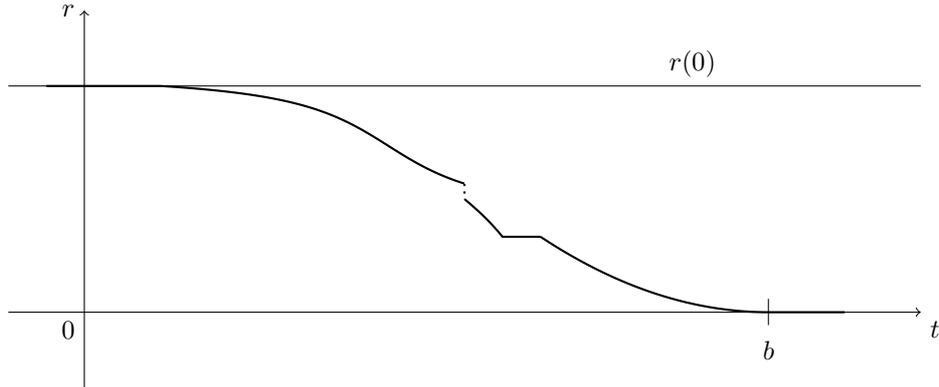


Figure 2: Graph of the function r .

Step 2: regular case.

We assume some hypotheses on r in order to have v regular enough to apply the classical coarea formula on W_p . Let r be a C^2 decreasing function on $]0, b[$. We assume there exist $\varepsilon > 0$ and $q \in]0, 1/4[$ such that

$$\forall t \in [b - \varepsilon, b], \quad r(t) = (b - t)^q$$

$$\forall t \in [0, \varepsilon], \quad r(t) = R - t^q$$

In that case, r is one to one and its derivative does not vanish on $]0, b[$. Then we have $v(x) = r^{-1}(|x|)$ and the function v is C^2 on $B(0, r(0)) \setminus \{0\}$. However, for $|x| < r(b - \varepsilon)$, we have $v(x) = b - |x|^{1/q} = b - (|x|^2)^{1/2q}$ which is C^2 on $x = 0$ because $1/2q > 2$. In the same way, for $r(\varepsilon) < |x| < R$, we have $v(x) = (R - |x|)^{1/q}$ which links with the zero function with C^2 regularity on $\mathbb{R}^d \setminus B(0, r(0))$. This allows us to have $v \in C_c^2(\Omega)$ and using the coarea formula (proposition 1.9), we get

$$W_p(v) = \int_0^{+\infty} W_p(\{v \geq t\}) dt.$$

Step 3: general case.

We will approximate the function r with a regular enough sequence to use the previous case and with a good enough convergence to pass to the limit.

We obtain regularity on r using a well chosen convolution. Indeed, the Willmore energy of a radius r ball is (up to a constant) r^{d-1-p} and, when $d-1-p < 0$, that means the energy sequence is the integral of the inverse-like functions $t \mapsto 1/r_n(t)^{|d-1-p|}$. In order to have convergence of the energy we will use the monotone convergence theorem, that is why we need a non increasing sequence (r_n) . In the case $d-1-p \geq 0$, the convergence is straightforward because the sequence is bounded and with compact support. Finally, we adjust this C^2 sequence of functions to add the t^q tails.

- Let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a mollifier such that

$$\begin{aligned} &\rho \text{ is of class } C^2, \\ &\rho > 0 \text{ on }]0, 1[, \\ &\text{supp}(\rho) = [0, 1] \\ &\text{and } \int_{\mathbb{R}} \rho(t) dt = 1. \end{aligned}$$

We extend r by R for $t < 0$ and by 0 for $t > b$. For $n \in \mathbb{N}^*$, picking $\rho_n(t) = n\rho(nt)$ and $\tilde{r}_n = r * \rho_n$. So \tilde{r}_n is C^2 on \mathbb{R} (because r is bounded and ρ is compact supported). Moreover, we have the formula :

$$\begin{aligned} \tilde{r}_n(t) &= \int_{\mathbb{R}} nr(t-s)\rho(ns)ds \\ &= \int_0^{1/n} nr(t-s)\rho(ns)ds \\ &= \int_0^1 r(t-s/n)\rho(s)ds. \end{aligned}$$

Thanks to that formula, we can prove that (\tilde{r}_n) is a non increasing sequence of non increasing functions. $\forall t \in \mathbb{R}, \forall s \in [0, 1]$:

$$\begin{aligned} &t - \frac{s}{n} \leq t - \frac{s}{n+1} \\ \Rightarrow &r\left(t - \frac{s}{n+1}\right) \leq r\left(t - \frac{s}{n}\right) \\ \Rightarrow &r\left(t - \frac{s}{n+1}\right)\rho(s) \leq r\left(t - \frac{s}{n}\right)\rho(s) \\ \Rightarrow &\tilde{r}_{n+1}(t) \leq \tilde{r}_n(t). \end{aligned}$$

So (\tilde{r}_n) is a non increasing sequence.

$\forall t \leq t', \forall s \in [0, 1]$:

$$\begin{aligned} &t - \frac{s}{n} \leq t' - \frac{s}{n} \\ \Rightarrow &r\left(t' - \frac{s}{n}\right) \leq r\left(t - \frac{s}{n}\right) \\ \Rightarrow &r\left(t' - \frac{s}{n}\right)\rho(s) \leq r\left(t - \frac{s}{n}\right)\rho(s) \\ \Rightarrow &\tilde{r}_n(t') \leq \tilde{r}_n(t). \end{aligned}$$

So \tilde{r}_n is a non increasing function.

Let us now define

$$b_n = \inf(t \in \mathbb{R} \mid \tilde{r}_n = 0)$$

and check that

$$b_n = b + \frac{1}{n}.$$

Indeed,

$\forall t > b + 1/n, \forall s \in [0, 1] :$

$$t - \frac{s}{n} > b \Rightarrow r\left(t - \frac{s}{n}\right) = 0 \Rightarrow \tilde{r}_n(t) = 0.$$

So $b_n \leq b + 1/n$.

$\forall t \in]b, b + 1/n[, \exists s_0 \in]0, 1[$ such that $t - s_0/n = b$. Thus, $r(t - s/n) = 0$ on $[0, s_0[$ and $r(t - s/n) > 0$ on $]s_0, 1]$. Thus $\tilde{r}_n(t) > 0$. As \tilde{r}_n is non increasing, we have

$$\forall t < b + 1/n, \tilde{r}_n(t) > 0.$$

So $b_n \geq b + 1/n$.

Moreover $\tilde{r}_n < R$ on \mathbb{R}^{+*} because $\forall t \in]0, 1/n[, \exists s_0 \in]0, 1[$ such that $t - s_0/n = 0$. Thus, $r(t - s/n) \leq r(0) < R$ on $[0, s_0[$ and $r(t - s/n) = R$ on $]s_0, 1]$. Thus $\tilde{r}_n(t) < R$. Remember \tilde{r}_n is non increasing for larger t .

