Math in the Cabin - Shape Analysis Workshop in Bad Gastein

Martin Bauer, Martins Bruveris, Philipp Harms, Boris Khesin, Stephen Marsland, Peter Michor, Klas Modin, Olaf Müller, Xavier Pennec, Stefan Sommer, et al.

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July 16 – July 22, 2014, Austria

The workshop "Math in the cabin" took place in Bad Gastein, in the period July 16 – July 22, 2014. The aim of the week was to bring together a group of researchers with diverse backgrounds — ranging from differential geometry to applied medical image analysis — to discuss questions of common interest, that can be vaguely summarized under the heading "shape analysis". These Proceedings contain a summary of selected discussions, that were held during this week.

Contents

1 General remarks

The aim of the week was to bring together a group of researchers with diverse backgrounds — ranging from differential geometry to applied medical image analysis — to discuss questions of common interest, that can be vaguely summarized under the heading "shape analysis".

Participants

1. Martin Bauer (University of Vienna)
2. Martins Bruveris (Brunel University London)
3. Philipp Harms (ETH Zurich)
4. Boris Khesin (University of Toronto)
5. Stephen Marsland (Massey University)
6. Peter Michor (University of Vienna)
7. Klas Modin (Chalmers University)
8. Olaf Müller (University of Regensburg)
9. Xavier Pennec (INRIA Sophia Antipolis)
10. Stefan Sommer (University of Copenhagen)
11. François-Xavier Vialard (University Paris-Dauphine)

Schedule

We kept the formal schedule of the week to a minimum, so that the participants would have time to talk and work with each other. Every day we had two or sometimes three talks or discussion sessions led by a participant. These talks or the resulting discussions are summarized by each participant in the second half of this report.

Monday, July 16

Arrival in Bad Gastein

Tuesday, July 17

Stephen Marsland, Torsion in image matching
Xavier Pennec, Statistics on Lie groups

Wednesday, July 18

Martin Bauer, Metrics on densities and Kähler potentials
Martins Bruveris, Completeness properties

Thursday, July 19

Olaf Müller, Volume-preserving embeddings
Peter Michor, Uniqueness of the Fisher–Rao metrics

Friday, July 20

Boris Khesin, The pentagram map
Klas Modin, Information geometry
François-Xavier Vialard, Shape splines

Saturday, July 21

Philipp Harms, Stochastics and shapes
Stefan Sommer, Flows with defects
Boris Khesin, Vortex filaments and the Hasimoto transform

Sunday, July 22

Departure from Bad Gastein

Support

This workshop is partially supported by the FWF-project P24625-N25.

2 Information geometry and the Fisher–Rao metric on the space of probability distributions

Klas Modin

Start with a probability distribution function $x \mapsto p(x, \theta)$ depending on parameters $\theta = (\theta^1, \ldots, \theta^k)$ (for example, a Gaussian distribution depending on mean and variance). Fisher's information matrix is a way of measuring the information about θ carried by a random variable with probability distribution $p(x, \theta)$; it is the expectation of the second moments of the *score*:

$$\mathcal{I}_{ij}(\theta) = E\left[\left(\frac{\partial}{\partial\theta^i}\ln p(\cdot,\theta)\right)\left(\frac{\partial}{\partial\theta^j}\ln p(\cdot,\theta)\right)\right]. \tag{2.1}$$

For example, if $p(x, \theta)$ is Gaussian with mean $\theta = \mu$, then the Fisher information is large if the variance is small and vice versa. Fisher's information matrix is the Hessian of the relative entropy, or Kullback–Leibler divergence, between two probability distributions within the family $p(x, \theta)$.

Rao [4] interpreted $\mathcal{I}_{ij}(\theta)$ as a Riemannian metric on the "manifold" of probability distributions parameterised by θ. As presented up until now, it appears curious that this *Fisher–Rao metric* is invariant under changes of variables $\theta \mapsto \theta'$. Having said that, the invariance properties become perfectly transparent when turning to the infinite dimensional space of all probability distributions. I will now explain this viewpoint, which, to my knowledge, was first pursued by Friedrich [1].

Let M be an n dimensional orientiable manifold (no Riemannian structure is needed). The space of probability densities on M is given by

$$\mathrm{Dens}(M) = \{\mu \in \Omega^n(M) \, : \, \int_M \mu = 1, \mu > 0\}, \tag{2.2}$$

where $\Omega^n(M)$ is the space of smooth n–forms and positiveness is taken with respect to a fixed orientation. $\mathrm{Dens}(M)$ is a manifold in the Fréchet topology of smooth functions, as explained, for example, by Hamilton [2]. More specifically, $\mathrm{Dens}(M)$ is an open subset of the affine space of n–forms with total volume 1. The tangent space at $\mu \in \mathrm{Dens}(M)$ is given by

$$T_\mu \mathrm{Dens}(M) = \{\alpha \in \Omega^n(M) \, : \, \int_M \alpha = 0\}.$$

If $\alpha \in T_\mu\mathrm{Dens}(M)$, then $\alpha/\mu \in C^\infty(M)$ is the smooth function defined by $\alpha = (\alpha/\mu)\mu$.

Definition. The *Fisher–Rao metric* is the Riemannian metric on $\mathrm{Dens}(M)$ given by

$$G^F_\mu(\alpha, \beta) = \frac{1}{4}\int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\,\mu. \tag{2.3}$$

Let $\mathrm{Diff}(M)$ denote the group of diffeomorphisms on M. Since G^F is defined without reference to a prefered Riemannian metric or volume form, it is invariant under $\mathrm{Diff}(M)$

acting by pullback $(\varphi, \mu) \mapsto \varphi^*\mu$ (the tangent-lifted action is also given by pullback). Explicitly,

$$G^F_\mu(\alpha, \beta) = \frac{1}{4}\int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu = \frac{1}{4}\int_M \varphi^*\left(\frac{\alpha}{\mu}\frac{\beta}{\mu}\mu\right) = G^F_{\varphi^*\mu}(\varphi^*\alpha, \varphi^*\beta).$$

We conjecture that G^F is the *only* metric on $\mathrm{Dens}(M)$ with this invariance property.

Fix a volume form $\mu_0 \in \mathrm{Dens}(M)$ and let $L^2(M)$ be the Hilbert space of real functions on M, square integrable with respect to μ_0. To reveal the geometry of $(\mathrm{Dens}(M), G^F)$, define $W\colon \mathrm{Dens}(M) \to L^2(M)$ by

$$W(\mu) = \sqrt{\frac{\mu}{\mu_0}}.$$

Notice that

$$\|W(\mu)\|_{L^2} = \int_M W(\mu)^2\mu_0 = \int_M \frac{\mu}{\mu_0}\mu_0 = \int_M \mu = 1.$$

Indeed, the image of W is an open subset of the unitary L^2–sphere, denoted $S^{L^2(M)}$. The tangent derivative of W is given by

$$T_\mu W \cdot \alpha = \frac{1}{2}\frac{\alpha}{\mu_0}\left(\frac{\mu}{\mu_0}\right)^{-1/2} = \frac{1}{2}\frac{\alpha}{\mu_0}\sqrt{\frac{\mu_0}{\mu}}.$$

Take a curve $\mu(t) \in \mathrm{Dens}(M)$ and define $f(t) \coloneqq W(\mu(t))$. Then $f(t) \in S^{L^2(M)}$ and

$$\begin{aligned}\|\dot f\|^2_{L^2} = \|T_\mu W \cdot \dot\mu\|^2_{L^2} &= \int_M \left(\frac{1}{2}\frac{\dot\mu}{\mu_0}\sqrt{\frac{\mu_0}{\mu}}\right)^2 \mu_0 \\ &= \frac{1}{4}\int_M \frac{\dot\mu}{\mu_0}\frac{\dot\mu}{\mu_0}\frac{\mu_0}{\mu}\mu_0 = \frac{1}{4}\int_M \frac{\dot\mu}{\mu}\frac{\dot\mu}{\mu}\mu = G^F_\mu(\dot\mu, \dot\mu).\end{aligned}$$

This calculation shows that W is a Riemannian isometry between $(\mathrm{Dens}(M), G^F)$ and an open subset of the unit sphere in $L^2(M)$. In particular, the sectional curvature of $(\mathrm{Dens}(M), G^F)$ is constant and positive. The geodesics on $\mathrm{Dens}(M)$ are therefore explicit, corresponding to great circles on $S^{L^2(M)}$ (for explicit formulas see, for example, Khesin, Lenells, Misiołek, et al. [3]).

