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Analogous to cliques of (m, n) -colored mixed graphs

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Abstract

A (m, n) -colored mixed graph is a graph having arcs of m different colors and edges of n different colors. A graph homomorphism of a (m, n) -colored mixed graph G to a (m, n) -colored mixed graph H is a vertex mapping such that if uv is an arc (edge) of color c in G , then $f(u)f(v)$ is also an arc (edge) of color c . The (m, n) -colored mixed chromatic number $\chi_{(m,n)}(G)$ of a (m, n) -colored mixed graph G , introduced by Nešetřil and Raspaud, is the order (number of vertices) of the smallest homomorphic image of G .

In this article, we define the analogue of clique for (m, n) -colored mixed graphs. A (m, n) -clique is a (m, n) -colored mixed graph C with $\chi_{(m,n)}(C) = |C|$. We show that almost all (m, n) -colored mixed graphs are (m, n) -cliques. We prove exact bound on the order of the biggest outerplanar (m, n) -clique and provide upper and lower bounds for the order of the biggest planar (m, n) -clique. Furthermore, we address a particular complexity problem related to $(0, 2)$ -colored mixed graphs and compare the parameters $\chi_{(1,0)}$ and $\chi_{(0,2)}$.

Keywords: colored mixed graphs, signed graphs, graph homomorphism, chromatic number, clique number, planar graphs.

1 Introduction

The notion of vertex coloring and chromatic number was studied in a generalized set up by Nešetřil and Raspaud [10] by defining colored homomorphism for (m, n) -colored mixed graphs. In this article we will define and study the notion of cliques and clique numbers for the same generalized type of graphs.

A (m, n) -colored mixed graph $G = (V, A \cup E)$ is a graph G with set of vertices V , set of arcs A and set of edges E where arcs are colored with m colors and the edges are colored with n colors and the underlying undirected graph is simple. Let $G = (V_1, A_1 \cup E_1)$ and $H = (V_2, A_2 \cup E_2)$ be two (m, n) -colored mixed graphs. A colored homomorphism of G to H is a function $f : V_1 \rightarrow V_2$ satisfying

$$uv \in A_1 \Rightarrow f(u)f(v) \in A_2,$$

$$uv \in E_1 \Rightarrow f(u)f(v) \in E_2,$$

and the color of the arc or edge linking $f(u)$ and $f(v)$ is the same as the color of the arc or the edge linking u and v [10]. We write $G \rightarrow H$ whenever there exists a homomorphism of G to H .

Given a (m, n) -colored mixed graph G let H be a (m, n) -colored mixed graph with minimum order (number of vertices) such that $G \rightarrow H$. Then the order of H is the (m, n) -colored mixed chromatic number $\chi_{(m,n)}(G)$ of G . The maximum (m, n) -colored mixed chromatic number taken over all (m, n) -colored mixed graphs having underlying undirected simple graph G is denoted by $\chi_{(m,n)}(G)$. Let \mathcal{F} be a family of undirected simple graphs. Then $\chi_{(m,n)}(\mathcal{F})$ is the maximum of $\chi_{(m,n)}(G)$ taken over all $G \in \mathcal{F}$.

Note that a $(0, 1)$ -colored mixed graph G is nothing but an undirected simple graph while $\chi_{(0,1)}(G)$ is the ordinary chromatic number. Similarly, the study of $\chi_{(1,0)}(G)$ is the study of oriented chromatic number which is considered by several researchers in the last two decades (for details please check the recent updated survey [13]). Alon and Marshall [1] studied the homomorphism of $(0, m)$ -colored mixed graphs with a particular focus on $m = 2$.

Nešetřil and Raspaud [10] showed that $\chi_{(m,n)}(G) \leq k(2m + n)^{k-1}$ where G is a k -acyclic colorable graph. As planar graphs are 5-acyclic colorable due to Borodin [3], the same authors implied $\chi_{(m,n)}(\mathcal{P}) \leq 5(2m + n)^4$ for the family \mathcal{P} of planar graphs as a corollary. This result, in particular, implies $\chi_{(1,0)}(\mathcal{P}) \leq 80$ and $\chi_{(0,2)}(\mathcal{P}) \leq 80$ (independently proved before in [11] and [1], respectively). It is observed that using similar techniques the same or close upper and lower bounds can be proved for both $\chi_{(1,0)}(\mathcal{F})$ and $\chi_{(0,2)}(\mathcal{F})$ for several graph families (details in Section 4). This made us wonder if there is an underlying general relation between the two kinds of graph colorings. Moreover, note that in the main results, each colored arcs are contributing twice as much each colored edge. It is natural to wonder if this is a universal trend or not. In this article we explore this speculation and actually end up proving the opposite, that is, we construct examples of undirected simple graphs for which $\chi_{(0,2)}(G) - \chi_{(1,0)}(G)$ is arbitrarily high or low.

A (m, n) -clique C is a (m, n) -colored mixed graph with $\chi_{(m,n)}(C) = |C|$ (where $|C|$ is the order of the graph C). The problem is to find a (m, n) -clique with maximum order contained as a subgraph in a (m, n) -colored mixed graph G . That maximum order is the (m, n) -absolute clique number $\omega_{a(m,n)}(G)$ of G . Our definition is a generalization using the intuition of oriented clique [5] and oriented absolute clique number [8] which is the case when $(m, n) = (1, 0)$. Note that $\omega_{a(0,1)}(G)$ is just the ordinary clique number of a simple undirected graph G .

