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On cliques of signed and switchable signed graphs

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Abstract

Vertex coloring of a graph G with n-colors can be equivalently thought to be a graph homomorphism (edge preserving vertex mapping) of G to the complete graph K_n of order n. So, in that sense, the chromatic number $\chi(G)$ of G will be the order of the smallest complete graph to which G admits a homomorphism to. As every graph, which is not a complete graph, admits a homomorphism to a smaller complete graph, we can redefine the chromatic number $\chi(G)$ of G to be the order of the smallest graph to which G admits a homomorphism to. Of course, such a smallest graph must be a complete graph as they are the only graphs with chromatic number equal to their order.

The concept of vertex coloring can be generalize for other types of graphs, namely, oriented graphs (directed graphs with no cycle of length 1 or 2), 2-edge-colored or signed graphs (graphs with positive or negative signs assigned to each edge) and switchable signed graphs (equivalence class of signed graph with respect to switching signs of edges incident to the same vertex) using the notion of graph homomorphism. Naturally, the chromatic number is defined to be the order of the smallest graph (of the same type) to which a graph admits homomorphism to. For the above mentioned type of graphs, the graphs with smallest order, that is, the graphs with order equal to their (so defined) "chromatic number" are called ocliques, scliques and [s]-cliques respectively. These "cliques" turns out to be much more complicated than their undirected counterpart and are interesting objects of study.

In this article, we mainly study different aspects of "cliques" for signed and switchable signed graphs. In particular, we show that it is NP-hard to decide if edges of a given undirected graph can be assigned positive and negative signatures such that it becomes an sclique or an [s]-clique. We also show that, asymptotically, almost all signed graphs are scliques or [s]-cliques. Furthermore, we prove a sufficient and necessary condition for a signed graph (or switchable signed graph) to be an sclique (or [s]-clique). We study the number of vertices that an sclique (or [s]-clique) can have when their underlying graph is planar and prove a tight upper bound of 15. We also study the same for outerplanar graphs and planar graphs with given girth (length of the smallest cycle). Finally, we generalize the concept of "cliques" for n-edge-colored graphs (graphs with one among n different colors assigned to each of its edge) and do a similar study for outerplanar and planar graphs.

Keywords: signed graphs, switchable signed graphs, graph homomorphism, cliques, chromatic number, oriented graph.

1 Introduction

A *signed graph* or *2-edge-colored graph* (G, Σ) is a graph G with an assignment of *positive* (black lines used to denote them in the figures) and *negative* (black "dashed" lines used to denote them in the figures) signs to its edges where Σ is the set of negative edges and G is its underlying graph. We denote the set of positive edges by Σ^c. When the set of negative edges Σ is understood, we can denote the signed graph (G, Σ) by (G). In general, the set of vertices and the set of edges of the signed graph (G, Σ) are denoted by $V(G)$ and $E(G)$. Two incident edges $uv \in \Sigma$ and $vw \in \Sigma^c$ are together called a *rainbow 2-path* with *terminal* vertices u, w and *internal* vertex v. In such cases, we say that the vertices u and w are connected by a rainbow 2-path.

A *signed k-coloring* [1] of a signed graph (G, Σ) is a mapping ϕ from the vertex set $V((G, \Sigma))$ to the set $\{1, 2, ..., k\}$ such that,

- (i) $\phi(u) \neq \phi(v)$ whenever u and v are adjacent and
- (ii) if uv is a positive edge and wx is a negative edge of (G, Σ), then $\phi(u) = \phi(w)$ implies $\phi(v) \neq \phi(x)$.

The *signed chromatic number* $\chi_s((G, \Sigma))$ of a signed graph (G, Σ) is the smallest integer k for which (G, Σ) has a signed k-coloring. The signed chromatic number $\chi_s(G)$ of an undirected graph G is the maximum of the signed chromatic numbers of all the signed graphs with underlying graph G. The signed chromatic number $\chi_s(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the signed chromatic numbers of the graphs from the family $\mathcal{F}$.

Alternatively, we can define the signed chromatic number using homomorphisms of signed graphs. Given two signed graphs (G, Σ) and (H, Λ), ϕ is a *homomorphism* of (G, Σ) to (H, Λ) if $\phi : V(G) \longrightarrow V(H)$ is a mapping such that every edge of (G, Σ) is mapped to an edge of the same sign of (H, Λ) (that is, $uv \in E(G)$ and $\phi(u)\phi(v) \in E(H)$ have the same sign). We write $(G, \Sigma) \to (H, \Lambda)$ whenever there exists a homomorphism of (G, Σ) to (H, Λ). The *signed chromatic number* $\chi_s((G, \Sigma))$ of a signed graph (G, Σ) is the minimum *order* (number of vertices) of a signed graph (H, Λ) such that (G, Σ) admits a homomorphism to (H, Λ).

Homomorphism of signed graphs were studied by Nešetřil and Raspaud [26], Alon and Marshall [1] and Montejano *et al.* [21]. In fact, initially, the notion of signed vertex coloring and chromatic number was defined using the notion of homomorphism, in a way similar to oriented vertex coloring and oriented chromatic number of oriented graphs [8, 28, 32, 31]. Intuitively, homomorphism of an oriented graph to another is an arc preserving vertex mapping, while the oriented chromatic number $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the minimum order of an oriented graph $\overrightarrow{H}$ to which $\overrightarrow{G}$ admits a homomorphism (detailed definitions are given in Section 3).

In practise, while studying signed coloring, it was observed that analogous versions of several results from the theory of oriented coloring can be proved using similar techniques. This made us wonder if there is an underlying general relation between the two kinds of graph colorings. In this article we study this idea and actually end up proving the opposite. We prove that given an (undirected) graph, its signed chromatic number and its oriented chromatic number can be arbitrarily different.

To *resign* a vertex v of a signed graph (G, Σ) is to change the signs of the edges incident to v. Two signed graphs (G, Σ) and (G, Σ') are in a *resign relation* if we can obtain (G, Σ') by resigning some vertices of (G, Σ). Note that the resign relation is an equivalence relation. A *switchable signed graph* $[G, \Sigma]$ is an equivalence class of signed graphs (where (G, Σ) is an element of the equivalence class) with respect to resign relation. Any element of the equivalence class

$[G, \Sigma]$ is a *presentation* of it. We use the notation $(G, \Sigma) \in [G, \Sigma]$ for (G, Σ) is a presentation of $[G, \Sigma]$.

The most important component in the study of switchable signed graphs are its two kinds of cycles, namely, *balanced cycles*: cycles with even number of negative edges and *unbalanced cycles*: cycles with odd number of negative edges. Note that the parity of the number of negative edges in a cycle of a switchable signed graph remains invariant under the resign operation and hence the above two definitions are consistent.

switchable signed graphs have been studied since the middle of the last century [12, 35]. Naserasr, Rollová and Sopena [25] recently introduced and studied homomorphisms of switchable signed graphs. In their beautifully written article they showed that many important theorem and conjectures of classical graph theory can be captured and extended using the notion of switchable signed homomorphism. That work has since inspired a significant number of works [23, 24, 9, 27, 30] on switchable signed homomorphism in a short period of time.

Given two switchable signed graphs $[G, \Sigma]$ and $[H, \Lambda]$, we say there is a *homomorphism* ϕ of $[G, \Sigma]$ to $[H, \Lambda]$ if ϕ is a homomorphism of a presentation $(G, \Sigma') \in [G, \Sigma]$ to a presentation $(H, \Lambda') \in [H, \Lambda]$. We write $[G, \Sigma] \to [H, \Lambda]$ whenever there exists a homomorphism of $[G, \Sigma]$ to $[H, \Lambda]$.

A *switchable signed k-coloring* of a switchable signed graph $[G]$ is a vertex coloring which is a signed k-coloring of a presentation of the graph. The *switchable signed chromatic number* $\chi_{[s]}([G])$ of a switchable signed graph $[G]$ is the minimum of the signed chromatic numbers of the elements of the equivalence class $[G]$. The switchable signed chromatic number of an undirected graph and of a family of graphs is defined similarly like in the case of signed chromatic number.

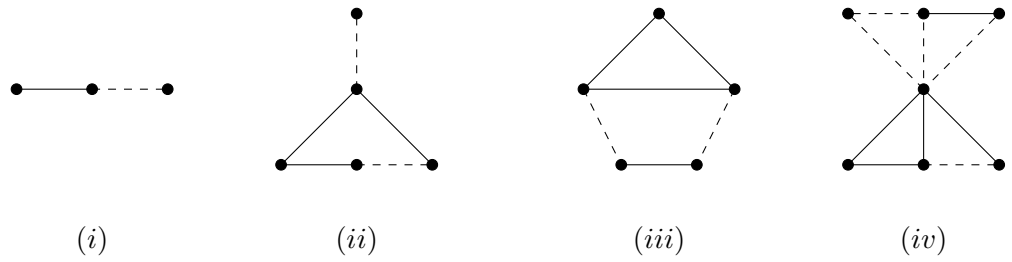
Alternatively, the switchable signed chromatic number $\chi_{[s]}([G])$ of the switchable signed graph $[G]$ is the minimum order of a switchable signed graph $[H]$ such that $[G]$ admits a homomorphism to $[H]$.

The *chromatic number* $\chi(G)$ of an undirected simple graph G is the minimum number of colors needed to color the vertices of G such that no two adjacent vertices receive the same color. A *clique* is a simple undirected graph whose order and chromatic number are equal. It is easy to notice that cliques are nothing but the complete graphs.

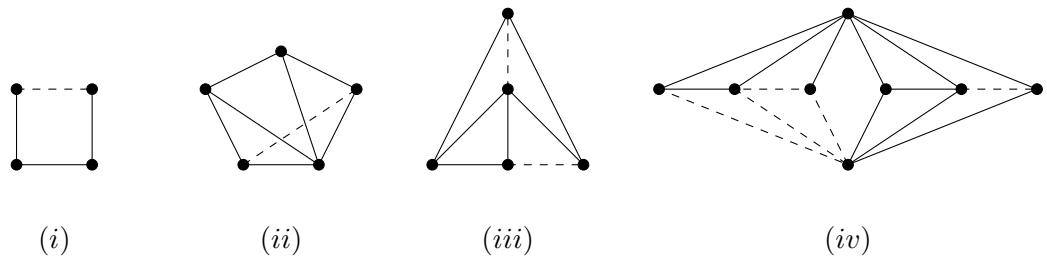
The notion analogous to "cliques" for oriented graphs, known as *oriented cliques* or *ocliques* (oriented graphs with oriented chromatic number equal to their order), was introduced by Klostermeyer and MacGillivray [15] and have been further studied in [29, 22, 3, 33]. The structure of ocliques seems way more difficult to understand than in the undirected case. For instance, the exact value of the minimum number of arcs in an oclique of order k is not known yet. Füredi, Horak, Pareek and Zhu [10], and Kostochka, Łuczak, Simonyi and Sopena [16], independently proved that this number is $(1 + o(1))k \log_2 k$. In the same paper, Füredi *et. al.* commented that a similar result can be proved for signed graphs as well using the exact same technique (they did not define signed cliques though).

For signed graphs, no similar study is published yet while switchable signed cliques were defined and studied in [25]. Here, in this article, we introduce signed cliques and study it along with switchable signed cliques. It seems that the structure of signed and switchable signed cliques are also quite difficult to comprehend and, hopefully, this study would open a new interesting line of research within the domain of graph coloring.

A *signed clique* or an *sclique* is a signed graph (G, Σ) for which $\chi_s((G, \Sigma)) = |V((G, \Sigma))|$. The *signed absolute clique number* $\omega_{as}((G, \Sigma))$ of a signed graph (G, Σ) is the maximum order of an sclique contained in (G, Σ) as a subgraph. Similarly, a *switchable signed clique* or simply an *[s]-clique* is a switchable signed graph $[G]$ for which $\chi_{[s]}([G]) = |V(G)|$. The *switchable signed absolute clique number* $\omega_{as}([G])$ of a switchable signed graph $[G]$ is the maximum order of an



(a) Example of scliques.



(b) Example of [s]-cliques.

Figure 1: (a) Examples of signed cliques. (b) Examples of switchable signed cliques.

[s]-clique contained in $[G]$ as a subgraph. The signed and switchable signed absolute clique numbers of an undirected graph and of a family of graphs is defined similarly like in the case of signed chromatic number.

These cliques are an important component in the study of signed or switchable signed homomorphism – as to find out the signed or switchable signed chromatic number of a graph we need to study homomorphism of that graph to a signed or a switchable signed clique. In this article we study different aspects of signed and switchable signed cliques, namely, asymptotic, complexity and structural aspects. Later we extend the concept of cliques to n-edge-colored graphs [26, 1, 21, 4] and briefly study them as well. Curiously our definition of an sclique coincides with the definition of graphs with *rainbow connection number 2* [13, 18, 6]. As rainbow connection number [19, 7, 5, 17, 34, 2] is a well studied topic, this coincidence increases the significance of this work.

In Section 2 we fix some notations and study the general structure of signed and switchable signed cliques. In Section 3 we compare 2-edge-colored coloring and oriented coloring. Then in Section 4 and Section 5 we study the asymptotic and complexity aspects of signed and switchable signed cliques, respectively. In Section 6 we investigate the structure of signed and switchable signed cliques for planar graphs. In Section 7 we introduce the definition of *n-edge-colored cliques* and study them briefly. Finally, in Section 8 we conclude the article.

2 Preliminary notations and basic properties

In order to get used to the concept of scliques and $[s]$-cliques we provide some examples in Fig. 1. To verify these examples the following characterizations are useful.

Proposition 2.1. *A signed graph (G, Σ) is an sclique if and only if its each pair of non-adjacent vertices are connected by a rainbow 2-path.*

Proof. Let (G, Σ) be a signed graph with its each pair of non-adjacent vertices connected by a rainbow 2-path. Clearly, from the definition of signed coloring, any two vertices must receive different colors under any signed coloring of (G, Σ). Hence (G, Σ) is an sclique. This proves the "if" part of the proposition.

To prove the converse, let (G, Σ) be an sclique. Let u and v be a pair of vertices in (G, Σ) which are not connected by a rainbow 2-path. Let f be a signed coloring with $f(u) = f(v)$ and $f(x) \neq f(y)$ for all distinct $x, y \in V(G)$ such that $\{x, y\} \neq \{u, v\}$. It is easy to note that f is indeed a signed coloring of (G, Σ). In fact, f is a $(|G| - 1)$-signed coloring of (G, Σ). This implies $\chi_s((G, \Sigma)) < |G|$ contradicting the fact that (G, Σ) is an sclique. This proves the "only if" part of the proposition. □

A similar characterization was proved for $[s]$-cliques in [25]. We recall it here.

Proposition 2.2. *A switchable signed graph $[G, \Sigma]$ is an $[s]$-clique if and only if its each pair of non-adjacent vertices are contained in an unbalanced 4-cycle.*

Two non-adjacent vertices are contained in an unbalanced 4-cycle means the two vertices are connected by two disjoint 2-paths among which exactly one 2-path is a rainbow path. Therefore, the condition for being an $[s]$-clique is stronger than the condition for being an sclique while there are scliques which are not $[s]$-cliques (a rainbow 2-path, for instance).