I will now disclose the relation between the Fisher–Rao metric (2.3) and Fisher's information matrix (2.1).

Definition. A *statistical manifold* is a finite dimensional submanifold of $\mathrm{Dens}(M)$.

If Γ is a statistical manifold of dimension k, then the Fisher–Rao metric on $\mathrm{Dens}(M)$ induces a Riemannian metric on Γ, also called Fisher–Rao. Here is the main claim: in local coordinates, the matrix for the induced Fisher–Rao metric is, up to scaling, Fisher's information matrix. Indeed, let $(\theta^1, \ldots, \theta^k) \mapsto p(\cdot, \theta)\mu_0 \in \mathrm{Dens}(M)$ be a local parameterisation of Γ. The expression $g_{ij}(\theta)$ for the Fisher–Rao metric expressed in coordinates $\theta^1, \ldots, \theta^k$ is then

$$\begin{aligned}
g_{ij}(\theta) &= G^F_{p(\cdot,\theta)\mu_0}\left(\frac{\partial p(\cdot,\theta)}{\partial\theta^i}\mu_0, \frac{\partial p(\cdot,\theta)}{\partial\theta^j}\mu_0\right) \\
&= \frac{1}{4}\int_M \frac{\frac{\partial}{\partial\theta^i}p(\cdot,\theta)}{p(\cdot,\theta)}\frac{\frac{\partial}{\partial\theta^j}p(\cdot,\theta)}{p(\cdot,\theta)}p(\cdot,\theta)\mu_0 \\
&= \frac{1}{4}\int_M \left(\frac{\partial}{\partial\theta^i}\ln p(\cdot,\theta)\right)\left(\frac{\partial}{\partial\theta^j}\ln p(\cdot,\theta)\right)p(\cdot,\theta)\mu_0 \\
&= \frac{1}{4}E\left[\left(\frac{\partial}{\partial\theta^i}\ln p(\cdot,\theta)\right)\left(\frac{\partial}{\partial\theta^j}\ln p(\cdot,\theta)\right)\right] = \frac{1}{4}\mathcal{I}_{ij}(\theta).
\end{aligned}$$

From this construction, the previously curious observation that the Fisher–Rao metric is invariant under reparameterisations $\theta \mapsto \theta'$ is now obvious: Let

$$\mathrm{Diff}_\Gamma(M) = \{\varphi \in \mathrm{Diff}(M); \varphi^*\mu \in \Gamma \quad \forall\, \mu \in \Gamma\}.$$

Then $\mathrm{Diff}_\Gamma(M)$ is a subgroup of $\mathrm{Diff}(M)$, so the Fisher–Rao metric on Γ is invariant under $\mathrm{Diff}_\Gamma(M)$, hence invariant under changes of coordinates $\theta \mapsto \theta'$ of Γ. Notice that, in general, the sectional curvature of the Fisher–Rao metric on a statistical manifold Γ is *not* constant, as it depends on how Γ is embedded in $\mathrm{Dens}(M)$.

As a final remark in this section, I will discuss how the Fisher–Rao metric on discrete probability distributions is related to the Fisher–Rao metric on $\mathrm{Dens}(M)$. Recall that a discrete probability distributions is a sequence $(p_1, \ldots, p_k) \in (\mathbb{R}^+)^k$ such that $\sum_i p_i = 1$. By the map

$$w\colon (p_1, \ldots, p_k) \mapsto (\sqrt{p_1}, \ldots, \sqrt{p_k}),$$

the set of all discrete probability distributions is mapped onto a portion of the radius one sphere $S^{k-1} \subset \mathbb{R}^k$ (see Figure 1). The *discrete Fisher–Rao metric* is the constant curvature metric inherited from S^k. The relation to G^F on $\mathrm{Dens}(M)$ goes as follows. Let $e_1, \ldots, e_k$ be an orthonormal set of elements in $L^2(M)$. Notice that $e^i \in S^{L^2(M)}$, since each element is normalised. Let Γ_k denote the subset of elements in $\mathrm{Dens}(M)$ of the form $\mu = \sum_{i=1}^k p_i e_i^2 \mu_0$ for some discrete probability distribution $(p_1, \ldots, p_k)$. Then Γ_k is a statistical manifold, isomorphic to the space of discrete probability distributions, and the Fisher–Rao metric on $\mathrm{Dens}(M)$ restricted to Γ_k induces the discrete Fisher–Rao metric.

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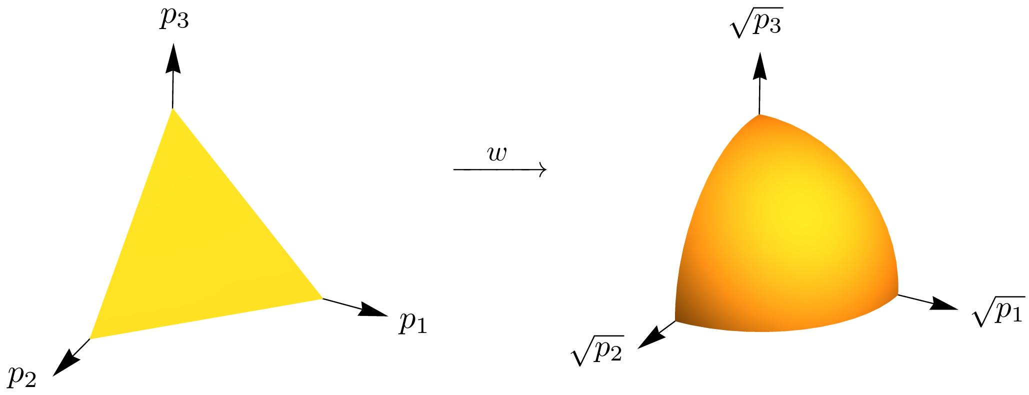

Figure 1: Illustration of the affine (left) and spherical (right) representations of the space of discrete probability distributions.

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3 Riemannian Metrics on spaces of densities, Kähler potentials, Riemannian metrics and diffeomorphisms Martin Bauer

3.1 Introduction and notation

Let M be a compact manifold of dimension m, that is equipped with a Riemannian metric $\bar{g}$. The corresponding volume density will be denoted by $\text{vol} = \text{vol}(\bar{g})$. We want to discuss relations between certain Riemannian metrics on the spaces in the below diagram:

$$\begin{tikzcd}
\mathrm{Diff}(M) \arrow[r, "\varphi\mapsto\varphi^*\bar{g}"] \arrow[d, "\varphi\mapsto\varphi^*\,\mathrm{vol}"'] & \mathrm{Met}(M) \arrow[dl, "g\mapsto\mathrm{vol}(g)"'] \\
\mathrm{Dens}(M) & \mathcal{H}(M) \arrow[u, "\psi\mapsto\bar{g}+\partial\bar{\partial}\psi"'] \arrow[l, "\psi\mapsto(1-\Delta\psi)\mathrm{vol}"]
\end{tikzcd}$$

Here $\mathrm{Diff}(M)$ denotes the Fréchet Lie group group of all smooth diffeomorphisms of the manifold M, $\mathrm{Met}(M)$ is the manifold of all smooth Riemannian metrics on the manifold M, $\mathrm{Dens}(M)$ is the manifold of smooth volume densities with fixed total volume 1 and $\mathcal{H}(M)$ denotes the space of smooth Kähler potentials.