While studying oriented absolute clique number, another related parameter, the oriented relative clique number, arose naturally [8]. An analogous parameter seems significant for the study of homomorphisms of (m, n) -colored mixed graphs as well. A *relative (m, n) -clique* R of a (m, n) -colored mixed graph G is a set of vertices such that for any two distinct vertices $u, v \in R$ we have $f(u) \neq f(v)$ for each homomorphism $f : G \rightarrow H$ and for each (m, n) -colored mixed graph H . That is, no two distinct vertices of a relative clique can be identified under any homomorphism. The *(m, n) -relative clique number* $\omega_{r(m,n)}(G)$ of a (m, n) -colored mixed graph G is the cardinality of a largest relative (m, n) -clique of G . For undirected simple graphs the two parameters coincide but they are different for all $(m, n) \neq (0, 1)$. The two parameters for an undirected simple graph and for a family of graphs are defined similarly as in the case of (m, n) -colored mixed chromatic number.

From the definitions it is clear that $\omega_{a(m,n)}(G) \leq \omega_{r(m,n)}(G) \leq \chi_{(m,n)}(G)$ for any (m, n) -colored mixed graph G . We will show that (m, n) -cliques are not rare objects by proving that almost all (m, n) -colored mixed graphs are (m, n) -cliques.

We also studied the parameter for the family \mathcal{O} of outerplanar and \mathcal{P} of planar graphs and provide exact bound of $\omega_{a(m,n)}(\mathcal{O}) = \omega_{r(m,n)}(\mathcal{O}) = 3(2m + n) + 1$ for outerplanar graph and quadratic upper and lower bound of $3(2m + n)^2 + (2m + n) + 1 \leq \omega_{a(m,n)}(\mathcal{P}) \leq 9(2m + n)^2 + 2(2m + n) + 2$ for planar graphs. Note that the present lower and upper bounds for the (m, n) -colored mixed chromatic number of these two families are $(2m + n)^2 + \epsilon(2m + n) + 1 \leq$

$\chi_{(m,n)}(\mathcal{O}) \leq 3(2m+n)^2$ and $(2m+n)^3 + \epsilon(2m+n)^2 + (2m+n) + \epsilon \leq \chi_{(m,n)}(\mathcal{P}) \leq 5(2m+n)^4$ where $\epsilon = 1$ for m odd or $m = 0$, and $\epsilon = 2$ for $m > 0$ even [7].

Given an undirected simple graph it is NP -hard to decide if it is underlying graph of an (m, n) -clique for $(m, n) = (1, 0)$ and $(0, 2)$ is known [2]. We prove a related similar result for an equivalence class of $(0, 2)$ -colored mixed graph.

An important related question: Naserasr, Rollova and Sopena [9] recently studied the homomorphism of a particular equivalence class of $(0, 2)$ -graphs which they called signed graphs. Using these notions they managed to reformulate and extend several classical theorems and conjectures including the Four-Color Theorem and the Hadwiger's conjecture. They also defined absolute and relative clique number of signed graphs in a similar fashion. We will avoid the definitions related to homomorphisms of signed graphs as it will make this article unnecessarily complicated. The specific question that we are interested in can be formulated using the notions already defined in this paper. The reader is encouraged to read the above mentioned paper for further details. In the "2nd Autumn meeting on signed graphs" (Thézac, France 2013) organized by Naserasr and Sopena the following question was asked by Naserasr:

Question 1.1. *A $(0, 2)$ -colored mixed graph is an signed clique if each pair of its non-adjacent vertices is part of a 4-cycle with three edges of the same color while the other edge has a different color. Given an undirected simple graph G what is the complexity of deciding if it is an underlying graph of a signed clique?*

Here we prove that this is an NP -complete problem.

2 On absolute and relative (m, n) -clique numbers

First we will characterize the (m, n) -cliques. Let G be an (m, n) -colored mixed graph. Let uvw be a 2-path in the underlying simple graph of G . Then uvw is a *special 2-path* of G if one of the following holds:

- (i) uv and vw are edges of different colors,
- (ii) uv and vw are arcs (possibly of the same color),
- (iii) uv and wv are arcs of different colors,
- (iv) vu and vw are arcs of different colors,
- (v) exactly one of uv and vw is an edge.

Proposition 2.1. *Let G be an (m, n) -colored mixed graph. Then two vertices u, v of G are part of a relative clique if and only if either they are adjacent or they are connected by a special 2-path.*

Proof. Let u, v be two vertices of a (m, n) -colored mixed graph G . If they are not part of any relative clique then there exists a (m, n) -colored mixed graph H and a homomorphism $f : G \rightarrow H$ such that $f(u) = f(v)$.

Firstly, suppose that u, v are adjacent. Then the image $f(u)$ of u and v will induce a loop in the underlying graph of H . Secondly, suppose that u, v are connected by a special 2-path. Then the image $f(u)$ of u and v will induce multiple edges in the underlying graph of H . But

we are working with (m, n) -colored mixed graphs with underlying simple graphs. Hence, u, v can neither be adjacent, nor be connected by a special 2-path.