• Now we modify this sequence in order to have decreasing functions, to be able to add the t^q tails after $t = b_n$ and at $t = 0$ and then to have a C^2 function v_n . Let $p^* = |d - 1 - p|$ if $d - 1 - p < 0$ and $p^* = 1$ else and take the auxiliary function (see Figure 3)

$$\forall t \in [0, b_n], \quad h_n(t) = \left(1 - \frac{1}{n^{1/2p^*}}\right) \frac{t}{b_n}.$$

And define

$$\forall t \in [0, b_n], \quad r_n(t) = R - h_n(t)(R - \tilde{r}_n(t)).$$

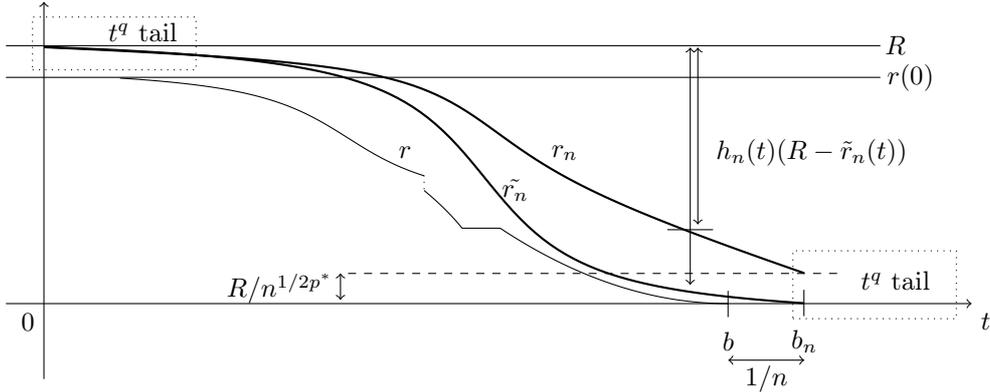


Figure 3: Regularization of r on $[0, b_n]$.

This sequence remains non increasing. Indeed, pick t in $[0, b_n]$, (b_n) is non increasing and non negative so $\left(\frac{t}{b_n}\right)$ is also non increasing and non positive. $\left(1 - \frac{1}{n^{1/2p^*}}\right)$ is non increasing and non negative so $(h_n(t))$ is non increasing and non negative. $(\tilde{r}_n(t))$ is non increasing and bounded by R . So $((R - \tilde{r}_n(t))h_n(t))$ is non decreasing. Thus $(r_n(t))$ is non increasing. Furthermore $0 \leq h_n(t) \leq 1 - 1/n^{1/2p^*}$ therefore the sequence is bounded by $R/n^{1/2p^*}$ and R .

Moreover, we easily have that r_n is decreasing thanks to the calculation :

$$\forall t \in]0, b_n], \quad r'_n(t) = - \underbrace{\left(1 - \frac{1}{n^{1/2p^*}}\right)}_{<0} \left(\underbrace{\frac{1}{b_n}}_{>0} \underbrace{(R - \tilde{r}_n(t))}_{>0} + \underbrace{\frac{t}{b_n}}_{>0} \underbrace{(-\tilde{r}'_n(t))}_{\geq 0} \right) < 0.$$

• Now we build the t^q tail beyond $t = b_n$. With $0 < q < \min(1, 1/2p^*)$, consider

$$\forall t \in [c_n, b_n + 1/n], \quad \eta(t) = \left(b + \frac{2}{n} - t\right)^q.$$

We took

$$c_n = b + \frac{2}{n} - \left(\frac{R}{2}\right)^{1/q} \frac{1}{n^{1/2pq}},$$

c_n was chosen to satisfy $\eta(c_n) = \frac{R}{2n^{1/2p^*}}$ and for $n > (R/2)^{\frac{1/q}{1/2p^*-1}}$ we have $b_n < c_n < b_n + 1/n$. Also

$$r_n(b_n) = \frac{R}{n^{1/2p^*}} > \frac{R}{2n^{1/2p^*}} = \eta(c_n),$$

$$r'_n(b_n) = -R \left(1 - \frac{1}{n^{1/2p^*}}\right) < 0 \quad \text{and} \quad \eta'(c_n) = -q \left(\left(\frac{R}{2}\right)^{1/q} \frac{1}{n^{1/2p^*}}\right)^{q-1} < 0.$$

Thanks to Lemma 2.4, we can reattach η et r_n between $[b_n, c_n]$ in a C^2 and decreasing way (see Figure 4).

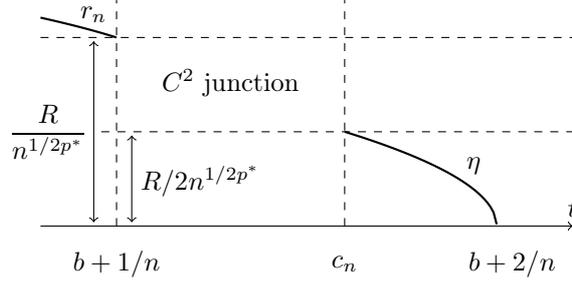


Figure 4: Regularization of r near b .

Now we extend r_n by 0 beyond $b_n + 1/n$ that allows it to be in the regular case.

- Once more we add the t^q tail near $t = 0$ considering

$$\forall t \in [0, 1/n], \quad \theta(t) = R + \frac{1}{n} - t^q.$$

Let $a_n = (1/2n)^{1/q}$ be such that $\theta(a_n) = R + 1/2n$. Notice we have $0 < a_n < 1/n$ when $n > (1/2)^{\frac{1/q}{1/q-1}}$. Furthermore

$$r_n(1/n) < R < \theta(a_n), \quad r'_n(1/n) < 0 \quad \text{and} \quad \theta'(a_n) = -q \left(\frac{1}{2n}\right)^{\frac{q-1}{q}} < 0.$$

Then, as before, we can do a C^2 reattachment using Lemma 2.4 (see Figure 5).

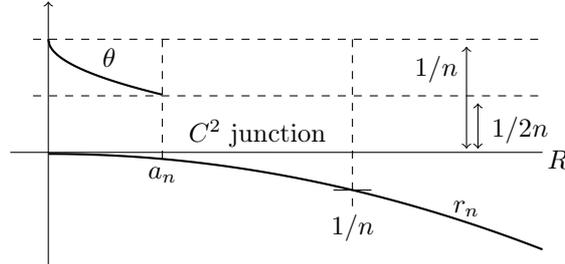


Figure 5: Regularization of r near a .

The sequence (r_n) built with the two tails satisfies the assumption of the regular case, therefore we have

$$W_p(v_n) = \int_0^{\sup v_n} W_p(B(0, r_n(t))) dt = \omega_d \int_0^{b+2/n} r_n(t)^{d-1-p} dt.$$

We just have to check that v_n converges to v in L^1 and that the integral above converges to the corresponding integral with r .

• Thanks to the convolution, \tilde{r}_n converges to r (expanded by R and 0) in L^1_{loc} . Thus, in the compact $[0, b+2]$, up to a subsequence,

$$\forall t \in [0, b+2] \text{ ae}, \quad \tilde{r}_n(t) \rightarrow r(t) \quad \text{when } n \rightarrow +\infty.$$

Moreover, for all $t \in]0, b]$, there exists a rank n such that $1/n < t$ and so, as t is far away from the tail in 0,

$$|r_n(t) - \tilde{r}_n(t)| \leq |R - \tilde{r}_n(t)| \left| \frac{t}{b+1/n} \right| \frac{1}{n^{1/2p}} \leq \frac{R}{n^{1/2p}} \xrightarrow{n \rightarrow +\infty} 0.$$

And for all $t > b$, there exists a rank n such that $b+2/n < t$ and so

$$r_n(t) = \tilde{r}_n(t) = r(t) = 0.$$

Thus, r_n converges to r almost everywhere on $[0, b+2]$. Take $v_n(x) = r_n^{-1}(|x|)$ and using lemma 2.3

$$\|v - v_n\|_{L^1} = \int_0^{b+2} \alpha_d |r_n(t)^d - r(t)^d| dt$$

with α_d the volume of the unit ball of \mathbb{R}^d .