3.2 Relations between the above spaces

In this part we want to shortly discuss the relations in the above diagram.

By Moser's trick the diffeomorphism group $\mathrm{Diff}(M)$ acts transitive on $\mathrm{Dens}(M)$. Thus the mapping π:

$$\pi : \begin{cases} \mathrm{Diff}(M) & \to \mathrm{Dens}(M) \\ \varphi & \mapsto \varphi^* \mathrm{vol}\,. \end{cases}$$

is surjective and defines a projection from $\mathrm{Diff}(M)$ to $\mathrm{Dens}(M)$.

A similar statement holds for the map from $\mathrm{Met}(M)$ to $\mathrm{Dens}(M)$. To see surjectivity we can simply construct a metric that has the prescribed volume density, i.e., for a given volume density $\mu \in \mathrm{Dens}(M)$ we can consider the metric $g = \mu^{2/m}\bar{g}$. Then $\mathrm{vol}(g) = \mu$.

The situation is more complicated for the action of the diffeomorphism on the space of all Riemannian metrics. This action is far from beeing transitive. However, it has been shown that the image of this map is a smooth submanifold of $\mathrm{Met}(M)$, see [7].

Finally we want to note that the isomorphism between $\mathrm{Dens}(M)$ and $\mathcal{H}(M)$ is exactly the Calabi-Yau mapping.

3.3 The Fisher–Rao, the Calabi and the Ebin metric

The Fisher–Rao metric.

The Fisher-Rao metric on the space of volume densities is defined as

$$G^{FR}_{\mu}(\alpha, \beta) = \int_M \frac{\alpha}{\mu}\frac{\beta}{\mu}\mu.$$

It is a cannocial metric on the space of volume densities, which means that it can defined without using any additional geometric structure. Furthermore it descibes an infinite dimensional sphere — it has constant positive curvature, see [10]

In the following sections we will discuss relations of the Fisher-Rao metric to two other prominent metrics on infinite dimensional spaces: the Calabi metric on the space of Kähler potentials and the Ebin metric on the manifold of all Riemannian metrics.

The Ebin metric.

On the manifold of all Riemannian metrics one can consider the reparametrization-invariant L^2-metric, given by:

$$G^E_g(h, k) = \int_M \mathrm{Tr}(g^{-1}hg^{-1}k)\,\mathrm{vol}(g).$$

This metric was first considered by Ebin [7, 8]. Subsequently the curvature of the space has been calculated, an analytic formula for minimizing geodesic has been derived and its metric completion has been determined see [2, 4, 9, 11].

The relation to the Fisher-Rao-metric is described in the following theorem:

Theorem. *Let π be the canonical projection from the manifold of metrics* $\mathrm{Met}(M)$ *to the space of densities* $\mathrm{Dens}(M)$*:*

$$\pi : \begin{cases} \mathrm{Met}(M) & \to \mathrm{Dens}(M) \\ g & \mapsto \mathrm{vol}(g). \end{cases}$$

Then π is a Riemannian submersion with respect to the Fisher-Rao and the Ebin metric.

The Calabi metric

The Calabi-metric is the restriction of the Ebin metric to the set of Kähler-metrics within a fixed Kähler class (assuming that the manifold has a complex structure). This set can be represented as the space of Kähler potentials. Thereon the Calabi–metric has the form:

$$G^C_\Phi(\Psi_1, \Psi_2) = \int_M \Delta\Psi_1 \Delta\Psi_2 (1 - \Delta\Phi)\,\mathrm{vol}$$

This connection between the Ebin–metric on $\mathrm{Met}(M)$ and the Calabi–metric on $\mathcal{H}$ was described in [5]. In particular they constructed a weighted version G^{Vol} of the Ebin–metric such that the space $\left(\mathcal{H}, G^C\right)$ is almost a totally geodesic subspace of the space $\left(\mathrm{Met}(M), G^{\mathrm{Vol}}\right)$.

Regarding the connection of the Calabi–metric and the Fisher-Rao metric we have the following result, which was noted in [12], see also [5]:

Theorem. *The spaces* $\left(\mathcal{H}, G^C\right)$ *and* $\left(\mathrm{Dens}(M), G^F\right)$ *are via the Calabi-Yau-map isometric to each other.*

Connections to right-invariant metrics on the diffeomorphism group

Finally we want to fit right-invariant metrics on the diffeomorphism group into the picture. Therefore we consider the projection $\pi : \mathrm{Diff}(M) \mapsto \mathrm{Dens}(M)$ given by $\pi(\varphi) = \varphi^* \mathrm{vol}$. On $\mathrm{Diff}(M)$ we consider the right invariant degenerated metric

$$G^1_{\mathrm{id}}(X, Y) = \int_M \operatorname{div} X \operatorname{div} Y \,\mathrm{vol}\ .$$

Note that the kernel of this bilinear form consists of all divergence free vector fields, which is exactly the tangent space to the fiber of π.

Theorem ([12])**.** *The metric G^1 on* $\mathrm{Diff}(M)$ *descends via π to the Fisher-Rao metric on the space of volume densities.*

This degenerate metric can be extended to a nondegenerate metric with the same descending property [13].

The property that a right-invariant metric on $\mathrm{Diff}(M)$ descends via π to a metric on $\mathrm{Dens}(M)$ is rather restrictive. In fact one needs a metric that is also left invariant with respect to the action of the group of all volume preserving diffeomorphisms. Due to the uniquness of the Fisher-Rao metric on the space of volume densities [1, 3] we obtain the following result:

Theorem. *Let G be any right–invariant metric on* $\mathrm{Diff}(M)$ *that decends via π to a metric $\tilde{G}$ on* $\mathrm{Dens}(M)$*. Then $\tilde{G}$ is invariant under reparametrizations and thus it is already a multiple of the Fisher-Rao metric.*

3.4 The Donaldson-metric

The previous hierachy of metrics is already pretty well-investigated. In the following we want to discuss the Donaldson metric. The goal of this part is to derive a similar hierachy of metrics as in the previous section – but related to the Donaldson metric.

The Donaldson metric

In the article [6] Donaldson considered the following metric on $\mathcal{H}$:

$$G^D_\Phi(\Psi, \Psi) := \int_M \Psi_1 \Psi_2 (1 - \Delta\Phi) \,\mathrm{vol}\ .$$

In this article he has shown that the curvature of the space $(\mathcal{H}, G^D)$ is non-positive. Note, that for $\dim(M) = 2$ this equals the Donaldson-Mabucci-Semmes metric on the space of Kähler potentials.

The Donaldson metric on $\mathrm{Dens}(M)$.

Using the Calabi-Yau mapping we can transport this metric to the space of volume densities.

