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Limiting spectral distribution of Gram matrices associated with functionals of β -mixing processes

Marwa Banna

Abstract

We give asymptotic spectral results for Gram matrices of the form $n^{-1}\mathcal{X}_n\mathcal{X}_n^T$ where the entries of \mathcal{X}_n are dependent across both rows and columns and that are functionals of absolutely regular sequences and have only finite second moments. We derive, under mild dependence conditions in addition to an arithmetical decay condition on the β -mixing coefficients, an integral equation of the Stieltjes transform of the limiting spectral distribution of $n^{-1}\mathcal{X}_n\mathcal{X}_n^T$ in terms of the spectral density of the underlying process. Applications to examples of positive recurrent Markov chains and dynamical systems are also given.

Key words: Random matrices, sample covariance matrices, Stieltjes transform, absolutely regular sequences, limiting spectral distribution, spectral density, Marčenko-Pastur distributions.

Mathematical Subject Classification (2010): 60F99, 60G10, 62E20.

1 Introduction

For a random matrix $\mathcal{X}_n \in \mathbb{R}^{N \times n}$, the study of the asymptotic behavior of the eigenvalues of the $N \times N$ Gram matrix $n^{-1}\mathcal{X}_n\mathcal{X}_n^T$ gained interest as it is employed in many applications in statistics, signal processing, quantum physics, finance, etc. In order to describe the distribution of the eigenvalues, it is convenient to introduce the empirical spectral measure defined by $\mu_{n^{-1}\mathcal{X}_n\mathcal{X}_n^T} = N^{-1} \sum_{k=1}^N \delta_{\lambda_k}$, where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of $n^{-1}\mathcal{X}_n\mathcal{X}_n^T$. This type of study was actively developed after the pioneering work of Marčenko and Pastur [11], who proved that under the assumption $\lim_{n \rightarrow +\infty} N/n = c \in (0, +\infty)$, the empirical spectral distribution of large dimensional Gram matrices with i.i.d. centered entries of finite variance converges almost surely to a non-random distribution. The limiting spectral distribution (LSD) obtained, i.e. the Marčenko-Pastur distribution, is given explicitly in terms of c and depends on the distribution of the entries of \mathcal{X}_n only through their common variance. The original Marčenko-Pastur theorem is stated for random variables having moment of forth order; for the proof under second moment only, we refer to Yin [17].

Since then, a large amount of study has been done aiming to relax the independence structure between the entries of \mathcal{X}_n . For example, Bai and Zhou [2] treated the case where the columns of \mathcal{X}_n are i.i.d. with their coordinates having a very general dependence structure and moments of forth order. Recently, Banna and Merlevède [3] extended along another direction the Marčenko-Pastur theorem to a large class of weakly dependent sequences of real random variables having only second moments. Letting $(X_k)_{k \in \mathbb{Z}}$ be a stationary process of the form $X_k = g(\dots, \varepsilon_{k-1}, \varepsilon_k, \varepsilon_{k+1}, \dots)$, where the ε_k 's are i.i.d. real valued random variables and $g: \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ is a measurable function, they consider the $N \times N$ sample covariance matrix $\mathbf{A}_n = \frac{1}{n} \sum_{k=1}^n \mathbf{X}_k \mathbf{X}_k^T$ with the \mathbf{X}_k 's being independent copies of the vector $\mathbf{X} = (X_1, \dots, X_N)^T$. Assuming only that X_0 has a moment of second order, and imposing a dependence condition expressed in terms of conditional expectation, they prove that if $\lim_{n \rightarrow \infty} N/n = c \in (0, \infty)$, then almost surely, $\mu_{\mathbf{A}_n}$ converges weakly to a non-random probability measure μ whose Stieltjes transform satisfies an integral equation that depends on c and on the spectral density of the underlying stationary process $(X_k)_{k \in \mathbb{Z}}$. In Theorem 2.4 of this paper, we let $(\varepsilon_k)_{k \in \mathbb{Z}}$ be an

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absolutely regular (β -mixing) sequence, and we get, under the absolute summability of the covariance and a near-epoch-dependence condition, the same limiting distribution as in Theorem 2.1 of [3].

In the above mentioned model, the random vector $\mathbf{X} = (X_1, \dots, X_N)^T$ can be viewed as an N -dimensional process repeated independently n times to obtain the \mathbf{X}_k 's. However, in practice it is not always possible to observe a high dimensional process several times. In the cases where only one observation of length Nn can be recorded, it seems reasonable to partition it into n dependent observations of length N , and to treat them as n dependent observations. Up to our knowledge this was first done by Pfaffel and Schlemm [12] who showed that this approach is valid and leads to the correct asymptotic eigenvalue distribution of the sample covariance matrix if the components of the underlying process are modeled as independent moving averages. They derive the LSD of a Gram matrix having the same form as the one defined in (2.3) and associated with a stationary linear process $(X_k)_{k \in \mathbb{Z}}$ with i.i.d. innovations of finite fourth moments under a polynomial decay condition of the coefficients of the underlying linear process.