As $|r_n(t)^d - r(t)^d|$ converges to 0 almost everywhere and $|r_n(t)^d - r(t)^d| \leq 2R^d$ then, by Lebesgue dominated convergence theorem,

$$v_n \xrightarrow{n \rightarrow +\infty} v \quad \text{in } L^1(\mathbb{R}^d).$$

• If $d-1-p \geq 0$ then, as $|r_n| \leq R$, by Lebesgue dominated convergence theorem, we have

$$W_p(v_n) = \omega_d \int_0^{b+2/n} r_n(t)^{d-1-p} dt \xrightarrow{n \rightarrow +\infty} \omega_d \int_0^b r(t)^{d-1-p} dt = \int_0^{+\infty} W_p(\{v \geq t\}) dt.$$

If $d-1-p < 0$ then we cut the integral on several intervals :

$$\frac{W_p(v_n)}{\omega_d} = \int_0^{b+2/n} \frac{1}{r_n(t)^{p^*}} dt = \underbrace{\int_0^{1/n} \frac{1}{r_n(t)^{p^*}} dt}_{I_1} + \underbrace{\int_{1/n}^b \frac{1}{r_n(t)^{p^*}} dt}_{I_2} + \underbrace{\int_b^{c_n} \frac{1}{r_n(t)^{p^*}} dt}_{I_3} + \underbrace{\int_{c_n}^{b+2/n} \frac{1}{r_n(t)^{p^*}} dt}_{I_4}.$$

Notice after both C^2 reattachments the sequence (r_n) may no more be decreasing but we need it only on $[1/n, b]$ where it is satisfied.

For I_1 , for all $t \in [0, 1/n] : r_n(t) \geq r_n(1/n) \geq r_n(b/2) \geq r(b/2) > 0$ because r_n is a non-increasing function and (r_n) a non-increasing sequence bounded by below by r . So

$$\forall t \in [0, 1/n], \quad \frac{1}{r_n(t)^{p^*}} \leq \frac{1}{r(b/2)^{p^*}}.$$

Then $I_1 \rightarrow 0$ when $n \rightarrow +\infty$.

For I_2 , $\left(1_{t > 1/n} \frac{1}{r_n(t)^{p^*}}\right)$ is a non-decreasing sequence of functions converging almost everywhere to $\frac{1}{r(t)^{p^*}}$ on $]0, b[$. By the monotone convergence theorem,

$$I_2 \xrightarrow{n \rightarrow +\infty} \int_0^b \frac{1}{r(t)^{p^*}} dt.$$

For I_3 , on $[b, c_n] : r_n(t) \geq R/n^{1/2p^*}$, so

$$I_3 \leq \frac{\sqrt{n}}{R^{p^*}} (c_n - b) \leq \frac{2}{R^{p^*}} \frac{1}{\sqrt{n}}.$$

Then $I_3 \rightarrow 0$ when $n \rightarrow +\infty$.

And for I_4 , on $[c_n, b + 2/n]$:

$$r_n(t) = \left(b + \frac{2}{n} - t\right)^q.$$

Using Cauchy-Schwarz inequality (and writing $A = \left(\frac{R}{2}\right)^{1/q} \frac{1}{n^{1/2qp^*}}$) :

$$I_4 = \int_0^A \frac{ds}{s^{qp^*}} \leq \sqrt{\int_0^1 1_{s < A}^2 ds} \sqrt{\int_0^1 \frac{ds}{s^{2qp^*}}}.$$

The first square root is equal to $\left(\frac{R}{2}\right)^{1/2q} \frac{1}{n^{1/4qp^*}} \rightarrow 0$ when $n \rightarrow +\infty$. The second one is finite because $2qp^* < 1$. Then $I_4 \rightarrow 0$ when $n \rightarrow +\infty$.

Finally we have, in every case :

$$W_p(v_n) = \omega_d \int_0^{b+2/n} r_n(t)^{d-1-p} dt \xrightarrow{n \rightarrow +\infty} \omega_d \int_0^b r(t)^{d-1-p} dt = \int_0^{+\infty} \overline{W}_p(\{v \geq t\}) dt.$$

As $v_n \rightarrow v$ in L^1 and by the definition of \overline{W}_p we have

$$\overline{W}_p(v) \leq \int_0^{+\infty} \overline{W}_p(\{v \geq t\}) dt.$$

The other inequality is always true (lemma 2.2), consequently

$$\overline{W}_p(v) = \int_0^{+\infty} \overline{W}_p(\{v \geq t\}) dt.$$

□

Remark 2.6. We can have a slightly better result by replacing v bounded by

$$\int_0^{+\infty} r(t)^{d-1-p} dt < +\infty$$

and $v \in L^1$. Indeed, take the sequence v_n by cutting v where its values are over n . The corresponding radius sequence are $t \mapsto r(t)1_{t < n}$. Apply Theorem 2.5 to v_n , the Willmore energies converge to the right limit thanks to the new hypothesis.

3 Minimizing movement of radial functions.

Now we can begin the study of the minimizing movement associated with the Willmore energy \overline{W} with a radially non-increasing initial data. Consider the functional space

$$L_{c+}^1(\mathbb{R}^d) = \{u \in L^1(\mathbb{R}^d) \mid u \geq 0, \text{ supp}(u) \text{ is compact}\}.$$

and let $u_0 \in L_{c+}^1(\mathbb{R}^d)$ be a radially non-increasing initial data. The description of the minimizing movement is done in two steps. First we study the minimization of

$$\mathcal{F}(u) = \overline{W}(u) + \frac{1}{2\tau} \|u - u_0\|_{L^1}^2$$

in $L_{c+}^1(\mathbb{R}^d)$ and we let u_1 be a minimizer. Iterate the scheme replacing u_0 by u_1 to construct a minimizing sequence (u_n) .

In a second step, we study the behaviour of the minimizing sequence when $\tau \rightarrow 0$: we write $u_\tau(t) = u_{[t/\tau]}$ and we look for a limit function $u : [0, +\infty[\rightarrow L_{c+}^1(\mathbb{R}^d)$ of u_τ when $\tau \rightarrow 0$, maybe up to a subsequence.

We will describe the minimizing movement throughout three results. The first one confirms a natural intuition that there exists a radially non-increasing minimizer of \mathcal{F} , i.e. it has the same structure as the initial data. Our second result states that this minimizer is a truncation of the initial data (all values larger than some λ are replaced with λ). And then we prove that the minimizing movement is an erosion of the initial data with speed given by an ordinary differential equation.

We denote by r_0 the radius function of the initial data u_0 .

3.1 Radially non-increasing minimizer.

First we prove that among all functions of $L^1_{c_+}(\mathbb{R}^d)$, the radially non-increasing ones are the best for our minimization problem, see Theorem 3.3 below.

Consider any candidate $u \in L^1_{c_+}(\mathbb{R}^d)$. Replace every level set by a ball with same volume. Doing that, one constructs a radially non-increasing function \tilde{u} with lower Willmore energy (using the previous coarea formula 2.5) and closer to the initial data because level sets are well included. That is why

$$\mathcal{F}(\tilde{u}) \leq \mathcal{F}(u).$$

Let us now write all the details. Let u be a candidate function in $L^1_{c_+}(\mathbb{R}^d)$, define the radius function of u by the relation

$$|\{u \geq t\}| = |B(0, r(t))|.$$

r is a non-increasing function such that

$$\text{supp } r = [0, \sup u]$$

and

$$r(0) < +\infty \text{ because } |B(0, r(0))| = |\text{supp } u|.$$

Then we define

$$\tilde{u}(x) = \sup\{t \in \mathbb{R} \mid r(t) \geq |x|\}$$

such that

$$\{\tilde{u} \geq t\} = \overline{B}(0, r(t)).$$

\tilde{u} is a radially non-increasing function with r as radius function.