Now assume that u, v are neither adjacent, nor connected by a special 2-path. Then simply identify the vertices u, v of G to obtain another (m, n) -colored mixed graph H' . Call this identified new vertex u for convenience. If in H' there are more than one edge of the same color between two vertices, delete all of them but one. Also, if in H' there are more than one arc of the same color in the same direction between two vertices, delete all of them but one. Doing this we obtain a new (m, n) -colored mixed graph H from H' . Note that the underlying graph of H is simple. Also it is easy to see that $g : G \rightarrow H$ where $g(x) = x$ for $x \neq u, v$ and $g(u) = g(v) = u$ is a homomorphism. \square

Corollary 2.2. *Let G be an (m, n) -colored mixed graph. Then G is a (m, n) -clique if and only if each pair of non-adjacent vertices of G is connected by a special 2-path.*

Proof. It follows from the definitions that (m, n) -colored mixed graph G is a (m, n) -clique if and only if all its vertices are part of a relative clique. Then the result follows by Proposition 2.1. \square

Using these characterizations we will first prove our asymptotic result. Consider the model of generating a random (m, n) -mixed graph on k vertices in which the adjacency type between each pair of vertices – one of the m arc types in forward or backward direction, one of the n edge types or non-adjacent – is chosen uniformly at random. We show that under such a model, as $k \rightarrow \infty$ the probability of generating an (m, n) -mixed clique approaches 1.

Theorem 2.3. *For $(m, n) \neq (0, 1)$ almost every graph is an (m, n) -mixed clique.*

Proof. Let \mathcal{G}_k be the set of all (m, n) -mixed graphs on k vertices (where $(m, n) \neq (0, 1)$), and let \mathcal{C}_k be the set of all (m, n) -cliques on k vertices.

$$|\mathcal{G}_k| = (2m + n + 1)^{\binom{k}{2}}.$$

Consider generating an (m, n) -mixed graph, G , on k vertices such that a particular pair of vertices $u, v \in V(G)$ are neither adjacent nor connected by a special 2-path. An easy counting argument gives that the number of such graphs is:

$$(k - 2)(6m + 3n + 1)(2m + n + 1)^{\binom{k-2}{2}}.$$

This is seen by considering the possibilities for the different sorts of adjacencies that can occur between u, v and each of the other $k - 2$ vertices. This implies directly the following upper bound on the cardinality $|\overline{\mathcal{C}}_k|$ of the set of (m, n) -mixed graph that are not (m, n) -cliques.

$$|\overline{\mathcal{C}}_k| < \binom{k}{2} (k - 2)(6m + 3n + 1)(2m + n + 1)^{\binom{k-2}{2}}.$$

For fixed values of m and n :

$$\lim_{k \rightarrow \infty} \frac{\binom{k}{2} (k - 2)(6m + 3n + 1)(2m + n + 1)^{\binom{k-2}{2}}}{(2m + n + 1)^{\binom{k}{2}}} = 0.$$

Thus,

$$\lim_{k \rightarrow \infty} \frac{|\overline{\mathcal{C}}_k|}{|\mathcal{G}_k|} = 0.$$

Therefore, as k grows large, the ratio of the number of (m, n) -mixed cliques on k vertices to the number of (m, n) -mixed graphs on k vertices approaches 1. \square

Now we will prove our result for the family \mathcal{O} of outerplanar graphs.

Theorem 2.4. *Let \mathcal{O} be the family of outerplanar graphs. Then $\omega_{a(m,n)}(\mathcal{O}) = \omega_{r(m,n)}(\mathcal{O}) = 3(2m + n) + 1$ for all $(m, n) \neq (0, 1)$.*

Proof. First we will show that $\omega_{a(m,n)}(\mathcal{O}) \geq 3(2m + n) + 1$ by explicitly constructing an outerplanar (m, n) -clique $H = (V, A \cup E)$ with $3(2m + n) + 1$ vertices as follows:

- the set of vertices $V = \{x\} \cup \{v_{i,j} | 1 \leq i \leq 2m + n, 1 \leq j \leq 3\}$,
- the set of edges $E = \{xv_{i,j} | 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{v_{i,1}v_{i,2}, v_{i,2}v_{i,3} | 1 \leq i \leq 2m + n\}$,
- the set of arcs $A = \{xv_{i,j}, v_{i+m,j}x | n + 1 \leq i \leq n + m, 1 \leq j \leq 3\}$,
- the edges $v_{i,1}v_{i,2}$ and $v_{i,2}v_{i,3}$ have different colors for all $i \in \{1, 2, \dots, n\}$,
- the edges $xv_{i,j}$ receives the i th edge color for all $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, 3\}$,
- the arcs $xv_{i+n,j}$ and $v_{i+n+m,j}x$ receives the i th arc color for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, 3\}$.

It is easy to check that the graph H is indeed an outerplanar (m, n) -clique with $3(2m + n)$ vertices. Thus, $\omega_{r(m,n)}(\mathcal{O}) \geq \omega_{a(m,n)}(\mathcal{O}) \geq 3(2m + n) + 1$

Now to prove the upper bound let $G = (V, A \cup E)$ be a minimal (with respect to number of vertices) (m, n) -colored mixed graph with underlying outerplanar graph having relative clique number $\omega_{r(m,n)}(\mathcal{O})$. Moreover, assume that we cannot add any more edges/arcs keeping the graph G outerplanar. We can assume this because adding more edges will not affect the relative clique number as it is already equal to $\omega_{r(m,n)}(\mathcal{O})$. Let R be a relative clique of cardinality $\omega_{r(m,n)}(\mathcal{O})$ of G . Let $S = V \setminus R$.