Lemma. *The pullback of the Donaldson metric to the space of volume densites* $\mathrm{Dens}(M)$ *via the Calabi-Yau map yields an H^{-2}–type metric given by:*

$$G^{-2}_\mu(\alpha, \beta) = \int_M \Delta^{-1}\left(\frac{\alpha}{\mu}\right) \Delta^{-1}\left(\frac{\beta}{\mu}\right) \mu$$

Here Δ is the Laplacian of some fixed background metric. Thus it is easy to see that the metric is not equivariant with respect to the action of the diffeomorphism group.

As a next step we want to consider a metric on the diffeomorphism group, that projects onto this metric.

Lemma. *Let G_0 be any Riemannian metric on the group of volume preserving diffeomorphisms and denote by X_μ the divergence free part of the vector field X On* $\mathrm{Diff}(M)$ *we consider the metric*

$$G_\varphi(X \circ \varphi, X \circ \varphi) = \int_M \Delta^{-1}(\operatorname{div} X) \Delta^{-1}(\operatorname{div} Y) \varphi^* \,\mathrm{vol} + G_0(X_\mu, X_\mu)\,.$$

Then G decends via π to the Donaldson metric on $\mathrm{Dens}(M)$.

Note that the metric on the diffeomorphism group is not right-invariant.

However, in the case of $\dim(M) = 2$ we can use a natural metric – related to the Hofer-metric on the group of symplectomorphisms – of the same order on on the group of volume preserving diffeomorphisms to complement the metric,

Therefore, we recall that in dimension 2 one can write the Hodge decomposition of a vector field X as

$$X = \mathrm{grad}(f_1) + \mathrm{sgrad}(f_2)$$

where sgrad denotes the symplectic gradient. Then we can consider the metric

$$G_\varphi^{-1}(X, X) = \int_M f_1^2 + f_2^2 \varphi^* \operatorname{vol} .$$

Since this metric fits into the family of metrics considered in the previous lemma it decends to the Donaldson metric.

3.5 Outlook and open questions

This is only a very informal first study of these relations and it would be interesting to study these connections more rigorously. Also it would be of interest to fit the space of all complex structures into the picture. Then the question arises which metrics would be induced on this space, by the previously studied metrics. Also, there is a variety of other prominent metrics on some of these spaces, e.g., the Wasserstein–metric or the Dirichlet–metric to name just two examples. It would be of interest to fit them also into this diagram.

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4 Information geometry: invariant metrics on families of densities and dual connections

Xavier Pennec

In Information geometry, the Fisher-Rao metric is considered to be the unique metric which is invariant by any smooth reparameterization, i.e. which is invariant under the action of the group of diffeomorphisms. However, the construction of the Fisher-Rao metric is usually done explicitly only for finite-dimensional and parametric families of densities. In the case of the Gaussian family with the same mean, this construction leads to classical invariant metric $\langle V, W\rangle_\Sigma = \mathrm{Tr}(\Sigma^{-1}V\Sigma^{-1}W)$ on the space of positive definite matrices (up to a scaling factor).

However, constructing all the metrics on the space of positive definite matrices which are invariant by the space reparameterizations that leave the family globally invariant (here the linear transformations) leads to a larger two parameter family $\langle V, W\rangle_\Sigma = \mathrm{Tr}(\Sigma^{-1}V\Sigma^{-1}W) + \beta\,\mathrm{Tr}(\Sigma^{-1}V)\,\mathrm{Tr}(\Sigma^{-1}W)$ with $\beta > -1/n$. All these metrics share the same connection and define isomorphic spaces [2]. One can wonder if the additional degrees of freedom in the invariant metric come from the non compacity of the original space (here $\mathbb{R}^n$) or if they come from a reduction principle since we only consider a subgroup of the group of diffeomorphisms.

Two connections ∇^1 and ∇^{-1} are said to be dual with respect to a metric if

$$\langle \nabla^1_X V, \nabla^{-1}_X W\rangle = \partial_X \langle V, W\rangle$$

for all vector fields X, V, W. In information geometry, dually flat connections are an important structure of interst because they give two dual affine biorthogonal coordinate systems (often called the exponential and mixture coordinate systems) with dual (Bregman) divergences (a distance without the symmetry). Other properties include the fact that the symmetrized Bregman centroid necessarily lies on the geodesic passing through the two sided centroids [1]. Better understanding the symmetries induced by these dual connections could lead to new algorithms not only for densities but also for other manifolds.

For instance, on a Lie group, there are three special Cartan-Schouten connections which can be expressed in the in the Lie algebra of left-invariant vector fields as $\nabla_x y = \lambda[x, y]$. For $\lambda = 0$ (left or - connection) and $\lambda = 1$ (right or + connection) the curvature is null but the torsion is $T(x, y) = \pm[x, y]$. These two flat connections are called the left and right (or + and -) Cartan connections. In the middle, for $\lambda = 1/2$, we get the canonical Cartan connection (also called mean or 0-connection) which is torsion free but generally curved ($R(x, y, z) = [[x, y], z]/4$). Among the Cartan-Schouten connections, the - connection is the unique one for which all the left-invariant vector fields are covariantly constant; the + connection is the only one for which all the right-invariant vector fields are covariantly constant; and the 0-connection is the only one which is torsion-free (its curvature tensor is covariantly constant). These connections are generally non-metric (unless the group is the direct product of compact and Abelian groups, in which case the mean connection is the Levi-Civita connection of all bi-invariant metrics), but one could wonder if the symmetry between left and right connections with respect to the mean connection could not be used to generalize information geometric algorithms on Lie groups.

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5 Flows with Defects
Stefan Sommer

Images are conventionally matched using diffeomorphisms, and often penalty terms inspired by elastic energies are used to regularize the matching problem [3]. However, in the context of computational anatomy, changes in the anatomy of human organs are rarely the results of elastic deformations; following cell division or cell death, the tissue is not under stress as implied by elastic models. Instead, stress free states can be hypothesized to be the result of a reorganization of the tissue structure. Here, we discuss an approach for incorporating such reorganization for image matching with the LDDMM framework.

In models of defects in crystals, mappings from one crystal state to a subsequent with defects are nonholonomic [1], i.e. with non-commuting second order partial derivatives. If material coordinates q are mapped to coordinates $x = \phi(q)$, the coordinate coframes are related by $dx^i = e^i{}_j(q)dq^j$. The mapping being nonholonomic is reflected in a non-zero difference $\partial_j e^i{}_k(q) - \partial_k e^i{}_j(q)$ contrary to the case if ϕ was C^2. Nonholonomic mappings are multi-valued and often specified in differential form using the matrices $e^i{}_j(q)$ that encode both possible stress and infinitesimal defects in the crystal structures. The covariant connection $\Gamma_{\lambda\kappa}{}^{\mu} = e_i{}^{\mu} e^i{}_{\kappa,\lambda}$ defined by the frames deviates from the Levi-Civita and it carries torsion.

In the LDDMM framework [2, 5], a diffeomorphism flow ϕ_t is defined from a time-dependent Eulerian vector field v_t, and ϕ_t defines a family of pullback geometries $g^{\phi_t} = \phi_t^* g$ in material coordinates. Here g is a fixed metric on the domain manifold M. Similarly, pullback connections $\nabla^{\phi_t} = \phi_t^* \nabla$ are induced. These connections are the Levi-Civita connections of g^{ϕ_t} and the induced stress mentioned above is captured in the deviation of g^{ϕ_t} from g [4].

To provide a model of changing anatomy that allows stress free configurations, we propose to model cell division and cell death similarly to the appearance or disappearance of atoms in crystals. Beyond appearance and death, the analogy to crystal defects also permits a range of discontinuous deformations to be represented, e.g. sliding motion of the lung boundary.