Use Lemma 2.3 to see that

$$\|\tilde{u}\|_{L^1} = \int_0^{+\infty} |\{\tilde{u} \geq t\}| dt = \int_0^{+\infty} |B(0, r(t))| dt = \int_0^{+\infty} |\{u \geq t\}| dt = \|u\|_{L^1}.$$

Now we use the following lemma (see the appendix for a proof)

Lemma 3.1. *Let $\lambda, \mu > 0$. The function $(A, B) \mapsto |A \Delta B|$, defined on the collection of pairs (A, B) such that $|A| = \lambda$ and $|B| = \mu$, reaches its minimum whenever $A \subset B$ or $B \subset A$ (up to a negligible set) and its minimum is $|A \Delta B| = |\lambda - \mu|$.*

On one hand $\{u \geq t\}$ and $\{\tilde{u} \geq t\}$ have the same volume. On the other hand $\{\tilde{u} \geq t\}$ and $\{u_0 \geq t\}$ are concentric balls. Thus, using the previous lemma

$$|\{u \geq t\} \Delta \{u_0 \geq t\}| \geq |\{\tilde{u} \geq t\} \Delta \{u_0 \geq t\}|.$$

Integrating over t on $[0, +\infty[$ and using Lemma 2.3, we have

$$\|u - u_0\|_{L^1} \geq \|\tilde{u} - u_0\|_{L^1}. \quad (4)$$

Moreover, for all $t > 0$, $\{u \geq t\}$ is compact, so with the lemma (proof on appendix)

Lemma 3.2. *Let E be a compact and non negligible subset of \mathbb{R}^d . Then*

$$\overline{W}(E) \geq \omega_d.$$

we have $|\{u \geq t\}| = 0$ or $\overline{W}(\{u \geq t\}) \geq \omega_d$. In the first case, that means $r(t) = 0$ and then $\overline{W}(B(0, r(t))) = 0$. In the second case, remark $\omega_d = \overline{W}(B(0, R)) = W(B(0, R))$ for all ball with non null radius R .

In any case

$$\overline{W}(\{u \geq t\}) \geq \overline{W}(\{\tilde{u} \geq t\}).$$

Integrating over t on $[0, +\infty[$, using the coarea formula (Theorem 2.5) for \tilde{u} and the coarea inequality (Lemma 2.2) for u , we have

$$\overline{W}(u) \geq \overline{W}(\tilde{u}). \quad (5)$$

Combine (4) and (5) to prove the following theorem

Theorem 3.3. For all $u \in L^1_{c+}(\mathbb{R}^d)$, there exists $\tilde{u} \in L^1_{c+}(\mathbb{R}^d)$ radially non-increasing such that

$$\mathcal{F}(\tilde{u}) \leq \mathcal{F}(u),$$

both values being possibly $+\infty$.

Remark 3.4. With the idea that the ball minimizes the perimeter with prescribed volume, we can prove with the coarea formula for functions with bounded variation that this construction also decreases the total variation. Indeed we have :

$$|D\tilde{u}| \leq |Du|.$$

3.2 Shape of a minimizer.

We know now that we can look for the minimum of \mathcal{F} among radially non-increasing functions of $L^1_{c+}(\mathbb{R}^d)$. We transform the minimization problem on \mathcal{F} onto a minimization problem on the radius. We introduce

$$\mathcal{A} = \{r : [0, +\infty[\rightarrow \mathbb{R}^+ \mid r \text{ is non-increasing, } r(0) < +\infty \text{ and } r \in L^d([0, +\infty[)\}.$$

Through any radially non-increasing function $u : \mathbb{R}^d \rightarrow \mathbb{R}^+$ can be associated with a radius function $r : [0, +\infty[\rightarrow \mathbb{R}^+$ by the relation

$$\{u \geq t\} = B(0, r(t)). \quad (6)$$

As a consequence : for all u , r is non-increasing and conversely for all non-increasing functions r , there exists a linked function u . Moreover $u \in L^1 \Leftrightarrow r \in L^d$ and $\text{supp } u$ is compact $\Leftrightarrow r(0) < +\infty$.

Property 3.5. For any pair of functions (u, r) satisfying relation (6) :

$$u \in L^1_{c+}(\mathbb{R}^d) \Leftrightarrow r \in \mathcal{A}.$$

We write then the expression of \mathcal{F} using the radius functions of u and u_0 :

$$\overline{W}(u) = \omega_d \int_0^{\sup u} r(t)^{d-1-(d-1)} dt = \omega_d \sup u = \omega_d |\text{supp } r|$$

and

$$\|u - u_0\|_{L^1} = \alpha_d \int_0^{+\infty} |r(t)^d - r_0(t)^d| dt.$$

So

$$\mathcal{F}(u) = F(r) = \omega_d |\text{supp } r| + \frac{\alpha_d^2}{2\tau} \left(\int_0^{+\infty} |r(t)^d - r_0(t)^d| dt \right)^2.$$

We deduce the following proposition

Proposition 3.6. For any pair of functions (u, r) satisfying relation (6), the following properties are equivalent :

- u is a minimizer of \mathcal{F} among radially non-increasing functions of $L^1_{c+}(\mathbb{R}^d)$
- r is a minimizer of F in \mathcal{A} .

Now we will see that a minimizer is in fact a truncation of the initial data r_0 . Let r be any candidate, consider $\tilde{r}(t) = r_0(t)1_{t < \text{supp } r}$ and compute $F(\tilde{r})$. As $\text{supp } \tilde{r} \subset \text{supp } r$ and $\tilde{r} = r_0$ on $\text{supp } r$ then

$$F(\tilde{r}) \leq F(r).$$

Thus, we look for a minimum with the shape of r_0 :

$$r(t) = r_0(t)1_{t < \lambda}.$$

Our non-trivial minimization problem on $L^1_{c+}(\mathbb{R}^d)$ is now reduced to a one parameter (λ) minimization problem. Note $a = |\text{supp } r_0|$ and observe that we can only consider $0 \leq \lambda \leq a$.

We study the minimization of

$$\begin{aligned} f & : [0, a] \rightarrow \mathbb{R}^+ \\ t & \mapsto \omega_d \lambda + \frac{\alpha_d^2}{2\tau} \left(\int_\lambda^a r_0(s)^d ds \right)^2. \end{aligned}$$

$r_0^d \in L^1$ so f is continuous (then admits a minimum) and almost everywhere differentiable :

$$f'(\lambda) = \omega_d - \frac{\alpha_d^2}{\tau} r_0(\lambda)^d \int_{\lambda}^a r_0(s)^d ds.$$

r_0 is non-increasing and non-negative and, if it is constant on an interval then the integral cannot be constant on the same interval, that is why f' is (strictly) increasing. Notice that $f'(a) = \omega_d > 0$ and

$$f'(0) = \omega_d - \frac{\alpha_d^2}{\tau} r_0(0)^d \|r_0\|_{L^d} < 0$$

for τ small enough. If r_0 is continuous then there exists a unique λ such that $f'(\lambda) = 0$ otherwise we can define

$$\lambda = \sup\{s \mid f'(s) \leq 0\}.$$

Thus, there exists a unique λ minimizing f and, if r_0 is continuous, it is defined by the relation

$$r_0(\lambda)^d \int_{\lambda}^a r_0(s)^d ds = \frac{\omega_d}{\alpha_d^2} \tau. \quad (7)$$

Notice that if τ is not small enough then the minimum λ is equal to zero. Moreover for all $z \in [0, a[$

$$r_0(z)^d \int_z^a r_0(s)^d ds > 0,$$

so there exists $\tau > 0$ small enough such that $f'(z) < 0$ and then $\lambda > z$. Consequently

$$\lambda \xrightarrow{\tau \rightarrow 0} a.$$

Remark 3.7. Notice that discontinuity points of u_0 and r_0 are not the same. Indeed, in the example of figure 6: the point B is a discontinuity point for u_0 (a discontinuity sphere actually) but it is a point where r_0 is locally constant. Conversely, u_0 is locally constant at point A and it is a discontinuity point for r_0 .

