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# New results about Quasi-Affine Transformations

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## Abstract

A Quasi-Affine Transformation (QAT) is a transformation on  $\mathbb{Z}^n$  which corresponds to the composition of an affine transformation and an integer part function. Some studies of QATs and their applications can be found in [1],[2],[7],[3] and [4]. The aim of this paper is to complete and improve the results of these works.

*Keywords:* Discrete Affine Transformations, Tilings.

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## 1. Introduction.

If we compose an affine transformation from  $\mathbb{Z}^n$  to  $\mathbb{Q}^n$  with an integer part function we obtain a Quasi-Affine Transformation (QAT) from  $\mathbb{Z}^n$  to  $\mathbb{Z}^n$ . We will only consider the integer part function with a positive remainder. We will denote it  $\lfloor \cdot \rfloor$ . If  $x$  and  $y$  are two integers,  $\lfloor \frac{x}{y} \rfloor$  denotes the quotient of the Euclidean division of  $x$  by  $y$ . The notations introduced in the definition 1 will be used in the rest of the paper.

**Definition 1.** Let  $A = (a_{i,j})_{1 \leq i,j \leq n}$  denote an integer matrix,  $V = (v_i)_{1 \leq i \leq n}$  an integer vector and  $\omega$  a positive integer. We denote  $g$  the affine transformation from  $\mathbb{Z}^n$  to  $\mathbb{Q}^n$ , defined by:

$$\begin{aligned} \mathbb{Z}^n &\longrightarrow \mathbb{Q}^n \\ X &\longmapsto \frac{1}{\omega}AX + V \end{aligned}$$

The Quasi-Affine Transformation or QAT associated with  $g$  is the transformation of the discrete plane  $\mathbb{Z}^n$  defined by the composition of  $g$  with the greatest integer part function noted  $\lfloor \cdot \rfloor$ . The QAT is then denoted  $G$ .

The present paper is structured in two distinct parts.

In the first part (section 2) we study the behaviour under iteration of a QAT: if  $g$  is a contracting linear transformation of  $\mathbb{R}^n$  then  $g$  has a unique fixed

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point and for each  $X \in \mathbb{R}^n$  the sequence  $g^n(X)$  tends toward this fixed point. We will say that a QAT  $G$  is contracting if  $g$  is contracting. But such a QAT has not necessarily a unique fixed point. The behaviour under iteration of 2D contracting QATs has been studied in [6], [10] and [9]. We will go on with this study in  $\mathbb{Z}^n$ . We will prove that such a QAT has a unique fixed point if and only if there are no cycle among the points of norm 1.

In the second part (section 3) we study tilings generated by QATs: a tile with index  $Y \in \mathbb{Z}^n$  is the set of points  $X \in \mathbb{Z}^n$  such that  $Y = G(X)$ . Such a tile can be empty, contain one or several points. In [1],[2], [3] and [4] the periodicity properties of these tilings are studied and used to transform discrete nD images by discrete affine transformations. We proved that there exists a finite number of distinct tiles and each tile can be obtained by translating one of the distinct tile by a given vector. In 2D and 3D, explicit formulas have been proposed to compute a minimal basis of the QAT periodic structure [1, 4]. In the n-dimensional case, an upper bound on the number of distinct tiles is given in [2] but this bound is not tight. We will give a better upper bound of the period and we give an algorithm which determines the period and the corresponding translation vectors.

## 2. Behaviour under iteration of a quasi-linear transformation

In this section we consider affine transformations  $g$  such that  $0 \leq v_i \leq \omega$ . We note  $g_1$  the linear transformation defined by the same matrix but the vector  $V = (0, 0, \dots, 0)$ . The associated QAT  $G$  corresponds to the composition of  $g_1$  with some other integer part function.

We study QATs associated to contracting transformations considering the infinite norm defined by:

$$\|g\|_\infty = \frac{1}{\omega} \max_{i=1, \dots, n} \left( \sum_{j=1, \dots, n} |a_{i,j}| \right). \quad (1)$$

It is well known that if  $g_1$  is a contracting linear transformation of  $\mathbb{R}^n$  then  $g_1$  has the origin as unique fixed point and for each  $X \in \mathbb{R}^n$  the sequence  $g_1^n(X)$  tends toward this fixed point. We will say that a QAT  $G$  is contracting if  $g$  is contracting. But such a QAT has not necessarily a unique fixed point. Consider the sequence  $Y_n = G(Y_{n-1})_{n \geq 0} = G^n(Y_0)_{n \geq 0}$  with  $Y_0 \in \mathbb{Z}^n$ , depending on  $Y_0$  we can obtain a cycle ( $Y_p = Y_q$  for some  $p, q \in \mathbb{N}$ ), fixed point ( $Y_n = Y_{n+1}$  for some  $n \in \mathbb{N}$ ), or leaves (points without antecedant) see figure 1. Figures 2 and 3 illustrate this behaviour in figure 3 the QAT is defined by  $\frac{1}{3} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$  and  $V = (0, 0)$ , it has a unique fixed point and the colour of a point is determined by the number of iterations necessary to reach the point  $(0, 0)$ . Figure 2 corresponds to QAT defined by  $\frac{1}{4495} \begin{pmatrix} 4187 & -1622 \\ 1622 & 4187 \end{pmatrix}$  and  $V = (0, 0)$ , this QAT has many cycles, the points of a cycle and those which reach this cycle have the same colours.