Proposition 2.3. *Every $[s]$-clique is an sclique but every sclique is not an $[s]$-clique.*

Therefore, the $[s]$-cliques depicted in Fig. 1(b) are also scliques.

From the definition of signed (or switchable signed) chromatic number and absolute clique number it immediately follows that the signed (or switchable signed) absolute clique number of a graph is bounded above by its signed (or switchable signed) chromatic number.

Proposition 2.4.

(*i*) *For any signed graph (G, Σ) we have $\omega_{as}((G, \Sigma)) \leq \chi_s((G, \Sigma))$.*

(*ii*) *For any switchable signed graph $[G, \Sigma]$ we have $\omega_{[as]}([G, \Sigma]) \leq \chi_{[s]}([G, \Sigma])$.*

We also note that the parameter signed (or switchable signed) absolute clique number respects homomorphism in the following sense:

Proposition 2.5.

(*i*) *Let $(G, \Sigma) \to (H, \Lambda)$. Then $\omega_{as}((G, \Sigma)) \leq \omega_{as}((H, \Lambda))$.*

(*ii*) *Let $[G, \Sigma] \to [H, \Lambda]$. Then $\omega_{[as]}([G, \Sigma]) \leq \omega_{[as]}([H, \Lambda])$.*

Now we will define some notations and a few parameters to be used in this article. For a signed graph (G, Σ) (or for an oriented graph $\overrightarrow{G}$), every parameter we introduce below is denoted using (G, Σ) (or $\overrightarrow{G}$) as a subscript. In order to simplify notation, this subscript will be dropped whenever there is no chance of confusion.

The set of all adjacent vertices of a vertex v in a signed graph (G, Σ) is called its set of *neighbors* and is denoted by $N_{(G,\Sigma)}(v)$. If $uv \in \Sigma$, then v is a *positive-neighbor* of u and if $uv \in \Sigma^c$, then v is a *negative-neighbor* of u. The set of all positive-neighbors and the set of all

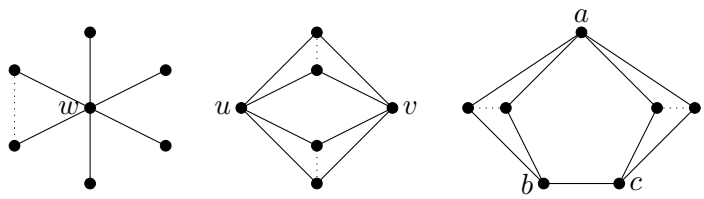


Figure 2: List of all triangle-free planar graphs with diameter 2 (Plesník (1975)).

negative-neighbors of v are denoted by $N^+_{(G,\Sigma)}(v)$ and $N^-_{(G,\Sigma)}(v)$ respectively. The *degree* of a vertex v in a signed graph (G,Σ), denoted by $d_G(v)$, is the number of neighbors of v in (G,Σ). Naturally, the *positive-degree* (resp. *negative-degree*) of a vertex v in a signed graph (G,Σ) is denoted by $d^+_{(G,\Sigma)}(v)$ (resp. $d^-_{(G,\Sigma)}(v)$) for v in (G,Σ). Two vertices u and v of a signed graph *agree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\alpha(v)$ for some $\alpha \in \{+,-\}$. Two vertices u and v of a signed graph *disagree* on a third vertex w of that graph if $w \in N^\alpha(u) \cap N^\beta(v)$ for some $\{\alpha,\beta\} = \{+,-\}$.

For an oriented graph $\overrightarrow{G}$ if there is an arc $\overrightarrow{uv}$, then u is an *in-neighbor* of v and v is an *out-neighbor* of u. The set of all in-neighbors and the set of all out-neighbors of v are denoted by $N^-_{\overrightarrow{G}}(v)$ and $N^+_{\overrightarrow{G}}(v)$, respectively.

We define some graph parameters, not related to coloring, and some results related to those parameters which will be used later.

The *distance* $d_G(x,y)$ (or $d(x,y)$ when there is no confusion) between two vertices x and y of a graph G is the smallest length of a path connecting x and y. The *diameter* $diam(G)$ of a graph G is the maximum distance between pairs of vertices of the graph.

Theorem 2.6. *[14] The triangle-free graphs with diameter 2 are precisely the graphs listed in Fig. 2.*

The graphs depicted in Fig. 2 are the stars, the complete bipartite graphs $K_{2,n}$ for some natural number n, and the graphs obtained by adding copies of two non-adjacent vertices of the 5-cycle.

A vertex subset D is a *dominating* set of a graph G if every vertex of G is either in D or adjacent to a vertex of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G.

Theorem 2.7 (Goddard and Henning [11])**.** *Any planar graph with diameter 2 has domination number at most 2, except for a particular planar graph on 9 vertices (depicted in Fig. 3) which has domination number 3.*

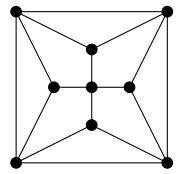


Figure 3: The unique planar graph with diameter 2 and domination number 3.

Now let us observe one curious example. The signed graph depicted in Fig. 4 has signed absolute clique number 3 but its four vertices with degree more than two can never receive the

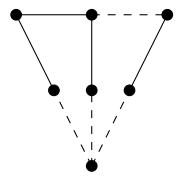


Figure 4: A signed graph with relative clique number 4 and absolute clique number 3.

same color under any signed coloring. One can note that whenever two vertices of a signed graph are adjacent or connected by a rainbow 2-path, they must receive different color. These kinds of examples motivated the definition of "relative cliques" in signed graphs, a notion that will be used for proving results in this article.

A *relative clique* of a signed graph (G, Σ) is a set $R \subseteq V((G, \Sigma))$ of vertices such that any two vertices from R are either adjacent or connected by a rainbow 2-path. The *relative clique number* $\omega_{rs}((G, \Sigma))$ of a signed graph (G, Σ) is the maximum order of a signed relative clique of (G, Σ). The term relative clique and the definition are given by following the similar term and definition used in [25] for switchable signed graphs. Naturally, we can extend Proposition 2.4(i) in the following manner:

Proposition 2.8. *For any signed graph (G, Σ) we have $\omega_{as}((G, \Sigma)) \leq \omega_{rs}((G, \Sigma)) \leq \chi_s((G, \Sigma))$.*

Here we also mention a result that we will recall in one of the succeeding sections.

Theorem 2.9. *Let (G, Σ) be a signed outerplanar graph. Then $\omega_{rs}((G, \Sigma)) \leq 7$.*

The proof follows from a generalized result (Theorem 7.3) proved in Section 7.

3 Signed vs oriented coloring

Colorings of oriented graphs first appeared in the work of Courcelle [8] on the monadic second order logic of graphs. Since then it has been considered by many researchers, following the work of Raspaud and Sopena [28] on oriented colorings of planar graphs.

An *oriented k-coloring* [32] of an oriented graph $\overrightarrow{G}$ is a mapping ϕ from the vertex set $V(\overrightarrow{G})$ to the set $\{1, 2, ..., k\}$ such that,

- (i) $\phi(u) \neq \phi(v)$ whenever u and v are adjacent and
- (ii) if $\overrightarrow{uv}$ and $\overrightarrow{wx}$ are two arcs in $\overrightarrow{G}$, then $\phi(u) = \phi(x)$ implies $\phi(v) \neq \phi(w)$.

The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the smallest integer k for which $\overrightarrow{G}$ has an oriented k-coloring.

Alternatively, we can define oriented chromatic number by defining homomorphisms of oriented graphs. The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the minimum order of an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{G}$ admits a homomorphism to $\overrightarrow{H}$.

The oriented chromatic number $\chi_o(G)$ of an undirected graph G is the maximum of the oriented chromatic numbers of all the oriented graphs with underlying graph G. The oriented chromatic number $\chi_o(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the oriented chromatic numbers of the graphs from the family $\mathcal{F}$.

It is observed that similar bounds hold for oriented chromatic number and signed chromatic number for several families of graphs, namely, paths, cycles, trees, graphs with bounded maximum degree, graphs with bounded acyclic chromatic number, outerplanar graphs, outerplanar graphs with given girth (length of the smallest cycle), k-trees, planar graphs, planar graphs with given girth etc. [21].

Naturally this made us curious if there is an underlying relation between the two chromatic numbers or not. Hence we studied the difference between the two parameters, that is, $\chi_s(G) - \chi_o(G)$ for a given graph G and managed to prove the following result.

Theorem 3.1. *Given any integer n, there exists an undirected graph G such that $\chi_s(G) - \chi_o(G) = n$.*

Proof. All the complete graphs have their oriented chromatic number equal to their signed chromatic number. So, we need to prove $\chi_s(G) - \chi_o(G) = n$ for non-zero integers n.

Let A and B be two undirected graphs. Let $A + B$ be the undirected graph obtained by taking disjoint copies of A and B and adding a new vertex ∞ adjacent to all the vertices of A and B. So, $A + B$ is the undirected graph dominated by the vertex ∞ and $N(\infty)$ is the disjoint union of A and B.

It is easy to observe that

$$\chi_s(A + B) \leq \chi_s(A) + \chi_s(B) + 1.$$

Let $\Sigma_A \subseteq E(A)$ and $\Sigma_B \subseteq E(B)$ be such that we have $\chi_s((A, \Sigma_A)) = \chi_s(A)$ and $\chi_s((B, \Sigma_B)) = \chi_s(B)$. Now choose $\Sigma_{A+B} \subseteq E(A + B)$ such that,

$$\Sigma_{A+B} = \Sigma_A \cup \Sigma_B \cup \{\infty b | b \in V(B)\}.$$

Clearly, the vertex ∞ must receive a color different from any other vertex in the graph in any signed coloring. Also, due to the choice of Σ_{A+B}, the signed graph induced by $N^+(\infty)$ is isomorphic to (A, Σ_A) and the signed graph induced by $N^-(\infty)$ is isomorphic to (B, Σ_B). Note that the vertices of $N^+(\infty)$ must receive colors different from the colors received by the vertices of $N^-(\infty)$ in any signed coloring. Therefore,

$$\begin{aligned}\chi_s((A + B, \Sigma_{A+B})) &\geq \chi_s((A, \Sigma_A)) + \chi_s((B, \Sigma_B)) + 1 \\ &= \chi_s(A) + \chi_s(B) + 1.\end{aligned}$$

This implies

$$\chi_s(A + B) = \chi_s(A) + \chi_s(B) + 1. \tag{1}$$

Similarly, consider orientations $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{A + B}$ of A, B and $A + B$ respectively such that $\chi_o(\overrightarrow{A}) = \chi_o(A)$, $\chi_o(\overrightarrow{B}) = \chi_o(B)$ and that the oriented graphs induced by $N^+(\infty)$ and $N^-(\infty)$ in $\overrightarrow{A + B}$ are isomorphic to $\overrightarrow{A}$ and $\overrightarrow{B}$ respectively.

With arguments similar to what we gave for proving equation 1 we have,

$$\chi_o(A+B) = \chi_o(A) + \chi_o(B) + 1. \tag{2}$$

Let H be an undirected graph. Then we define, by induction, the graph $H_k = H + H_{k-1}$ for $k \geq 2$ where $H_1 = H$. Note that,

$$\chi_s(H_k) = k \times \chi_s(H) + (k-1) \text{ and } \chi_o(H_k) = k \times \chi_o(H) + (k-1).$$

The above two equations imply

$$\chi_s(H_k) - \chi_o(H_k) = k \times (\chi_s(H) - \chi_o(H)). \tag{3}$$

It is easy to observe that, a path P_5 of length 5 has oriented chromatic number equal to 3 while its signed chromatic number is 4. On the other hand, a cycle P_5 of length 5 has oriented chromatic number equal to 5 while its signed chromatic number is 4. Hence we have $\chi_s(P_5) - \chi_o(P_5) = 1$ and $\chi_s(C_5) - \chi_o(C_5) = -1$.

Now by replacing H with P_5 and C_5 in equation 3 we are done. □

4 Asymptotic aspects

It is well known that almost all graphs have diameter 2 [18]. As both scliques and $[s]$-cliques are graphs with diameter 2, it is interesting to find how "rare" these objects are. Upon a study to this end, we can show that almost all signed graphs are $[s]$-cliques, which clearly implies that almost all signed graphs are scliques. Both the results are stronger than proving that almost all graphs have diameter 2.

To show that almost all signed graphs are $[s]$-cliques, we require the following result.

Lemma 4.1. *For a fixed set of n vertices V and a pair $u, v \in V$, there are no more than $2 \cdot 5(n-2)3^{\binom{n}{2}-2(n-2)-1}$ signed graphs with vertex set V such that u and v are not adjacent and are not part of an unbalanced 4-cycle.*

Proof. We begin by bounding the number of signed graphs in which u and v are the terminal vertices of a rainbow 2-path but not the terminal vertices of a 2-path with edges of the same type. Each of the other $n-2$ vertices are either adjacent to either exactly one of u and v or to neither u nor v or to both u and v by edges of the different types. This gives a total of 5 possibilities for the edges between the remaining vertices and u and v. As such there are no more than

$$5(n-2)3^{\binom{n}{2}-2(n-2)-1}$$

signed graphs with vertex set V such that u and v are not adjacent and are terminal vertices of a rainbow 2-path but not ends of a 2-path with edges of the same type.

A similar argument shows that the number of signed graphs in which u and v are the terminal vertices of a 2-path which is not a rainbow 2-path is no more than

$$5(n-2)3^{\binom{n}{2}-2(n-2)-1}$$

Therefore the total number of signed graphs on vertex set V in which u and v are not adjacent and and are not part of an unbalanced 4-cycle is at most

$$2 \cdot 5(n-2)3^{\binom{n}{2}-2(n-2)-1}$$

. □

Theorem 4.2. *Almost all signed graphs are* $[s]$*-cliques.*

Proof. The total number of signed graphs on n vertices is $3^{\binom{n}{2}}$ and the number of n vertex signed graphs which are not $[s]$-cliques is at most $\binom{n}{2} \cdot 2 \cdot 5(n-2)3^{\binom{n}{2}-2(n-2)-1}$.

$$\lim_{n\to\infty} \frac{\binom{n}{2} \cdot 2 \cdot 5(n-2)3^{\binom{n}{2}-2(n-2)-1}}{3^{\binom{n}{2}}} = 0.$$

As n goes to infinity, the ratio of the number signed graphs on n vertices which are not $[s]$-cliques to the number of n vertex signed graphs goes zero. This implies that almost all signed graphs are $[s]$-cliques. □

The above result clearly implies the following:

Theorem 4.3. *Almost all signed graphs are scliques.*

5 Complexity aspects

We herein focus on complexity aspects related to scliques and $[s]$-cliques. Due to Propositions 2.1 and 2.2, recall that deciding whether a given signed or switchable signed graph is an sclique or $[s]$-clique, respectively, can be done in polynomial time. So the next interesting question is to think in terms of underlying graphs: can we easily decide whether a graph G underlies (that is, is underlying graph of) an sclique or an $[s]$-clique? Or, rephrased differently, when can we assign signs to the edges of G so that we obtain an sclique or an $[s]$-clique?