In this work, we study the same model of random matrices as in [12] but considering the case where the entries come from a non causal stationary process $(X_k)_{k \in \mathbb{Z}}$ of the form $X_k = g(\dots, \varepsilon_{k-1}, \varepsilon_k, \varepsilon_{k+1}, \dots)$ where $(\varepsilon_k)_{k \in \mathbb{Z}}$ is an absolutely regular sequence and $g : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ is a measurable function such that X_k is a proper centered random variable of finite second moment. Under an arithmetical decay condition on the β -mixing coefficients, we prove in Theorem 2.1 that the Stieltjes transform is concentrated almost surely around its expectation as n tends to infinity. Then, along a similar direction as in [3] and assuming the absolute summability of the covariance plus a near-epoch-dependence condition, we prove in Theorem 2.2 that almost surely, $\mu_{\mathbf{B}_n}$ converges weakly to the same non-random limiting probability measure μ obtained in the cases mentioned above.

We recall now that the absolutely regular (β -mixing) coefficient between two σ -algebras \mathcal{A} and \mathcal{B} is defined by

$$\beta(\mathcal{A}, \mathcal{B}) = \frac{1}{2} \sup \left\{ \sum_{i \in I} \sum_{j \in J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)| \right\},$$

where the supremum is taken over all finite partitions $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ that are respectively \mathcal{A} and \mathcal{B} measurable (see Rozanov and Volkonskii [13]). For $n > 0$, the coefficients $(\beta_n)_{n \geq 0}$ of β -mixing of a sequence $(\varepsilon_i)_{i \in \mathbb{Z}}$ are defined by

$$\beta_0 = 1 \quad \text{and} \quad \beta_n = \sup_{k \in \mathbb{Z}} \beta(\sigma(\varepsilon_\ell, \ell \leq k), (\varepsilon_{\ell+n}, \ell \geq k)) \quad \text{for } n \geq 1. \quad (1.1)$$

Moreover, $(\varepsilon_i)_{i \in \mathbb{Z}}$ is said to be absolutely regular or β -mixing if $\beta_n \rightarrow 0$ as $n \rightarrow \infty$.

Outline. In Section 2, we specify the models studied and state the limiting results for the Gram matrices associated with the process defined in (2.1). The proofs shall be deferred to Section 4, whereas applications to examples of Markov chains and dynamical systems shall be introduced in Section 3.

Notation. For any real numbers x and y , $x \wedge y := \min(x, y)$ whereas $x \vee y := \max(x, y)$. Moreover, the notation $[x]$ denotes the integer part of x . For any non-negative integer q , a null row vector of dimension q will be denoted by $\mathbf{0}_q$. For a matrix A , we denote by A^T its transpose matrix and by $\text{Tr}(A)$ its trace. Finally, we shall use the notation $\|X\|_r$ for the \mathbb{L}^r -norm ($r \geq 1$) of a real valued random variable X .

For any square matrix A of order N with only real eigenvalues, its empirical spectral measure and distribution are respectively defined by

$$\mu_A = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k} \quad \text{and} \quad F^A(x) = \frac{1}{N} \sum_{k=1}^N \mathbf{1}_{\{\lambda_k \leq x\}},$$

where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of A . The Stieltjes transform of μ_A is given by

$$S_A(z) := S_{\mu_A}(z) = \int \frac{1}{x-z} d\mu_A(x) = \frac{1}{N} \text{Tr}(A - z\mathbf{I})^{-1},$$

where $z = u + iv \in \mathbb{C}^+$ (the set of complex numbers with positive imaginary part), and \mathbf{I} is the identity matrix of order N .

2 Results

We consider a non causal stationary process $(X_k)_{k \in \mathbb{Z}}$ defined as follows: let $(\varepsilon_i)_{i \in \mathbb{Z}}$ be an absolutely regular process with β -mixing sequence $(\beta_k)_{k \geq 0}$ and let $g : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be a measurable function such that, for any $k \in \mathbb{Z}$,

$$X_k = g(\xi_k) \quad \text{with} \quad \xi_k = (\dots, \varepsilon_{k-1}, \varepsilon_k, \varepsilon_{k+1}, \dots) \quad (2.1)$$

is a proper two-sided random variable, $\mathbb{E}(g(\xi_k)) = 0$ and $\|g(\xi_k)\|_2 < \infty$.