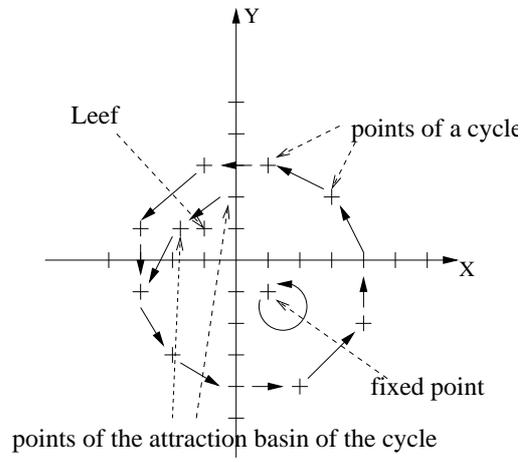


Figure 1: Example of cycle, fixed point and basin

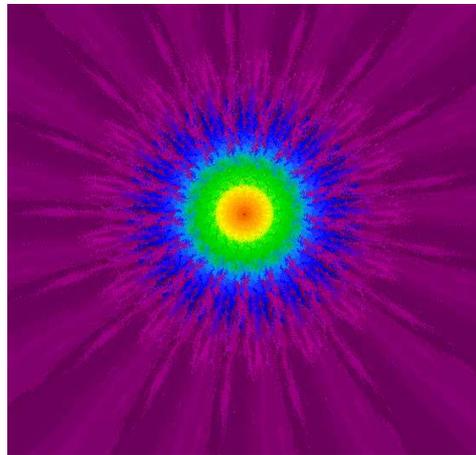


Figure 2: QAT with many cycles

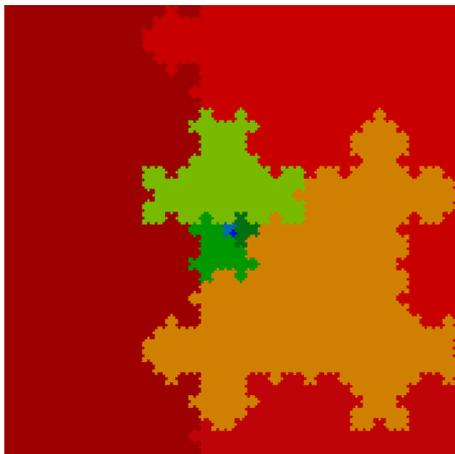


Figure 3: QAT with a unique fixed point

**Definition 2.** A *consistent* (with respect to the continuous affine function) Quasi-Affine Transformation is a QAT which has the origin as unique fixed point such that for each discrete point  $Y$  the sequence  $(G^n(Y))_{n \geq 0}$  tends toward this unique fixed point.

The following theorem, proved in [6], states conditions such that a 2D-QAT is a consistent QAT.

**Theorem 1.** Let  $G$  be a QAT such that  $V = (0, 0)$  and  $\|g\|_\infty < 1$ ,  $G$  is a consistent QAT if and only if one of the three following conditions are satisfied:

- (1)  $b \leq 0$ ,  $a + b \leq 0$ ,  $c > 0$  and  $d \leq 0$
- (2)  $a \leq 0$ ,  $b > 0$ ,  $c \leq 0$  and  $c + d \leq 0$
- (3)  $a \leq 0$ ,  $b \leq 0$ ,  $c \leq 0$  and  $d \leq 0$

**Remark 1.** The case where  $\|g\|_\infty = 1$  is also studied in [6]: The author states also the conditions such that a QAT of norm 1 is a consistent QAT.

In the following we will only consider QATs such that  $\|g\|_\infty \leq 1$ . The first results concern the localisation of the cycles.

**Lemma 1.** Let  $D = G(X) - g_1(X)$  and  $s = \max(\frac{\|V\|_\infty}{\omega}, 1 - \frac{\|V\|_\infty}{\omega})$ , then  $\|D\|_\infty \leq s$  if  $s \leq \frac{1}{2}$  and  $\|D\|_\infty < s$  if  $s > \frac{1}{2}$ .

PROOF. Consider  $X = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ ,  $Y = (y_1, y_2, \dots, y_n) = G(X) \in \mathbb{Z}^n$ ,  $Y' = (y'_1, y'_2, \dots, y'_n) = g_1(X) \in \mathbb{Q}^n$  and  $D = (d_1, d_2, \dots, d_n) = Y - Y'$ . For

all  $i = 1, 2, \dots, n$  we have

$$\begin{aligned}
y_i - y'_i &= \left\lfloor \frac{\sum_{1 \leq j \leq n} (a_{i,j} x_j) + v_i}{\omega} \right\rfloor - \frac{\sum_{1 \leq j \leq n} (a_{i,j} x_j)}{\omega} \\
&= \frac{\sum_{1 \leq j \leq n} (a_{i,j} x_j) + v_i}{\omega} - r_i - \frac{\sum_{1 \leq j \leq n} (a_{i,j} x_j)}{\omega} \text{ with } 0 \leq r_i < 1 \\
&= \frac{v_i}{\omega} - r_i \text{ with } 0 \leq r_i < 1
\end{aligned}$$