Sclique Signing
Input: a graph G.
Question: is G underlying graph of an sclique?

[S]-clique Signing
Input: a graph G.
Question: is G underlying graph of an $[s]$-clique?

It was shown that Sclique Signing is NP-hard in context of "rainbow connection number" (we omit the definition) in [6]. It is interesting that our newly defined concept has already been dealt with in a different context. Here we will prove that [S]-clique Signing is also NP-complete in general, hence implying that an easy characterization of scliques and $[s]$-cliques in terms of underlying graph should not exist (unless P=NP). As the NPness of [S]-clique Signing is obvious (given a signature Σ of G, one can check in polynomial time, due to Proposition 2.2, whether $[G, \Sigma]$ is an $[s]$-clique or not), we now just focus on the NP-hardness of the problem. Due to the proof given in [6] we have the following result.

Theorem 5.1. Sclique Signing *is NP-complete.*

Now we prove below that [S]-clique Signing is NP-hard.

Theorem 5.2. [S]-clique Signing *is NP-complete.*

Proof. We prove the NP-hardness of [S]-CLIQUE SIGNING by reduction from the following NP-complete problem.

MONOTONE NOT-ALL-EQUAL 3-SATISFIABILITY
Instance: a 3CNF formula F over variables $x_1, x_2, ..., x_n$ and clauses $C_1, C_2, ..., C_m$ involving no negated variables.
Question: is F *not-all-equal satisfiable*, that is, does there exist a truth assignment to the variables under which every clause has at least one true and one false variable?

Due to the NP-completeness of the 2-COLOURING OF 3-UNIFORM HYPERGRAPH problem (see [20]), it is easily seen that MONOTONE NOT-ALL-EQUAL 3-SATISFIABILITY remains NP-complete when every clause of F has its three variables being different. So this additional restriction is understood throughout. From a 3CNF formula F, we construct a graph G_F such that

$$F \text{ is not-all-equal satisfiable}$$
$$\Leftrightarrow$$
$$G_F \text{ underlies an } [s]\text{-clique.}$$

The construction of G_F is achieved in two steps. We first construct, from F, a graph H_F such that F is not-all-equal satisfiable if and only if there exists a signature Σ_H of H under which only some *representative pairs* of non-adjacent vertices belong to unbalanced 4-cycles. This equivalence is obtained by designing H_F in such a way that every representative pair belongs to a unique 4-cycle, with some of these 4-cycles overlapping to force some edges to have the same or different signs by Σ_H. Then we obtain G_F by adding some vertices and edges to H_F in such a way that no new 4-cycles including representative pairs are created, and there exists a partial signature of the edges in $E(G_F) \setminus E(H_F)$ for which every non-representative pair is included in an unbalanced 4-cycle. In this way, the equivalence between G_F and F is only dependent of the equivalence between H_F and F, which has not been altered when constructing G_F from H_F.

Step 1. Start by adding two vertices r_1 and r_2 to H_F. Then, for every variable x_i of F, add two vertices u_i and u'_i to H_F, and link these vertices to both r_1 and r_2. Now, for every $i \in \{1, 2, ..., n\}$, assuming the variable x_i belongs to the (distinct) clauses $C_{j_1}, C_{j_2}, ..., C_{j_{n_i}}$, add n_i new vertices $v_{i,j_1}, v_{i,j_2}, ..., v_{i,j_{n_i}}$ to H_F, and join all these new vertices to both r_1 and r_2. Finally, for every clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F, add a new vertex w_j to H_F, and join it to all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$.

The representative pairs are the following. For every variable x_i of F, all pairs $\{u_i, u'_i\}$ and those of the form $\{u'_i, v_{i,j}\}$ are representative. Also, for every clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F, the pairs $\{v_{i_1,j}, v_{i_2,j}\}$, $\{v_{i_1,j}, v_{i_3,j}\}$ and $\{v_{i_2,j}, v_{i_3,j}\}$ are representative.

We below prove some claims about the existence of a *good signature* of H_F, i.e. a signature under which every representative pair of H_F belongs to an unbalanced 4-cycle.

Claim 1. *Let Σ_H be a good signature of H_F, and let x_i be a variable appearing in clauses $C_{j_1}, C_{j_2}, ..., C_{j_{n_i}}$ of F. If r_1 and r_2 agree (resp. disagree) on u_i, then r_1 and r_2 agree (resp. disagree) on $v_{i,j_1}, v_{i,j_2}, ..., v_{i,j_{n_i}}$.*

Proof. Because $\{u_i, u'_i\}$ is representative and $u_i r_1 u'_i r_2 u_i$ is the only 4-cycle containing u_i and u'_i, the vertices r_1 and r_2 agree on u_i and disagree on u'_i in (H_F, Σ_H) without loss of generality. Now, because every pair $\{u'_i, v_{i,j}\}$ is representative, the only 4-cycle including u'_i and $v_{i,j}$ is $u'_i r_1 v_{i,j} r_2 u'_i$, and r_1 and r_2 disagree on u'_i in (H_F, Σ_H), necessarily r_1 and r_2 agree on $v_{i,j}$. □

Claim 2. *Let Σ_H be a good signature of H_F, and let $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ be a clause of F. Then r_1 and r_2 cannot agree or disagree on all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$.*

Proof. First note that the only 4-cycles of H_F containing, say, $v_{i_1,j}$ and $v_{i_2,j}$ are $v_{i_1,j}r_1v_{i_2,j}r_2v_{i_1,j}$, $v_{i_1,j}w_jv_{i_2,j}r_1v_{i_1,j}$ and $v_{i_1,j}w_jv_{i_2,j}r_2v_{i_1,j}$. The claim then follows from the fact that if r_1 and r_2, say, agree on all of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$, then (H_F, Σ_H) has no unbalanced 4-cycle including r_1 and r_2 and two of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$. So, since Σ_H is a good signature, necessarily there are at least three unbalanced 4-cycles containing w_j and every two of $v_{i_1,j}, v_{i_2,j}, v_{i_3,j}$, but one can easily convince himself that this is impossible.

Assume on the contrary that e.g. r_1 and r_2 agree on $v_{i_1,j}$ and disagree on $v_{i_2,j}$ and $v_{i_3,j}$. So far, note that $r_1v_{i_1,j}r_2v_{i_2,j}r_1$ and $r_1v_{i_1,j}r_2v_{i_3,j}r_1$ are unbalanced 4-cycles of (H_F, Σ_H). Then there is no contradiction against the fact that Σ_H is good, since e.g. $v_{i_2,j}$ and $v_{i_3,j}$ can agree on w_j (and, in such a situation, $v_{i_2,j}w_jv_{i_3,j}r_1v_{i_2,j}$ is an unbalanced 4-cycle). The important thing to have in mind is that signing the edges incident to w_j can only create unbalanced 4-cycles containing the representative pairs $\{v_{i_1,j}, v_{i_2,j}\}$, $\{v_{i_1,j}, v_{i_3,j}\}$ and $\{v_{i_2,j}, v_{i_3,j}\}$. So signing the edges incident to w_j to make $v_{i_2,j}$ and $v_{i_3,j}$ belong to some unbalanced 4-cycle does not compromise the existence of other unbalanced 4-cycles including farther vertices from another representative pair. □

We claim that we have the desired equivalence between not-all-equal satisfying F and finding a good signature Σ_H of H_F. To see this holds, just assume, for every variable x_i of F, that having r_1 and r_2 agreeing (resp. disagreeing) on u_i simulates the fact that variable x_i of F is set to *true* (resp. *false*) by some truth assignment, and that having r_1 and r_2 agreeing (resp. disagreeing) on some vertex $v_{i,j}$ simulates the fact that variable x_i provides value *true* (resp. *false*) to the clause C_j of F. Then the property described in Claim 1 depicts the fact that if x_i is set to some truth value by a truth assignment, then x_i provides the same truth value to every clause containing it. The property described in Claim 2 depicts the fact that every clause C_j is considered satisfied by some truth assignment if and only if C_j is supplied different truth values by its variables. So we can deduce a good signature of H_F from a truth assignment not-all-equal satisfying F, and vice-versa.

Step 2. As described above, we now construct G_F from H_F in such a way that

- every 4-cycle of G_F including the vertices of a representative pair is also a 4-cycle in H_F,
- the edges of $E(G_F) \setminus E(H_F)$ can be switchable signed so that every two vertices which do not form a representative pair belong to an unbalanced 4-cycle.

In this way, G_F will underlie an $[s]$-clique if and only if H_F admits a good signature, which is true if and only if F can be not-all-equal satisfied. The result will then hold by transitivity.

For every vertex u of H_F, add the edges ua_u and ub_u, where a_u and b_u are two new vertices. Now, for every two distinct vertices u and v of H_F, if $\{u, v\}$ is not a representative pair, then add another vertex $c_{u,v}$ to the graph, as well as the edges $uc_{u,v}$ and $vc_{u,v}$. Finally turn the subgraph induced by all newly added vertices into a clique. The resulting graph is G_F. As claimed above, note that the only 4-cycles of G_F containing two vertices u and v forming a representative pair are those of H_F. Every other such new cycle has indeed length at least 6. Namely, every such new cycle starts from u, then has to enter the clique by either a_u or b_u, cross the clique to either a_v or b_v, reach v, before finally going back to u.

Consider the following signing to the edges in $E(G_F) \setminus E(H_F)$. For every vertex $u \in V(H_F)$, let ub_u be negative. Similarly, for every two distinct vertices u and v of H_F such that $\{u, v\}$ is

not representative (i.e. $c_{u,v}$ exists), let $c_{u,v}v$ be negative. Let finally all other edges be positive. Clearly, under this partial signature of G_F, every two vertices u and v of G_F not forming a representative pair are either adjacent or belong to some unbalanced 4-cycle:

- if u and v do not belong to H_F, then they belong to the clique and are hence adjacent;
- if u belongs to H_F but v does not, then observe that either u and v are adjacent (in this situation v is either a_u, b_u or $c_{u,i}$ for some i), or ua_uvb_uu is an unbalanced 4-cycle;
- if u and v are vertices of H_F and $\{u, v\}$ is not representative, then e.g. $uc_{u,v}va_va_uu$ is an unbalanced 4-cycle.

According to all previous arguments, finding a truth assignment not-all-equal satisfying F is equivalent to finding an $[s]$-clique overlying G_F, as claimed. So [S]-CLIQUE SIGNING is NP-hard, and hence NP-complete. □

Note that the analogous result for oriented graph is proved in [3].

6 Signed and switchable signed absolute clique numbers

In this section we study the parameters signed and switchable signed absolute clique numbers for some specific classes of graphs. We start off with the easy classes, namely, paths, forests and cycles. As every sclique has diameter at most 2, we have the following results.

Theorem 6.1.

(i) Let G be a path. Then $\omega_{as}(G) \leq 3$ and $\omega_{[as]}(G) \leq 2$.

(ii) Let G be a forest. Then $\omega_{as}(G) \leq 3$ and $\omega_{[as]}(G) \leq 2$.

(iii) Let G be a cycle. Then $\omega_{as}(G) \leq 4$ and $\omega_{[as]}(G) \leq 4$.

Now we shift our focus to the family of outerplanar graphs. Let $\mathcal{O}_k$ denote the family of outerplanar graphs with girth (length of the smallest cycle) at least k. Then we have the following results.

Theorem 6.2.

(a) $\omega_{as}(\mathcal{O}_3) = 7$ and $\omega_{[as]}(\mathcal{O}_3) = 4$.

(b) $\omega_{as}(\mathcal{O}_4) = 4$ and $\omega_{[as]}(\mathcal{O}_4) = 4$.

(c) $\omega_{as}(\mathcal{O}_k) = 3$ and $\omega_{[as]}(\mathcal{O}_k) = 2$ for $k \geq 5$.

Proof. (a) The proof for $\omega_{as}(\mathcal{O}_3) \leq 7$ follows from Theorem 2.9. Now notice that the graph depicted in Fig. 1(a)(iv) is an outerplanar sclique of order 7. Hence we have $\omega_{as}(\mathcal{O}_3) = 7$.

We know that the switchable signed chromatic number for outerplanar graphs is at most 5 [25]. It is easy to check that there exists no outerplanar switchable signed graph of order 5. Also note that an unbalanced 4-cycle is an $[s]$-clique. Hence we have $\omega_{[as]}(\mathcal{O}_3) = 4$.

(b,c) We know that an $[s]$-clique has diameter at most 2. Hence the rest of the proof of the above theorem easily follow from the list of triangle-free planar graphs with diameter 2 given by Plesnik in Theorem 2.6. □

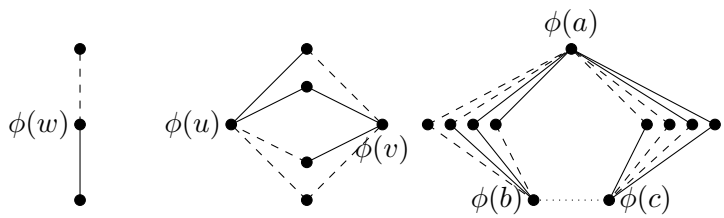


Figure 5: Planar signed graphs with diameter 2 and girth at least 4 (the dotted edge bc refers to an edge which can be negative or positve as per necessity).

Finally, we study the parameter for planar graphs. Let $\mathcal{P}_k$ denote the family of planar graphs with girth (length of the smallest cycle) at least k. Then we have the following results.

Theorem 6.3.

(a) $\omega_{as}(\mathcal{P}_3) = 15$ *and* $\omega_{[as]}(\mathcal{P}_3) = 8$. *[25]*

(b) $\omega_{as}(\mathcal{P}_4) = 6$ *and* $\omega_{[as]}(\mathcal{P}_4) = 4$.

(c) $\omega_{as}(\mathcal{P}_k) = 3$ *and* $\omega_{[as]}(\mathcal{P}_k) = 2$ *for* $k \geq 5$.

The proof of $\omega_{[as]}(\mathcal{P}_3) = 8$ was given by Naserasr, Rollová and Sopena [25]. The proof of $\omega_{as}(\mathcal{P}_3) = 15$ is big and is proved separately in the end of this section.

Proof of Theorem 6.3(b,c)

(b) In 1975, Plesník [14] characterized and listed all triangle-free planar graphs with diameter 2. They are precisely the graphs depicted in Fig. 2 (see Theorem 2.6). First we will prove that $\omega_{as}(\mathcal{P}_4) = 6$.

Note that any signed graph with the graphs from Fig. 2 as underlying graphs admits a homomorphism to the graphs depicted in Fig. 5 respectively (that is, the first signed graph depicted in Fig. 5 is a universal bound for the first family of graphs described in Fig. 2; the second ... etc.).

To prove the homomorphisms we map the vertices w, u, v, a, b, c from Fig. 2 to the corresponding vertices $\phi(w), \phi(u), \phi(v), \phi(a), \phi(b), \phi(c)$ in Fig 5 respectively. Choose the sign of the edge $\phi(b)\phi(c)$ the same as the sign of the edge bc.

