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► **To cite this version:**

Julio Araujo, Nathann Cohen, Susanna de Rezende, Frédéric Havet, Phablo Moura. On the proper orientation number of bipartite graphs. 9th International colloquium on graph theory and combinatorics, Jun 2014, Grenoble, France. hal-01076904

HAL Id: hal-01076904

<https://hal.science/hal-01076904>

Submitted on 23 Oct 2014

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On the proper orientation number of bipartite graphs[☆]

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Abstract

An *orientation* of a graph G is a digraph D obtained from G by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *indegree* of v in D , denoted by $d_D^-(v)$, is the number of arcs with head v in D . An orientation D of G is *proper* if $d_D^-(u) \neq d_D^-(v)$, for all $uv \in E(G)$. The *proper orientation number* of a graph G , denoted by $\vec{\chi}(G)$, is the minimum of the maximum indegree over all its proper orientations. It is well-known that $\vec{\chi}(G) \leq \Delta(G)$, for every graph G . In this paper, we first prove that $\vec{\chi}(G) \leq \left\lfloor \left(\Delta(G) + \sqrt{\Delta(G)} \right) / 2 \right\rfloor + 1$ if G is a bipartite graph, and $\vec{\chi}(G) \leq 4$ if G is a tree. We then prove that deciding whether $\vec{\chi}(G) \leq \Delta(G) - 1$ is an \mathcal{NP} -complete problem. We also show that it is \mathcal{NP} -complete to decide whether $\vec{\chi}(G) \leq 2$, for planar *subcubic* graphs G . Moreover, we prove that it is \mathcal{NP} -complete to decide whether $\vec{\chi}(G) \leq 3$, for planar bipartite graphs G with maximum degree 5.

Keywords: proper orientation, graph colouring, bipartite graph, hardness.

1. Introduction

In this paper, all graphs are *simple*, that is without loops and multiple edges. We follow standard terminology as used in [1].

An *orientation* D of a graph G is a digraph obtained from G by replacing each edge by just one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *indegree* of v in D , denoted by $d_D^-(v)$, is the number of arcs with head v in D . We use the notation $d^-(v)$ when the orientation D is clear from the context. The orientation D of G is *proper* if $d^-(u) \neq d^-(v)$, for all $uv \in E(G)$. An orientation with maximum indegree at most k is called a *k-orientation*. The *proper orientation number* of a graph G , denoted by $\vec{\chi}(G)$, is the minimum integer k such that G admits a proper k -orientation. This graph parameter was introduced by Ahadi and Dehghan [2]. It is well-defined for any graph G since one can always obtain a proper $\Delta(G)$ -orientation (see [2]). In other words, $\vec{\chi}(G) \leq \Delta(G)$. Note that every proper orientation of a graph G induces a proper vertex colouring of G . Thus, $\vec{\chi}(G) \geq \chi(G) - 1$. Hence, we have the following sequence of inequalities: $\omega(G) - 1 \leq \chi(G) - 1 \leq \vec{\chi}(G) \leq \Delta(G)$.

These inequalities are best possible in the sense that, for a complete graph K , $\omega(K) - 1 = \chi(K) - 1 = \vec{\chi}(K) = \Delta(K)$. However, one might expect better upper bounds on some parameters by taking a convex

[☆]Research supported by CNPq-Brazil (477203/2012-4), FUNCAP/CNRS INC-0083-00047.01.00/13, FAPESP-Brazil (Proc. 2011/16348-0, 2013/19179-0, 2013/03447-6) and ANR Blanc STINT ANR-13-BS02-0007-03.

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combination of two others. Reed [3] showed that there exists $\epsilon_0 > 0$ such that $\chi(G) \leq \epsilon_0 \cdot \omega(G) + (1 - \epsilon_0)\Delta(G)$ for every graph G and conjectured the following.

Conjecture 1 (Reed [3]). For every graph G , $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

If true, this conjecture would be tight. Johannson [4] settled Conjecture 1 for $\omega(G) = 2$ and $\Delta(G)$ sufficiently large.

Likewise, one may wonder if similar upper bounds might be derived for the proper orientation number.

Problem 1.