As it is not possible to increase the number of edges/arcs of G keeping the graph outerplanar $d(v) \geq 2$ for all $v \in V$. Then, as G is outerplanar, there exists a vertex $u_1 \in V$ with $d(u_1) = 2$. Note that if $u_1 \in S$ then we can delete u_1 and connect the neighbors of u_1 with an edge (if they are not already adjacent) to obtain a graph with same relative clique number contradicting the minimality of G . Thus, $u_1 \in R$. Fix an outerplanar embedding of G with the outer (facial) cycle having vertices u_1, u_2, \dots, u_R of R embedded in a clockwise manner on the cycle. Also let a and b be the two neighbors of u_1 . Note that a and b are adjacent as it is not possible to increase the number of edges/arcs of G keeping it outerplanar.

Every vertex of $R \setminus \{u_1, a, b\}$ is connected to u_1 through a or b by a special 2-path. Note that, a and b can have at most one common neighbor other than u_1 . Let that common neighbor, if it exists, be x . So, all the vertices from $R \setminus \{u_1, a, b, x\}$ are adjacent to exactly one of a, b . Also from the first part of the proof we know that $\omega_{r(m,n)}(\mathcal{O}) \geq \omega_{a(m,n)}(\mathcal{O}) \geq 3(2m + n) + 1$ thus, $|R \setminus \{u_1, a, b, x\}| \geq 3$ for all $(m, n) \neq (0, 1)$. Suppose neither a nor b is adjacent to all the vertices of $R \setminus \{u_1, a, b, x\}$. In that case, there are two vertices in $R \setminus \{u_1, a, b, x\}$ that are neither adjacent nor connected by a 2-path. Therefore, either a or b must be adjacent to all the vertices of $R \setminus \{a, b\}$.

Assume without loss of generality that a is adjacent to all the vertices of $R \setminus \{a\}$. Let $u_i, u_j \in (R \setminus \{a\})$ be two vertices with $i \neq j$. If both u_i and u_j are adjacent to a with same color of edges, or same color of arcs with the same direction then $|i - j| \leq 2$ modulo $|R|$ as otherwise

they can be neither adjacent nor connected by a special 2-path in G . Hence the number of vertices from $R \setminus \{a\}$ adjacent to a with the same color of edges or same color of arcs with the same direction is at most three.

From this we can conclude that

$$\begin{aligned} |R| &\leq 2 \sum_{k=1}^m 3 + \sum_{k=1}^n 3 + |\{a\}| \\ &\leq 3(2m+n) + 1. \end{aligned}$$

This completes the proof. \square

Furthermore, using the above result, we will prove lower and upper bounds for the (m, n) -absolute clique number of the family \mathcal{P} of planar graphs.

Theorem 2.5. *Let \mathcal{P} be the family of planar graphs. Then $3(2m+n)^2 + (2m+n) + 1 \leq \omega_{a(m,n)}(\mathcal{P}) \leq 9(2m+n)^2 + 2(2m+n) + 2$ for all $(m, n) \neq (0, 1)$.*

Proof. First we will show that $\omega_{a(m,n)}(\mathcal{P}) \geq 3(2m+n)^2 + (2m+n) + 1$ by explicitly constructing a planar (m, n) -clique $H^* = (V^*, A^* \cup E^*)$ with $3(2m+n)^2 + (2m+n) + 1$ vertices. First recall the example of the outerplanar (m, n) -clique H from the previous proof (proof of Theorem 2.4). Take $2m+n$ copies of H and call them $H_1, H_2, \dots, H_{2m+n}$. Let the vertices of H_i be $V_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,3(2m+n)+1}\}$ for all $i \in \{1, 2, \dots, 2m+n\}$. The graph H^* is constructed as follows:

- the set of vertices $V^* = \{x\} \cup (\cup_{i=1}^{2m+n} V_i)$,
- the edges $xv_{i,j}$ has the i th edge color for $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, 3(2m+n) + 1\}$,
- the arcs $xv_{i+n,j}$ and $v_{i+n+m,j}x$ has the i th arc color for $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, 3(2m+n) + 1\}$.

It is easy to check that the graph H^* is indeed a planar (m, n) -clique with $3(2m+n)^2 + (2m+n) + 1$ vertices.

Now to prove the upper bound first notice that any (m, n) -clique has diameter at most 2. Let $G = (V, A \cup E)$ be a planar (m, n) -clique with more than $\omega_{a(m,n)}(G) > 3(2m+n)^2 + (2m+n) + 1$. Assume that G is triangulated. As deleting edges do not increase the (m, n) -absolute clique number, it is enough to prove this result for triangulated G .

Suppose that the set of vertices adjacent to a vertex u by an edge with i th edge color is $N_i(u)$ for all $i \in \{1, 2, \dots, n\}$. Also, let the set of vertices v that are adjacent to u with an arc uv with j th arc color is $N_j^+(u)$ and the set of vertices v that are adjacent to u with an arc vu with j th arc color is $N_j^-(u)$ for $j \in \{1, 2, \dots, m\}$.

We know that a diameter two planar graph is dominated by at most two vertices except for a particular graph on nine vertices due to Goddard and Henning [4]. We have already shown in the first part of the proof that $\omega_{a(m,n)}(\mathcal{P}) \geq 3(2m+n)^2 + (2m+n) + 1 \geq 15$ for all $(m, n) \neq (0, 1)$. Thus, we can assume that G is dominated by at most two vertices.