Now to complete the first homomorphism, map the vertices of $N^\alpha(w)$ to the unique vertex in $N^\alpha(\phi(w))$ for $\alpha \in \{+, -\}$.

To complete the second homomorphism, map the vertices of $N^\alpha(u) \cap N^\beta(u)$ to the unique vertex in $N^\alpha(\phi(u)) \cap N^\beta(\phi(v))$ for $\alpha, \beta \in \{+, -\}$.

To complete the third homomorphism, map the vertices of $N^\alpha(a) \cap N^\beta(t)$ to the unique vertex in $N^\alpha(\phi(a)) \cap N^\beta(\phi(t))$ for $\alpha, \beta \in \{+, -\}$ and $t \in \{b, c\}$.

Now note that the first two signed graphs depicted in Fig. 5 are scliques of order 3 and 6 respectively, while the third graph is not an sclique but clearly has signed relative clique number 5.

Hence, there is no triangle-free planar sclique of order more than 6. Also, the only example of a triangle-free sclique of order 6 is the second graph depicted in Fig. 5.

The result $\omega_{[as]}(\mathcal{P}_4) = 4$ follows similarly from Theorem 2.6 and from the fact that an unbalanced 4-cycle is an $[s]$-clique. $\square$

(c) It is easy to check the upper bounds using Theorem 2.6 and the lower bounds follows from the fact that the rainbow 2-path is an sclique of order 3 and an edge is ab $[s]$-clique of order 2. $\square$

Proof of Theorem 6.3(a)

(a) Take two copies of the signed outerplanar sclique depicted in Fig. 1(a)(i) and a vertex ∞. Join ∞ and the vertices of the first copy of the signed outerplanar sclique depicted in Fig. 1(a)(i) with positive edges. Join ∞ and the vertices of the second copy of the signed outerplanar sclique depicted in Fig. 1(a)(i) with negative edges. This so-obtained graph is an sclique of order 15. Note that the graph is also planar. This proves the lower bound.

For proving the upper bound, first consider an sclique (H) with domination number 1. Suppose (H) is dominated by the vertex v. As (H) is an sclique, the set of vertices $N^+(v)$ are part of a relative clique in the signed outerplanar graph $((H[N(v)])$. Therefore, by Theorem 2.9 we have $|N^+(v)| \leq 7$.

Similarly we have $|N^-(v)| \leq 7$. Hence, $|N(v)| \leq 14$. This implies that the order of the graph (H) is at most 15.

Goddard and Henning [11] (see Theorem 2.7) proved that every planar graph of diameter 2 has domination number at most 2 except for a particular graph on nine vertices.

Hence, to prove the theorem, it will be enough to prove that any planar sclique with domination number 2 must have order at most 15. More precisely, we need to prove the following result.

Lemma 6.4. *Let (H) be a planar sclique with domination number 2. Then $|V((H))| \leq 15$.*

If two vertices u and v are adjacent (or connected by a rainbow 2-path) we say they are at *rainbow distance* 1 (or 2) and denote it by $rd(u,v) = 1$ (or $rd(u,v) = 2$). In any other case we say $rd(u,v) > 2$. If every pair of vertices of a signed graph (G) are at rainbow distance at most 2 then we say that the signed graph has *rainbow diameter 2*. It is easy to note that the signed graphs with rainbow diameter 2 are precisely scliques.

Let (G,Γ) be a planar sclique with $|V(G)| > 15$. Assume that G is triangulated and has domination number 2.

We define the partial order $\prec$ for the set of all dominating sets of order 2 of G as follows: for any two dominating sets $D = \{x,y\}$ and $D' = \{x',y'\}$ of order 2 of (G), $D' \prec D$ if and only if $|N(x') \cap N(y')| < |N(x) \cap N(y)|$.

Let $D = \{x,y\}$ be a maximal dominating set of order 2 of G with respect to $\prec$. Also for the rest of this proof, $t, t', \alpha, \overline{\alpha}, \beta, \overline{\beta}$ are variables satisfying $\{t,t'\} = \{x,y\}$ and $\{\alpha,\overline{\alpha}\} = \{\beta,\overline{\beta}\} = \{+,-\}$.

Now, we fix the following notations (Fig: 6):

$$C = N(x) \cap N(y),\ C^{\alpha\beta} = N^{\alpha}(x) \cap N^{\beta}(y),\ C_t^{\alpha} = N^{\alpha}(t) \cap C,$$
$$S_t = N(t) \setminus C,\ S_t^{\alpha} = S_t \cap N^{\alpha}(t) \text{ and } S = S_x \cup S_y.$$

Hence we have,

$$16 \leq |(G,\Gamma)| = |D| + |C| + |S|. \tag{4}$$

Let (H) be the signed graph obtained from the induced subgraph $(G)[D \cup C]$ of (G) by deleting all the edges between the vertices of D and all the edges between the vertices of C.

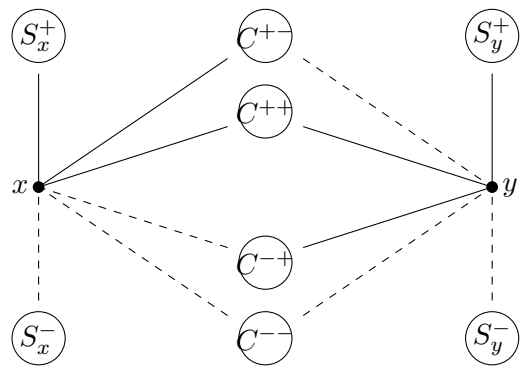


Figure 6: Structure of G (not a planar embedding)

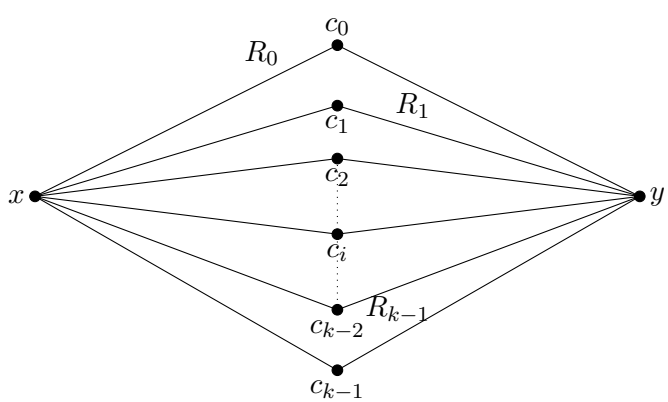


Figure 7: A planar embedding of $und(H)$

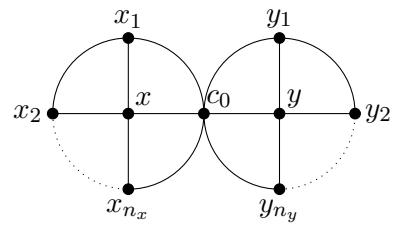


Figure 8: For $|C| = 1$ while x and y are non-adjacent

Note that it is possible to extend the planar embedding of H given in Fig 7 to a planar embedding of G for some particular ordering of the elements of, say $C = \{c_0, c_1, ..., c_{k-1}\}$.

Notice that H has k faces, namely the unbounded face F_0 and the faces F_i bounded by edges $xc_{i-1}, c_{i-1}y, yc_i, c_ix$ for $i \in \{1, ..., k-1\}$. Geometrically, H divides the plane into k connected components. The *region* R_i is the ith connected component (corresponding to the face F_i) of the plane. *Boundary points* of a region R_i are c_{i-1} and c_i for $i \in \{1, ..., k-1\}$ and, c_0 and c_{k-1} for $i = 0$. Two regions are *adjacent* if they have at least one common boundary point (hence, a region is adjacent to itself).

Now for the different possible values of $|C|$, we want to show that H cannot be extended to a planar sclique of order at least 16. Note that, for extending H to (G), we can add new vertices only from S. Any vertex $v \in S$ will be inside one of the regions R_i. If there is at least one vertex of S in a region R_i, then R_i is *non-empty* and *empty* otherwise. In fact, when there is no chance of confusion, R_i may also be used to represent the set of vertices of S contained in the region R_i.

As any two distinct non-adjacent vertices of (G) must be connected by a rainbow 2-path, we have the following three lemmas:

Lemma 6.5. *(a) If $(u, v) \in S_x \times S_y$ or $(u, v) \in S_t^\alpha \times S_t^\alpha$, then u and v are in adjacent regions.*

(b) If $(u, c) \in S_t^\alpha \times C_t^\alpha$, then c is a boundary point of a region adjacent to the region containing u.

Lemma 6.6. *Let R, R^1, R^2 be three distinct regions such that R is adjacent to R^i with common boundary point c^i while the other boundary points of R^i is $\overline{c^i}$ for all $i \in \{1, 2\}$. If $v \in S_t^\alpha \cap R$ and $u^i \in ((S_t^\alpha \cup S_{t'}) \cap R^i) \cup (\{\overline{c^i}\} \cap C_t^\alpha)$, then v disagrees with u^i on c^i, where $i \in \{1, 2\}$. If both u^1 and u^2 exist, then $|S_t^\alpha \cap R| \leq 1$.*

Lemma 6.7. *For any edge uv in (G), we have $|N^\alpha(u) \cap N^\beta(v)| \leq 3$.*

Now we ask the question "How small $|C|$ can be?" and try to prove possible lower bounds of $|C|$. The first result regarding the lower bound of $|C|$ is proved below.

Lemma 6.8. *We have $|C| \geq 2$.*

Proof. We know that x and y are either connected by a rainbow 2-path or by an edge. If x and y are adjacent, then as (G) is triangulated, we have $|C| \geq 2$. If x and y are non-adjacent, then $|C| \geq 1$. Hence it is enough to show that we cannot have $|C| = 1$ while x and y are non-adjacent.

If $|C| = 1$ and x and y are non-adjacent, then the triangulation will force the configuration depicted in Fig 8 as a subgraph of G, where $C = \{c_o\}$, $S_x = \{x_1, ..., x_{n_x}\}$ and $S_y = \{y_1, ..., y_{n_y}\}$. Without loss of generality we may assume $|S_y| \geq |S_x|$. Then by equation (4) we have,

$$n_y = |S_y| \geq \lceil (16 - 2 - 1)/2 \rceil = 7.$$

Clearly $n_x \geq 3$ as otherwise $\{c_0, y\}$ is a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$ which contradicts the maximality of D.

For $n_x = 3$, we know that c_0 is not adjacent to x_2 as otherwise $\{c_0, y\}$ is a dominating set with at least two common neighbors $\{y_1, y_{n_y}\}$ contradicting the maximality of D. But then x_2 should be adjacent to y_i for some $i \in \{1, ..., n_y\}$ as otherwise $d(x_2, y) > 2$. Now the triangulation will force x_2 and y_i to have at least two common neighbors. Also x_2 cannot be adjacent to y_j for any $j \neq i$, as it will create a dominating set $\{x_2, y\}$ with at least two common neighbors $\{y_i, y_j\}$ contradicting the maximality of D. Hence, x_2 and y_i are adjacent to both x_1 and x_3. Note that t_{ℓ_t} and t_{ℓ_t+k} are adjacent if and only if $k = 1$, as otherwise $d(t_{\ell_t+1}, t') > 2$ for $1 \leq \ell_t < \ell_t + k \leq n_t$. In this case, by equation (4) we have,

$$n_y = |S_y| \geq 16 - 2 - 1 - 3 = 10.$$

Assume $i \geq 5$. Hence, c_0 is adjacent to y_j for all $j = 1, 2, 3$, as otherwise $d(y_j, x_3) > 2$. This implies $d(y_2, x_2) > 2$, a contradiction. Similarly $i < 5$ will also force a contradiction. Hence $n_x \geq 4$.

For $n_x = 4$, c_0 cannot be adjacent to both x_3 and $x_{n_x-2} = x_2$ as it creates a dominating set $\{c_0, y\}$ with at least two common neighbors $\{y_1, y_{n_y}\}$ contradicting the maximality of D. For $n_x \geq 5$, c_0 is adjacent to x_3 implies, either for all $i \geq 3$ or for all $i \leq 3$, x_i is adjacent to c_0, as otherwise $d(x_i, y) > 2$. Either of these cases will force c_0 to become adjacent to y_j, as otherwise we will have either $d(x_1, y_j) > 2$ or $d(x_{n_x}, y_j) > 2$ for all $j \in \{1, 2, ..., n_y\}$. But then we will have a dominating set $\{c_0, x\}$ with at least two common neighbors contradicting the maximality of D. Hence for $n_x \geq 5$, c_0 is not adjacent to x_3. Similarly we can show, for $n_x \geq 5$, that c_0 is not adjacent to either x_3 or x_{n_x-2}.

So, for $n_x \geq 4$, without loss of generality we can assume that c_0 is not adjacent to x_3. We know that $d(y_1, x_3) \leq 2$. We have already noted that t_{l_t} and t_{l_t+k} are adjacent if and only if $k = 1$ for any $0 \leq l_t < l_t + k \leq n_t$. Hence to have $d(y_1, x_3) \leq 2$, we must have one of the following edges: y_1x_2, y_1x_3, y_1x_4 or y_2x_3. The first edge will imply the edges x_2y_j as otherwise $d(x_1, y_j) > 2$ for all $j = 3, 4, 5$. These three edges will imply $d(x_4, y_3) > 2$. Hence we do not have y_1x_2.

The other three edges, assuming we cannot have y_1x_2, will force the edges x_2c_0 and x_1c_0 for having $d(x_2, y) \leq 2$ and $d(x_1, y) \leq 2$. This will imply $d(x_1, y_4) > 2$, a contradiction. Hence we cannot have the other three edges also.

Hence we are done. □

Now we will prove that, for $2 \leq |C| \leq 5$, at most one region of (G) can be non-empty. Later, using this result, we will improve the lower bound of $|C|$.

Lemma 6.9. *If $2 \leq |C| \leq 5$, then at most one region of (G) is non-empty.*

Proof. For pictorial help one can look at Fig 7. For $|C| = 2$, if x and y are adjacent, then the region that contains the edge xy is empty, as otherwise triangulation will force x and y to have a common neighbor other than c_0 and c_1. So for the rest of the proof we can assume x and y to be non-adjacent for $|C| = 2$.

Step 0: First we shall show that it is not possible to have either $S_x = \emptyset$ or $S_y = \emptyset$ and have at least two non-empty regions. Without loss of generality assume that $S_x = \emptyset$. Then x and y are non-adjacent, as otherwise y will be a dominating vertex which is not possible.

For $|C| = 2$, if both $S_y \cap R_0$ and $S_y \cap R_1$ are non-empty, then triangulation will force, either multiple edges c_0c_1 (one in each region) or a common neighbor of x, y other than c_0, c_1, a contradiction.