It follows that  $\frac{v_i}{\omega} - 1 < y_i - y'_i \leq \frac{v_i}{\omega}$ , but  $0 \leq v_i \leq \omega$  so  $|\frac{v_i}{\omega} - 1| > |y_i - y'_i|$  and  $|y_i - y'_i| \leq \frac{v_i}{\omega}$ . We conclude that  $G(X) = g_1(X) + D$  with  $\|D\|_\infty \leq s$  the inequality being strict if  $\max(\frac{\|V\|_\infty}{\omega}, 1 - \frac{\|V\|_\infty}{\omega}) \neq \max(\frac{\|V\|_\infty}{\omega})$  that is to say  $\|V\|_\infty < \frac{\omega}{2}$ .  $\square$

**Corollary 1.** *The image of a point of norm  $m$  has a norm less than or equal to  $m$ .*

PROOF. Let denote  $X$  a point such that  $\|X\|_\infty = m$ , we have :

$$\begin{aligned}
\|G(X)\|_\infty &\leq \|g_1(X)\|_\infty + \|D\|_\infty \leq \|g_1\|_\infty m + \|D\|_\infty \\
\text{so, if } s > \frac{1}{2} : \quad &\|G(X)\|_\infty < m + s \leq m + 1 \\
\text{and, if } s \leq \frac{1}{2} : \quad &\|G(X)\|_\infty \leq m + s \leq m + \frac{1}{2} < m + 1.
\end{aligned}$$

We conclude that  $\|G(X)\| \leq m$ .  $\square$

**Theorem 2.** *Consider a QAT  $G$  such that  $\|g\|_\infty \leq 1$ , and note:*

$$s = \max\left(\frac{\|V\|_\infty}{\omega}, 1 - \frac{\|V\|_\infty}{\omega}\right). \quad (2)$$

*If  $X$  is a point of a cycle then  $\|X\|_\infty \leq \frac{s}{1 - \|g\|_\infty}$ , the inequality being strict if  $s > \frac{1}{2}$ .*

PROOF. We proved in lemma 1 that  $G(X) = g_1(X) + D_1$  with  $\|D_1\|_\infty \leq s$ , so  $G^2(X) = g_1(g_1(X) + D_1) + D_2 = g_1^2(X) + g_1(D_1) + D_2$  with  $\|D_2\|_\infty \leq s$ , and recurrently  $G^k(X) = g_1^k(X) + g_1^{k-1}(D_1) + g_1^{k-2}(D_2) + \dots + g_1(D_{k-1}) + D_k$  with  $\|D_i\|_\infty \leq s$  for  $i = 1, 2, \dots, k$  each inequality being strict if  $s > \frac{1}{2}$ . It follows that

$$\begin{aligned}
&\|G^k(X)\|_\infty \\
&\leq \|g_1^k(X)\|_\infty + \|g_1^{k-1}(D_1)\|_\infty + \|g_1^{k-2}(D_2)\|_\infty + \dots + \|g_1(D_{k-1})\|_\infty + \|D_k\|_\infty \\
&\leq \|g_1\|_\infty^k \|X\|_\infty + \|g_1\|_\infty^{k-1} \|D_1\|_\infty + \dots + \|g_1\|_\infty \|D_{k-1}\|_\infty + \|D_k\|_\infty \\
&< \|g_1\|_\infty^k \|X\|_\infty + \|g_1\|_\infty^{k-1} s + \dots + \|g_1\|_\infty s + s
\end{aligned}$$

If  $X$  belongs to a cycle it exists  $k \in \mathbb{N}$  such that  $G^k(X) = X$  and so

$$\|X\|_\infty(1 - \|g_1\|_\infty^k) < \frac{1 - \|g_1\|_\infty^k}{1 - \|g_1\|_\infty} s$$

$$\text{Finally: } \|X\|_\infty < \frac{s}{1 - \|g_1\|_\infty}. \quad \square$$

**Remark 2.** A consequence of corollary 1 is that if  $X$  belongs to a cycle than for all  $k \geq 0$ ,  $\|G^k(X)\|_\infty = \|X\|_\infty$ .

**Lemma 2.** Consider  $X = (x_1, x_2, \dots, x_n)$  a point of  $\mathbb{Z}^n$ , note  $m = \|X\|_\infty$ . Let  $g$  a QAT such that  $\|g\|_\infty < 1$  and  $V = (0, 0, \dots, 0)$ . If  $X$  is the image of  $Y$  such that  $\|Y\|_\infty = m$  then, for all  $i = 1, 2, \dots, n$ ,  $|x_i| < m$  or  $x_i = -m$ .

PROOF. Let note  $X = (x_1, x_2, \dots, x_n) = G(Y)$  this point. We have

$$\begin{aligned} x_i &= \sum_{j=1, \dots, n} a_{i,j} y_j \\ &= \sum_{\substack{j=1, \dots, n \\ a_{i,j} y_j > 0}} a_{i,j} y_j + \sum_{\substack{j=1, \dots, n \\ a_{i,j} y_j \leq 0}} a_{i,j} y_j \\ &< \sum_{\substack{j=1, \dots, n \\ a_{i,j} y_j > 0}} |a_{i,j} y_j| \\ &< \sum_{\substack{j=1, \dots, n \\ a_{i,j} y_j > 0}} |a_{i,j}| m \\ &< m \end{aligned}$$

As  $m = \|X\|_\infty$ , we conclude that  $|x_i| < m$  or  $x_i = -m$ . □

**Corollary 2.** If we consider the round up function that is to say  $V = (v_1, v_2, \dots, v_n)$  with  $v_i = \frac{\omega}{2}$  and a QAT  $G$  such that  $\|g\|_\infty < \frac{3}{4}$ , the only points that can belong to a cycle are the points with norm equal to 1.