3.3 Minimizing movement.

According to the minimizing movement principle, we will construct a minimizing sequence. For each iteration, the minimized function is very similar to the function f . We suppose now that the radius function r_0 is continuous except in a (latter we will need the discontinuity in a)

Let $\lambda_0 = a$, consider the functions

$$f_n(\lambda) = \omega_d \lambda + \frac{\alpha_d^2}{2\tau} \left(\int_{\lambda}^{\lambda_n} r_0(s)^d ds \right)^2$$

and iterate the (unique) minimizer of f_n on $[0, \lambda_n]$ to find λ_{n+1} . The sequence (λ_n) is non-increasing. The value of τ is fixed once for all. As long as it is small enough, λ_n is decreasing and satisfy (7) :

$$r_0(\lambda_{n+1})^d \int_{\lambda_{n+1}}^{\lambda_n} r_0(s)^d ds = \frac{\omega_d}{\alpha_d^2} \tau.$$

And after first time τ is not small enough (not necessarily existing), we have $\lambda_{n+1} = 0$ and so for all terms beyond.

As r_0 is non-increasing, we have

$$r_0(\lambda_{n+1})^d r_0(\lambda_n)^d (\lambda_n - \lambda_{n+1}) \leq \frac{\omega_d \tau}{\alpha_d^2} \leq r_0(\lambda_{n+1})^{2d} (\lambda_n - \lambda_{n+1}).$$

In order to have a global estimation on λ_n , we need to suppose $r_0(a) > 0$ which yields the discontinuity mentioned above. We obtain the double inequality :

$$\frac{\omega_d \tau}{r_0(0)^{2d} \alpha_d^2} \leq \lambda_n - \lambda_{n+1} \leq \frac{\omega_d \tau}{r_0(a)^{2d} \alpha_d^2}. \quad (8)$$

Remark 3.8. The first inequality above shows that, beyond some rank, we have $\lambda_n = 0$. Indeed, by summing we deduce that

$$\lambda_n \leq a - \omega_d r_0(0)^{2d} \alpha_d^2 n \tau.$$

As we necessarily have $\lambda_n \geq 0$, that means

$$n \tau \leq \frac{a r_0(0)^{2d} \alpha_d^2}{\omega_d}.$$

That gives us a bound, depending on τ , of the rank when λ_n vanishes.

The second inequality allows us to prove the minimizing movement convergence. Define

$$\begin{aligned} \lambda_\tau &: \mathbb{R}^+ \rightarrow [0, a] \\ t &\mapsto \lambda_n + \frac{s - n\tau}{\tau} (\lambda_{n+1} - \lambda_n) \quad \text{for } s \in [n\tau, (n+1)\tau]. \end{aligned}$$

λ_τ is a family of continuous and piecewise affine functions satisfying, using (8)

$$|\lambda'_\tau(t)| \leq \frac{\omega_d}{r_0(a)^{2d} \alpha_d^2}.$$

By the remark, we know that $\lambda_\tau(t) = 0$ for $t \geq \frac{a r_0(0)^{2d} \alpha_d^2}{\omega_d}$ and then, using Ascoli theorem :

Proposition 3.9. *There exists a limit function $\lambda : \mathbb{R}^+ \rightarrow [0, a]$ such that $\lambda_\tau \rightarrow \lambda$ uniformly when, up to a subsequence, $\tau \rightarrow 0$. λ is lipschitz continuous and non-increasing.*

Thus, the minimizing movement exists, is continuous and consists of an erosion of the level of the initial data.

Remark 3.10. For convenience, we rather used continuous piecewise affine function but the conclusion remains the same since

$$|\lambda_{[t/\tau]} - \lambda_\tau(t)| \leq \frac{\omega_d}{r_0(a)^{2d} \alpha_d^2} \tau.$$

Furthermore, we can have more information on the limit function λ . Let t such that $\lambda(t) > 0$ and $\varepsilon > 0$ such that $\lambda(t + \varepsilon) > 0$. For all $\tau > 0$, write

$$t = p\tau + \eta \quad \text{and} \quad t + \varepsilon = (p+q)\tau + \eta'$$

with $p, q \in \mathbb{N}$ and $0 \leq \eta, \eta' < \tau$. Then

$$\lambda_p \rightarrow \lambda(t) \quad \text{and} \quad \lambda_{p+q} \rightarrow \lambda(t + \varepsilon)$$

when $\tau \rightarrow 0$.

For all integers between p and $p+q$, use the double inequality (8) with λ_p and λ_{p+q} instead of 0 and a :

$$\frac{\omega_d \tau}{r_0(\lambda_{p+q})^{2d} \alpha_d^2} \leq \lambda_n - \lambda_{n+1} \leq \frac{\omega_d \tau}{r_0(\lambda_p)^{2d} \alpha_d^2}.$$

Summing on n from p to $p+q-1$, we obtain

$$\frac{\omega_d}{r_0(\lambda_{p+q})^{2d} \alpha_d^2} q \tau \leq \lambda_p - \lambda_{p+q} \leq \frac{\omega_d}{r_0(\lambda_p)^{2d} \alpha_d^2} q \tau.$$

Writing $q\tau$ in term of ε , η and η' and dividing by ε , one can see that

$$\frac{\omega_d}{r_0(\lambda_{p+q})^{2d} \alpha_d^2} \frac{\varepsilon + \eta - \eta'}{\varepsilon} \leq \frac{\lambda_p - \lambda_{p+q}}{\varepsilon} \leq \frac{\omega_d}{r_0(\lambda_p)^{2d} \alpha_d^2} \frac{\varepsilon + \eta - \eta'}{\varepsilon}.$$

Passing to the limit $\tau \rightarrow 0$ we have

$$\frac{\omega_d}{r_0(\lambda(t+\varepsilon))^{2d} \alpha_d^2} \leq -\frac{\lambda(t+\varepsilon) - \lambda(t)}{\varepsilon} \leq \frac{\omega_d}{r_0(\lambda(t))^{2d} \alpha_d^2}.$$

And now take $\varepsilon \rightarrow 0$ to observe that λ has a right derivative on t . Processing exactly the same way (taking care of the sign), we obtain

$$\frac{\omega_d}{r_0(\lambda(t))^{2d}\alpha_d^2} \leq -\frac{\lambda(t-\varepsilon) - \lambda(t)}{-\varepsilon} \leq \frac{\omega_d}{r_0(\lambda(t-\varepsilon))^{2d}\alpha_d^2}.$$

And we have the same left derivative.

Proposition 3.11. *At every point t where $\lambda(t) > 0$, λ is differentiable and satisfies*

$$\lambda'(t) = -\frac{\omega_d}{\alpha_d^2} \frac{1}{r_0(\lambda(t))^{2d}}. \quad (9)$$

Thus now we understand the minimizing movement for a radially non-increasing function where its highest non empty level set is not reduced to a point. The ordinary differential equation (9) contains the difficulty of $r_0(a) = 0$ because, in that case, the initial data is truncated with an infinite initial speed. After the starting point, the movement is clear. Indeed, on every interval on the form $[T, +\infty[$ with $T > 0$, the minimizing movement exists (we have the double inequality (8) except for the first times $\lambda_0, \lambda_1, \dots$ and so the uniform convergence remains) and follows the law (9).