At the workshop, we discussed how a change of the pullback connection ∇^{ϕ_t} to a torsion carrying connection can be used to incorporate crystal-like defects into the flow of maps ϕ_t. The connection can be concretely represented by frames at each flowing particle given by time-dependent matrices $e^i{}_{j,t}(q)$ as in the crystal case. Open questions that we worked on include (1) how to properly pass from the time-discrete mappings used in the crystal setting to the time-continuous flows in LDDMM, (2) how regularization of the flow using differential operators can be formulated in the torsion carrying geometry, and (3) how spatial regularization of the defects can be imposed.

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6 The role of torsion in shape analysis Stephen Marsland

Torsion is the great unloved of differential geometry. The principal reason for this is the privileged position of the Levi-Civiti connection in Riemannian geometry, which is the unique torsion-free connection that is always used for parallel transport. The price that is paid for avoiding torsion is curvature, which has many nasty effects, but problems with doing statistics is a key one: any generalisation of an extrinsic mean (such as the Karcher mean) has computational and theoretical difficulties, and the development of any analogue of Principal Components is limited at best, and wrong everywhere except at the origin.

However, the Levi-Civita connection is by far from the only choice of connection; in general, in Riemann-Cartan space there is the capacity to consider any form of connection that admits both torsion and curvature, while at the other end of the continuum is Weitzenböck space, which has torsion, but not curvature. While this leads to its own problems in terms of interpretation and more particularly, analysis, it has advantages for statistics provided that it is directions that we are interested in rather than position.

Over the week I had interesting conversations about torsion with many of the attendees, most particularly Xavier Pennec (with whom I first talked about this 8 years ago!), Stefan Sommers, and Peter Michor. My own research in the area of torsion has focussed on its use for statistics of shape, which has meant focusing on the underlying Weitzenböck space with its flat Euclidean metric. It has particularly looked at relations to Principal Curves, and to the auto-parallel formulation of General Relativity, which aimed to explain action at a distance. It was this very interesting to discuss the ideas of Xavier and Stefan, whose own interpretation of torsion has come from crystallography (in particular, the work of Kleinert in explaining stress in crystals). This has some links to medical imaging, where one can imagine such stress appearing as the motion of an organ relative to another part of the body (such as the lung moving with respect to the ribs). More interestingly for me, though it has links to dynamics, where points of singular torsion are related to anholonomy. We had several interesting discussions during the week concerning how to link these two pictures of the same underlying ideas.

With Peter I was interested in discussing the 'solder form', which is the differential forms version of torsion, and linked to the Ehresmann connection that I have been studying recently. It was very useful to get Peter's insight on this problem and to discuss it in detail with him.

Finally, amongst many very interesting discussions on a wide variety of topics during the week, I was particularly struck by work on stochastic shape, especially with Philipp Harms and F-X Vialard. I have previously done some small work in this area with a collaborator in Bath, Tony Shardlow, but had not followed up on it since seeing F-X's paper. I was therefore very pleased to meet him and discuss the work, together with the paper by von Renesse and Sturm on 'Entropic Measure and Wasserstein Diffusion'.

This meeting was one of the most stimulating weeks of mathematics I have been privileged to be involved in and I thank the organisers, and wish to strongly endorse the idea that meeting like this are far more productive than conferences.

7 Geodesic evolution on the space of volume preserving immersions
Olaf Müller

Let M be a compact connected oriented finite dimensional manifold, and let $(N, \bar{g})$ be a Riemannian manifold of bounded geometry. Let $\mathrm{Emb}(M, N)$ be the space of all smooth embeddings $M \to N$. It is a smooth manifold modelled on Frechét spaces. The tangent space at f of $\mathrm{Emb}(M, N)$ equals $\Gamma(f^*TN)$, the space of sections of TN along f.

In our article (joint work with Martin Bauer and Peter Michor) we will study certain submanifolds of these infinite dimensional manifolds. Therefore we introduce the spaces

$$\mathrm{Emb}_\mu(M, N) \subset \mathrm{Emb}_{\mathrm{Vol}}(M, N) \subset \mathrm{Emb}(M, N)$$
$$\mathrm{Imm}_\mu(M, N) \subset \mathrm{Imm}_{\mathrm{Vol}}(M, N) \subset \mathrm{Imm}(M, N)\,.$$

Here $\mathrm{Emb}_\mu(M, N)$ and $\mathrm{Imm}_\mu(M, N)$ denotes the spaces of all smooth embeddings (resp. immersions), that preserve a fixed volume form μ. The larger space $\mathrm{Emb}_{\mathrm{Vol}}(M, N)$ and $\mathrm{Imm}_{\mathrm{Vol}}(M, N)$ denote the spaces of all immersions/embeddings that have a fixed total volume Vol.

We are interested in local and global well-posedness of the geodesic equation on these spaces w.r.t. natural metrics of Sobolev type. The first step consists in identifying an appropriate submanifold structure for $\mathrm{Imm}_\mu(M, N)$ and $\mathrm{Imm}_{\mathrm{Vol}}(M, N)$ as subspaces of $\mathrm{Imm}(M, N)$. Let $\mathrm{Emb}^*_\mu(M, N)$ resp. $\mathrm{Imm}^*_\mu(M, N)$ denote the subspace of $\mathrm{Imm}_\mu(M, N)$ resp. $\mathrm{Emb}_\mu(M, N)$ consisting of those elements having nowhere vanishing second fundamental form. In the article [1] it has been shown that the space $\mathrm{Emb}^*_\mu(M, N)$ (respectively, by the same proof, $\mathrm{Imm}^*_\mu(M, N)$) is a smooth tame splitting submanifold of $\mathrm{Emb}(M, N)$ (respectively, of $\mathrm{Imm}(M, N)$). The choice of this space is not very fortunate for our questions concerning completeness, as the H^s metrics cannot prevent vanishing mean curvature somewhere or even totally geodesic parts of the immersions, so we want to ged rid of that additional condition. A first step is to show a similar statement for the spaces above. Like in [1], the proof of this statement will be an application of the Nash-Moser inverse function theorem.

As our ultimate focus is geodesic completeness, our main subject will be spaces of *immersions*, but note that the proofs of most theorems, e.g. of the following one, immediately carry over to the smaller spaces of *embeddings*.

Theorem. *The space* $\mathrm{Imm}_\mu(M, N)$ *is a tame splitting Fréchet submanifold of the space* $\mathrm{Imm}(M, N)$.

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8 Metrics on spaces of immersions where horizontality equals normality

Philipp Harms

We discussed a recent preprint [1] about metrics on shape space of immersions that have a particularly simple horizontal bundle. One considers reparametrization invariant Sobolev type metrics G on the space $\operatorname{Imm}(M,N)$ of immersions of a compact manifold M in a Riemannian manifold $(N,\overline{g})$. The tangent space $T_f \operatorname{Imm}(M,N)$ at each immersion f has two natural splittings to be described below. The first splitting can be easily calculated numerically, while the second splitting is important because it mirrors the geometry of shape space and geodesics thereon. Therefore, the case where the two splittings coincide is of particular interest.

Splitting into tangential/normal components

Letting $\pi_N : TN \to N$ denote the projection of a tangent vector onto its foot point, the splitting is given by

$$T \operatorname{Imm}(M,N) = \operatorname{Tan} \oplus \operatorname{Nor}\,,$$

where

$$\begin{aligned} T_f \operatorname{Imm}(M,N) &= \{h \in C^\infty(M,TN) : \pi_N \circ h = f\}\,, \\ \operatorname{Tan}_f &= \{Tf \circ X \mid X \in \mathfrak{X}(M)\}\,, \\ \operatorname{Nor}_f &= \{h \in T_f \operatorname{Imm}(M,N) \mid \forall k \in \operatorname{Tan}_f : \overline{g}(h,k) = 0\}\,. \end{aligned}$$

This splitting always exists and depends only on the geometry of $(N,\overline{g})$.