Now, let $N := N(n)$ be a sequence of positive integers and consider the $N \times n$ random matrix \mathcal{X}_n defined by

$$\mathcal{X}_n = ((\mathcal{X}_n)_{ij})_{i,j} = (X_{(j-1)N+i})_{i,j} = \begin{pmatrix} X_1 & X_{N+1} & \cdots & X_{(n-1)N+1} \\ X_2 & X_{N+2} & \cdots & X_{(n-1)N+2} \\ \vdots & \vdots & & \vdots \\ X_N & X_{2N} & \cdots & X_{nN} \end{pmatrix} \in \mathcal{M}_{N \times n}(\mathbb{R}) \quad (2.2)$$

and note that its entries are dependent across both rows and columns. Let \mathbf{B}_n be its corresponding Gram matrix given by

$$\mathbf{B}_n = \frac{1}{n} \mathcal{X}_n \mathcal{X}_n^T. \quad (2.3)$$

In what follows, \mathbf{B}_n will be referred to as the Gram matrix associated with $(X_k)_{k \in \mathbb{Z}}$. Our purpose is to study the limiting distribution of the empirical spectral measure $\mu_{\mathbf{B}_n}$ defined by

$$\mu_{\mathbf{B}_n}(x) = \frac{1}{N} \sum_{k=1}^N \delta_{\lambda_k},$$

where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of \mathbf{B}_n . We start by showing that if the β -mixing coefficients decay arithmetically then the Stieltjes transform of \mathbf{B}_n concentrates almost surely around its expectation as n tends to infinity.

Theorem 2.1 *Let \mathbf{B}_n be the matrix defined in (2.3) and associated with $(X_k)_{k \in \mathbb{Z}}$ defined in (2.1). If $\lim_{n \rightarrow \infty} N/n = c \in (0, \infty)$ and*

$$\sum_{n \geq 1} \log(n)^{\frac{3\alpha}{2}} n^{-\frac{1}{2}} \beta_n < \infty \quad \text{for some } \alpha > 1, \quad (2.4)$$

the following convergence holds: for any $z \in \mathbb{C}^+$,

$$S_{\mathbf{B}_n}(z) - \mathbb{E}(S_{\mathbf{B}_n}(z)) \rightarrow 0 \quad \text{almost surely, as } n \rightarrow +\infty.$$

As the entries of \mathcal{X}_n are dependent across both rows and columns, classical arguments as those in Theorem 1 (ii) of [7] are not sufficient to prove the above convergence. In fact, we use maximal coupling for absolutely regular sequences in order to break the dependence structure of the matrix.

Theorem 2.2 Let \mathbf{B}_n be the Gram matrix defined in (2.3) and associated with $(X_k)_{k \in \mathbb{Z}}$ defined in (2.1). Provided that $\lim_{n \rightarrow \infty} N/n = c \in (0, \infty)$ and assuming that (2.4) is satisfied and that

$$\sum_{k \geq 0} |\text{Cov}(X_0, X_k)| < \infty \quad (2.5)$$

and

$$\lim_{n \rightarrow +\infty} n \|X_0 - \mathbb{E}(X_0 | \varepsilon_{-n}, \dots, \varepsilon_n)\|_2^2 = 0 \quad (2.6)$$

then, with probability one, $\mu_{\mathbf{B}_n}$ converges weakly to a probability measure whose Stieltjes transform $S = S(z)$ ($z \in \mathbb{C}^+$) satisfies the equation

$$z = -\frac{1}{\underline{S}} + \frac{c}{2\pi} \int_0^{2\pi} \frac{1}{\underline{S} + (2\pi f(\lambda))^{-1}} d\lambda, \quad (2.7)$$

where $\underline{S}(z) := -(1-c)/z + cS(z)$ and $f(\cdot)$ is the spectral density of $(X_k)_{k \in \mathbb{Z}}$.

Remark 2.3

1. Condition (2.5) implies that the spectral density $f(\cdot)$ of $(X_k)_{k \in \mathbb{Z}}$ exists, is continuous and bounded on $[0, 2\pi)$. Moreover, it follows from Proposition 1 in Yao [16] that the limiting spectral distribution is compactly supported.
2. Condition (2.6) is a so-called \mathbb{L}^2 -near-epoch-dependent condition.

Now, for a positive integer n , we consider n independent copies of the sequence $(\varepsilon_k)_{k \in \mathbb{Z}}$ that we denote by $(\varepsilon_k^{(i)})_{k \in \mathbb{Z}}$ for $i = 1, \dots, n$. Setting $\xi_k^{(i)} = (\dots, \varepsilon_{k-1}^{(i)}, \varepsilon_k^{(i)}, \varepsilon_{k+1}^{(i)}, \dots)$ and $X_k^{(i)} = g(\xi_k^{(i)})$, it follows that $(X_k^{(1)})_{k \in \mathbb{Z}}, \dots, (X_k^{(n)})_{k \in \mathbb{Z}}$ are n independent copies of $(X_k)_{k \in \mathbb{Z}}$. Define now for any $i \in \{1, \dots, n\}$, the random vector $\mathbf{X}_i = (X_1^{(i)}, \dots, X_N^{(i)})^T$ and set

$$\mathfrak{X}_n = (\mathbf{X}_1 | \dots | \mathbf{X}_n) \quad \text{and} \quad \mathbf{A}_n = \frac{1}{n} \mathfrak{X}_n \mathfrak{X}_n^T = \frac{1}{n} \sum_{k=1}^n \mathbf{X}_k \mathbf{X}_k^T. \quad (2.8)$$

Theorem 2.4 Let \mathbf{A}_n be the Gram matrix defined in (2.8) and associated with $(X_k)_{k \in \mathbb{Z}}$ defined in (2.1). Provided that $\lim_{n \rightarrow \infty} N/n = c \in (0, \infty)$ and $\lim_{n \rightarrow \infty} \beta_n = 0$ and assuming that (2.5) and (2.6) are satisfied, then, with probability one, $\mu_{\mathbf{A}_n}$ converges weakly to a probability measure whose Stieltjes transform $S = S(z)$ ($z \in \mathbb{C}^+$) satisfies equation (2.7).