For $|C| = 3, 4$ and 5, triangulation implies the edges $c_0c_1, ..., c_{k-2}c_{k-1}, c_{k-1}c_0$. Hence every $v \in S_y$ must be connected to x by a rainbow 2-path through c_i for some $i \in \{1, 2, ..., k-1\}$. Now assume $|S_y^\alpha| \geq |S_y^{\overline{\alpha}}|$ for some $\alpha \in \{+, -\}$. Then by equation (4) we have,

$$|S_y^\alpha| \geq \lceil (16 - 2 - 5)/2 \rceil = 5.$$

Now by Lemma 6.5, we know that the vertices of S_y^α will be contained in two adjacent regions for $|C| = 4, 5$. For $|C| = 3$, $S_y^\alpha \cap R_i$ for all $i \in \{0, 1, 3\}$ implies $|S_y^\alpha| \leq 3$ by Lemma 6.6. Hence, without loss of generality, we may assume $S_y^\alpha \subseteq R_1 \cup R_2$. If both $S_y^\alpha \cap R_1$ and $S_y^\alpha \cap R_2$ are non-empty, then by Lemma 6.6, each vertex of $S_y^\alpha \cap R_1$ disagrees with each vertex of $S_y^\alpha \cap R_2$ on c_1. Then $\{c_1, y\}$ becomes a dominating set with at least six common neighbors (c_0, c_2 and four vertices from S_y^α) which contradicts the maximality of D.

Hence, all the vertices of S_y^α must be contained in one region, say R_1. Then each of them should be connected to x by a rainbow 2-path with internal vertex either c_0 or c_1. However, the vertices that are connected to x by a rainbow 2-path with internal vertex c_0 should have rainbow distance at most 2 with the vertices connected to x by a rainbow 2-path with internal vertex c_1. It is not possible to connect them unless they are all adjacent to either c_0 or c_1. But then it will contradict the maximality of D.

Hence both S_x and S_y are non-empty.

Step 1: Now we will prove that at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, for all $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$. It is trivial for $|C| = 2$. For $|C| = 4$ and 5, the statement follows from Lemma 6.5. For $|C| = 3$, we consider the following two cases:

(i) Assume $S_t \cap R_i \neq \emptyset$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. Then by Lemma 6.6 we have, $|S_t \cap R_i| \leq 1$ for all $t \in \{x, y\}$ and for all $i \in \{0, 1, 2\}$. Then by equation (4) we have,

$$16 \leq |(G)| = 2 + 3 + 4 = 9.$$

This is a contradiction.

(ii) Assume that five out of the six sets $S_t \cap R_i$ are non-empty and the other one is empty, where $t \in \{x, y\}$ and $i \in \{0, 1, 2\}$. Without loss of generality we can assume $S_x \cap R_0 = \emptyset$. By Lemma 6.6 we have $|S_t \cap R_i| \leq 1$ for all $(t, i) \in \{(x, 1), (x, 2), (y, 0)\}$. In particular, $|S_x| \leq 2$.

Now, all vertices of $S_t \cap R_i$ is adjacent to c_1, for being at rainbow distance at most 2 from each other, by Lemma 6.6. That means, every vertex of S_x is adjacent to c_1. Hence, there can be at most three vertices in $(S_y \cap R_1) \cup (S_y \cap R_2)$ as otherwise the dominating set $\{c_1, y\}$ will contradict the maximality of D. Hence $|S_y| \leq 4$.

Therefore by equation (4) we have,

$$16 \leq |(G)| = 2 + 3 + (2 + 4) = 11.$$

This is a contradiction.

Hence at most four sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$.

Step 2: Now assume that exactly four sets out of the sets $S_t \cap R_i$ are non-empty, for all $t \in \{x, y\}$ and $i \in \{0, ..., k-1\}$. Without loss of generality we have the following three cases (by Lemma 6.5):

(i) Assume the four non-empty sets are $S_x \cap R_1, S_y \cap R_0, S_y \cap R_1$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). We have the edges c_0c_{k-1} and c_1c_2 by triangulation. Lemma 6.6 implies that $S_x \cap R_1 = \{x_1\}$ and that the vertices of $S_y \cap R_0$ and the vertices of $S_y \cap R_2$ disagree with x_1 on c_0 and c_1 respectively. Hence by Lemma 6.7, we have $|S_y \cap R_0|, |S_y \cap R_2| \leq 3$.

For $|C| = 3$, if every vertex from $S_y \cap R_1$ is adjacent to either c_0 or c_1, then $\{c_0, c_1\}$ will be a dominating set with at least four common neighbors $\{x, y, x_1, c_2\}$ contradicting the maximality of D. If not, then triangulation will force x_1 to be adjacent to at least two vertices y_1, y_2 (say) from S_y. But then $\{x_1, y\}$ will become a dominating set with at least four common neighbors $\{y_1, y_2, c_0, c_1\}$ contradicting the maximality of D.

For $|C| = 4$ and 5, Lemma 6.5 implies that vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_2$ disagree with each other on y. Now by Lemma 6.6, any vertex of $S_y \cap R_1$ is adjacent to either c_0 (if it agrees with the vertices of $S_y \cap R_0$ on y) or c_1 (if it agrees with the vertices of $S_y \cap R_2$ on y). Also vertices of $S_y \cap R_0$ and $S_y \cap R_2$ are connected to x_1 by a rainbow 2-path through c_0 and c_1 respectively.

Now by equation (4) we have,

$$|S_y| \geq (16 - 2 - 5 - 1) = 8.$$

Hence, without loss of generality, at least four vertices y_1, y_2, y_3, y_4 of S_y are adjacent to c_0. Hence $\{c_0, y\}$ is a dominating set with at least five common neighbors $\{y_1, y_2, y_3, y_4, c_{k-1}\}$ contradicting the maximality of D for $|C| = 4$.

For $|C| = 5$, each vertex of $S_y \cap R_1$ disagree with c_3 by Lemma 6.5 and hence without loss of generality are all adjacent to c_0. Now $|S_y \cap R_2| \leq 3$ and $|S_y| \geq 8$ implies $|S_y \cap (R_0 \cup R_1)| \geq 5$. But every vertex of $S_y \cap (R_0 \cup R_1)$ and c_4 are adjacent to c_0. Hence $\{c_0, y\}$ is a dominating set with at least six common neighbors, contradicting the maximality of D for $|C| = 5$.

(ii) Assume the four non-empty sets are $S_x \cap R_0, S_x \cap R_1, S_y \cap R_0$ and $S_y \cap R_1$. For $|C| = 2$ every vertex in S is adjacent to either c_0 or c_1 (by Lemma 6.6). So, $\{c_0, c_1\}$ is a dominating set. Hence no vertex $w \in S$ can be adjacent to both c_0 and c_1 because otherwise $\{c_0, c_1\}$ will be a dominating set with at least three common neighbors $\{x, y, w\}$ contradicting the maximality of D. By equation (4) we have,

$$|S| \geq 16 - 2 - 2 = 12.$$

Hence, without loss of generality, we may assume $|S_x \cap R_0| \geq 3$. Assume $\{x_1, x_2, x_3\} \subseteq S_x \cap R_0$. Now all vertices of $S_x \cap R_0$ must be adjacent to c_0 (or c_1), as otherwise it will force all vertices of $S_y \cap R_1$ to be adjacent to both c_0 and c_1 (by Lemma 6.6). Without loss of generality assume all vertices of $S_x \cap R_0$ are adjacent to c_0. Then all $w \in S_y$ will be adjacent to c_0, as otherwise $d(w, x_i) > 2$ for some $i \in \{1, 2, 3\}$. But then $\{c_0, x\}$ will be a dominating set with at least three common vertices $\{x_1, x_2, x_3\}$ contradicting the maximality of D.

For $|C| = 3, 4$, every vertex of S will be adjacent to c_0 (by Lemma 6.6). By equation (4) we have,

$$|S| \geq (16 - 2 - 4) = 10.$$

Hence, without loss of generality, $|S_x| \geq 5$. Hence $\{c_o, x\}$ is a dominating set with at least five common neighbors $S_x \cup \{y\}$ contradicting the maximality of D for $|C| = 3, 4$.

For $|C| = 5$, every vertex of $S_t \cap R_i$ disagree with c_{i+2} on t and hence $|S_t \cap R_i| \leq 3$ for $i \in \{0, 1\}$ by Lemma 6.5. Assume, $|S_x \cap R_0| = 3$ and $S_x \cap R_0 = \{x_1, x_2, x_3\}$. Now assume without loss of generality that $c_2 \in N^{\alpha}(x)$. Hence, we must have $\{x_1, x_2, x_3\} \subseteq N^{\overline{\alpha}}(x)$.

Note that x_1, x_2, x_3 must agree on c_0 in order to be at rainbow distance at most 2 with the vertices of $S_y \cap R_1$. Further assume that $\{x_1, x_2, x_3\} \subseteq N^{\beta}(c_0)$. But then as all the three vertices $\{x_1, x_2, x_3\}$ are adjacent to both x and c_0, the only way each of them can be at rainbow distance 2 with c_3 is by a rainbow 2-path with internal vertex x. Hence we have $c_3 \in N^{\alpha}(x)$. This implies that $x_4 \in N^{\overline{\alpha}}(x)$ for any vertex $x_4 \in S_x \cap R_1$. But then the vertices of $S_x \cap R_1$ must disagree with vertices of $S_x \cap R_0$ on c_0 making it impossible for the vertices of $S_1 \cap R_0$ to be at rainbow distance at most 2 with x_1, x_2, x_3 and with the vertices of $S_x \cap R_1$. Hence we must have $|S_x \cap R_0| \leq 2$.

Similarly we can prove $|S_t \cap R_i| \leq 2$ for $i \in \{0, 1\}$.

Now we will show that it is not possible to have $|S_t \cap R_i| = 2$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$.

Suppose we have $|S_t \cap R_i| = 2$ for all $(t, i) \in \{x, y\} \times \{0, 1\}$. Then clearly, the vertices of $S_t \cap R_i$ disagree with c_{i+2} and c_{i+3} on t. Hence, the vertices of $S_t \cap R_0$ agree with the vertices of $S_t \cap R_1$ on t. Therefore, the vertices of $S_t \cap R_0$ must disagree with the vertices of $S_t \cap R_1$ on c_0.

Then it will not be possible to have both the vertices of $S_x \cap R_0$ at rainbow distance at most 2 with all the four vertices of S_y.

Therefore, we have $|S| \leq 7$. Hence by equation (4) we have,

$$16 \leq |(G)| \leq 2 + 5 + 7 = 14.$$

This is a contradiction. Hence we are done.

(iii) Assume the four non-empty sets are $S_x \cap R_1, S_x \cap R_2, S_y \cap R_0$ and $S_y \cap R_1$ (only possible for $|C| = 3$). Now Lemma 6.6 implies that every vertex of $(S_x \cap R_1) \cup (S_y \cap R_0)$ is adjacent to c_0 and every vertex of $(S_x \cap R_2) \cup (S_y \cap R_1)$ is adjacent to c_1.

Moreover triangulation forces the edges c_0c_2 and c_1c_2. Triangulation also forces some vertex $v_1 \in S_y \cap R_1$ to be adjacent to c_0. This will create the dominating set $\{c_0, c_1\}$ with at least four common neighbors $\{x, y, v_1, c_2\}$ contradicting the maximality of D.

Hence at most three sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$.

Step 3: Now assume that exactly three sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, ..., k-1\}$. Without loss of generality we have the following two cases (by Lemma 6.5):

(i) Assume the three non-empty sets are $S_x \cap R_0, S_y \cap R_0$ and $S_y \cap R_1$. Triangulation implies the edge c_0c_1 inside the region R_1.

For $|C| = 2$, there exists $u \in S_y \cup R_1$ such that u is adjacent to both c_0 and c_1 by triangulation. Now if $|S_y \cup R_1| \geq 2$, then some other vertex $v \in S_y \cup R_1$ must be adjacent to either c_0 or c_1. Without loss of generality we may assume that v is adjacent to c_0. Then every $w \in S_x \cap R_0$ will be adjacent to c_0 to have $d(v, w) \leq 2$. But then $\{c_0, y\}$ will be a dominating set with at least three common neighbors $\{c_1, u, v\}$ contradicting the maximality of D.

So we must have $|S_y \cup R_1| = 1$. Now let us assume that $S_y \cup R_1 = \{u\}$. Then any $w \in S_x \cap R_0$ is adjacent to either c_0 or c_1. If $|S_x| \geq 5$, then without loss of generality we can assume that at

least three vertices of S_x are adjacent to c_0. Now to have at most distance 2 with all those three vertices, every vertex of S_y will be adjacent to c_0. This will create the dominating set $\{c_0, x\}$ with at least three common neighbors contradicting the maximality of D.

Also $|S_x| = 1$ clearly creates the dominating set $\{c_0, y\}$ (as x_1 is adjacent to c_0 by triangulation) with at least three common neighbors (a vertex from $S_y \cap R_0$ by triangulation, u and c_1) contradicting the maximality of D.

For $2 \leq |S_x| \leq 4$, c_0 (or c_1) can be adjacent to at most two vertices of $S_y \cap R_0$ because otherwise there will be one vertex $v \in S_y \cap R_0$ which will force c_0 (or c_1) to be adjacent to all vertices of $w \in S_x$ in order to satisfy $d(v, w) \leq 2$ and create a dominating set $\{c_0, y\}$ that contradicts the maximality of D.

Also, not all vertices of S_x can is adjacent to c_0 (or c_1) as otherwise $\{c_o, y\}$ (or $\{c_1, y\}$) will be a dominating set with at least three common neighbors (u, c_1 (or c_0) and a vertex from $S_y \cap R_0$) contradicting the maximality of D.

Note that, by equation (4), we have,

$$|S_y \cap R_0| \geq 10 - S_x.$$

Assume $S_x = \{x_1, ..., x_n\}$ with triangulation forcing the edges c_0x_1, x_1x_2, ..., $x_{n-1}x_n$, x_nc_1 for $n \in \{2, 3, 4\}$.

For $|S_x| = 2$, at most four vertices of $S_y \cap R_0$ can be adjacent to c_0 or c_1. Hence there will be at least four vertices of $S_y \cap R_0$ each connected to x by a rainbow 2-path through x_1 or x_2. Without loss of generality x_1 will be adjacent to at least 2 vertices of S_y and hence $\{x_1, y\}$ will be a dominating set contradicting the maximality of D.

For $|S_x| = 3$, without loss of generality assume that x_2 is adjacent to c_0. To satisfy $d(x_1, v) \leq 2$ for all $v \in S_y \cap R_0$, at least four vertices of S_y will be adjacent connected to x_1 by a rainbow 2-path through x_2 (as, according to previous discussions, at most two vertices of S_y can be adjacent to c_0). This will create the dominating set $\{x_2, y\}$ contradicting the maximality of D.

For $|S_x| = 4$ we have x_2c_0 and x_3c_1 as otherwise at least three vertices of S_x will be adjacent to either c_0 or c_1 which is not possible (because it forces all vertices of S_y to be adjacent to c_0 or c_1). Now each vertex $v \in S_y \cap R_0$ must be adjacent to either c_0 or x_2 (to satisfy $d(v, x_1) \leq 2$) and also to either c_1 or x_3 (to satisfy $d(v, x_4) \leq 2$) which is not possible to do keeping the graph planar.