We assumed the continuity of r_0 on $[0, a[$. Nevertheless, (9) remains valid even if r_0 is not continuous but only non-increasing. Indeed may r_0 be continuous or not, we still have the existence of the sequence (λ_n) and by definition :

$$\lambda_{n+1} = \sup\{s \mid f'_n(s) \leq 0\}.$$

Notice that r_0 is non-increasing and therefore left and right continuous. In the sequel we will write $r_0(s+)$ and $r_0(s-)$ for the right and left limit at any point s . Noticing (λ_n) is decreasing (for non null terms) and f'_n increasing we have :

for (almost) every $s \in]\lambda_{n+1}, \lambda_n[$, $f'_n(s) \geq 0$ thus

$$r_0(s)^d \int_s^{\lambda_n} r_0(u)^d du \leq \frac{\tau\omega_d}{\alpha_d^2}.$$

So

$$r_0(\lambda_n)^{2d}(\lambda_n - s) \leq \frac{\tau\omega_d}{\alpha_d^2}.$$

Now take $s \rightarrow \lambda_{n+1}$ to have

$$\lambda_n - \lambda_{n+1} \leq \frac{\omega_d\tau}{r_0(\lambda_n)^{2d}\alpha_d^2}.$$

Identically, for (almost) every $s < \lambda_{n+1}$, $f'_n(s) \leq 0$ and then

$$r_0(s)^d \int_s^{\lambda_n} r_0(u)^d du \geq \frac{\tau\omega_d}{\alpha_d^2}.$$

So

$$r_0(s)^{2d}(\lambda_n - s) \geq \frac{\tau\omega_d}{\alpha_d^2}.$$

Letting $s \rightarrow \lambda_{n+1}$, one gets

$$\lambda_n - \lambda_{n+1} \geq \frac{\omega_d\tau}{r_0(\lambda_{n+1}-)^{2d}\alpha_d^2}.$$

For global estimation, we have thus exactly double inequality(8) and then the existence of the minimizing movement.

Similarly, to satisfy (9), the same technique works also up to a small modification as follow. Notice that $r_0(\lambda_{p+q}-) \leq r_0(\lambda_{p+q+1})$ and $\lambda_{p+q+1} \rightarrow \lambda(t+\varepsilon)$ because $|\lambda_{p+q} - \lambda_{p+q+1}| \leq \frac{\omega_t}{r_0(a)^{2d}\alpha_d^2}\tau$. Therefore

$$\frac{\omega_d}{r_0(\lambda_{p+q+1})^{2d}\alpha_d^2} \frac{\varepsilon + \eta - \eta'}{\varepsilon} \leq \frac{\lambda_p - \lambda_{p+q}}{\varepsilon} \leq \frac{\omega_d}{r_0(\lambda_p)^{2d}\alpha_d^2} \frac{\varepsilon + \eta - \eta'}{\varepsilon}. \quad (10)$$

For $m = \lceil s/\tau \rceil$ we do not know how $\lambda_m \rightarrow \lambda(s)$ when τ vanishes (the dependence on τ of λ_m is not only in $m = \lceil s/\tau \rceil$) so we use the inequalities :

$$\frac{1}{r_0(\lambda(s)-)} \leq \liminf_{\tau \rightarrow 0} \frac{1}{r_0(\lambda_m)} \leq \limsup_{\tau \rightarrow 0} \frac{1}{r_0(\lambda_m)} \leq \frac{1}{r_0(\lambda(s)+)}.$$

Passing to lim inf in the left hand side of double inequality (10) (with $s = t + \varepsilon$ and $m = p + q + 1$) and passing to the lim sup in the right hand side (with $s = t$ and $m = p$) we have

$$\frac{\omega_d}{r_0(\lambda(t+\varepsilon)-)^{2d}\alpha_d^2} \leq \frac{\lambda(t) - \lambda(t+\varepsilon)}{\varepsilon} \leq \frac{\omega_d}{r_0(\lambda(t)+)^{2d}\alpha_d^2}.$$

Thanks to monotonicity ($t \mapsto \lambda(t)$ is non-increasing) we have

$$\frac{1}{r_0(\lambda(t+2\varepsilon))} \leq \frac{1}{r_0(\lambda(t+\varepsilon)-)} \quad \text{and} \quad \frac{1}{r_0(\lambda(t)+)} \leq \frac{1}{r_0(\lambda(t-\varepsilon))}.$$

And so

$$\frac{\omega_d}{r_0(\lambda(t+2\varepsilon))^{2d}\alpha_d^2} \leq \frac{\lambda(t) - \lambda(t+\varepsilon)}{\varepsilon} \leq \frac{\omega_d}{r_0(\lambda(t-\varepsilon))^{2d}\alpha_d^2}.$$

As $t + 2\varepsilon$ decreases to t when $\varepsilon \rightarrow 0^+$ then $\lambda(t + 2\varepsilon)$ increases to $\lambda(t)$ and so $r_0(\lambda(t + 2\varepsilon)) \rightarrow r_0(\lambda(t)-)$ when $\varepsilon \rightarrow 0^+$. Similarly $r_0(\lambda(t - \varepsilon)) \rightarrow r_0(\lambda(t)+)$ when $\varepsilon \rightarrow 0^+$. Thus we can pass to lim inf and lim sup in the previous inequalities to have

$$-\frac{\omega_d}{r_0(\lambda(t)+)^{2d}\alpha_d^2} \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{\lambda(t+\varepsilon) - \lambda(t)}{\varepsilon} \leq \limsup_{\varepsilon \rightarrow 0^+} \frac{\lambda(t+\varepsilon) - \lambda(t)}{\varepsilon} \leq -\frac{\omega_d}{r_0(\lambda(t)-)^{2d}\alpha_d^2}.$$

Using exactly the same arguments for the left derivative we obtain

$$-\frac{\omega_d}{r_0(\lambda(t)+)^{2d}\alpha_d^2} \leq \liminf_{\varepsilon \rightarrow 0^+} \frac{\lambda(t-\varepsilon) - \lambda(t)}{-\varepsilon} \leq \limsup_{\varepsilon \rightarrow 0^+} \frac{\lambda(t-\varepsilon) - \lambda(t)}{-\varepsilon} \leq -\frac{\omega_d}{r_0(\lambda(t)-)^{2d}\alpha_d^2}.$$

At the points where r_0 is continuous, we obtain exactly (9). At any other points, we do not know if λ is derivable but the growth ratio is controlled.

4 Conclusion.

Let us recall our objective: we were looking for a way to decrease the L^1 relaxation of the scale invariant Willmore energy

$$W(u) = \frac{1}{(d-1)^{d-1}} \int_{\mathbb{R}^d} \left| \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right|^{d-1} |\nabla u| dx.$$

We considered a radially non-increasing initial function u_0 and its radius function r_0 as in figure 6.

We proved that the minimizing movement in the space

$$L_{c+}^1(\mathbb{R}^d) = \{u \in L^1(\mathbb{R}^d) \mid u \geq 0, \operatorname{supp}(u) \text{ is compact}\}$$

is given by a truncation of the initial function.

Theorem 4.1. *Let $u_0 \in L_{c+}^1(\mathbb{R}^d)$ be a bounded and radially non-increasing initial condition and r_0 its radius function. If the highest non empty level set of u_0 is not reduced to $\{0\}$ (meaning $r_0(\sup u_0) > 0$) then there exists a minimizing movement for \overline{W} starting from u_0 .*

Moreover, there exists $\lambda : [0, +\infty[\rightarrow \mathbb{R}^+$ such that the minimizing movement is given by the map

$$\begin{aligned} \mathbb{R}^+ &\rightarrow L_{c+}^1(\mathbb{R}^d) \\ t &\mapsto u_t \end{aligned}$$

with

$$u_t(x) = \min(u_0(x), \lambda(t)).$$

and the function λ is Lipschitz continuous, non-increasing, satisfy $\lambda(0) = \sup u_0$ and vanishes in finite time.