Remark 2.5 The proof of Theorem 2.4 is quite similar to that of Theorem 2.2. Since the columns of the $N \times n$ matrix \mathfrak{X}_n considered in (2.8) are independent copies of the random vector $(X_1, \dots, X_N)^T$, then, as a consequence of Theorem 1 (ii) of Guntuboyina and Leeb [7], we can approximate directly $S_{\mathbf{B}_n}(z)$ by its expectation and there is no need to any coupling arguments as those in Theorem 2.1 and thus there is no need to the arithmetic decay condition (2.4) on the absolutely regular coefficients. The rest of the proof follows exactly as that of Theorem 2.2 after simple modifications of indices.

3 Applications

In this section we shall apply the results of Section 2 to the Harris recurrent Markov chain and to some uniformly expanding maps in dynamical systems.

3.1 Harris recurrent Markov chain

The following example is a symmetrized version of the Harris recurrent Markov chain defined by Doukhan *et al.* [6]. Let $(\varepsilon_n)_{n \in \mathbb{Z}}$ be a stationary Markov chain taking values in $E = [-1, 1]$ and let K be its Markov kernel defined by

$$K(x, \cdot) = (1 - |x|)\delta_x + |x|\nu,$$

with ν being a symmetric atomless law on E and δ_x denoting the Dirac measure at point x . Assume that $\theta = \int_E |x|^{-1}\nu(dx) < \infty$ then there exists a unique invariant measure π given by

$$\pi(dx) = \theta^{-1}|x|^{-1}\nu(dx)$$

and $(\varepsilon_n)_{n \in \mathbb{Z}}$ is positively recurrent. We shall assume in what follows that the measure ν satisfies for any $x \in [0, 1]$,

$$\frac{d\nu}{dx}(x) \leq c x^a \quad \text{for some } a, c > 0. \quad (3.9)$$

Now, let g be a measurable function defined on E such that

$$X_k = g(\varepsilon_k) \quad (3.10)$$

is a centered random variable with finite moment of second order.

Corollary 3.1 *Let \mathbf{A}_n and \mathbf{B}_n be the matrices defined in (2.8) and (2.3) respectively and associated with $(X_k)_{k \in \mathbb{Z}}$ defined in (3.10). Assume that for any $x \in E$, $g(-x) = -g(x)$ and $|g(x)| \leq C|x|^{1/2}$ with C being a positive constant. Then, provided that ν satisfies (3.9) and $N/n \rightarrow c \in (0, \infty)$, the conclusion of Theorem 2.4 holds for $\mu_{\mathbf{A}_n}$. In addition, if (3.9) holds with $a > 1/2$ then the conclusion of Theorem 2.2 holds for $\mu_{\mathbf{B}_n}$.*

Proof. Notice that in this case (2.6) holds directly. Hence, to prove the above corollary, it suffices to show that conditions (2.5) and (2.4) are satisfied. Noting that g is an odd function we have

$$\mathbb{E}(g(\varepsilon_k)|\varepsilon_0) = (1 - |\varepsilon_0|)^k g(\varepsilon_0) \quad \text{a.s.}$$

Therefore, by the assumption on g and (3.9), we get for any $k \geq 0$,

$$\begin{aligned} \mathbb{E}(X_0 X_k) &= \mathbb{E}(g(\varepsilon_0)\mathbb{E}(g(\varepsilon_k)|\varepsilon_0)) = \theta^{-1} \int_E g^2(x)(1 - |x|)^k |x|^{-1}\nu(dx) \\ &\leq C^2 c \theta^{-1} \int_E \frac{x^{a+1}}{|x|} (1 - |x|)^k dx. \end{aligned} \quad (3.11)$$

By the properties of the Beta and Gamma functions, it follows that $|\mathbb{E}(X_0 X_k)| = O\left(\frac{1}{k^{a+1}}\right)$ and thus (2.5) holds. Moreover, it has been proved in Section 4 of [6] that if (3.9) is satisfied then $(\varepsilon_k)_{k \in \mathbb{Z}}$ is an absolutely regular sequence with $\beta_n = O(n^{-a})$ as $n \rightarrow \infty$ and thereby all the conditions of Theorem 2.4 are satisfied. However, if in addition $a > 1/2$ then (2.4) is also satisfied and Theorem 2.2 follows as well.