If we consider the interger part function with positive remainder that is to says  $V = (0, 0, \dots, 0)$  and a QAT  $G$  such that  $\|g\|_\infty < \frac{1}{2}$ , the only points that can belong to a cycle are the points  $X = (x_1, x_2, \dots, x_n)$  such that  $x_i = 0$  or  $-1$ .

PROOF. The first assertion is a direct consequence of theorem 2. The second assertion is a consequence of lemma 2 and theorem 2. □

In [6] we used these properties to determine, in 2D, the conditions such that a QAT with norm less than  $\frac{1}{2}$  is aconsistent QAT. We established that this conditions were the same for all QAT with norm less than 1. We will now prove that it holds in dimension  $n$ .

**Theorem 3.** Let  $G$  be a QAT such that  $\|g\|_\infty \leq 1$ , if  $G$  has a cycle than it has a cycle among the points of norm 1.

In order to prove the theorem we need the following lemma.

**Lemma 3.** Let consider a couple of points  $X = (x_1, x_2, \dots, x_n)$  and  $P = (p_1, p_2, \dots, p_n)$  such that  $\|X\|_\infty = m$  and

$$\begin{cases} p_i = 1 & \text{if } x_i = m \\ p_i = -1 & \text{if } x_i = -m \\ p_i = 0 \text{ or } 1 & \text{if } x_i = m - 1 \\ p_i = 0 \text{ or } -1 & \text{if } x_i = 1 - m \\ p_i = 0, -1 \text{ or } 1 & \text{if } |x_i| < m - 1 \end{cases}$$

Let us denote  $Y = (y_1, y_2, \dots, y_n) = G(X)$  and  $Q = (q_1, q_2, \dots, q_n) = G(P)$ . If  $\|Y\|_\infty = m$  then  $\|Q\|_\infty = 1$  and the couple of points  $Y$  and  $Q$  verify the conditions of  $X$  and  $P$ .

PROOF. As  $P$  is of norm 1, the lemma 1 induce that  $q_j = 0, 1$  or  $-1$ .

Consider  $X' = (x'_1, x'_2, \dots, x'_n)$  such that  $x'_i = x_i - p_i$ , it is easy to see that  $|x'_i| < m$  and so

$$-(m-1)\omega \leq \sum_{1 \leq k \leq n} a_{i,k} x'_i \leq (m-1)\omega.$$

Moreover, we have for all  $j = 1, 2, \dots, n$ :

$$q_j = \sum_{1 \leq k \leq n} a_{j,k} p_k + v_j = \sum_{1 \leq k \leq n} a_{j,k} x_k + v_j - \sum_{1 \leq k \leq n} a_{j,k} x'_k$$

As  $Y$  is of norm  $m$ , it exists  $1 \leq j \leq n$  such that  $y_j = -m$  or  $y_j = m$  and so

- if  $y_j = -m$ :

$$-m\omega \leq \sum_{1 \leq k \leq n} a_{j,k} x_k + v_j < (-m+1)\omega.$$

So  $q_j < (-m+1)\omega + (m-1)\omega$  which induce that  $q_j = -1$ .

- if  $y_j = m$ :

$$m\omega \leq \sum_{1 \leq k \leq n} a_{j,k} x_k + v_j < (m+1)\omega.$$

So  $q_j \geq m\omega - (m-1)\omega$  and so  $q_j = 1$ .

Assume that it exists  $k$  such that  $y_k = m - 1$  then

$$(m-1)\omega \leq \sum_{1 \leq k \leq n} a_{j,k} x_k + v_j < m\omega.$$

So  $q_k \geq (m-1)\omega - (m-1)\omega$ :  $q_k$  is positive and so  $q_k = 0$  or  $1$ .

Assume that it exists  $k$  such that  $y_k = 1 - m$  then

$$(1 - m)\omega \leq \sum_{1 \leq k \leq n} a_{j,k}x_k + v_j < (2 - m)\omega.$$

So  $q_k < (2 - m)\omega + (m - 1)\omega$ :  $q_k = 0$  or  $-1$ . □

PROOF OF THEOREM ???. Consider  $X$  a point of a cycle and a point  $P$  defined as in lemma 3 and consider the sequences  $X_i = G^i(X)$  and  $P_i = G^i(P)$ . For each  $i \in \mathbb{N}$  the points  $X_i$  and  $P_i$  verify the condition of lemma 3. The sequence  $P_i$  is then an infinite sequence of points of norm 1. But it exists a finite number of points of norm 1 so there is a cycle among these points. □

**Remark 3.** If  $V = (0, 0, \dots, 0)$ , the lemma 2 states that a point  $X = (x_1, x_2, \dots, x_n)$  of norm 1 can belong to a cycle only if  $x_i \neq 1$ . Theorem 3 implies that such a QAT has a cycle if and only if it has a cycle among the points  $X = (x_1, x_2, \dots, x_n)$  such that  $x_i = 0$  or  $-1$ .

In this section we studied the behaviour under iteration of QATs. In [6] theorem 2 was proved for 2D QATs, we extend this result in nD. Moreover, we conjectured that a 2D-QAT is consistent if and only if there are no cycle among the points of norm 1. In this section we proved this conjecture even for nD-QATs.