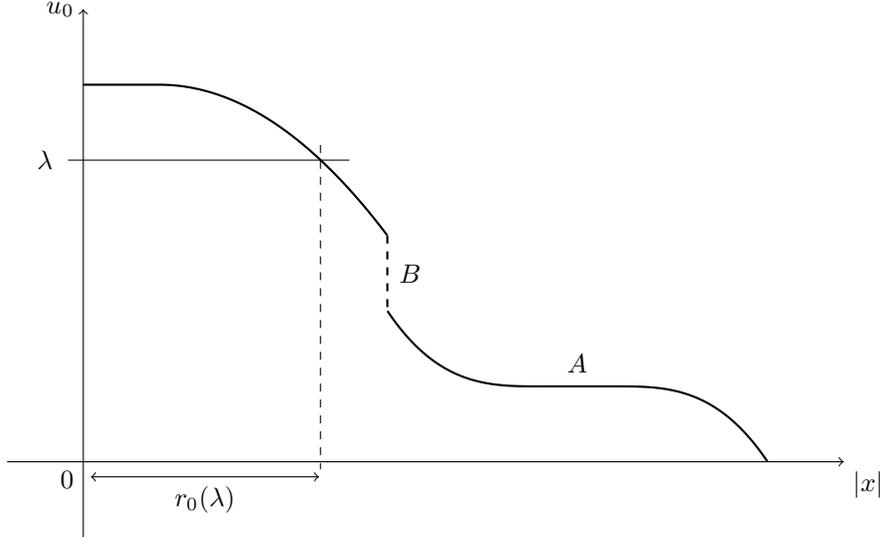


Figure 6: Initial condition u_0 .

Moreover, we obtained some interesting information on λ depending on the continuity of r_0 . With the notations of the previous theorem.

Theorem 4.2. *With the notations of Theorem 4.1, for all $t \in \mathbb{R}^+$ such that $\lambda(t) > 0$:*

- *if $\lambda(t)$ is a continuity point of r_0 then λ is differentiable on t and*

$$\lambda'(t) = -\frac{\omega_d}{\alpha_d^2} \frac{1}{r_0(\lambda(t))^{2d}}$$

- *otherwise, we have the following control on the growth ratio:*

$$-\frac{\omega_d}{r_0(\lambda(t)^+)^{2d} \alpha_d^2} \leq \liminf_{\varepsilon \rightarrow 0} \frac{\lambda(t+\varepsilon) - \lambda(t)}{\varepsilon} \leq \limsup_{\varepsilon \rightarrow 0} \frac{\lambda(t+\varepsilon) - \lambda(t)}{\varepsilon} \leq -\frac{\omega_d}{r_0(\lambda(t)^-)^{2d} \alpha_d^2}$$

for both $\varepsilon \rightarrow 0^+$ and $\varepsilon \rightarrow 0^-$.

An illustration of the minimizing movement is shown in figure 7.

In zone B , the erosion speed is locally constant and in zone A , it possibly jumps.

Perspectives.

We were not able to see what happens if $r_0(\sup u_0) = 0$. Indeed, the minimizing sequence convergence is given by Ascoli's Theorem and we need a uniform control on the speed decrease of (λ_n) . Looking at the ODE (Theorem 4.2), the initial speed is non-finite if $r_0(\sup u_0) = 0$ therefore, we cannot hope such a control. For now, we do not know if, in that case, the flow consists on a erosion with a non-finite initial speed. Otherwise, the flow may consist on cutting off a small height and then the erosion begins following the ODE.

More generally, we can wonder what is the flow of the characteristic function of any set E . Is the flow an erosion? Or do the level sets evolve? Maybe both? We proved in this paper that for a ball $E = B(0, R)$, the gradient flow starting from $u_0 = 1_{B(0, R)}$ is an erosion given by

$$u_t = (1 - c_R t) 1_{B(0, R)}$$

where $c_R = \frac{\omega_d}{\alpha_d^2 R^{2d}}$. However, balls have minimal scale invariant Willmore energy which can explain that the level sets do not evolve. The only way to reduce its energy is then to be smaller and smaller. If the set E is not minimal then we can have both behaviours: decrease of the height and/or deformation of the level sets. Perhaps if the Willmore energy is small enough then it is better to be rigid on level sets and to diminish the height. This will be the purpose of future work.

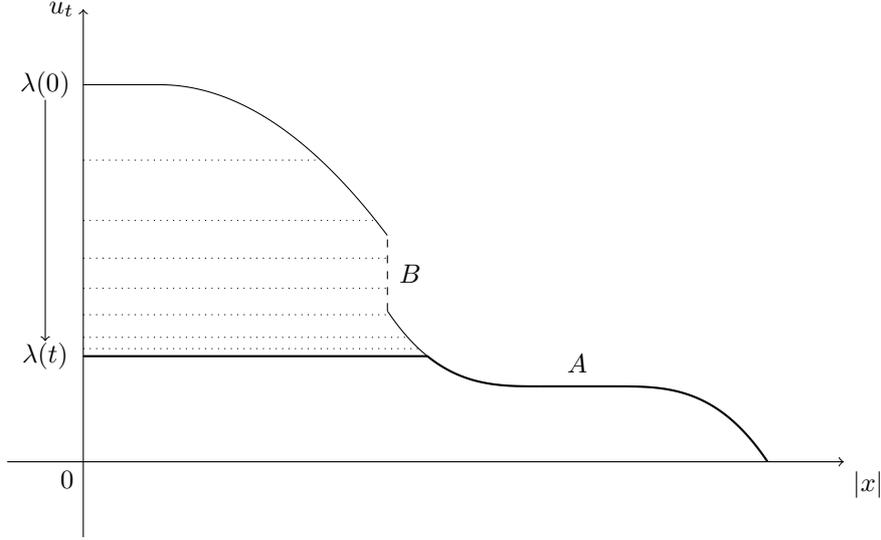


Figure 7: Minimizing movement u_t .

A Appendix : Proof of the lemmas.

We prove the technical lemmas.

Lemma 2.2 (coarea inequality). *Let $u \in L^1(\Omega)$, we have*

$$\int_{-\infty}^{+\infty} \overline{W}_p(\{u \geq t\}) dt \leq \overline{W}_p(u).$$

Proof. It is obvious if $\overline{W}_p(u) = +\infty$. If not, let (u_n) be a sequence of $L^1(\Omega)$ with C_c^2 regularity such that

$$u_n \rightarrow u \text{ in } L^1(\Omega) \quad \text{and} \quad W_p(u_n) \rightarrow \overline{W}_p(u).$$

Thanks to lemma 2.3, up to a subsequence :

$$\forall t \in \mathbb{R} \text{ ae}, \quad \{u_n \geq t\} \rightarrow \{u \geq t\} \text{ in } L^1.$$

Moreover, by Sard's lemma, for almost every t , $\{u_n \geq t\}$ is a regular set of \mathbb{R}^d . By definition of \overline{W}_p , we have :

$$\overline{W}_p(\{u \geq t\}, \Omega) \leq \liminf_{n \rightarrow +\infty} W_p(\{u_n \geq t\}).$$

Integrating according to t and using Fatou's lemma and the coarea formula (proposition 1.9), we get :

$$\begin{aligned} \int_{-\infty}^{+\infty} \overline{W}_p(\{u \geq t\}, \Omega) dt &\leq \liminf \int_{-\infty}^{+\infty} W_p(\{u_n \geq t\}) dt \\ &= \liminf W_p(u_n, \Omega) \\ &= \overline{W}_p(u). \end{aligned}$$

□

Lemma 2.3 (L^1 norm). *For measurable functions $u, v : \mathbb{R}^d \mapsto \mathbb{R}$ we have*

$$\int_{\mathbb{R}^d} |u - v| dx = \int_{-\infty}^{+\infty} |\{u \geq t\} \Delta \{v \geq t\}| dt.$$

With $\{u \geq t\}$ the t -height level set of u .