3.2 Uniformly expanding maps

Functionals of absolutely regular sequences occur naturally as orbits of chaotic dynamical systems. For instance, for uniformly expanding maps $T : [0, 1] \rightarrow [0, 1]$ with absolutely continuous invariant measure μ , one can write $T^k = g(\varepsilon_k, \varepsilon_{k+1}, \dots)$ for some measurable function $g : \mathbb{R}^{\mathbb{Z}} \rightarrow \mathbb{R}$ where $(\varepsilon_k)_{k \geq 0}$ is an absolutely regular sequence. We refer to Section 2 of [9] for more details and

for a precise definition of such maps (see also Example 1.4 in [5]). Hofbauer and Keller prove in Theorem 4 of [9] that the mixing rate of $(\varepsilon_k)_{k \geq 0}$ decreases exponentially, i.e.

$$\beta_k \leq C e^{-\lambda k}, \quad \text{for some } C, \lambda > 0. \quad (3.12)$$

Moreover, setting for any $k \geq 0$,

$$X_k = f \circ T^k - \mu(f), \quad (3.13)$$

where $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous Hölder function, we have by Theorem 5 of [9] that $(X_k)_{k \geq 0}$ satisfies (2.5) and (2.6). Finally, as (2.4) holds by (3.12) then the conclusions of Theorem 2.4 and Theorem 2.2 hold for the associated matrices \mathbf{A}_n and \mathbf{B}_n respectively.

4 Proof of Theorem 2.1

Let m be a positive integer (fixed for the moment) such that $m \leq \sqrt{N}/2$ and let $(X_{k,m})_{k \in \mathbb{Z}}$ be the sequence defined for any $k \in \mathbb{Z}$ by,

$$X_{k,m} = \mathbb{E}(X_k | \varepsilon_{k-m}, \dots, \varepsilon_{k+m}). \quad (4.1)$$

Consider the $N \times n$ matrix $\mathcal{X}_{n,m} = ((\mathcal{X}_{n,m})_{ij})_{i,j} = (X_{(j-1)N+i,m})_{i,j}$ and finally set

$$\mathbf{B}_{n,m} = \frac{1}{n} \mathcal{X}_{n,m} \mathcal{X}_{n,m}^T. \quad (4.2)$$

The proof will be done in two principal steps. First, we shall prove that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |S_{\mathbf{B}_n}(z) - S_{\mathbf{B}_{n,m}}(z)| = 0 \quad \text{a.s.} \quad (4.3)$$

and then

$$\lim_{n \rightarrow \infty} |S_{\mathbf{B}_{n,m}}(z) - \mathbb{E}(S_{\mathbf{B}_{n,m}}(z))| = 0 \quad \text{a.s.} \quad (4.4)$$

We note that for any two $N \times n$ random matrices \mathbf{A} and \mathbf{B} , we have

$$|S_{\mathbf{A}\mathbf{A}^T}(z) - S_{\mathbf{B}\mathbf{B}^T}(z)| \leq \frac{\sqrt{2}}{Nv^2} |\text{Tr}(\mathbf{A}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T)|^{1/2} |\text{Tr}(\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B})^T|^{1/2}. \quad (4.5)$$

For a proof, the reader can check Inequalities (4.18) and (4.19) of [3]. Thus, we get, for any $z = u + iv \in \mathbb{C}^+$,

$$|S_{\mathbf{B}_n}(z) - S_{\mathbf{B}_{n,m}}(z)|^2 \leq \frac{2}{v^4} \left(\frac{1}{N} \text{Tr}(\mathbf{B}_n + \mathbf{B}_{n,m}) \right) \left(\frac{1}{Nn} \text{Tr}(\mathcal{X}_n - \mathcal{X}_{n,m})(\mathcal{X}_n - \mathcal{X}_{n,m})^T \right) \quad (4.6)$$

Recall that mixing implies ergodicity and note that as $(\varepsilon_k)_{k \in \mathbb{Z}}$ is an ergodic sequence of real-valued random variables then $(X_k)_{k \in \mathbb{Z}}$ is also so. Therefore, by the ergodic theorem,

$$\lim_{n \rightarrow +\infty} \frac{1}{N} \text{Tr}(\mathbf{B}_n) = \lim_{n \rightarrow +\infty} \frac{1}{Nn} \sum_{k=1}^{Nn} X_k^2 = \mathbb{E}(X_0^2) \quad \text{a.s.} \quad (4.7)$$

Similarly,

$$\lim_{n \rightarrow \infty} \frac{1}{N} \text{Tr}(\mathbf{B}_{n,m}) = \mathbb{E}(X_{0,m}^2) \quad \text{a.s.} \quad (4.8)$$

Starting from (4.6) and noticing that $\mathbb{E}(X_{0,m}^2) \leq \mathbb{E}(X_0^2)$, it follows that (4.3) holds if we prove

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \left| \frac{1}{Nn} \text{Tr}(\mathcal{X}_n - \mathcal{X}_{n,m})(\mathcal{X}_n - \mathcal{X}_{n,m})^T \right| = 0 \quad \text{a.s.} \quad (4.9)$$