### 3. Periodicity of the tiles

In order to obtain efficient computation for discrete image transformation by a discrete affine transformation we can use some properties of QAT. In [7] the 2D case has been treated, in [4] the properties were extended in 3D. More precisely, we have seen in these papers that periodicity properties of tilings associated to QATs improve considerably computation for discrete image transformation by a discrete affine transformation. This periodicity has been studied in the n-dimensional case in [2] and [3], but the study was not complete, now we go on with this study.

#### 3.1. Definitions and results of [2], [3] and [4]

We remember here the definitions and results of [2], [3] and [4] that will be used in next section.

The affine transformation  $g$  extends to  $\mathbb{R}^n$  and for each point  $Y$  of  $\mathbb{R}^n$ , it exists a unique point  $X$  of  $\mathbb{R}^n$  such that  $Y = g(X)$ . This is not the same for a QAT. Indeed, each point of  $\mathbb{Z}^n$  can have either none, one or several antecedents. The set of antecedents of  $X \in \mathbb{Z}^n$  is then called tile with index  $X$ . In other words, a tile of a QAT is the set of antecedents of a point by this QAT.

**Definition 3.** We define a p-tile ( $p \in \mathbb{N}$ ) related to the point  $Y \in \mathbb{Z}^n$  and denote  $P_{G,Y}^p$  the set of points  $X$  of  $\mathbb{Z}^n$  such that  $G^p(X) = Y$  where  $G^p$  denotes the iterative application of  $G$ .

In the sequel we will always consider the QAT  $G$  defined by  $A, \omega$  and  $V$ . To simplify the notations, the tile  $P_Y$  will denote the tile related to the point  $Y$  of the QAT  $G$ .

**Definition 4.** • Let  $X \in \mathbb{Z}^n$  with coordinates  $x_j$  for  $j = 1, 2, \dots, n$ . We define the remainder modulo  $G$  of  $X$  to be  $R_X = (R_1, R_2, \dots, R_n)$  where  $R_j = a_{j,1}x_1 + a_{j,2}x_2 + \dots + a_{j,n}x_n$  modulo  $\omega$  for  $j = 1, 2, \dots, n$ . We will also note  $AX = \omega Y + R$  (we then have  $X \in P_Y$ ).

- Let  $P_Y = \{X_1, X_2, \dots, X_m\}$  (with  $Y \in \mathbb{Z}^n$  and  $X_i \in \mathbb{Z}^n$  for  $i = 1, 2, \dots, m$ ) be a non empty tile. We then refer to the remainder modulo  $G$  of  $P_Y$ , noted  $R_Y$ , the set  $R_Y = \{R_{X_1}, R_{X_2}, \dots, R_{X_m}\}$ .
- Two tiles  $P_Y$  and  $P_{Y'}$  are said to be identical modulo  $G$  when their remainders modulo  $G$  are the same. We then denote  $P_Y \equiv P_{Y'}$ .

**Definition 5.** Two tiles  $P_Y$  and  $P_{Y'}$  are geometrically identical when  $P_{Y'}$  is an image of  $P_Y$  by the translation of an integer vector.

**Proposition 1.** • *Two non-empty tiles  $P_Y$  and  $P_{Y'}$  are identical modulo  $G$  if and only if there exist two points  $X$  and  $X'$  respectively in  $P_Y$  and  $P_{Y'}$  such that:*

$$AX = AX' + \omega(Y - Y').$$

- *Two tiles identical modulo  $G$  are geometrically identical. The reciprocal is false.*

In figure 4 we show some tiles of the QAT defined by  $A' = \frac{1}{6} \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$  (a point of  $\mathbb{Z}^2$  is represented by a unit square whose bottom-left corner corresponds to the represented point). For each point in a tile we provide its corresponding remainder. Tiles  $P_{(2,1)}$  and  $P_{(0,0)}$  are identical modulo  $G$ , therefore they are also geometrically identical. It should also be noted that tiles  $P_{(1,0)}$  and  $P_{(1,1)}$  are geometrically identical but they are not identical modulo  $G$ .

**Definition 6 (Periodicity notations).** For all  $0 \leq i < n$ , We define the set  $\mathcal{A}_i$  as follows:

$$\mathcal{A}_i = \left\{ \alpha \in \mathbb{N}^* / \exists (\beta_j)_{0 \leq j < i} \in \mathbb{Z}^i, \forall (y_0, \dots, y_{n-1}) \in \mathbb{Z}^n, \right. \\ \left. P_{y_0, \dots, y_i + \alpha, \dots, y_{n-1}} \equiv P_{y_0 + \beta_0, \dots, y_{i-1} + \beta_{i-1}, y_i, \dots, y_{n-1}} \right\}$$

Furthermore, let us consider  $\alpha_i = \min(\mathcal{A}_i)$ . We define  $\{\beta_j^i\}_{0 \leq j < i} \in \mathbb{Z}^i$  and  $U_i \in \mathbb{Z}^n$  such that

$$\forall (y_0, \dots, y_{n-1}) \in \mathbb{Z}^n, P_{y_0, \dots, y_i + \alpha_i, \dots, y_{n-1}} = T_{U_i} P_{y_0 + \beta_0^i, \dots, y_{i-1} + \beta_{i-1}^i, y_i, \dots, y_{n-1}}.$$