For $|C| = 3, 4, 5$ by Lemma 6.6, each vertex of S_x disagree with each vertex of $S_y \cap R_1$ on c_0. We also have the edge x_1c_2 for some $x_1 \in S_x$ by triangulation. Now by equation (4) we have,

$$|S| \geq (16 - 2 - |C|) = 13 - |C|.$$

Hence $|S_x| \leq 2$ for $|C| = 3, 4$, as otherwise every vertex $u \in S_y$ will be adjacent to c_0 creating a dominating set $\{c_0, t\}$ with at least $(|C| + 1)$ common neighbors $S_t \cup \{c_1\}$ for some $t \in \{x, y\}$ contradicting the maximality of D. For $|C| = 5$, as every vertex in $S_x \cap R_0$ agree with each other on x (as they all must disagree with c_2 on x) and on c_0 (as they all disagree with vertices of $S_y \cap R_1$ on c_0). So, by Lemma 6.7, we have $|S_x \cap R_0| \leq 3$. But if $|S_x \cap R_0| = 3$ then every vertex of S_y will be adjacent to c_0 creating a dominating set $\{c_0, y\}$ with at least six common neighbors $S_y \cup \{c_1\}$ contradicting the maximality of D.

Hence $|S_x| \leq 2$ for $|C| = 3, 4$ and 5.

Now for $|C| = 3$, we can assume x and y are non-adjacent as otherwise $\{c_0, y\}$ will be a dominating set with at least four common neighbors (x, c_1 and, two other vertices each from the

sets $S_y \cap R_0$, $S_y \cap R_1$ by triangulation) contradicting the maximality of D. Hence triangulation will imply the edge c_1c_2. Now for $|S_x| \leq 2$, either $\{c_0, c_2\}$ is a dominating set with at least four common neighbors $\{x, y, c_1, x_1\}$ contradicting the maximality of D or x_1 is adjacent to at least two vertices $y_1, y_2 \in S_y \cap R_0$ creating a dominating set $\{x_1, y\}$ (the other vertex in S_x must be adjacent to x_1 by triangulation) with at least four common neighbors $\{y_1, y_2, c_0, c_2\}$ contradicting the maximality of D.

For $|C| = 4$ we have $|S_y \cap R_1| \leq 2$ as otherwise we will have the dominating set $\{c_0, y\}$ with at least five common neighbors (c_1, vertices of $S_y \cap R_1$ and one vertex of $S_y \cap R_0$ by triangulation) contradicting the maximality of D. Now by equation (4) we have,

$$\begin{aligned} |S_y \cap R_0| &\geq (16 - |D| - |C| - |S_x| - |S_y \cap R_1|) \\ &\geq (16 - 2 - 4 - 2 - 2) = 6. \end{aligned}$$

Now, at most two vertices of $S_y \cap R_0$ can be adjacent to c_0 as otherwise $\{c_0, y\}$ will be a dominating set with at least five common neighbors (c_1, vertices of $S_y \cap R_0$ and one vertex of $S_y \cap R_1$ by triangulation) contradicting the maximality of D.

Also by triangulation in R_3 we either have the edge xy or have the edge c_2c_3. But, if we have the edge xy, then $|S_y \cap R_1| = 1$ as otherwise the dominating set $\{c_0, y\}$ will contradict the maximality of D. Hence, by triangulation, and to have rainbow distance at most 2 with the vertices of S_x, each vertex of $S_y \cap R_0$ will be adjacent either to c_3 or to x_1. This will create a dominating set $\{x_1, y\}$ or $\{c_3, y\}$ that contradicts the maximality of D. Hence, we do not have the edge xy (not even in other regions) and have the edge c_2c_3.

For $|S_x| \leq 2$, the vertices of $S_y \cap R_0$ will be adjacent to either c_3 or c_0 or x_1 to have rainbow distance at most 2 with x. But then triangulation will force at least one vertex of $S_y \cap R_0$ to be common neighbor of c_3 and x_1 and another vertex of $S_y \cap R_0$ to be common neighbor of c_3 and x_1 or the edge c_0c_3. It is not difficult to check, casewise, (drawing a picture for individual cases will help in understanding the scenario) that one of the sets $\{c_0, y\}$, $\{c_3, y\}$ or $\{x_1, y\}$ will be a dominating set contradicting the maximality of D.

For $|C| = 5$ by Lemma 6.5, each vertex of $S_y \cap R_i$ must disagree with c_{i+2} on y. If vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ agree with each other on y, then they must disagree with each other on c_0 which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$. If vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ disagree with each other on y, then vertices of $S_y \cap R_i$ must agree with c_{3-i} on y. Then, by Lemma 6.6, each vertex of $S_y \cap R_i$ must be connected to c_{3-i} by a rainbow 2-path through c_{4-3i} which implies $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 1\}$.

Assume, we have $|S_y \cap R_0| = 3$ and $|S_y \cap R_1| = 3$. Then each vertex of $S_y \cap R_i$ must disagree with both c_{i+2} and c_{i+3} on y. This will imply that the vertices of $S_y \cap R_0$ and vertices of $S_y \cap R_1$ disagree with each other on c_0. Now there will be no way to have rainbow distance at most 2 between a vertex of S_x and all the six vertices of S_y.

Hence we must have $|S_y| \leq 5$. Then by equation (4) we have,

$$16 \leq |(G)| \leq 2 + 5 + (2 + 5) = 14.$$

This is a contradiction. This concludes this particular subcase.

(ii) Assume the three non-empty sets are $S_x \cap R_1, S_y \cap R_0$ and $S_y \cap R_2$ (only possible for $|C| \geq 3$). By Lemma 6.6, we have $S_x = \{x_1\}$ and the fact that each vertex of $S_y \cap R_i$ disagrees with $c_{i^2/4}$ on x_1 for $i \in \{0, 2\}$. Triangulation implies the edges x_1c_0, x_1c_1, $c_{k-1}c_0$, c_0c_1 and c_1c_2.

For $|C| = 3$, $\{c_0, c_1\}$ is a dominating set with at least four common neighbors $\{x, y, c_2, x_1\}$ contradicting the maximality of D. For $|C| = 4$ and 5 we have, every vertex of $S_y \cap R_0$ disagree with every vertex of $S_y \cap R_2$ on y. Hence, by Lemma 6.7, we have $|S_y \cap R_i| \leq 3$ for all $i \in \{0, 2\}$. Hence by equation (4) we have

$$\begin{aligned} 16 \leq |(G)| &= |D| + |C| + |S| \\ &\leq [2 + 5 + (1 + 3 + 3)] = 14. \end{aligned}$$

This is a contradiction.

Step 4: Hence at most two sets out of the $2k$ sets $S_t \cap R_i$ can be non-empty, where $t \in \{x, y\}$ and $i \in \{0, 1, ..., k-1\}$.

Now assume that exactly two sets out of the sets $S_t \cap R_i$ are non-empty, where $t \in \{x, y\}$ and $i \in \{0, ..., k-1\}$, yet there are two non-empty regions. Without loss of generality assume that the two non-empty sets are $S_x \cap R_0$ and $S_y \cap R_1$. Triangulation will force x and y to have a common neighbor other than c_0 and c_1 for $|C| = 2$ which is a contradiction.

For $|C| = 3, 4, 5$ triangulation implies the edges $c_{k-1}c_0$ and c_0c_1. By Lemma 6.6, we know that each vertex of S is adjacent to c_0. By equation (4) we have,

$$|S| \geq (16 - 2 - 5) = 9.$$

Hence, without loss of generality, we may assume $|S_x| \geq 4$. Then $\{c_0, x\}$ will be a dominating set with at least six common neighbors $S_x \cup \{c_{k-1}, c_1\}$ contradicting the maximality of D.

Hence we are done. □

The lemma proved above was one of the key steps to prove the theorem. Now we will improve the lower bound of $|C|$.

Lemma 6.10. *We have* $|C| \geq 6$.

Proof. For $|C| = 2, 3, 4, 5$ without loss of generality by Lemma 6.9, we may assume R_1 to be the only non-empty region. Then triangulation will force the configuration depicted in Fig 9 as a subgraph of G, where $C = \{c_o, ..., c_{k-1}\}$, $S_x = \{x_1, ..., x_{n_x}\}$ and $S_y = \{y_1, ..., y_{n_y}\}$. Without loss of generality we may assume,

$$|S_y| = n_y \geq n_x = |S_x|.$$

Then by equation (4) we have,

$$n_y = |S_y| \geq (16 - 2 - |C| - |S_x|) = 14 - |C| - |S_x|. \tag{5}$$

First of all assume $n_x = 0$. Then x is non-adjacent to y as otherwise y will dominate the whole graph. So we have the edges $c_0c_1, c_1c_2, ..., c_{k-1}c_0$ by triangulation. Then by equation 5 we have,

$$|S_y| \geq 14 - 5 = 9.$$

Now to have $rd(x, y_i) \leq 2$, every y_i must be connected to x by a rainbow 2-path with internal vertex either c_0 or c_1. Hence at least four vertices of S_y must be adjacent to either c_0 or c_1.

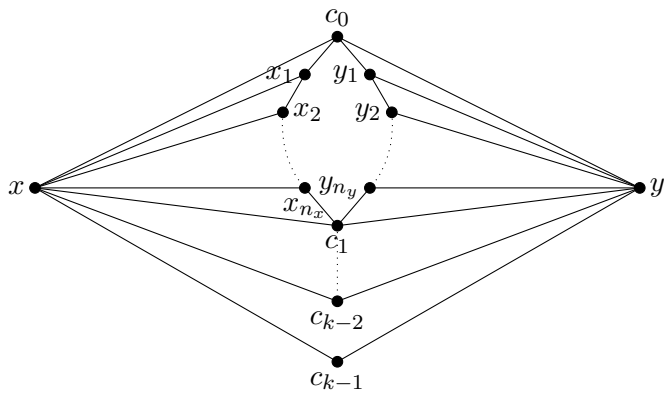


Figure 9: The only non-empty region is R_1

Note that c_0 is also adjacent to c_{k-1}, c_1 and that c_1 is also adjacent to c_0, c_2. So, the dominating set $\{c_0, y\}$ or $\{c_1, y\}$ will contradict the maximality of D. Hence $n_x \geq 1$.

Claim 1: $|C| = 5$ is not possible.

Proof of claim 1: Assume that $|C| = 5$. Then by equation 5 we have,

$$|S_y| \geq 14 - 5 - n_x = 9 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$. Now every vertex of S_y disagree with c_3 on y. They also must disagree with y on c_2 as otherwise all of them will be connected to c_2 by rainbow 2-paths with internal vertex c_1 and imply $rd(y_1, y_4) > 2$. For similar reason, the vertices of S_y must disagree with c_4 on y.

Moreover, the edge c_0c_1 does not exist because it will force each vertex of S_y to be connected to vertices of S_x by rainbow 2-paths with internal vertex either c_0 or c_1. In fact, for $n_x \geq 2$, as not all vertices of S_x can be adjacent to both c_0 and c_1, every vertex of S_y will be connected to the vertices of S_x by rainbow 2-paths with internal vertex being exactly one of c_0, c_1 implying $rd(y_1, y_4) > 2$. For $n_x = 1$, as $n_y \geq 8$, at least four vertices of S_y will be connected to the vertices of S_x by rainbow 2-paths with internal vertex being exactly one of c_0, c_1 implying $rd(y_i, y_{i+3}) > 2$ for some $i \in \{1, 2, ..., n_y\}$. Hence the edge c_0c_1 does not exist.

Also, if we have the edge y_1y_4 and without loss of generality assume the edge y_1y_3 by triangulation, then every vertex of S_x must be connected to y_2 by rainbow 2-paths with internal vertex y_1. In this case $\{y_1, y\}$ is a dominating set with at least n_y common neighbors (c_0 and $n_y - 1$ common neighbors from S_y). Hence, to avoid contradicting the maximality of D, we must have $n_y \leq 5$. Then we must also have $n_x \geq 4$. But then, as every vertex of S_x agree on y_1 and on x (as they all disagree with c_3 on x) we have $rd(x_1, x_4) > 2$, a contradiction. Hence we do not have the edge y_1y_4.

Therefore, y_1 and y_4 must be connected by a rainbow 2-path with an internal vertex x_j from S_x for some $j \in \{1, 2, .., n_x\}$. As we cannot have the edge y_1y_4, this will imply that every vertex of $S \setminus \{x_j\}$ will be adjacent to x_j to be at rainbow distance at most 2 from each other. Then we can arrive to a contradiction exactly like the case described in the paragraph above.

This proves the claim. $\Diamond$

Claim 2: $|C| = 4$ is not possible.

Proof of claim 2: Assume that $|C| = 4$. Then by equation 5 we have,

$$|S_y| \geq 14 - 4 - n_x = 10 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 5$.

Now we will show that every vertex of S_y disagree with c_2 and c_3 on y. First note that no vertex can agree with both c_2 and c_3 on y as otherwise it must be adjacent to both c_0 and c_1 which is impossible as $n_y \geq 5$. So, basically, if the claim is not true, then some vertices of S_y will agree with c_2 on y and the other vertices of S_y will agree with c_3 on y.

Also at most three vertices of S_y can agree with c_2 (or c_3) on y. So, $n_y \leq 6$. Hence, $n_x \geq 4$.

Now, three vertices agree on, say, c_2, then they will all disagree with c_2 on c_1 and every vertex (there are at least four such vertices) of S_x will disagree with those three vertices on c_1. Then, to have rainbow distance at most 2 with the vertices of S_x, the other vertices (there are at least two such vertices) of S_y should be adjacent to c_1 which is not possible as they are already connected to c_3 with rainbow 2-paths with internal vertex c_0.

The rest of the proof is similar to the proof Claim 1. Using similar arguments it is possible to show that the edge c_0c_1 does not exist, the edge y_1y_4 does not exist and it is not possible to have a rainbow 2-path with internal vertex from S_x connecting y_1 and y_4. $\Diamond$

Claim 3: $|C| = 3$ is not possible.

Proof of claim 3: Assume that $|C| = 3$. Then by equation 5 we have,

$$|S_y| \geq 14 - 3 - n_x = 11 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 6$.

First note that it is not possible to have the edge c_0c_1 as this will force some three vertices of S_y to be connected to vertices of S_x by rainbow 2-paths with internal vertex c_0 (or c_1) making $\{c_0, y\}$ (or $\{c_1, y\}$) a dominating set that contradicts the maximality of D.

For $n_x \geq 7$, there are at least 4 vertices in S_y that agree with each other on y. We need to have rainbow distance at most 2 between them. Let those four vertices be y_i, y_j, y_k, y_l with $i > j > k > l$.

Now assume we have the edge y_iy_l. Then every vertex of S_x will be adjacent to either y_i or y_l. Without loss of generality assume that every vertex of S_x is adjacent to y_i. But then $\{y_i, y\}$ will be a dominating set with at least 4 common neighbors contradicting the maximality of D. Hence $n_y \leq 6$. Therefore we must have $n_x \geq 4$.