By the construction of \mathcal{X}_n and $\mathcal{X}_{n,m}$ and again the ergodic theorem, we get

$$\lim_{n \rightarrow \infty} \frac{1}{Nn} \text{Tr}(\mathcal{X}_n - \mathcal{X}_{n,m})(\mathcal{X}_n - \mathcal{X}_{n,m})^T = \lim_{n \rightarrow \infty} \frac{1}{Nn} \sum_{k=1}^{Nn} (X_k - X_{k,m})^2 = \mathbb{E}(X_0 - X_{0,m})^2 \quad \text{a.s.}$$

(4.9) follows by applying the usual martingale convergence theorem in \mathbb{L}^2 , from which we infer that $\lim_{m \rightarrow +\infty} \|X_0 - \mathbb{E}(X_0 | \varepsilon_{-m}, \dots, \varepsilon_m)\|_2 = 0$ (see Corollary 2.2 by Hall and Heyde [8]).

We turn now to the proof of (4.4). With this aim, we shall prove that for any $z = u + iv$ and $x > 0$,

$$\mathbb{P}(|S_{\mathbf{B}_{n,m}}(z) - \mathbb{E}S_{\mathbf{B}_{n,m}}(z)| > 4x) \leq 4 \exp \left\{ -\frac{x^2 v^2 N^2 (\log n)^\alpha}{320 n^2} \right\} + \frac{32 n^2 (\log n)^\alpha}{x^2 v^2 N^2} \beta_{\left[\frac{Nn}{(\log n)^\alpha} \right]}, \quad (4.10)$$

for some $\alpha > 1$. Noting that

$$\sum_{n \geq 2} (\log n)^\alpha \beta_{\lfloor n^2 / (\log n)^\alpha \rfloor} < +\infty \text{ is equivalent to (2.4)}$$

and applying Borel-Cantelli Lemma, (4.4) follows by (2.4) and the fact that $\lim_{n \rightarrow \infty} N/n = c$. To prove (4.10), we start by noting that

$$\begin{aligned} & \mathbb{P}(|(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E}(S_{\mathbf{B}_{n,m}}(z))| > 4x) \\ & \leq \mathbb{P}(|\Re(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E}\Re(S_{\mathbf{B}_{n,m}}(z))| > 2x) + \mathbb{P}(|\Im(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E}\Im(S_{\mathbf{B}_{n,m}}(z))| > 2x) \end{aligned}$$

For a row vector $\mathbf{x} \in \mathbb{R}^{Nn}$, we partition it into n elements of dimension N and write $\mathbf{x} = (x_1, \dots, x_n)$ where x_1, \dots, x_n are row vectors of \mathbb{R}^N . Now, let $A(\mathbf{x})$ and $B(\mathbf{x})$ be respectively the $N \times n$ and $N \times N$ matrices defined by

$$A(\mathbf{x}) = (x_1^T | \dots | x_n^T) \quad \text{and} \quad B(\mathbf{x}) = \frac{1}{n} A(\mathbf{x}) A(\mathbf{x})^T. \quad (4.11)$$

Also, let $h_1 := h_{1,z}$ and $h_2 := h_{2,z}$ be the functions defined from \mathbb{R}^{Nn} into \mathbb{R} by

$$h_1(\mathbf{x}) = \int f_{1,z} d\mu_{B(\mathbf{x})} \quad \text{and} \quad h_2(\mathbf{x}) = \int f_{2,z} d\mu_{B(\mathbf{x})},$$

where $f_{1,z}(\lambda) = \frac{\lambda - u}{(\lambda - u)^2 + v^2}$ and $f_{2,z}(\lambda) = \frac{v}{(\lambda - u)^2 + v^2}$ and note that $S_{B(\mathbf{x})}(z) = h_1(\mathbf{x}) + ih_2(\mathbf{x})$. Now, denoting by $\mathbf{X}_{1,m}^T, \dots, \mathbf{X}_{n,m}^T$ the columns of $\mathcal{X}_{n,m}$ and setting \mathbf{A} to be the row random vector of \mathbb{R}^{Nn} given by

$$\mathbf{A} = (\mathbf{X}_{1,m}, \dots, \mathbf{X}_{n,m}),$$

we note that $B(\mathbf{A}) = \mathbf{B}_{n,m}$ and $h_1(\mathbf{A}) = \Re(S_{\mathbf{B}_{n,m}}(z))$. Moreover, setting $\mathcal{F}_i = \sigma(\varepsilon_k, k \leq iN + m)$ for $1 \leq i \leq n$ with the convention that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_{\lfloor n/q \rfloor q + q} = \mathcal{F}_n$, we note that $\mathbf{X}_{1,m}, \dots, \mathbf{X}_{i,m}$ are \mathcal{F}_i -measurable. Finally, we let q be a positive integer less than n and write the following decomposition:

$$\begin{aligned} \Re(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E}\Re(S_{\mathbf{B}_{n,m}}(z)) &= h_1(\mathbf{X}_{1,m}, \dots, \mathbf{X}_{n,m}) - \mathbb{E}h_1(\mathbf{X}_{1,m}, \dots, \mathbf{X}_{n,m}) \\ &= \sum_{i=1}^{\lfloor n/q \rfloor + 1} (\mathbb{E}(h_1(\mathbf{A}) | \mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}) | \mathcal{F}_{(i-1)q})). \end{aligned}$$