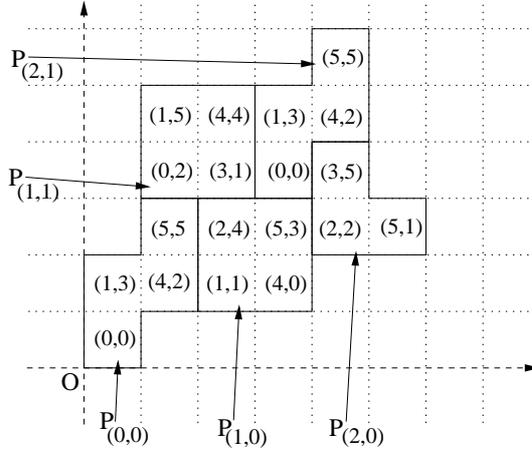


Figure 4: Example of tiles and remainders

**Theorem 4 (Periodicity).** *The set of QAT tiles is  $n$ -periodic, in other words*

$$\forall 0 \leq i < n, \mathcal{A}_i \neq \emptyset$$

Let us suppose that quantities  $\alpha_i$ ,  $\beta_j^i$  and  $U_i$  are given. The following theorem allows us to obtain  $P_Y$  as the translation of a canonical tile  $P_{Y_0}$ .

**Theorem 5.**  $\forall (y_0, \dots, y_{n-1}) \in \mathbb{Z}^n$ , we have  $P_{y_0, \dots, y_{n-1}} = T_W P_{y_0^0, \dots, y_{n-1}^0}$  with

$$W = \sum_{i=0}^{n-1} \gamma_i U_i \quad \text{and} \quad \forall n > i \geq 0, \begin{cases} \gamma_i &= \lfloor \frac{y_i + \sum_{j=i+1}^{n-1} \gamma_j \beta_j^i}{\alpha_i} \rfloor \\ y_i^0 &= \lfloor \frac{y_i + \sum_{j=i+1}^{n-1} \gamma_j \beta_j^i}{\alpha_i} \rfloor \end{cases}$$

Let  $AU = T$  denote the Hermite Normal Form (with  $\det(U) = \pm 1$  and  $T$  upper triangular), this form always exists for nonsingular integer square matrix and  $T$  is an upper triangular matrix such that  $t_{i,i} > 0$  (see [5, 12]). In the two following propositions we denote  $G$  the QAT defined by  $\frac{1}{\omega}A$  and  $G'$  the QAT defined by  $\frac{1}{\omega}T$ .

We then have  $G(X) = Y$  if and only if  $X = UX'$  and  $G'(X') = Y$ . It follows that the tiles of  $G'$  are in bijection with the tiles of  $G$ .

**Proposition 2.** *Let  $P_Y$  and  $P_{Y_1}$  be two tiles of  $G$  and assign  $P'_Y$  and  $P'_{Y_1}$  to the two corresponding tiles of  $G'$ .  $P_Y$  and  $P_{Y_1}$  are identical modulo  $G$  if and only if  $P'_Y$  and  $P'_{Y_1}$  are identical modulo  $G'$ . More precisely, the tiles of  $G$  and  $G'$  related to the same point have the same remainder.*

**Proposition 3.** *Let us denote  $\delta$  as the determinant of  $T$ . The number of distinct tiles modulo  $G'$  is at least equal to  $\frac{\delta}{\gcd(\omega, t_{1,1}) \gcd(\omega, t_{2,2}) \dots \gcd(\omega, t_{n,n})}$  and at most equal to  $\frac{\delta}{\gcd(\omega, t_{1,1})}$ .*

### 3.2. Periodicity of the tiles

In [2] we proved the proposition 3 which gives an upper and a lower bound for the number of distinct tiles.

In this section we give a better upper bound of the number of distinct tiles, more precisely we give an upper bound for each  $\alpha_i$  defined in definition 6. We also give an algorithm which allows to determine the translation vectors  $U_i$  defined in definition 6 and essential for a practical use of theorem 5.

The results of proposition 2 states that to study the periodicity of the tiles of a  $G$  it is sufficient to study the periodicity of the tiles of a  $G'$ . Therefore, in following proposition and lemma, we will only consider upper triangular matrix  $T$ . Moreover, it is trivial that the number of distinct tiles and the periodicity does not depend on the value of  $V$  (see proposition 1 to be convinced), therefore we will assume in the following proposition and lemma that  $V = (0, 0, \dots, 0)$ . In order to simplify the notations, we will denote  $O$  the point  $(0, 0, \dots, 0)$  of  $\mathbb{Z}^n$ .