First assume that G is dominated by a single vertex x . Let G' be the graph obtained by deleting the vertex x from G . Note that G' is an outerplanar graph. Furthermore, the graph induced by $N_i(x)$ is a relative (m, n) -clique of G' . Thus, by Theorem 2.4 $|N_i(x)| \leq 3(2m+n) + 1$ for all $i \in \{1, 2, \dots, n\}$. Similarly, $|N_j^+(x)|, |N_j^-(x)| \leq 3(2m+n) + 1$ for all $j \in \{1, 2, \dots, m\}$. Thus,

$$\begin{aligned}\omega_{an}(G) &\leq |\cup_{i=1}^n N_i(x)| + |\cup_{j=1}^m N_j^+(x)| + |\cup_{j=1}^m N_j^-(x)| + |\{x\}| \\ &\leq 3(2m+n)^2 + (2m+n) + 1.\end{aligned}$$

Now let G has domination number 2. Let x, y be such that they dominates G and has maximum number of common neighbors among all pairs of dominating vertices of G . Now we fix some notations to prove the rest of this result.

- $C = N(x) \cap N(y)$ and $C_{ij} = N_i(x) \cap N_j(y)$ for all $i, j \in \{1, 2, \dots, n\}$,
- $C_{ij}^{*\alpha} = N_i(x) \cap N_j^\alpha(y)$ and $C_{ji}^{\alpha*} = N_j^\alpha(x) \cap N_i(y)$ for all $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m\}$ and $\alpha \in \{+, -\}$,
- $C_{ij}^{\alpha\beta} = N_i^\alpha(x) \cap N_j^\beta(y)$ for all $i, j \in \{1, 2, \dots, m\}$ and $\alpha, \beta \in \{+, -\}$,
- $S_{x_i} = N_i(x) \setminus C$, $S_{y_i} = N_i(y) \setminus C$, $S_{x_j}^\alpha = N_j^\alpha(x) \setminus C$ and $S_{y_j}^\alpha = N_j^\alpha(y) \setminus C$ for all $i \in \{1, 2, \dots, n\}$, $j \in \{1, 2, \dots, m\}$ and $\alpha \in \{+, -\}$,
- $S_x = N(x) \setminus C$, $S_y = N(y) \setminus C$ and $S = S_x \cup S_y$.

Note that, for $|C| \geq 6$ we must have $|C_{ij}|, |C_{ik}^{*\beta}|, |C_{lj}^{\alpha*}|, |C_{lk}^{\alpha\beta}| \leq 3$ for all $i, j \in \{1, 2, \dots, n\}$, $l, k \in \{1, 2, \dots, m\}$ and $\alpha, \beta \in \{+, -\}$ as otherwise it is not possible to have pairwise distance at most two between the vertices of C which has the same incidence rule with both x and y keeping the graph planar. So, we can conclude that

$$|C| \leq \left| \bigcup_{1 \leq i, j \leq n} C_{ij} \right| + \left| \bigcup_{\substack{1 \leq i \leq n, \\ 1 \leq k \leq m, \\ \alpha \in \{+, -\}}} C_{ik}^{*\beta} \right| + \left| \bigcup_{\substack{1 \leq l \leq m, \\ 1 \leq j \leq n, \\ \alpha \in \{+, -\}}} C_{lj}^{\alpha*} \right| + \left| \bigcup_{\substack{1 \leq l, k \leq m, \\ \alpha, \beta \in \{+, -\}}} C_{lk}^{\alpha\beta} \right| \leq 3(2m+n)^2.$$

As $3(2m+n)^2 \geq 6$ for all $(m, n) \neq (0, 1)$ we can say that $|C| \leq 3(2m+n)^2$ in general.

If $|C| \geq 2$ then pick two vertices $u, v \in C$. Note that the cycle induced by x, y, u, v divides the plane into two regions: denote the interior by R_1 and the exterior by R_2 . Consider a planar embedding of G . Observe that if we delete the vertices x, y and all the vertices placed in R_2 , then the resultant graph, denoted by G_1 , is outerplanar. Similarly, if we delete the vertices x, y and all the vertices placed in R_1 , then the resultant graph, denoted by G_2 , is outerplanar.

Observe that if we restrict ourselves to G_1 , then the set $[S_{x_i} \cup S_{y_i}]_{G_1}$ of vertices of $(S_{x_i} \cup S_{y_i})$ which are also part of G_1 is a relative clique. Thus, by Theorem 2.4 $[S_{x_i} \cup S_{y_i}]_{G_1} \leq 3(2m+n) + 1$ for all $i \in \{1, 2, \dots, n\}$. Similarly, we can show that $[S_{x_j}^\alpha \cup S_{y_j}^\alpha]_{G_1} \leq 3(2m+n) + 1$ for all $j \in \{1, 2, \dots, m\}$ and $\alpha \in \{+, -\}$. Thus, in G_1 we have $|[S]_{G_1}| \leq 3(2m+n)^2 + (2m+n)$.

Similarly, in G_2 we have $|[S]_{G_2}| \leq 3(2m+n)^2 + (2m+n)$. Thus, in G we have $|S| \leq 6(2m+n)^2 + 2(2m+n)$.

For $|C| = 1$, the graph obtained by deleting the vertices x and y is outerplanar. So, repeating the same argument as above we get $|S| \leq 3(2m+n)^2 + (2m+n)$ in this case. Hence, no matter what, we have $|S| \leq 6(2m+n)^2 + 2(2m+n)$.

Therefore,

$$|G| = |C| + |S| + |\{x, y\}| \leq 9(2m + n)^2 + 2(2m + n) + 2.$$

Hence we are done. \square

Finally, we would like to make the following conjecture regarding the (m, n) -absolute clique number of planar graphs.