- (a) Does there exist a positive ϵ_1 such that $\vec{\chi}(G) \leq \epsilon_1 \cdot \omega(G) + (1 - \epsilon_1)\Delta(G)$?
- (b) Does there exist a positive ϵ_2 such that $\vec{\chi}(G) \leq \epsilon_2 \cdot \chi(G) + (1 - \epsilon_2)\Delta(G)$?

Observe that both questions are intimately related. Indeed if the answer to (a) is positive for ϵ_1 , then the answer to (b) is also positive for ϵ_1 . On the other hand, if the answer to (b) is positive for ϵ_2 , then the answer to (a) is also positive for $\epsilon_1 = \epsilon_0 \cdot \epsilon_2$ by the above-mentioned result of Reed.

In Section 2, we answer Problem 1 positively in the case of bipartite graphs by showing that: if G is bipartite, then $\vec{\chi}(G) \leq \left\lfloor \frac{\Delta(G)+\sqrt{\Delta(G)}}{2} \right\rfloor + 1$. We also argue that this bound is tight for $\Delta(G) \in \{2, 3\}$.

In Section 3, we prove that $\vec{\chi}(T) \leq 4$, for every tree T . Moreover, we show that $\vec{\chi}(T) \leq 3$ if $\Delta(T) \leq 6$, and $\vec{\chi}(T) \leq 2$ if $\Delta(T) \leq 3$. We also argue that all these bounds are tight.

In Section 4, we study the computational complexity of computing the proper orientation number of a bipartite graph. In their seminal paper, Ahadi and Dehghan proved that it is \mathcal{NP} -complete to decide whether $\vec{\chi}(G) = 2$ for planar graphs G . We first improve their reduction and show that it is \mathcal{NP} -complete to decide whether $\vec{\chi}(G) \leq 2$, for planar *subcubic* graphs G . Moreover, we prove that deciding whether $\vec{\chi}(G) \leq \Delta(G) - 1$ is an \mathcal{NP} -complete problem for general graphs G . Finally, we show that it is also \mathcal{NP} -complete to decide whether $\vec{\chi}(G) \leq 3$ for planar bipartite graphs G with maximum degree 5.

Due to space limitation, we omit the proofs of these results.

2. General upper bound

Theorem 1. *Let G be a bipartite graph and let k be a positive integer. If $\Delta(G) > 2k + \frac{\sqrt{1+8k+1}}{2}$, then $\vec{\chi}(G) \leq \Delta(G) - k$.*

Sketch of proof. In order to prove this theorem, we describe an algorithm (see Algorithm 1) that produces a proper $(\Delta(G) - k)$ -orientation. Let $G = (X \cup Y, E)$ be a bipartite graph as in the statement of Theorem 1. The algorithm consists of two phases.

The first phase (lines 1 to 8 in Algorithm 1) produces an orientation, not necessarily proper, of the edges of G in such a way that the indegree of each vertex in X is at most k and the indegree of each vertex in Y is at most $\Delta(G) - k$. It proceeds as follows. We first orient all edges $xy \in E(G)$ from x to y , where $x \in X$ and $y \in Y$. Then we define k matchings as described subsequently.

Let $G_1 = G$, and let M_1 be a matching in G_1 that covers all vertices of maximum degree. For each $i \in \{2, \dots, k\}$, let G_i be the graph obtained from G_{i-1} by removing the edges in M_{i-1} , that is $G_i = G_{i-1} \setminus M_{i-1}$, and let M_i be a matching in G_i that covers all vertices of degree $\Delta(G_i)$. Such a M_i exists since it is well known that every bipartite graph H has a proper $\Delta(H)$ -edge-colouring. Clearly, we have $\Delta(G_i) = \Delta(G_{i-1}) - 1$, for each $i \in \{2, 3, \dots, k\}$. Let $M := \bigcup_{i=1}^k M_i$. Observe that if a vertex has degree $\Delta(G) - k + j$ in G , where $j \in \{1, 2, \dots, k\}$, then it is incident to at least j edges in M . Hence, for all $j \in \{1, 2, \dots, k\}$ and for each vertex y in Y of degree $\Delta(G) - k + j$ in G , we reverse the orientation of exactly j edges in M incident to y . This ends the first phase.