**Lemma 4.** *Two tiles  $P_Y$  and  $P_{Y'}$  are identical modulo  $G'$  if and only if  $P_{Y-Y'}$  and  $P_O$  are identical modulo  $G'$ .*

PROOF. Let us prove that  $(P_Y \equiv P_{Y'}) \Rightarrow (P_{Y-Y'} \equiv P_O)$  :

$$\begin{aligned} & P_Y \equiv P_{Y'} \\ \Rightarrow & \exists X \in P_Y, X' \in P_{Y'} \text{ such that } TX = TX' + \omega(Y - Y') \\ \Rightarrow & T(X - X') = \omega(Y - Y') \\ \Rightarrow & X - X' \in P_{Y-Y'} \text{ and } R_{X-X'} = 0 \\ \Rightarrow & P_{Y-Y'} \equiv P_O \end{aligned}$$

Reciprocally, let us prove that  $(P_{Y-Y'} \equiv P_O) \Rightarrow (P_Y \equiv P_{Y'})$  :

$$P_{Y'} \equiv P_O \Rightarrow \exists Z \in P_{Y-Y'} \text{ such that } TZ = \omega(Y - Y').$$

Let  $X \in P_Y$  and  $X' = X - Z$ , we have :

$$TX' = TX - TZ = \omega Y + R_X - \omega(Y - Y') = \omega Y' + R_X$$

This implies that  $X' \in P_{Y'}$  and  $R_{X'} = R_X : P_Y \equiv P_{Y'}$ .

□

**Proposition 4.** *Let us denote  $\delta$  as the determinant of  $T$  and for  $i = 1, 2, \dots, n$ ,  $t'_{i,i} = \frac{t_{i,i}}{\gcd(\omega, t_{1,i}, t_{2,i}, \dots, t_{i,i})}$ . For  $i = 1, 2, \dots, n$ , the period  $\alpha_i$  (defined in definition 6) is at most equal to  $t'_{i,i}$ . It follows that the number of distinct tiles modulo  $G$  is at most equal to  $t'_{1,1} t'_{2,2} \dots t'_{n,n}$ .*

PROOF. In order to determine the periodicity of the tiles we have to determine  $\alpha_i = \min(\mathcal{A}_i)$  where

$$\mathcal{A}_i = \left\{ \alpha \in \mathbb{N}^* / \exists (\beta_j)_{0 \leq j < i} \in \mathbb{Z}^i, \forall (y_0, \dots, y_{n-1}) \in \mathbb{Z}^n, \right. \\ \left. P_{y_0, \dots, y_i + \alpha, \dots, y_{n-1}} \equiv P_{y_0 + \beta_0, \dots, y_{i-1} + \beta_{i-1}, y_i, \dots, y_{n-1}} \right\}$$

A consequence of lemma 4 is that

$$\mathcal{A}_i = \left\{ \alpha \in \mathbb{N}^* / \exists (\beta_j)_{0 \leq j < i} \in \mathbb{Z}^i, P_{\beta_0, \beta_1, \dots, \beta_{i-1}, \alpha, 0, \dots, 0} \equiv P_O \right\}$$

Let us denote  $w_k = \frac{\omega}{\gcd(\omega, t_{1,k}, t_{2,k}, \dots, t_{k,k})}$ ,  $t'_{i,k} = \frac{t_{i,k}}{\gcd(\omega, t_{1,k}, t_{2,k}, \dots, t_{k,k})}$  and consider the points  $X_k = (x_1, x_2, \dots, x_n)$  such that  $x_k = \omega_k$  and  $x_i = 0$  for  $i \neq k$ . Let  $Y_k = G'(X_k) = (y_1, y_2, \dots, y_n)$  and  $R_k = (r_1, r_2, \dots, r_n)$  be the remainder modulo  $G'$  of  $X_k$ . By construction of  $X_k$ , we have  $y_i = \lfloor \frac{t_{i,k}\omega_k}{\omega} \rfloor$  and thus:

$$\begin{cases} y_i = \lfloor \frac{t_{i,k}\omega_k}{\omega} \rfloor = \lfloor \frac{t'_{i,k}\omega}{\omega} \rfloor = t'_{i,k} & \text{for } i \leq k \\ y_i = 0 & \text{for } i > k \text{ (since } t_{i,k} = 0 \text{ for } i > k) \end{cases}$$

which implies that  $X_k$  belongs to  $P'_{t'_{1,k}t'_{2,k}\dots t'_{k,k}, 0, \dots, 0}$ .

Moreover,  $r_i = t_{i,k}\omega_k$  modulo  $\omega = t'_{i,k}\omega$  modulo  $\omega = 0$ , this proves that  $X_k$  is a point of  $P'_{t'_{1,k}t'_{2,k}\dots t'_{k,k}, 0, \dots, 0}$  and whose remainder equals  $(0, 0, \dots, 0)$ . For all  $k = 1, 2, \dots, n$ , the tile  $P'_{t'_{1,k}t'_{2,k}\dots t'_{k,k}, 0, \dots, 0}$  *equiv*  $P_O$  modulo  $G$  so that for each  $k$  the period  $\alpha_k$  is at most equal to  $t'_{k,k}$ .  $\square$

**Remark 4.** The above proposition does not gives the minimal period, but the result has another impact. Indeed in [4] we used the periodicity to improve the transformation of a 3D images by a linear transformations. In order to use these periodicity we need to determine the periods  $\alpha_k$  and the translation vectors  $U_k$  such that  $P'_{(\beta_1^k, \beta_2^k, \dots, \beta_{k-1}^k, 0, 0, \dots, 0)} = \mathcal{T}_{U_k} P_{(0, 0, \dots, 0, \alpha_k, 0, 0, \dots, 0)}$  (see proposition 5). In the following we will give an algorithm which determines the optimal period and the corresponding tranlation vectors.