For $n_y = 5, 6$, one can show that these cases are not possible without creating a dominating set that contradicts the maximality of D. If one just tries to have rainbow distance at most 2 between the vertices of S, the proof will follow. The proof of this part is also similar to the ones done before and, though a bit tedious, is not difficult to check. $\Diamond$

Claim 4: $|C| = 2$ is not possible.

Proof of claim 4: Assume that $|C| = 2$. Then by equation 5 we have,

$$|S_y| \geq 14 - 2 - n_x = 12 - n_x.$$

Therefore, as $n_y \geq n_x$, we have $n_y \geq 6$.

This is actually the easiest of the four claims. The case $n_y \geq 7$ can be argued as in the previous proof. For $n_y = 6$, we must have $n_x \geq 5$. If one just tries to have rainbow distance at

most 2 between the vertices of S, the proof will follow. The proof of this part is also similar to the ones done beforeand, though a bit tedious, is not difficult to check. $\diamondsuit$

This completes the proof of the lemma. $\square$

So, now we have proved that the value of $|C|$ is at least 6. This is an answer to our question "how small $|C|$ can be?". Now we will ask the question "How big $|C|$ can be?" and try to provide upper bounds for the value of $|C|$. The following lemma will help us to do so.

Lemma 6.11. *If $|C| \geq 6$, then the following holds:*

(a) We must have $|C^{\alpha\beta}| \leq 3$, $|C_t^\alpha| \leq 6$, $|C| \leq 12$. Moreover, if $|C^{\alpha\beta}| = 3$, then $(G)[C^{\alpha\beta}]$ is a rainbow 2-path.

(b) If $|C_t^\alpha| \geq 5$ (respectively $4, 3, 2, 1, 0$), then $|S_t^\alpha| \leq 0$ (respectively $1, 3, 4, 5, 6$).

Proof. **(a)** If $|C^{\alpha\beta}| \geq 4$, then there will be two vertices $u, v \in C^{\alpha\beta}$ with $d(u, v) > 2$ which is a contradiction. Hence we have the first inequality which implies the other two.

Also if $|C^{\alpha\beta}| = 3$, then the only way to connect the two non-adjacent vertices u, v of $C^{\alpha\beta}$ is to connected them with a rainbow 2-path through the other vertex (other than u, v) of $C^{\alpha\beta}$.

(b) Lemma 6.5(b) implies that if all elements of C_t^α do not belong to the set of four boundary points of any three consecutive regions (like R, R^1, R^2 in Lemma 6.6), then $|S_t^\alpha| = 0$. Hence we have $|C_t^\alpha| \geq 5$ implies $|S_t^\alpha| \leq 0$.

By Lemma 6.6, if all the elements of C_t^α belong to the set of four boundary points $c^1, c^2, \overline{c^1}, \overline{c^2}$ of three consecutive regions R, R^1, R^2 (like in Lemma 6.6) and contains both $\overline{c^1}, \overline{c^2}$, then $|S_t^\alpha| \leq 1$. Also $S_t^\alpha \subseteq R$ by Lemma 6.6. Hence we have,

$$|C_t^\alpha| \geq 4 \text{ implies } |S_t^\alpha| \leq 1.$$

Now assume that all the elements of C_t^α belongs to the set of three boundary points $c^1, c^2, \overline{c^1}$ of two adjacent regions R, R^1 (like in Lemma 6.6) and contains both $\overline{c^1}, c^2$. Then by Lemma 6.5, $v \in S_t^\alpha$ implies v is in R or R^1.

Now if both $S_t^\alpha \cap R$ and $S_t^\alpha \cap R^1$ are non-empty, then each vertex of $(S_t^\alpha \cap R) \cup \{c^2\}$ disagrees with each vertex of $(S_t^\alpha \cap R^1) \cup \{\overline{c^1}\}$ on c^1 (by Lemma 6.6).

Hence by Lemma 6.7 we have,

$$|(S_t^\alpha \cap R) \cup \{\overline{c^1}\}|, |(S_t^\alpha \cap R^1) \cup \{c^2\}| \leq 3.$$

This clearly implies,

$$|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| \leq 2 \text{ and } |S_t^\alpha| \leq 4.$$

Now suppose we have $|S_t^\alpha| = 4$ and hence also $|S_t^\alpha \cap R|, |S_t^\alpha \cap R^1| = 2$. Then $S_{t'} = \emptyset$ as the only way for a vertex of $S_{t'}$ to have rainbow distance at most 2 with every vertex of S_t is by being connected by a rainbow 2-path with internal vertex c_1, which is impossible as the vertices of $S_t^\alpha \cap R$ disagree with the vertices $S_t^\alpha \cap R^1$ on c_1.

In fact, for the same reason, it is impossible to have rainbow distance at most 2 between all the vertices of S_t and t' unless we have the edge tt' (that is the edge xy). But then the edge tt' makes t a vertex that dominates the whole graph contradicting the domination number of the

graph being 2. Therefore, it is not possible to have $|S_t^\alpha| = 4$. Hence we have $|S_t^\alpha| \leq 3$ in this case.

Also if one of $S_t^\alpha \cap R$ and $S_t^\alpha \cap R^1$ is empty then we must have $|S_t^\alpha| \leq 3$ by Lemma 6.6 and 6.7.

Hence we have

$$|C_t^\alpha| \geq 3 \text{ implies } |S_t^\alpha| \leq 3.$$

Let $R, R^1, R^2, c^1, c^2, \overline{c^1}, \overline{c^2}$ be like in Lemma 6.6 and assume $C_t^\alpha = \{c^1, c^2\}$. By Lemma 6.5, $v \in S_t^\alpha$ implies v is in R, R^1 or R^2 and also that both $S_t^\alpha \cap R^1$ and $S_t^\alpha \cap R^2$ can not be non-empty. Hence, without loss of generality, assume $S_t^\alpha \cap R^2 = \emptyset$.

Then by Lemma 6.6, vertices of $S_t^\alpha \cap R^1$ disagree with vertices of $(S_t^\alpha \cap R) \cup \{c^2\}$ on c^1. Hence by Lemma 6.7 we have,

$$|S_t^\alpha \cap R^1|, |(S_t^\alpha \cap R) \cup \{c^2\}| \leq 3.$$

This implies $|S_t^\alpha| \leq 5$.

Now if $S_t^\alpha \cap R^1 = \emptyset$, then we have $S_t^\alpha = S_t^\alpha \cap R$. Let $|S_t^\alpha \cap R| \geq 6$. Now consider the induced graph $(O) = (G)[(S \cap R) \cup \{c^1, c^2\}]$. In this graph the vertices of $(S_t^\alpha \cap R) \cup \{c^1, c^2\}$ are pairwise at rainbow distance at most 2. Hence $\omega_{rs}((O)) \geq 8$. But this is a contradiction as (O) is an outerplanar graph and every outerplanar graph has a signed relative clique number at most 7 (see Theorem 2.9 for details). Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 5.$$

Now suppose we have $|S_t^\alpha| = 5$. Then we must have $S_{t'} = \emptyset$ as otherwise it is not possible to have rainbow distance at most 2 between the vertices of S.

We also do not have the edge xy as it will contradict the domination number of the graph being 2 (t will dominate the graph). So, by triangulation we have the edges c^1c^2 and $c^{\overline{1}}c^1$. So, each vertex of S_t must be connected to t' with a rainbow 2-path with internal vertices from $\{c^{\overline{1}}, c^1, c^2\}$. But then it will not be possible to have rainbow distance at most 2 between the five vertices of S_t.

Hence,

$$|C_t^\alpha| \geq 2 \text{ implies } |S_t^\alpha| \leq 4.$$

In general S_t^α is contained in two distinct adjacent regions by Lemma 6.5. Without loss of generality assume $S_t^\alpha \subseteq R_1 \cup R_2$. If both $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ are non-empty, then by Lemma 6.6 we know that vertices of $S_t^\alpha \cap R_1$ disagree with vertices of $S_t^\alpha \cap R_2$ on c_1. Hence $|S_t^\alpha \cap R_1|, |S_t^\alpha \cap R_2| \leq 3$ which implies $|S_t^\alpha| \leq 6$.

Now assume only one of the two sets $S_t^\alpha \cap R_1$ and $S_t^\alpha \cap R_2$ is non-empty. Without loss of generality assume $S_t^\alpha \cap R_1 \neq \emptyset$. If $c_0, c_1 \notin C_t^\alpha$ and $|C_t^\alpha| = 1$, then we have $|S_t^\alpha \cap R_1| \leq 3$ by Lemma 6.6 and 6.7.

Otherwise, in the induced outerplanar graph $(O) = (G)[(S \cap R_1) \cup \{c_1, c_2\}]$ vertices of $S_t^\alpha \cup (C_t^\alpha \cap \{c_1, c_2\})$ are pairwise at rainbow distance at most 2. Therefore

$$|S_t^\alpha \cup (C_t^\alpha \cap \{c_1, c_2\})| \leq \chi_s((O)) \leq 7$$

which implies

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 6 \text{ (respectively 7).}$$

Now, when both the equalities hold, we must have $S_{t'} = \emptyset$ as otherwise $G[C_t^\alpha \cup S_t \cup S_{t'}]$ will contain a signed outerplanar graph with signed chromatic number at least 8, which is not possible, in order to have all the vertices of S at rainbow distance at most 2.

Now, $S_{t'} = \emptyset$ will imply that the edge xy is not there as otherwise t will dominate the whole graph. Hence, each vertex of S_t must be connected to t' by a rainbow 2-path with internal vertex c_i for some $i \in \{0, 1, 2\}$. But this will force $|S_t| \leq 5$ as otherwise the vertices of S_t will no longer be at rainbow distance at most 2 from each other.

Hence,

$$|C_t^\alpha| \geq 1 \text{ (respectively 0) implies } |S_t^\alpha| \leq 5 \text{ (respectively 6).}$$

Hence we are done. □

Now we will prove that the value of $|C|$ can be at most 5 which contradicts our previously proven lower bound of $|C|$.

Lemma 6.12. *We have* $|C| \leq 5$.

Proof. Without loss of generality we can suppose $|C_x^\alpha| \geq |C_y^\beta| \geq |C_y^{\overline{\beta}}| \geq |C_x^{\overline{\alpha}}|$ (the last inequality is forced). We know that $|C| \leq 12$ and $|C_x^\alpha| \leq 6$ (Lemma 6.11(a)). So it is enough to show that $|S| \leq 13 - |C|$ for all possible values of $(|C|, |C_x^\alpha|, |C_y^\beta|)$ since it contradicts (4).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (12, 6, 6)$, $(11, 6, 6)$, $(10, 6, 6)$, $(10, 6, 5)$, $(10, 5, 5)$, $(9, 5, 5)$, $(8, 4, 4)$ we have $|S| \leq 13 - |C|$ using Lemma 6.11(b).

For $(|C|, |C_x^\alpha|, |C_y^\beta|) = (8, 6, 6)$, $(7, 6, 6)$, $(7, 6, 5)$,$(6, 6, 6)$, $(6, 6, 5)$, $(6, 6, 4)$, $(6, 5, 5)$ we are forced to have,

$$|C^{\alpha\beta}| > 3.$$

This is a contradiction by Lemma 6.11(a).

So, $(|C|, |C_x^\alpha|, |C_y^\beta|) \neq (12, 6, 6)$, $(11, 6, 6)$, $(10, 6, 6)$, $(10, 6, 5)$, $(10, 5, 5)$, $(9, 5, 5)$, $(8, 4, 4)$, $(8, 6, 6)$, $(7, 6, 6)$, $(7, 6, 5)$,$(6, 6, 6)$, $(6, 6, 5)$, $(6, 6, 4)$, $(6, 5, 5)$.

We will be done if we prove that $(|C|, |C_x^\alpha|, |C_y^\beta|)$ cannot take the other possible values also. That leaves us checking a lot of cases. We will check just a few cases and observe that the other cases can be checked using similar logic.

Case 1: Assume $(|C|, |C_x^\alpha|, |C_y^\beta|) = (9, 6, 6)$.

Then we are forced to have, $|C^{\alpha\beta}| = |C^{\alpha\overline{\beta}}| = |C^{\overline{\alpha}\beta}| = 3$ in order to satisfy the first inequality of Lemma 6.11(a). So $(G)[C^{\alpha\beta}]$, $(G)[C^{\alpha\overline{\beta}}]$ and $(G)[C^{\overline{\alpha}\beta}]$ are rainbow 2-paths by Lemma 6.11(a). Without loss of generality we can assume $C^{\alpha\overline{\beta}} = \{c_0, c_1, c_2\}$ and $C^{\overline{\alpha}\beta} = \{c_3, c_4, c_5\}$. Hence by Lemma 6.5 we have $u \in R_1 \cup R_2$ and $v \in R_4 \cup R_5$ for any $(u, v) \in S_y^{\overline{\beta}} \times S_x^{\overline{\alpha}}$. Hence by Lemma 6.5, either $S_y^{\overline{\beta}}$ or $S_x^{\overline{\alpha}}$ is empty. Without loss of generality assume $S_y^{\overline{\beta}} = \emptyset$. Therefore we have, $|S| = |S_x| = |S_x^{\overline{\alpha}}| \leq 3$ (by Lemma 6.11(b)). So this case is not possible.

Case 2: Assume $(|C|, |C_x^\alpha|, |C_y^\beta|) = (7, 6, 4)$.

So, without loss of generality, we can assume that $(G)[C^{\alpha\beta}]$ and $(G)[C^{\alpha\overline{\beta}}]$ are rainbow 2-paths and, $C^{\alpha\beta} = \{c_0, c_1, c_2\}$, $C^{\alpha\overline{\beta}} = \{c_3, c_4, c_5\}$ and $C^{\overline{\alpha}\beta} = \{c_6\}$.

By Lemma 6.11 we have $|S_x| \leq 5$ and $|S_y| \leq 3 + 1 = 4$. So we are done if either $S_x = \emptyset$ or $S_y = \emptyset$.

So assume both S_x and S_y are non-empty. First assume that $S_y^\beta \neq \emptyset$. Then by Lemma 6.5 we have $S_y^\beta \subseteq R_5$, $S_x^{\overline{\alpha}} \subseteq R_5 \cup R_6$ and hence $S_y^{\overline{\beta}} = \emptyset$. By Lemma 6.6, vertices of S_y^β and vertices of $S_x^{\overline{\alpha}} \cap R_5$ must disagree with c_6 on c_5 while disagreeing with each other on c_5, which is not possible. Hence, $S_x^{\overline{\alpha}} \cap R_5 = \emptyset$. Also $|S_x^{\overline{\alpha}} \cap R_6| \leq 3$ as they all disagree on c_5 with the vertex of S_y^β. So $|S| \leq 4$ when $S_y^\beta \neq \emptyset$.