Splitting into vertical/horizontal components

This splitting depends on the metric G and on the projection

$$\pi : \operatorname{Imm}(M,N) \to B_i(M,N) = \operatorname{Imm}(M,N)/\operatorname{Diff}(M)$$

onto the shape space $B_i(M,N)$ of unparametrized immersions. The vertical and horizontal bundles at an immersion f are defined by

$$\operatorname{Ver}_f = \ker T_f\pi, \quad \operatorname{Hor}_f = (\ker T_f\pi)^{\perp,G}\,.$$

Depending on the metric G, these spaces might or might not span the entire tangent space $T_f \operatorname{Imm}$. If they do, then the splitting is given by

$$T \operatorname{Imm}(M,N) = \operatorname{Ver} \oplus \operatorname{Hor}\,.$$

Results and perspectives

A characterisation of all metrics G with the property that the two splittings coincide was obtained in [1].

In our discussion, we explored various possiblities to take advantage of this property when numerically solving the initial or boundary value problem for geodesics on shape space. In both cases, the algorithms do not require the inversion of an elliptic differential operator. The simplicity of the numerical scheme set up in [1] for the solution of the boundary value problem for geodesics suggests that this can be a major advantage over the general case.

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9 Introduction to pentagram maps
Boris Khesin

The pentagram map was originally defined by Schwartz [6] as a map on plane convex polygons considered up to their projective equivalence, where a new polygon is spanned by the shortest diagonals of the initial one. This map is the identity for pentagons, it is an involution for hexagons, while for polygons with more vertices it was shown to exhibit quasi-periodic behaviour under iterations. The pentagram map was extended to the case of twisted polygons and its integrability in 2D was proved in Ovsienko, Schwartz, and Tabachnikov [5], see also Soloviev [7].

A natural requirement for possible generalizations of this map from 2D to higher dimensions is their integrability. It turns out that there is no natural generalization of this map to polyhedra, but one can suggest natural integrable generalizations of the pentagram map to the space of generic space closed and twisted polygons.

Example. Here is an example of a pentagram map in $\mathbf{RP}^3$. Given an n-gon (v_k) we define its *diagonal plane* P_k as the plane passing through 3 vertices $P_k := (v_{k-2}, v_k, v_{k+2})$. Now the corresponding *pentagram map* T on polygons (v_k) in $\mathbf{RP}^3$ is defined by intersecting 3 consecutive diagonal planes:

$$Tv_k := P_{k-1} \cap P_k \cap P_{k+1} \,.$$

This map turns out to be a discrete integrable system [2].

Similarly, can define generalized pentagram maps $T_{I,J}$ on (projective equivalence classes of) polygons in $\mathbf{RP}^d$, associated with tuples of numbers I and J: the tuple I defines which vertices to take in the definition of the diagonal hyperplanes P_k, while the tuple J determines which of the hyperplanes to intersect in order to get the image point Tv_k. Many integrable and non-integrable cases were found in Beffa [1] and Khesin and Soloviev [2, 3, 4]. In general, their integrability is yet unknown and is an interesting open question.

Remark. In Khesin and Soloviev [2, 4] it was also proved that the continuous limit of any higher or dented pentagram map (and more generally, of any generalized pentagram map) in $\mathbf{RP}^d$ is the $(2, d+1)$-KdV flow of the Adler-Gelfand-Dickey hierarchy on the circle. For 2D this is the classical Boussinesq equation on the circle: $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$, which appears as the continuous limit of the 2D pentagram map [5].

Open Question. Is it possible to include all (n, m)-KdV flows of the Adler-Gelfand-Dickey hierarchy into this scheme, as appropriate continuous limits of the pentagram maps?

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10 Vortex filaments and the Hasimoto transform

Boris Khesin

Vortex filaments

The *vortex filament (or binormal) equation* is the evolution equation

$$\partial_t \gamma = \gamma' \times \gamma'' ,$$

of an arc-length parametrized space curve $\gamma(\cdot, t) \subset \mathbf{RP}^3$, where $\gamma' := \partial\gamma/\partial\theta$. For an arbitrary parametrization the filament equation reads $\partial_t \gamma = k \cdot \mathbf{b}$, where k and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ stand, respectively, for the curvature value and binormal unit vector of the curve γ at the corresponding point.

This binormal equation is known to be Hamiltonian with the Hamiltonian function given by the length functional $H(\gamma) = \text{length}(\gamma) = \int_\gamma \|\gamma'(\theta)\| \, d\theta$ and relative to the

Marsden-Weinstein symplectic structure on non-parametrized oriented space curves in $\mathbf{R}^3$, see e.g. Arnold and Khesin [1] and Marsden and Weinstein [4]. At a curve γ this symplectic structure is

$$\omega_\gamma^{MW}(V,W) := \int_\gamma i_V i_W \mu = \int_\gamma \mu(V,W,\gamma')\, d\theta$$

where V and W are two vector fields attached to the curve γ and regarded as variations of this curve, while the volume form μ is evaluated on the three vectors V, W and γ'. Equivalently, the Marsden-Weinstein symplectic structure can be defined by means of the operator J of almost complex structure on curves: any variation V is rotated by the operator J in the planes orthogonal to γ by $\pi/2$ in the positive direction (which makes a skew-gradient from a gradient field).

Furthermore, the Hasimoto transformation at any time t sends a curve $\gamma(\theta)$ with curvature $k(\theta)$ and torsion $\tau(\theta)$ to the wave function $\psi(\theta) = k(\theta)\exp\{i\int_{\theta_0}^{\theta}\tau(\zeta)\,d\zeta\}$ satisfying the 1-dimensional focusing NLS:

$$i\partial_t\psi + \psi'' + |\psi|^2\psi/2 = 0\,.$$

In particular, the binormal equation is an infinite-dimensional integrable system.

Extension to higher dimensions

A natural extension of the binormal equation to higher dimensions is as follows. Consider a closed oriented embedded submanifold (membrane) P of codimension 2 in $\mathbf{R}^n$ (or more generally, in a Riemannian manifold M^n) with $n \geq 3$. The Marsden-Weinstein symplectic structure ω^{MW} on membranes of codimension 2 in $\mathbf{R}^n$ (or in any n-dimensional manifold) with a volume form μ is defined similar to the 3-dimensional case: two variations of a membrane P are regarded as a pair of normal vector fields attached to the membrane P and the value of the symplectic structure on them is

$$\omega_P^{MW}(V,W) := \int_P i_V i_W \mu\,.$$

Here $i_V i_W \mu$ is an $(n-2)$-form integrated over P. Note that this symplectic structure can be thought of as the "total" averaging of the symplectic structures in each normal space N_pP to P. (The Marsden-Weinstein structure in higher dimensions was studied in Arnold and Khesin [1] and Haller and Vizman [2].)

Furthermore, define the Hamiltonian function on those membranes by taking their $(n-2)$-volume:

$$H(P) = \mathrm{v}olume(P) = \int_P \mu_P\,,$$

where μ_P is the volume form of the metric induced from $\mathbf{R}^n$ to P. (For a closed curve γ in $\mathbf{R}^3$ this Hamiltonian is the length functional discussed above.)