In [2] we give an algorithm that determines the points of a tile, this algorithm explores each point of this tile in a precise and predictable way. Indeed, let denote by  $X$  and  $X'$  the first points of  $P_Y$  and  $P_{Y'}$  determined by this algorithm. Then  $P_Y \equiv P_{Y'}$  if and only if  $X$  and  $X'$  have the same remainder. We used this algorithm to give another algorithm that determines the number of distinct tiles, a set of point  $S'$  such that  $\{P_Z/Z \in S'\}$  contains exactly once each distinct tiles and the remainder of the first point of each tile. We will now use these results to determine the periods  $\alpha_k$  and the translation vectors  $U_k$ .

We know that  $\alpha_k \leq t'_{k,k}$ , let us denote  $Y'_k = (y_1, y_2, \dots, y_n)$  with  $y_k = m \in \mathbb{N}$  and  $y_i = 0$  for  $i \neq k$  so  $\alpha_k = \min\{m \in \mathbb{Z}/\exists Y' \in S', P_{Y'} \equiv P_{Y_k}, Y_k \neq Y'\}$ . Assume that we know  $\alpha_k$ , let us denote  $Y_k = (y_1, y_2, \dots, y_n)$  with  $y_k = \alpha_k$  and  $y_i = 0$  for  $i \neq k$ , we can determine the first point of  $P_{Y_k}$ , we note  $X_k$  this point and  $R_{X_k}$  its remainder. Than it exists  $Y' \in S'$  such that  $P_{Y'} \equiv P_{Y_k}$ , if  $X'$  is the first point of  $P_{Y'}$  than  $X_k - X'$  corresponds to the translation vector. Therefore we obtain the following algorithm.

**Input:** the matrix  $T$ , the coefficients  $t_{k,k}$ ,  $S'$  list on index of distinct tiles, first points of these tiles, and corresponding remainders

**Output:**  $\alpha_k$  the periods and  $U_k$  the translation vector

```

for  $k = 1$  to  $n$  do
  for  $i = 1$  to  $n$  do
     $y_i \leftarrow 0$ 
  end
  ;
   $y_k \leftarrow 1$ ;
  found  $\leftarrow$  False;
  while  $y_k \leq t'_{k,k}$  And Not found do
     $Y \leftarrow (y_1, y_2, \dots, y_n)$ ;
    determine  $X$  the first point of  $P_Y$ , and  $R$  its remainder;
    determine  $Y'$  the point of  $S'$  such that  $P_Y \equiv P_{Y'}$  and  $X'$  the first
    point of  $P_{Y'}$ ;
    if  $Y' \neq Y$  then
      found  $\leftarrow$  True;
       $\alpha_k \leftarrow y_k$ ;
       $U_k = X - X'$ 
    end
  end
end

```

Algorithm 1: Algorithm that determines the periods and the translation vectors

#### 4. Conclusion

In this paper we completed and improved some theoretical results about QATs. We give a better upper bound for the periodicity of tilings generated by QATs, but this bound is not optimal. As a future work it would be interesting to find the optimal bound, to use these properties for the computation of transformation of discrete nD images by discrete linear transformations and of course compare this methods with other existing methods.

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#### References

- [1] M.-A. Jacob-Da Col : Applications quasi-affines et pavages du plan discret. Theoretical Computer Science, Vol. 259, Num. 1-2, pp 245-269 (2001) Available in English at <https://dpt-info.u-strasbg.fr/~jacob>.
- [2] M.-A. Jacob-Da Col, P. Tellier : Quasi-linear transformations and discrete tilings Theoretical Computer Science, Vol. 410, Elsevier, pp 2126-2134 (2009)

- [3] V. Blot, D. Coeurjolly : Quasi-Affine Transformation in Higher Dimension. In 15th International Conference on Discrete Geometry for Computer Imagery, LNCS, vol. 5810, Springer, Montreal (2009)
- [4] D. Coeurjolly, V. Blot, M.-A. Jacob-Da Col : Quasi-Affine Transformation in 3-D: Theory and Algorithms. In 13th International Workshop on Combinatorial Image Analysis, LNCS, vol. 5852, Springer, Mexico (2009)
- [5] H. Cohen : Algorithmic Number Theory. In Proceedings of the Second International Symposium, ANTS-II, France, LNCS, Vol. 1122, 1996.
- [6] M.-A. Jacob : Applications quasi-affines. PhD thesis, Université Louis Pasteur, Strasbourg. (1993)
- [7] M.-A. Jacob : Transformation of Digital Images by Discrete Affine Applications. Comput. & Graphics, Vol. 19, Elsevier, pp 373-389 (1995)
- [8] P. Nehlig, J.-P. Reveilles : Fractals and Quasi-Affine Transformations. Computer Graphics Forum, vol. 14, Blackwell, pp. 147-158 (1995)
- [9] P. Nehlig : Applications quasi-affines : pavages par images réciproques. Theoretical Computer Science, vol. 156, Elsevier, pp 1-38 (1995)
- [10] P. Nehlig : Applications Affines Discrètes et Antialiassage. PhD thesis, Université Louis Pasteur, Strasbourg. (1992)
- [11] J.-P. Reveilles. *Géométrie discrète, calcul en nombres entiers et algorithmique*.Thèse d'Etat, Université Louis Pasteur, Strasbourg. (1991)
- [12] A. Storjohann, G. Labahn : Asymptotically fast computation of hermite normal forms of integer matrices. In: ISSAC 96: Proceedings of the 1996 international symposium on Symbolic and algebraic computation, ACM, pp 259266 (1996)