Now, let $(\mathbf{A}_i)_{1 \leq i \leq [n/q]+1}$ be a sequence of row random vectors of \mathbb{R}^{Nn} defined for any $i \in \{1, \dots, [n/q]\}$ by

$$\mathbf{A}_i = (\mathbf{X}_{1,m}, \dots, \mathbf{X}_{(i-1)q,m}, \underbrace{\mathbf{0}_N, \dots, \mathbf{0}_N}_{2q \text{ times}}, \mathbf{X}_{(i+1)q+1,m}, \dots, \mathbf{X}_{n,m}),$$

and for $i = [n/q] + 1$ by

$$\mathbf{A}_{[n/q]+1} = (\mathbf{X}_{1,m}, \dots, \mathbf{X}_{[n/q]q,m}, \underbrace{\mathbf{0}_N, \dots, \mathbf{0}_N}_{n-[n/q]q \text{ times}}).$$

Noting that $\mathbb{E}(h_1(\mathbf{A}_{[n/q]+1})|\mathcal{F}_n) = \mathbb{E}(h_1(\mathbf{A}_{[n/q]+1})|\mathcal{F}_{[n/q]q})$, we write

$$\begin{aligned} & \Re\mathfrak{e}(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E} \Re\mathfrak{e}(S_{\mathbf{B}_{n,m}}(z)) \\ &= \sum_{i=1}^{[n/q]+1} \left(\mathbb{E}(h_1(\mathbf{A}) - h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}) - h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q}) \right) \\ & \quad + \sum_{i=1}^{[n/q]} \left(\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q}) \right) \\ & := M_{[n/q]+1,q} + \sum_{i=1}^{[n/q]} \left(\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q}) \right). \end{aligned} \quad (4.12)$$

Thus, we get

$$\begin{aligned} & \mathbb{P} \left(\left| \Re\mathfrak{e}(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E} \Re\mathfrak{e}(S_{\mathbf{B}_{n,m}}(z)) \right| > 2x \right) \\ & \leq \mathbb{P}(|M_{[n/q]+1,q}| > x) + \mathbb{P} \left(\left| \sum_{i=1}^{[n/q]} \left(\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q}) \right) \right| > x \right). \end{aligned} \quad (4.13)$$

Note that $(M_{k,q})_k$ is a centered martingale with respect to the filtration $(\mathcal{G}_{k,q})_k$ defined by $\mathcal{G}_{k,q} = \mathcal{F}_{kq}$. Moreover, for any $k \in \{1, \dots, [n/q] + 1\}$,

$$\begin{aligned} \|M_{k,q} - M_{k-1,q}\|_\infty &= \|\mathbb{E}(h_1(\mathbf{A}) - h_1(\mathbf{A}_k)|\mathcal{F}_{kq}) - \mathbb{E}(h_1(\mathbf{A}) - h_1(\mathbf{A}_k)|\mathcal{F}_{(k-1)q})\|_\infty \\ &\leq 2\|h_1(\mathbf{A}) - h_1(\mathbf{A}_k)\|_\infty \end{aligned}$$

Noting that $\|f'_{1,z}\|_1 = 2/v$ then by integrating by parts, we get

$$\begin{aligned} |h_1(\mathbf{A}) - h_1(\mathbf{A}_k)| &= \left| \int f_{1,z} d\mu_{B(\mathbf{A})} - \int f_{1,z} d\mu_{B(\mathbf{A}_k)} \right| \leq \|f'_{1,z}\|_1 \|F^{B(\mathbf{A})} - F^{B(\mathbf{A}_k)}\|_\infty \\ &\leq \frac{2}{vN} \text{Rank}(A(\mathbf{A}) - A(\mathbf{A}_k)), \end{aligned} \quad (4.14)$$

where the second inequality follows from Theorem A.44 of [1]. As for any $k \in \{1, \dots, [n/q]\}$, $\text{Rank}(A(\mathbf{A}) - A(\mathbf{A}_k)) \leq 2q$ and $\text{Rank}(A(\mathbf{A}) - A(\mathbf{A}_{[n/q]+1})) \leq q$, then overall we derive that

$$\|M_{k,q} - M_{k-1,q}\|_\infty \leq \frac{8q}{vN} \quad \text{and} \quad \|M_{[n/q]+1,q} - M_{[n/q]q}\|_\infty \leq \frac{4q}{vN} \quad \text{a.s.}$$

and hence applying the Azuma-Hoeffding inequality for martingales we get for any $x > 0$,

$$\mathbb{P}(|M_{[n/q]+1,q}| > x) \leq 2 \exp \left\{ -\frac{x^2 v^2 N^2}{160 q n} \right\}. \quad (4.15)$$