Theorem (Haller and Vizman [2], Khesin [3], and Shashikanth [5])**.** *In any dimension $n \geq 3$ the Hamiltonian vector field for the Hamiltonian H and the Marsden-Weinstein symplectic structure on codimension 2 membranes $P \subset \mathbf{R}^n$ is*

$$v_H(p) = C_n \cdot J(\mathbf{MC}(p))\,,$$

where C_n is a constant, J is the operator of positive $\pi/2$ rotation in every normal space N_pP to P, and $\mathbf{MC}(p)$ is the mean curvature vector to P at the point p.

Recall that the mean curvature vector $\mathbf{MC}(p)$ at a point p of a smooth submanifold P in any dimension can be defined either as the normalized trace of the second fundamental form at p, or the mean value of the vectors of curvature of geodesics in P passing through the point p when we average over the sphere S^{l-1} of all possible unit tangent vectors in T_pP for these geodesics.

The *higher vortex filament equation* on submanifolds of codimension 2 in $\mathbf{R}^n$ is given by the *binormal (or skew) mean-curvature flow*:

$$\partial_t P(p) = -J(\mathbf{MC}(p))\,.$$

For dimension $n = 3$ the mean curvature vector is the curvature vector $k \cdot \mathbf{n}$ of a curve γ: $\mathbf{MC} = k \cdot \mathbf{n}$, while the skew mean-curvature flow becomes the binormal equation: $\partial_t \gamma = -J(k \cdot \mathbf{n}) = k \cdot \mathbf{b}$. Unlike the case $n = 3$, for larger $n \geq 4$ the skew mean-curvature flow is apparently non-integrable.

Open Question. It would be very interesting to find an analogue of the Hasimoto transformation for any n relating the binormal mean-curvature flow with the higher-dimensional (and already non-integrable) nonlinear Schrödinger equation (NLS). For any d apparently curvature k becomes mean curvature vector $\mathbf{MC}$, while an analogue of the torsion τ for a surface of codimension 2 should be a flat $U(1)$ connection (i.e. a closed or exact 1-form) on P, for which $\mathbf{MC}$ is a horizontal section. Then the same formula $\psi(q) = k(q) \exp\{i \int_{q_0}^{q} \tau(q)\, dq\}$ for any t would work, as the integral depends locally only on the point q, but not on the path from q_0 to q in P. Would such a wave function $\psi(q,t)$ satisfy some analog of the 1D NLS equation?

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11 Diffeomorphisms groups generated by Gaussian vector fields François-Xavier Vialard

This note gives a partial answer to a question asked during the workshop.

We will denote by $\mathcal{H}_\sigma$ the reproducing kernel Hilbert space of vector fields on $\mathbb{R}^d$ generated by a Gaussian kernel $k_\sigma(x,y) = e^{-\|x-y\|^2/\sigma^2}\,\mathrm{Id}_{\mathbb{R}^d}$ for a positive real parameter σ. Let us first recall an analytical characterization of the space $\mathcal{H}_\sigma$, denoting $\hat{f}$ the Fourier transform of $f \in L^2(\mathbb{R}^d,\mathbb{R}^d)$:

$$\mathcal{H}_\sigma = \left\{ f \in L^2(\mathbb{R}^d,\mathbb{R}^d) \,\middle|\, \|f\|^2_{\mathcal{H}_\sigma} = \frac{\sigma^d}{2^d \pi^{d/2}} \int_{\mathbb{R}^d} |\hat{f}(\omega)|^2 \exp\left(\frac{\sigma^2|\omega|^2}{4}\right) \mathrm{d}\omega < \infty \right\}. \quad (11.1)$$

The group $\mathcal{G}_{\mathcal{H}_\sigma}$ consists of all flows that can be generated by $\mathcal{H}_\sigma$ vector fields,

$$\mathcal{G}_{\mathcal{H}_\sigma} = \left\{ \varphi(1) \,:\, \varphi(t) \text{ is the solution of (11.2) with } u \in L^1([0,1],\mathcal{H}_\sigma) \right\}.$$

Given a time-dependent vector field $u \in L^1([0,1],\mathcal{H}_\sigma)$, there exists a unique curve $\varphi \in C([0,1],\mathrm{Diff}^1(\mathbb{R}^d))$ solving

$$\partial_t \varphi(t) = u(t) \circ \varphi(t)\,, \qquad \varphi(0) = \mathrm{Id}\,, \quad (11.2)$$

for $t \in [0,1]$ almost everywhere. Let us recall what is proved in [9]:

- Since the space $\mathcal{H}_\sigma$ can be continuously embedded in the space of $H^n(\mathbb{R}^d)$ vector fields (Sobolev space of order $n \geq 1$), the flow is contained in $\mathrm{Diff}^\infty(\mathbb{R}^d)$.
- Since the kernel k_σ is positive definite on $\mathbb{R}^d$, the group $\mathcal{G}_{\mathcal{H}_\sigma}$ acts $n-$transitively on $\mathbb{R}^d$ if $d \geq 2$, i.e. for any two ordered sets of n distinct points $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ in $\mathbb{R}^d$ there exists an element $\varphi \in \mathcal{G}_{\mathcal{H}_\sigma}$ such that for each $i \in [\![1,n]\!]$, $\varphi(x_i) = y_i$.

These groups are widely used in application such as diffeomorphic image matching [2, 6, 7], although their understanding is less developed than the group of Sobolev diffeomorphisms [3, 4, 5]. A possible generalization of the previous property is the following question:

Question. Does the group $\mathcal{G}_{\mathcal{H}_\sigma}$ acts transitively on the space of compactly supported smooth densities $\mathrm{Dens}^\infty(\mathbb{R}^d) := \{\rho \in C^\infty(\mathbb{R}^d,\mathbb{R}) \mid \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1\}$?

Recall that this property is well-known in the case of smooth diffeomorphisms by the so-called *Moser trick*. As we will prove in this note, the answer to the above question is negative. Indeed, we have:

Proposition. *The group $\mathcal{G}_{\mathcal{H}_\sigma}$ is contained in the group of real analytic diffeomorphisms of $\mathbb{R}^d$, $\mathcal{G}_{\mathcal{H}_\sigma} \subset \mathrm{Diff}^\omega(\mathbb{R}^d)$. More precisely, any element of $\mathcal{G}_{\mathcal{H}_\sigma}$ admits a holomorphic extension on a cylindrical open set of $\mathbb{R}^d$ in $\mathbb{C}^d$, namely $C(r) = \{z \in \mathbb{C}^d \,|\, \forall i \in [\![1,d]\!]\, |\mathrm{Im}(z_i)| \leq r\}$ for $r > 0$ sufficiently small.*

Before proving the proposition, we briefly describe a counter-example to the above question. Let us define the set of singular points of a function by being the complementary of the set of points where the function is analytic. By definition, this set is closed; we will denote it by $S(\rho)$ for a given density ρ. The topology of $S(\rho)$ is preserved under the action of the group $\mathcal{G}_{\mathcal{H}_\sigma}$. Indeed, its action on densities only involves multiplication and composition by real analytic functions which preserve analyticity. Last, there exists densities whose singular set do not have the same topology (for instance connectedness).

Proof of the proposition. The Gaussian kernel has a complex extension (see [8])

$$k_\sigma^{\mathbb{C}}(z,z') = \exp\left(-\frac{1}{\sigma^2}\sum_{i=1}^d (z_i - \overline{z_i'})^2\right) \tag{11.3}$$

where $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and we denote the associated reproducing kernel Hilbert space by $\mathcal{H}_\sigma^{\mathbb{C}}(\mathbb{C}^d)$ or simply $\mathcal{H}_\sigma^{\mathbb{C}}$. This space can be explicitely described by

$$\mathcal{H}_\sigma^{\mathbb{C}} = \{f : \mathbb{C}^d \to \mathbb{C}^d \,|\, f \text{ is holomorphic and } \|f\|_\sigma < \infty\}$$

where

$$\|f\|_\sigma^2 = \int_{\mathbb{C}^d} \|f(z)\|^2 e^{\frac{1}{\sigma^2}\sum_{i=1}^d (z_i - \overline{z_i})^2}\, dz\,.$$

Note that $\|f\|_\sigma$ is the norm of f in the space $\mathcal{H}_\sigma^{\mathbb{C}}$ up to a multiplicative constant.