Conjecture 2.6. *For the family \mathcal{P} of planar graphs $\omega_{a(m,n)}(\mathcal{P}) = 3(2m + n)^2 + (2m + n) + 1$ for all $(m, n) \neq (0, 1)$.*

It is known that the conjecture is true for $(m, n) = (1, 0)$ [8] and $(0, 2)$ [12].

3 Complexity aspects

We know that the complexity of deciding whether, given an undirected simple graph G , we can assign colors to the arcs and edges of G to make it an (m, n) -clique is NP-hard for $(m, n) = (1, 0)$ and $(0, 2)$ [2]. Here we address a new related problem concerning signed graphs, an equivalence class of $(0, 2)$ -colored mixed graphs [9]. We will not discuss signed graphs here but will encourage the reader to have a look at [9] instead for finding motivation of the problem. We will formulate the problem in terms of the definitions and notions given in this article to keep this paper self-contained.

In a $(0, 2)$ -colored mixed graph G , by an *unbalanced 4-cycle* we refer to a 4-cycle of G having an odd number of edges of the same color. We call G a *signed clique* if each pair of vertices of G is part of an unbalanced 4-cycle. An undirected simple graph G is being *2-edge-colored* if we assign a color to each of its edges from a fixed set of two colors.

Our main result of this section reads as follows: It is NP-complete to decide whether a given undirected graph can be 2-edge-colored so that we obtain a signed clique. In other words, there should not be an easy characterization of signed cliques in terms of their underlying undirected graphs.

SIGNED CLIQUE EDGE-COLOR ASSIGNMENT

Input: An undirected graph G .

Question: Can G be 2-edge-colored so that we get a signed clique?

Theorem 3.1. SIGNED CLIQUE EDGE-COLOR ASSIGNMENT is NP-complete.

Proof. We prove the NP-hardness of SIGNED CLIQUE EDGE-COLOR ASSIGNMENT by reduction from the following NP-complete problem.

MONOTONE NOT-ALL-EQUAL 3-SATISFIABILITY

Instance: A 3CNF formula F over variables x_1, x_2, \dots, x_n and clauses C_1, C_2, \dots, C_m involving no negated variables.

Question: Is F *not-all-equal satisfiable*, that is, does there exist a truth assignment to the variables under which every clause has at least one true and one false variable?

Due to the NP-completeness of the 2-COLOURING OF 3-UNIFORM HYPERGRAPH problem (see [6]), it is easily seen that MONOTONE NOT-ALL-EQUAL 3-SATISFIABILITY remains NP-complete when every clause of F has its three variables being different. So this additional restriction is understood throughout. From a 3CNF formula F , we construct an undirected graph G_F such that

F is not-all-equal satisfiable

\Leftrightarrow

G_F can be 2-edge-colored in a signed clique way.

The construction of G_F is achieved in two steps. We first construct, from F , an undirected graph H_F such that F is not-all-equal satisfiable if and only if there exists a 2-edge-coloring c_H of H under which only some *representative pairs* of non-adjacent vertices belong to unbalanced 4-cycles. This equivalence is obtained by designing H_F in such a way that every representative pair belongs to a unique 4-cycle, with some of these 4-cycles overlapping to force some edges to have the same or different colors by c_H . Then we obtain G_F by adding some vertices and edges to H_F in such a way that no new 4-cycles including representative pairs are created, and there exists a partial 2-edge-coloring of the edges in $E(G_F) \setminus E(H_F)$ for which every non-representative pair is included in an unbalanced 4-cycle. In this way, the equivalence between G_F and F is only dependent of the equivalence between H_F and F , which has not been altered when constructing G_F from H_F .

Step 1. Start by adding two vertices r_1 and r_2 to H_F . Then, for every variable x_i of F , add two vertices u_i and u'_i to H_F , and link these vertices to both r_1 and r_2 . Now, for every $i \in \{1, 2, \dots, n\}$, assuming the variable x_i belongs to the (distinct) clauses $C_{j_1}, C_{j_2}, \dots, C_{j_{n_i}}$, add n_i new vertices $v_{i,j_1}, v_{i,j_2}, \dots, v_{i,j_{n_i}}$ to H_F , and join all these new vertices to both r_1 and r_2 . Finally, for every clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F , add a new vertex w_j to H_F , and join it to all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$.

The representative pairs are the following. For every variable x_i of F , all pairs $\{u_i, u'_i\}$ and those of the form $\{u'_i, v_{i,j}\}$ are representative. Also, for every clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F , the pairs $\{v_{i_1,j}, v_{i_2,j}\}$, $\{v_{i_1,j}, v_{i_3,j}\}$ and $\{v_{i_2,j}, v_{i_3,j}\}$ are representative.

We below prove some claims about the existence of a *good 2-edge-coloring* c_H of H_F , i.e. a 2-edge-coloring under which every representative pair of H_F belongs to an unbalanced 4-cycle. By (H_F, c_H) , we refer to the 2-edge-colored graph obtained from H_F by coloring its edges as indicated by c_H . Given two edges u_1u and u_2u of H_F meeting at u , we say below that u_1 and u_2 *agree* (resp. *disagree*) on u (by c_H) if u_1u and u_2u are assigned the same color (resp. distinct colors) by c_H .