The second phase reverses the orientation of some edges in $E(G) \setminus M$, step by step, in order to obtain a $(\Delta(G) - k)$ -orientation. This orientation is proper under the assumption of Theorem 1. \square

Algorithm 1: Proper Orientation of Bipartite Graphs

Input: Bipartite graph $G = (X \cup Y, E)$ and $k \in \mathbb{N}$ s.t. $\Delta(G) > 2k + \frac{\sqrt{1+8k+1}}{2}$.
Output: Proper $(\Delta(G) - k)$ -orientation for G .

- 1 $G_1 \leftarrow G$
- 2 Orient all edges in G from X to Y
- 3 **for** $i = 1, \dots, k$ **do**
- 4 $M_i \leftarrow$ matching of G_i saturating all vertices of degree $\Delta(G_i)$
- 5 $G_{i+1} \leftarrow G_i - M_i$
- 6 $M \leftarrow \bigcup_{i=1}^k M_i$
- 7 **foreach** $y \in Y$ **do**
- 8 \perp reverse the orientation of $\max\{0; d_G(y) - \Delta(G) + k\}$ edges of M incident to y
- 9 $\tilde{X} \leftarrow X$
- 10 **for** $\ell = \Delta(G) - k - 1, \dots, 2$ **do**
- 11 **while** $\exists x \in X$ s.t. $|N_{\leq \ell}(x)| \geq \ell - d^-(x)$ and $|N_{=\ell}(x)| \leq \ell - d^-(x)$ **do**
- 12 $\tilde{Y} \leftarrow$ set of $\ell - d^-(x)$ vertices of highest indegree in $N_{\leq \ell}(x)$
- 13 **foreach** $y \in \tilde{Y}$ **do**
- 14 \perp Reverse the orientation of xy (i.e. re-orient xy towards x)
- 15 $\tilde{X} \leftarrow \tilde{X} \setminus \{x\}$

Theorem 2. If G is a bipartite graph, then $\vec{\chi}(G) \leq \left\lfloor \frac{\Delta(G) + \sqrt{\Delta(G)}}{2} \right\rfloor + 1$.

Sketch of proof. By Theorem 1, for every $k \in \mathbb{N}$, if $\Delta(G) > 2k + \frac{\sqrt{1+8k+1}}{2}$, then $\vec{\chi}(G) \leq \Delta(G) - k$. In order to obtain a good upper bound for $\vec{\chi}(G)$, we must find the largest positive integer k such that the condition of Theorem 1 holds for a given graph G .

Solving the inequality for k , we obtain that $k < \frac{\Delta(G) - \sqrt{\Delta(G)}}{2}$. Since k is integer, we conclude that $\vec{\chi}(G) \leq \Delta(G) - \left\lfloor \frac{\Delta(G) - \sqrt{\Delta(G)}}{2} \right\rfloor + 1$, and the result follows. \square

Note that if G is bipartite and $\Delta(G) \in \{2, 3, 4\}$, then the bound of Theorem 2 is equal to the trivial upper bound $\vec{\chi}(G) \leq \Delta(G)$. For $\Delta(G) = 1$ and $\Delta(G) = 2$, this bound is tight due to the paths with 2 and 4 vertices, respectively. In addition, there exists a bipartite graph G with $\Delta(G) = 3$ and $\vec{\chi}(G) = 3$.

3. Trees

Theorem 3. If T is a tree, then the following statements hold:

- (1) if $\Delta(T) \leq 3$, then $\vec{\chi}(T) \leq 2$;
- (2) if $\Delta(T) \leq 6$, then $\vec{\chi}(T) \leq 3$;
- (3) $\vec{\chi}(T) \leq 4$.

Sketch of proof. We prove the three statements by using similar arguments. For $i \in \{1, 2, 3\}$, we consider a minimal counter-example M_i to statement (i) with respect to the number of vertices, and derive a contradiction that implies that no counter-example exists. Since M_i is a minimal counter-example, we have $\vec{\chi}(M_i) > i + 1$, but $\vec{\chi}(T) \leq i + 1$, for any proper subtree T of M_i . We use the latter fact to derive a proper $(i + 1)$ -orientation of M_i , which contradicts $\vec{\chi}(M_i) > i + 1$. \square

The three statements of the theorem are tight in the following sense: there is a tree with maximum degree 4 and proper orientation number 3, and a tree with maximum degree 7 and proper orientation number 4.