Proof. Pick $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. We have

$$|1_{\{u \geq t\}}(x) - 1_{\{v \geq t\}}(x)| = \begin{cases} 1 & \text{if } \min(u(x), v(x)) < t < \max(u(x), v(x)) \\ 0 & \text{else.} \end{cases}$$

Then

$$\int_{-\infty}^{+\infty} |1_{\{u \geq t\}}(x) - 1_{\{v \geq t\}}(x)| dt = \max(u(x), v(x)) - \min(u(x), v(x)) = |u(x) - v(x)|.$$

Integrating on $x \in \mathbb{R}^d$, using Fubini theorem and noticing that

$$\int_{\mathbb{R}^d} |1_{\{u \geq t\}}(x) - 1_{\{v \geq t\}}(x)| dx = |\{u \geq t\} \Delta \{v \geq t\}|,$$

the result is proved. \square

Lemma 2.4 (reattachment). *Let $a, b, \alpha, \beta \in \mathbb{R}$ such that $a, b < 0$. There exists a decreasing function $f \in C^2([0, 1])$ such that*

$$\begin{aligned} f(0) &= 1 & f(1) &= 0 \\ f'(0) &= a & f'(1) &= b \\ f''(0) &= \alpha & f''(1) &= \beta. \end{aligned}$$

Proof. Step 1: first we construct a C^2 non increasing function f_0 satisfying

$$\begin{aligned} f_0(0) &= 1 & f_0(1) &> 0 \\ f'_0(0) &= a & f'_0(1) &= b \\ f''_0(0) &= \alpha & f''_0(1) &= \beta. \end{aligned}$$

Let $\varepsilon > 0$, we define $f'_0 = 0$ on $[\varepsilon, 1 - \varepsilon]$ and we reattach the border conditions on f'_0 and f''_0 with polynomials. Then we will use

$$f_0(t) = 1 + \int_0^t f'_0(s) ds$$

and choose ε small enough to define f_0 .

We introduce the following polynomials :

$$\begin{aligned} P &= (\lambda_1 + \mu_1(X - \varepsilon))(X - \varepsilon)^2 & \text{with } \lambda_1 &= 3a/\varepsilon^2 + \alpha/\varepsilon \\ & & \mu_1 &= 2a/\varepsilon^3 + \alpha/\varepsilon^2 \\ Q &= (\lambda_2 + \mu_2(1 - \varepsilon - X))(1 - \varepsilon - X)^2 & \text{with } \lambda_2 &= 3b/\varepsilon^2 - \beta/\varepsilon \\ & & \mu_2 &= 2b/\varepsilon^3 - \beta/\varepsilon^2 \end{aligned}$$

These two polynomials satisfy the conditions:

$$\begin{aligned} P(0) &= a & P(\varepsilon) &= 0 & Q(1 - \varepsilon) &= 0 & Q(1) &= b \\ P'(0) &= \alpha & P'(\varepsilon) &= 0 & Q'(1 - \varepsilon) &= 0 & Q'(1) &= \beta. \end{aligned}$$

Moreover, taking ε small enough: $\varepsilon < \min(3|a|/|\alpha|, 3|b|/|\beta|)$, then these polynomials are negative where we need them to be.

$$\begin{aligned} P &< 0 & \text{on } [0, \varepsilon[\\ Q &< 0 & \text{on }]1 - \varepsilon, 1]. \end{aligned}$$

An easy calculation shows that

$$\int_0^\varepsilon P(t) dt + \int_{1-\varepsilon}^1 Q(t) dt = \frac{a+b}{2}\varepsilon + \frac{\alpha-\beta}{12}\varepsilon^2 < 0.$$

For $\varepsilon < 1/2$, let us define the function :

$$g(t) = \begin{cases} P(t) & \text{if } t \in [0, \varepsilon] \\ 0 & \text{if } t \in [\varepsilon, 1 - \varepsilon] \\ Q(t) & \text{if } t \in [1 - \varepsilon, 1]. \end{cases}$$

The function g is C^1 , satisfies the border first and second order conditions (with one order less) and is non positive. Now we choose $\varepsilon < \min(1/2, 3|a|/|\alpha|, 3|b|/|\beta|)$ such that

$$1 + \frac{a+b}{2}\varepsilon + \frac{\alpha-\beta}{12}\varepsilon^2 > 0$$

and we define

$$f_0(t) = 1 + \int_0^t g(s)ds.$$

Step 2: now we use a mollifier φ on $[0, 1]$ such that φ is positive on $]0, 1[$, vanishes at $t = 0$ and $t = 1$ at both orders 0 and 1 and $\int_0^1 \varphi(t)dt = 1$. Let $A > 0$ such that

$$1 + \frac{a+b}{2}\varepsilon + \frac{\alpha-\beta}{12}\varepsilon^2 - A = 0.$$

Take

$$f(t) = f_0(t) - A \int_0^t \varphi(s)ds.$$

This function verifies every condition. □

Lemma 3.1. *Let $\lambda, \mu > 0$. The function $(A, B) \mapsto |A\Delta B|$, defined on the collection of pairs (A, B) such that $|A| = \lambda$ and $|B| = \mu$, reaches its minimum whenever $A \subset B$ or $B \subset A$ (up to a negligible set) and its minimum is $|A\Delta B| = |\lambda - \mu|$.*

Proof.

$$|A\Delta B| = |A \setminus B| + |B \setminus A| = |A| + |B| - 2|A \cap B| = \lambda + \mu - 2|A \cap B|.$$

We have $|A \cap B| \leq \min(|A|, |B|)$ with equality when one set is included into the other (up to a negligible set). Then $(A, B) \mapsto |A\Delta B|$ achieves its minimum for $A \subset B$ or $B \subset A$.

If $A \subset B$ then $|A \cap B| = |A|$ and $\mu \geq \lambda$ so $|A\Delta B| = \mu - \lambda$.

If $B \subset A$ then $|A \cap B| = |B|$ and $\mu \leq \lambda$ so $|A\Delta B| = \lambda - \mu$. □

Lemma 3.2. *Let E be a compact and non negligible subset of \mathbb{R}^d . Then*

$$\overline{W}(E) \geq \omega_d.$$

Proof. It is a direct consequence of well known results for the scale invariant Willmore energy (see [27] and references therein for instance): for all regular compact hypersurfaces M of \mathbb{R}^d , we have

$$\int_M |H|^{d-1} d\mathcal{H}^{d-1} \geq \omega_d$$

with equality only for spheres.

$\overline{W}(E) < +\infty$ implies there exists a sequence E_n of C^2 and compact subset such that $E_n \rightarrow E$ in L^1 and $W(E_n) \rightarrow \overline{W}(E)$ when $n \rightarrow +\infty$. Using the result on the boundary ∂E_n , we have

$$W(E_n) \geq \omega_d.$$

Passing to the limit $n \rightarrow +\infty$, we have

$$\overline{W}(E) \geq \omega_d. □$$

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