Claim 1. *Let c_H be a good 2-edge-coloring of H_F , and let x_i be a variable appearing in clauses $C_{j_1}, C_{j_2}, \dots, C_{j_{n_i}}$ of F . If r_1 and r_2 agree (resp. disagree) on u_i , then r_1 and r_2 agree (resp. disagree) on $v_{i,j_1}, v_{i,j_2}, \dots, v_{i,j_{n_i}}$.*

Proof. Because $\{u_i, u'_i\}$ is representative and $u_i r_1 u'_i r_2 u_i$ is the only 4-cycle containing u_i and u'_i , the vertices r_1 and r_2 agree on u_i and disagree on u'_i in (H_F, c_H) without loss of generality. Now, because every pair $\{u'_i, v_{i,j}\}$ is representative, the only 4-cycle including u'_i and $v_{i,j}$ is $u'_i r_1 v_{i,j} r_2 u'_i$, and r_1 and r_2 disagree on u'_i in (H_F, c_H) , necessarily r_1 and r_2 agree on $v_{i,j}$. \square

Claim 2. *Let c_H be a good 2-edge-coloring of H_F , and let $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ be a clause of F . Then r_1 and r_2 cannot agree or disagree on all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$.*

Proof. First note that the only 4-cycles of H_F containing, say, $v_{i_1,j}$ and $v_{i_2,j}$ are $v_{i_1,j} r_1 v_{i_2,j} r_2 v_{i_1,j}$, $v_{i_1,j} w_j v_{i_2,j} r_1 v_{i_1,j}$ and $v_{i_1,j} w_j v_{i_2,j} r_2 v_{i_1,j}$. The claim then follows from the fact that if r_1 and r_2 , say, agree on all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$, then (H_F, c_H) has no unbalanced 4-cycle including r_1 and r_2 and two of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$. So, since c_H is a good 2-edge-coloring, necessarily there are at least three unbalanced 4-cycles containing w_j and every two of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$, but one can easily convince himself that this is impossible.

Assume on the contrary that *e.g.* r_1 and r_2 agree on $v_{i_1,j}$ and disagree on $v_{i_2,j}$ and $v_{i_3,j}$. So far, note that $r_1v_{i_1,j}r_2v_{i_2,j}r_1$ and $r_1v_{i_1,j}r_2v_{i_3,j}r_1$ are unbalanced 4-cycles of (H_F, c_H) . Then there is no contradiction against the fact that c_H is good, since *e.g.* $v_{i_2,j}$ and $v_{i_3,j}$ can agree on w_j (and, in such a situation, $v_{i_2,j}w_jv_{i_3,j}r_1v_{i_2,j}$ is an unbalanced 4-cycle). The important thing to have in mind is that coloring the edges incident to w_j can only create unbalanced 4-cycles containing the representative pairs $\{v_{i_1,j}, v_{i_2,j}\}$, $\{v_{i_1,j}, v_{i_3,j}\}$ and $\{v_{i_2,j}, v_{i_3,j}\}$. So coloring the edges incident to w_j to make $v_{i_2,j}$ and $v_{i_3,j}$ belong to some unbalanced 4-cycle does not compromise the existence of other unbalanced 4-cycles including farther vertices from another representative pair. \square

We claim that we have the desired equivalence between not-all-equal satisfying F and finding a good 2-edge-coloring c_H of H_F . To see this holds, just assume, for every variable x_i of F , that having r_1 and r_2 agreeing (resp. disagreeing) on u_i simulates the fact that variable x_i of F is set to *true* (resp. *false*) by some truth assignment, and that having r_1 and r_2 agreeing (resp. disagreeing) on some vertex $v_{i,j}$ simulates the fact that variable x_i provides value *true* (resp. *false*) to the clause C_j of F . Then the property described in Claim 1 depicts the fact that if x_i is set to some truth value by a truth assignment, then x_i provides the same truth value to every clause containing it. The property described in Claim 2 depicts the fact that every clause C_j is considered satisfied by some truth assignment if and only if C_j is supplied different truth values by its variables. So we can deduce a good 2-edge-coloring of H_F from a truth assignment not-all-equal satisfying F , and vice-versa.

Step 2. As described above, we now construct G_F from H_F in such a way that

- every 4-cycle of G_F including the vertices of a representative pair is also a 4-cycle in H_F ,
- the edges of $E(G_F) \setminus E(H_F)$ can be colored with two colors so that every two vertices which do not form a representative pair belong to an unbalanced 4-cycle.

In this way, G_F will be the support of a signed clique if and only if H_F admits a good 2-edge-coloring, which is true if and only if F can be not-all-equal satisfied. The result will then hold by transitivity.

For every vertex u of H_F , add the edges ua_u and ub_u , where a_u and b_u are two new vertices. Now, for every two distinct vertices u and v of H_F , if $\{u, v\}$ is not a representative pair, then add another vertex $c_{u,v}$ to the graph, as well as the edges $uc_{u,v}$ and $vc_{u,v}$. Finally turn the subgraph induced by all newly added vertices into a clique. The resulting graph is G_F . As claimed above, note that the only 4-cycles of G_F containing two vertices u and v forming a representative pair are those of H_F . Every other such new cycle has indeed length at least 6. Namely, every such new cycle starts from u , then has to enter the clique by either a_u or b_u , cross the clique to either a_v or b_v , reach v , before finally going back to u .