4. \mathcal{NP} -completeness

Ahadi and Dehgan [2] showed that it is \mathcal{NP} -complete to decide whether $\vec{\chi}(G) \leq 2$ for planar graphs G by using a reduction from the PLANAR 3-SAT problem. We first improve this result by showing that it is \mathcal{NP} -complete to decide whether the proper orientation number of planar subcubic graphs is at most 2.

Theorem 4. *The following problem is \mathcal{NP} -complete:*

INPUT : A planar graph G with $\Delta(G) = 3$ and $\delta(G) = 2$.

QUESTION : $\vec{\chi}(G) \leq 2$?

Sketch of proof. We show a reduction from the problem of deciding whether a planar 3-SAT formula is satisfiable. It is known that the PLANAR 3-SAT problem is \mathcal{NP} -complete [5].

Let $\phi = (X, \mathcal{C})$ be an instance of this problem, where $X = \{x_1, \dots, x_n\}$ is the set of variables and $\mathcal{C} = \{C_1, \dots, C_m\}$ is the set of clauses. Using the variable and clause gadgets depicted in Figures 1(a) and 1(b), respectively, we construct a planar graph $G'(\phi)$ such that $\vec{\chi}(G'(\phi)) \leq 2$ if, and only if, ϕ is satisfiable. \square

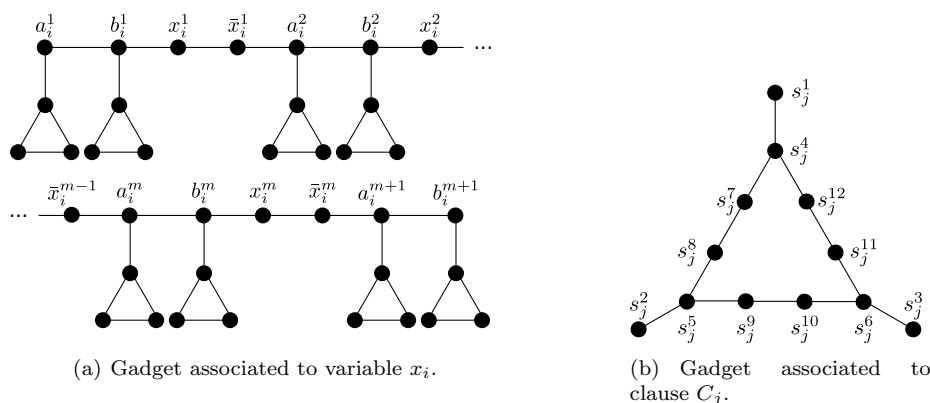


Figure 1: The variable and clause gadgets.

Recall that $\vec{\chi}(G) \leq \Delta(G)$, for any graph G . On the other hand, the following theorem shows that, for any integer $k \geq 3$, it is already \mathcal{NP} -complete to determine whether $\vec{\chi}(G) < k$, for graphs G with $\Delta(G) = k$.

Theorem 5. *Let k be an integer such that $k \geq 3$. The following problem is \mathcal{NP} -complete:*

INPUT : A graph G with $\Delta(G) = k$ and $\delta(G) = k - 1$.

QUESTION : $\vec{\chi}(G) \leq k - 1$?

Finally, we show that it is \mathcal{NP} -complete to decide whether $\vec{\chi}(G) \leq 3$, for planar bipartite graphs G .

Theorem 6. *The following problem is \mathcal{NP} -complete:*

INPUT : A planar bipartite graph G with $\Delta(G) = 5$.

QUESTION : $\vec{\chi}(G) \leq 3$?

Sketch of proof. We show a reduction from the problem of deciding whether a planar monotone 3-SAT formula is satisfiable. This problem was recently shown to be \mathcal{NP} -complete [6]. The idea of our reduction is roughly the same as in Theorems 4 and 5. \square

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