The restriction of $k_\sigma^{\mathbb{C}}$ to the real line is k_σ which means (see paragraph 5 of part 1 in [1]) that the space $\mathcal{H}_\sigma$ can be described as a subspace of $\mathcal{H}_\sigma^{\mathbb{C}}(\mathbb{C}^d)$. More precisely, for every $f \in \mathcal{H}_\sigma$ there exists a unique $\tilde{f} \in \mathcal{H}_\sigma^{\mathbb{C}}$ minimizing the norm $\|\tilde{f}\|_{\mathcal{H}_\sigma^{\mathbb{C}}}$ among the functions $\tilde{f} \in \mathcal{H}_\sigma^{\mathbb{C}}$ such that $\tilde{f}_{|\mathbb{R}^d} = f$. In particular, this shows that every element of $\mathcal{H}_\sigma$ is an analytic function.

We now consider a time dependent vector field $v_t \in L^2([0,1], \mathcal{H}_\sigma)$ and we denote by $\tilde{v}_t$ its lift in $L^2([0,1], \mathcal{H}_\sigma^{\mathbb{C}})$. The flow associated with $\tilde{v}_t$ may not exist since the Lipschitz constant of v_t may be infinite on $\mathbb{C}^d$, however this Lipschitz constant is uniformly bounded on any cylindrical neighborhood of $\mathbb{R}^d$: As proven in [8] in *proof of lemma 3.1*

$$\|f(z)\|^2 \leq \frac{c}{(2\sigma^2)^d}\|f\|_\sigma^2 \tag{11.4}$$

with $c(r) = \max\{e^{-\frac{1}{\sigma^2}\sum_{i=1}^d (z_i - \overline{z_i'})^2} \,|\, \max_{i=1,\ldots,d} \mathrm{Im}(z_i)| \leq r\}$, which implies the Lipschitz property on $C(r)$ using the Cauchy formula.

In order to show the result, we consider a smooth function η on $\mathbb{C}^d$ such that $\eta(z) = 1$ for $z \in C(1)$ and $\eta(z) = 0$ for $z \notin C(2)$. The time dependent vector field $u_t(z) = \eta(z)v_t(z)$ is holomorphic on $C(1)$ and globally Lipschitz with a constant that depends linearly on the Lipschitz constant of v_t on $C(2)$. Then, the flow ϕ_t of u_t is well defined and applying Gronwall's lemma we have:

$$\|\phi_t(z) - \phi_t(z')\| \leq \|z - z'\| \exp\left(\int_0^t \|u_s\|_{1,\infty}\, ds\right)\,, \tag{11.5}$$

where $\|u\|_{1,\infty}$ denotes the sup norm of u and its first derivative. Since there exists a constant M such that $\|u_t\|_{1,\infty} \leq M\|\tilde{v}_t\|_\sigma$, we have

$$\|\phi_t(z) - \phi_t(z')\| \leq \|z - z'\| \exp\left(\int_0^t M\|\tilde{v}_t\|_\sigma \, ds\right) , \tag{11.6}$$

In particular, there exists $\varepsilon > 0$ such that for all $t \in [0,1]$, $\phi_t(z) \in C(1)$ if $z \in C(\varepsilon)$. Since $u_t(z) = \tilde{v}_t(z)$ for $z \in C(1)$, ϕ_1 is holomorphic on $C(\varepsilon)$ being the flow of a vector field v_t which is holomorphic $\phi_t(C(\varepsilon)) \subset C(1)$. □

In the proof we used the complex extension of the Gaussian kernel but it is probably possible to prove the analyticity by direct estimations.

As a conclusion, the initial question might be reformulated as follows

Question. Does the group $\mathcal{G}_{\mathcal{H}_\sigma}$ acts transitively on the space of analytical densities $\mathrm{Dens}^\omega(\mathbb{R}^d) := \{\rho \in C^\omega(\mathbb{R}^d, \mathbb{R}) \mid \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1\}$?

However, the answer to this question is no in dimension 1 since this would imply that $\mathcal{G}_{\mathcal{H}_\sigma} = \mathrm{Diff}^\omega(\mathbb{R})$. Indeed, let F denote the cumulative distribution function of a Gaussian density which is an analytical diffeomorphism between $\mathbb{R}$ and $]0,1[$ and $\varphi \in \mathrm{Diff}^\omega(\mathbb{R})$ to which we associate the analytical cumulative distribution function $F \circ \varphi$. Using transitivity of $G_{\mathcal{H}_\sigma}$ on analytical densities, there exists $\psi \in G_{\mathcal{H}_\sigma}$ such that $F \circ \varphi \circ \psi = F$. This implies $\varphi \circ \psi = \mathrm{Id}$ and the result.

Therefore, this would even be interesting to answer the following simple question:

Question. Even if $\mathcal{H}_\sigma \subsetneq \mathcal{H}_{\sigma'}$ when $\sigma' < \sigma$, is it true that $\mathcal{G}_{\mathcal{H}_\sigma} \subsetneq \mathcal{G}_{\mathcal{H}_{\sigma'}}$?

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12 Open Questions in mathematical shape analysis Martins Bruveris

The following is a collection of mathematical questions that are inspired by applications to shape analysis and were mentioned at the workshop.

Geodesic completeness

Completeness properties of infinite dimensional manifolds are not easy to establish. It is known, that there exist geodesically complete Riemannian metrics on the space of curves as well as complete right–invariant metrics on the diffeomorphism group. Do other spaces also permit complete Riemannian metrics?

Let M be a compact manifold without boundary and

$$\operatorname{Met}(M) = \Gamma(S^2_+ TM)$$

the space of (smooth) Riemannian metrics on M.

Open Question. Does there exist a geodesically complete, smooth Riemannian metric on $\operatorname{Met}(M)$, that is invariant under the natural $\operatorname{Diff}(M)$ action?

The same question can be asked for the spaces of immersions. Let $(N, \bar{g})$ be a Riemannian manifold and μ a volume form on M. Denote by $\operatorname{Imm}(M, N)$ the space of immersions from M to N and by

$$\operatorname{Imm}_\mu(M, N) = \{f \in \operatorname{Imm}(M, N) \,:\, \operatorname{vol}(f^*\bar{g}) = \mu\}$$

the space of volume form–preserving immersions.

Open Question. Are $\operatorname{Diff}(M)$ invariant Sobolet–type metrics on $\operatorname{Imm}_\mu(M, N)$ geodesically complete for high enough Sobolev order? For example, consider the metrics

$$G_f(h, k) = \int_M \bar{g}\left((\operatorname{Id} + \Delta^g)^p h, k\right) \operatorname{vol}(f^*\bar{g})\,.$$

Are they geodesically complete for p high enough?

Remark. The conjecture is that $p > n/2 + 1$ is sufficient, where $n = \dim M$.

Remark. The same question can be posed for the space $\mathrm{Imm}(M, N)$ of all immersions. In this case the conjecture is that $p > n/2 + 2$ is sufficient. For the special cases $M = S^1$ or $M = N$ the bound $p > n/2 + 1$ has been established.