Consider the following coloring with two colors of the edges in $E(G_F) \setminus E(H_F)$. For every vertex $u \in V(H_F)$, let ub_u be colored 1. Similarly, for every two distinct vertices u and v of H_F such that $\{u, v\}$ is not representative (*i.e.* $c_{u,v}$ exists), let $c_{u,v}v$ be colored 1. Let finally all other edges be colored 2. Clearly, under this partial 2-edge-coloring of G_F , every two vertices u and v of G_F not forming a representative pair are either adjacent or belong to some unbalanced 4-cycle:

- if u and v do not belong to H_F , then they belong to the clique and are hence adjacent;
- if u belongs to H_F but v does not, then observe that either u and v are adjacent (in this situation v is either a_u , b_u or $c_{u,i}$ for some i), or ua_uvb_uu is an unbalanced 4-cycle;

- if u and v are vertices of H_F and $\{u, v\}$ is not representative, then *e.g.* $uc_{u,v}va_v a_u u$ is an unbalanced 4-cycle.

According to all previous arguments, finding a truth assignment not-all-equal satisfying F is equivalent to finding a signed clique from G_F , as claimed. So SIGNED CLIQUE EDGE-COLOR ASSIGNMENT is NP-hard, and hence NP-complete. \square

4 Comparing mixed chromatic number of $(1, 0)$ -colored and $(0, 2)$ -colored mixed graphs

This comparison is motivated from the fact that for several graph classes the two parameters $\chi_{(1,0)}$ and $\chi_{(0,2)}$ have the same or similar bounds, some of them tight [13, 7]. In most cases the same or similar bounds are proved using similar techniques. The similarity is probably due to the fact that in both cases adjacency can be of two types. Thus, there was a reasonable amount of hints indicating an underlying relation between the two parameters. Here we end up proving the opposite, that is, the parameters can be arbitrarily different.

Theorem 4.1. *Given any integer n , there exists an undirected graph G such that $\chi_{(0,2)}(G) - \chi_{(1,0)}(G) = n$.*

Proof. All the complete graphs have their oriented chromatic number equal to their signed chromatic number. So, we need to prove $\chi_{(0,2)}(G) - \chi_{(1,0)}(G) = n$ for non-zero integers n .

Let A and B be two undirected graphs. Let $A + B$ be the undirected graph obtained by taking disjoint copies of A and B and adding a new vertex ∞ adjacent to all the vertices of A and B . So, $A + B$ is the undirected graph dominated by the vertex ∞ and $N(\infty)$ is the disjoint union of A and B .

It is easy to observe that

$$\chi_{(0,2)}(A + B) \leq \chi_{(0,2)}(A) + \chi_{(0,2)}(B) + 1.$$

Let the edges connecting ∞ and the vertices of A receive the first color. Let the edges connecting ∞ and the vertices of B receive the second color. Also retain the edge-colors of the edges of A and B in $A + B$ for which they attain their respective chromatic numbers. Note that we have to use different set of colors for coloring the vertices of A and the vertices of B in this sort of color assignment of edges. Also, we need to use at least $\chi_{(0,2)}(A)$ colors to color the vertices of A and at least $\chi_{(0,2)}(B)$ colors to color the vertices of B . Furthermore, the vertex ∞ must receive a color different from the colors used for any other vertices of $A + B$. Thus,

$$\chi_{(0,2)}(A + B) = \chi_{(0,2)}(A) + \chi_{(0,2)}(B) + 1.$$

Similarly, consider the orientations \vec{A} , \vec{B} for which we have $\chi_{(1,0)}(\vec{A}) = \chi_{(1,0)}(A)$ and $\chi_{(1,0)}(\vec{B}) = \chi_{(1,0)}(B)$. Now orient the edges of A and B in $A + B$ like in \vec{A} and \vec{B} . Now orient the edges incident to the vertex ∞ in such a way that we have the arcs ∞a for all $a \in A$ and $b\infty$ for all $b \in B$. Note that we have to use different set of colors for coloring the vertices of A and the vertices of B for this sort of an orientation. Thus,

$$\chi_{(1,0)}(A + B) = \chi_{(1,0)}(A) + \chi_{(1,0)}(B) + 1. \tag{1}$$

Let H be an undirected graph. Then we define, by induction, the graph $H_k = H + H_{k-1}$ for $k \geq 2$ where $H_1 = H$. Note that,

$$\chi_{(0,2)}(H_k) = k \times \chi_{(0,2)}(H) + (k - 1) \text{ and } \chi_{(1,0)}(H_k) = k \times \chi_{(1,0)}(H) + (k - 1).$$

The above two equations imply

$$\chi_{(0,2)}(H_k) - \chi_{(1,0)}(H_k) = k \times (\chi_{(0,2)}(H) - \chi_{(1,0)}(H)). \quad (2)$$

It is easy to observe that, a path P_5 of length 5 (number of edges) has $\chi_{(0,2)}(P_5) = 4$ and $\chi_{(1,0)}(P_5) = 3$. On the other hand, a cycle C_5 of length 5 has $\chi_{(0,2)}(C_5) = 4$ and $\chi_{(1,0)}(C_5) = 5$. Hence we have $\chi_{(0,2)}(P_5) - \chi_{(1,0)}(P_5) = 1$ and $\chi_{(0,2)}(C_5) - \chi_{(1,0)}(C_5) = -1$.

Now by replacing H with P_5 and C_5 in equation 2 we are done. \square

This result shows that the two parameters does not have any additive relation between them. Even though we are not ruling out the possibility of other types of relation between the two parameters. For exapmle, one can try to figure out if the ratio $\chi_{(0,2)}(G)/\chi_{(1,0)}(G)$ is bounded or not where G is an undirected simple graph.

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