Now assume $S_y^\beta = \emptyset$ hence $S_y^{\overline{\beta}} \neq \emptyset$. Then by Lemma 6.5 we have $S_y^{\overline{\beta}} \subseteq R_1 \cup R_2$, $S_x^{\overline{\alpha}} \subseteq R_0 \cup R_1$ and hence $S_y^\beta = \emptyset$. Assume $S_y^{\overline{\beta}} \cap R_2 = \emptyset$ as otherwise vertices of $S_x^{\overline{\alpha}}$ will be adjacent to both c_0 and c_1 (to be connected to c_6 and vertices of $S_y^{\overline{\beta}} \cap R_2$ by a rainbow 2-path) implying $|S_x^{\overline{\alpha}}| \leq 1$ implying $|S| \leq 5$. If $S_x^{\overline{\alpha}} \cap R_0 \neq \emptyset$, then $|S_y^{\overline{\beta}} \cap R_1| = 1$, $|S_y^{\overline{\alpha}} \cap R_1| \leq 1$ and $|S_y^{\overline{\alpha}} \cap R_0| \leq 3$ by Lemma 6.6 and hence $|S| \leq 5$. If $S_x^{\overline{\alpha}} \cap R_0 = \emptyset$ then we have $|S_y^{\overline{\beta}} \cap R_1| \leq 2$, $|S_y^{\overline{\alpha}} \cap R_1| \leq 3$ and hence $|S| \leq 5$. So this case is not possible.

In a similar way one can handle the other cases. □

This proves Lemma 6.4 and implies Theorem 6.3(a).

7 Cliques for n-edge-colored graphs

An *n-edge-colored graph* $(G, E_1, E_2, ..., E_n)$ is a graph with a partition of its edges into k disjoint parts $E_1, E_2, ..., E_n$. The graph G is called the *underlying* graph of $(G, E_1, E_2, ..., E_n)$. For $uv \in E_i$ we say that the edge uv is *colored* with i for $i \in \{1, 2, ..., n\}$. When there is no chance of confusion we can denote an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ by (G). Two incident edges uv and vw with different colors are together called a *rainbow 2-path* with *terminal* vertices u, w and *internal* vertex v.

For an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ the set of neighbors of a vertex v is denoted by $N(v)$ and the degree of a vertex v is denoted by $d(v)$. Also $N_i(v)$ will denote the set of neighbors of v with which v is adjacent by an edge from E_i for all $i \in \{1, 2, ..., n\}$.

An *n-edge-colored k-coloring* [1] of an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ is a mapping ϕ from the vertex set $V((G, E_1, E_2, ..., E_n))$ to the set $\{1, 2, ..., k\}$ such that,

- (i) $\phi(u) \neq \phi(v)$ whenever u and v are adjacent and
- (ii) if $uv \in E_i$ and $wx \in E_j$ for $i \neq j$ where $i, j \in \{1, 2, ..., n\}$, then $\phi(u) = \phi(w)$ implies $\phi(v) \neq \phi(x)$.

The *n-edge-colored chromatic number* $\chi_n((G, E_1, E_2, ..., E_n))$ of an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ is the smallest integer k for which $(G, E_1, E_2, ..., E_n)$ has an n-edge-colored k-coloring. The n-edge-colored chromatic number $\chi_n(G)$ of an undirected graph G is the maximum of the n-edge-colored chromatic numbers of all the n-edge-colored graphs with underlying graph G. The n-edge-colored chromatic number $\chi_n(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the n-edge-colored chromatic numbers of the graphs from the family $\mathcal{F}$.

An *n-edge-colored clique* is an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ for which $\chi_n((G, E_1, E_2, ..., E_n)) = |V((G, E_1, E_2, ..., E_n))|$. The *$n$-edge-colored absolute clique number* $\omega_{an}((G, E_1, E_2, ...,$

E_n)) of an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ is the maximum order of an n-edge-colored clique contained in $(G, E_1, E_2, ..., E_n)$ as a subgraph.

Proposition 2.1 can be easily extended for n-edge-colored graphs.

Proposition 7.1. *An n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ is an n-edge-colored clique if and only if its each pair of non-adjacent vertices are connected by a rainbow 2-path.*

A *relative clique* of an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ is a set $R \subseteq V((G, E_1, ..., E_n))$ of vertices such that any two vertices from R are either adjacent or connected by a rainbow 2-path. The *relative clique number* $\omega_{rn}((G, E_1, E_2, ..., E_n))$ of an n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ is the maximum order of an n-edge-colored relative clique of $(G, E_1, E_2, ..., E_n)$.

Note that, in particular, 2-edge-colored graphs are nothing but signed graphs while the notion of coloring, chromatic number, absolute and relative clique number remains consistent with the restriction.

Also note that an extension of Proposition 2.8 follows directly from the definitions.

Proposition 7.2. *For any n-edge-colored graph $(G, E_1, E_2, ..., E_n)$ we have*

$$\omega_{an}((G, E_1, E_2, ..., E_n)) \leq \omega_{rn}((G, E_1, E_2, ..., E_n)) \leq \chi_n((G, E_1, E_2, ..., E_n)).$$

Our objective here is to study n-edge-colored cliques for planar graphs. We want to extend Theorem 6.3(a) for n-edge-colored graphs. Even though we failed to provide tight bound, we managed to provide close bounds for n-edge-colored absolute clique number for planar graphs while providing tight bounds for n-edge-colored relative and absolute clique number for outerplanar graphs.

Theorem 7.3. *For the family $\mathcal{O}_3$ of outerplanar graphs we have $\omega_{an}(\mathcal{O}_3) = \omega_{rn}(\mathcal{O}_3) = 3n + 1$ for $n \geq 2$.*

Proof. First we will show that $\omega_{an}(\mathcal{O}_3) \geq 3n + 1$ by explicitly constructing an n-edge-coloured outerplanar absolute clique, $(H, E_1, ..., E_n)$, with $3n + 1$ vertices as follows:

- the set of vertices $V(H) = \{x\} \cup \{v_{ij} | 1 \leq i \leq n, 1 \leq j \leq 3\}$,
- the set of edges $E(H) = \{xv_{ij}, v_{i1}v_{i2}, v_{i2}v_{i3} | 1 \leq i \leq n, 1 \leq j \leq 3\}$,
- the edges $v_{i1}v_{i2} \in E_1$ for all $i \in \{1, 2, ..., n\}$,
- the edges $v_{i2}v_{i3} \in E_2$ for all $i \in \{1, 2, ..., n\}$,
- the edges $xv_{ij} \in E_i$ for all $i \in \{1, 2, ..., n\}$.

It is easy to check that the graph $(H, E_1, ..., E_n)$ is indeed an n-edge-coloured outerplanar absolute clique with $3n + 1$ vertices.

Now to prove the upper bound let $(G, E_1, E_2, ..., E_n)$ be a minimal (with respect to number of vertices) n-edge-colored outerplanar graph with relative clique number $\omega_{rn}(\mathcal{O}_3)$. Moreover, without loss of generality, we can assume that (G) is maximal (that is, we cannot add any more edges keeping the graph outerplanar). We can assume this because adding more edges will not affect the relative clique number as it is already equal to $\omega_{rn}(\mathcal{O}_3)$. Let R be a relative clique of cardinality $\omega_{rn}(\mathcal{O}_3)$ of (G). Let $S = V(G) \setminus R$.

As (G) is maximal outerplanar $d(v) \geq 2$ for all $v \in V(G)$. Then, as (G) is outerplanar, there exists a vertex $u_0 \in V(G)$ with $d(u_0) = 2$. Fix an outerplanar embedding of (G) with the outer (facial) cycle having vertices $u_1, u_1, u_2, ..., u_R$ of R embedded in a clockwise manner on the cycle.

Note that if $u_1 \in S$ then we can delete u_1 and connect the neighbors of u_1 with an edge (if they are not already adjacent) to obtain a graph with same relative clique number contradicting the minimality of (G). Hence $u_1 \in R$ with neighbors a and b (say).

Then every vertex of $R \setminus \{u_1, a, b\}$ is connected to u_1 through internal vertex a or b. Now let $u_i, u_j \in (R \setminus \{u_1, a, b\}) \cap (N_k(a) \cup N_k(b))$ be two vertices with $i \neq j$ for some $k \in \{1, 2, ..., n\}$. Then it is easy to notice that $|i-j| \leq 2$ as otherwise they can be neither adjacent nor connected by a rainbow 2-path in (G). For the same reason if $u_i \in N_k(a)$ and $u_j \in N_k(b)$ for $i \neq j$ then we must have $|i-j| \leq 2$ for any $k \in \{1, 2, ..., n\}$.

From this we can conclude that $|(R \setminus \{u_1\}) \cap (N_k(a) \cup N_k(b))| \leq 3$ for any $k \in \{1, 2, ..., n\}$. Now as a and b are adjacent a and b are also contained in the set $\cup_{i=1}^{n}(R \setminus \{u_1\}) \cap (N_k(a) \cup N_k(b))$. Hence we have

$$\begin{aligned} R &\leq |\cup_{k=1}^{n} (R \setminus \{u_1\}) \cap (N_k(a) \cup N_k(b))| + |\{u_1\}| \\ &= \Sigma_{k=1}^{n} |(R \setminus \{u_1\}) \cap (N_k(a) \cup N_k(b))| + 1 \\ &\leq 3n + 1. \end{aligned}$$

This completes the proof. □

Now we will use the above theorem to prove the following result for the class planar graphs.

Theorem 7.4. *For the family $\mathcal{P}_3$ of planar graphs we have $3n^2+n+1 \leq \omega_{an}(\mathcal{P}_3) \leq 9n^2+2n+2$ for $n \geq 3$.*

Proof. First we will show that $\omega_{an}(\mathcal{P}_3) \geq 3n^2 + n + 1$ by explicitly constructing an n-edge-coloured planar absolute clique, $(H*, E_1, ..., E_n)$, with $3n^2 + n + 1$ vertices. For this, recall the example of n-edge-colored outerplanar absolute clique $(H, E_1, E_2, ..., E_n)$ from the previous proof (proof of Theorem 7.3). Now put the adjacencies of (G) in a way so that the following hold:

- the set of vertices $V(H*) = \{x\} \cup \{v_{ij} | 1 \leq i \leq n, 1 \leq j \leq 3n+1\}$,
- the induced subgraph $(H*)[\{v_{i1}, v_{i2}, ..., v_{i(3n+1)}]$ of (G) is isomorphic to (H),
- the edges $xv_{ij} \in E_i$ for all $i \in \{1, 2, ..., n\}$ and for all $j \in \{1, 2, ..., 3n+1\}$.

It is easy to check that the graph $(G, E_1, ..., E_n)$ is indeed an n-edge-coloured planar absolute clique with $3n^2 + n + 1$ vertices.

Now to prove the upper bound first notice that any n-edge-colored absolute clique has diameter at most 2. Let $(G, E_1, E_2, ..., E_n)$ be an n-edge-colored planar absolute clique. Assume that (G) is triangulated. As deleting edges do not increase the absolute clique number, it is enough to prove this result for triangulated (G). Now by Theorem 7.3 we know that (G) is dominated by at most two vertices.

First assume that (G) is dominated by a single vertex x. Then the graph $N_i(x)$ is an n-edge-colored relative clique in the induced graph $(G)[N(x)]$ (the graph obtained by deleting x from (G)) for each $i \in \{1, 2, ..., n\}$. But the graph $(G)[N(x)]$ is an outerplanar graph. So by Theorem 7.3 we have $|N_i(x)| \leq 3n + 1$. Hence if (G) is dominated by one vertex then

$$\begin{aligned}\omega_{an}((G)) &\leq |\cup_{k=1}^{n} N_i(x)| + |\{x\}| \\ &= \Sigma_{i=1}^{n}|N_i(x)| + 1 \\ &\leq 3n^2 + n + 1.\end{aligned}$$

Now let (G) is dominated by two vertices x and y. Now we fix some notations to prove the rest of this result.

- $C = N(x) \cap N(y)$ and $C_{ij} = N_i(x) \cap N_j(y)$ for all $i, j \in \{1, 2, ..., n\}$,
- $S_{x_i} = N_i(x) \setminus C$, $S_{y_i} = N_i(y) \setminus C$ for all $i, j \in \{1, 2, ..., n\}$,
- $S_x = N(x) \setminus C$, $S_y = N(y) \setminus C$.

Note that, for $|C| \geq 6$ we must have $|C_{ij}| \leq 3$ for all $i, j \in \{1, 2, ..., n\}$ as otherwise it is not possible to have pairwise distance at most two between the vertices of C_{ij} keeping the graph planar. So, we can conclude that

$$|C| \leq |\cup_{i,j=1}^{n} C_{ij}| = \Sigma_{i=1}^{n}\Sigma_{j=1}^{n}|C_{ij}| \leq 3n^2.$$

Now note that the graph obtained by deleting the vertices x and y from (G) is an outerplanar graph. Moreover, note that S_{x_i} is a relative clique in that outerplanar graph. Hence by Theorem 7.3 we have $|S_{x_i}| \leq 3n + 1$. Hence

$$|S_{x_i}| \leq |\cup_{i=1}^{n} S_{x_i}| = \Sigma_{i=1}^{n}|S_{x_i}| \leq 3n^2 + n.$$

Similarly, we have $|S_{x_i}| \leq 3n^2 + n$. Hence we can provide an upper bound of the number of vertices in (G).

$$|(G)| = |C| + |S_x| + |S_y| + |\{x, y\}| \leq 9n^2 + 2n + 2.$$

Hence we are done. □

Finally, we would like to make the following conjecture regarding the n-edge-colored absolute clique number of planar graphs.

Conjecture 7.5. *For the family $\mathcal{P}_3$ of planar graphs $\omega_{an}(\mathcal{P}_3) = 3n^2 + n + 1$ for $n \geq 2$.*

We have already shown that the conjecture is true for $n = 2$ (see Theorem 6.3(a)). We think that using similar techniques the conjecture can be proved for some smaller values of n and the bound provided in the above theorem can be improved. However, to prove the conjecture for all values of n might require a different technique.

8 Conclusion

In this article we brought together a few different concepts, namely, signed graphs, switchable signed graphs, oriented graphs, n-edge-colored graphs, rainbow connection (implicitly), while our focus mainly remained on absolute clique number (introduced in this article) of signed and switchable signed graphs. We also introduced relative clique and used it as a tool to prove our main result. We extended our results to n-edge-colored relative and absolute cliques (introduced in this article).

We propose the following questions and directions for further contributions in this topic:

(1) Let $f(n)$ (or $g(n)$) be the minimum number of edges in an sclique (or an [s]-clique) of order n. Is f (or g) a (strictly) increasing function?

(2) Study the absolute and relative clique numbers for different class of graphs.

(3) Is there some interesting class of graphs for which the difference of oriented and signed chromatic number is not arbitrarily large?

(4) One can make a similar study for (m, n)-mixed graphs [21] (that is, graphs with edges with m different colors and arcs with n different colors) and generalize our results.

We also made a conjecture on what the absolute clique number for n-edge colored graphs can be and solved it for $n = 2$. The case solved in this article is analogous to the conjecture made by Klostermeyer and MacGillivray [15] regarding oriented cliques which was settled positively [29] recently.

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