Now to control the second term of (4.12), we have, by Markov's inequality and orthogonality, for any $x > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^{\lfloor n/q \rfloor} (\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q}))\right| > x\right) \leq \frac{1}{x^2} \sum_{i=1}^{\lfloor n/q \rfloor} \|\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q})\|_2^2. \quad (4.16)$$

Fixing $i \in \{1, \dots, \lfloor n/q \rfloor\}$, we construct by Berbee's maximal coupling lemma [4], a sequence $(\varepsilon'_k)_{k \in \mathbb{Z}}$ distributed as $(\varepsilon_k)_{k \in \mathbb{Z}}$ and independent of \mathcal{F}_{iq} such that for any $j > iqN + m$,

$$\mathbb{P}(\varepsilon'_k \neq \varepsilon_k, \text{ for some } k \geq j) = \beta_{j-iqN-m}. \quad (4.17)$$

Let $(X'_{k,m})_{k \geq 1}$ be the sequence defined for any $k \geq 1$ by $X'_{k,m} = \mathbb{E}(X_k | \varepsilon'_{k-m}, \dots, \varepsilon'_{k+m})$ and let $\mathbf{X}'_{i,m}$ be the row vector of \mathbb{R}^N defined by $\mathbf{X}'_{i,m} = (X'_{(i-1)N+1,m}, \dots, X'_{iN,m})$. Finally, we define for any $i \in \{1, \dots, \lfloor n/q \rfloor\}$ the row random vector \mathbf{A}'_i of \mathbb{R}^{Nn} by

$$\mathbf{A}'_i = (\mathbf{X}_{1,m}, \dots, \mathbf{X}_{(i-1)q,m}, \underbrace{\mathbf{0}_N, \dots, \mathbf{0}_N}_{2q \text{ times}}, \mathbf{X}'_{(i+1)q+1,m}, \dots, \mathbf{X}'_{n,m}).$$

As $\mathbf{X}'_{(i+1)q+1,m}, \dots, \mathbf{X}'_{n,m}$ are independent of \mathcal{F}_{iq} then $\mathbb{E}(h_1(\mathbf{A}'_i)|\mathcal{F}_{iq}) = \mathbb{E}(h_1(\mathbf{A}'_i)|\mathcal{F}_{(i-1)q})$. Thus we write

$$\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q}) = \mathbb{E}(h_1(\mathbf{A}_i) - h_1(\mathbf{A}'_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i) - h_1(\mathbf{A}'_i)|\mathcal{F}_{(i-1)q}).$$

and infer that

$$\begin{aligned} \|\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q})\|_2 \\ \leq \|\mathbb{E}(h_1(\mathbf{A}_i) - h_1(\mathbf{A}'_i)|\mathcal{F}_{iq})\|_2 + \|\mathbb{E}(h_1(\mathbf{A}_i) - h_1(\mathbf{A}'_i)|\mathcal{F}_{(i-1)q})\|_2 \\ \leq 2\|h_1(\mathbf{A}_i) - h_1(\mathbf{A}'_i)\|_2 \end{aligned} \quad (4.18)$$

Similarly as in (4.14), we have

$$\begin{aligned} |h_1(\mathbf{A}_i) - h_1(\mathbf{A}'_i)| &\leq \frac{2}{vN} \text{Rank}(A(\mathbf{A}_i) - A(\mathbf{A}'_i)) \leq \frac{2}{vN} \sum_{\ell=(i+1)q+1}^n \mathbf{1}_{\{\mathbf{x}'_{\ell,m} \neq \mathbf{x}_{\ell,m}\}} \\ &\leq \frac{2n}{vN} \mathbf{1}_{\{\varepsilon'_k \neq \varepsilon_k, \text{ for some } k \geq (i+1)qN+1-m\}}. \end{aligned}$$

Hence by (4.17), we infer that

$$\|h_1(\mathbf{A}_i) - h_1(\mathbf{A}'_i)\|_2^2 \leq \frac{4n^2}{v^2 N^2} \beta_{qN+1-2m} \leq \frac{4n^2}{v^2 N^2} \beta_{(q-1)N}, \quad (4.19)$$

Starting from (4.16) together with (4.18) and (4.19), it follows that

$$\mathbb{P}\left(\left|\sum_{i=1}^{\lfloor n/q \rfloor} (\mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{iq}) - \mathbb{E}(h_1(\mathbf{A}_i)|\mathcal{F}_{(i-1)q}))\right| > x\right) \leq \frac{16n^3}{x^2 v^2 q N^2} \beta_{(q-1)N}. \quad (4.20)$$

Therefore, considering (4.13) and gathering the upper bounds in (4.15) and (4.20), we get

$$\mathbb{P}(|\Re(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E} \Re(S_{\mathbf{B}_{n,m}}(z))| > 2x) \leq 2 \exp\left\{-\frac{x^2 v^2 N^2}{160 q n}\right\} + \frac{16n^3}{x^2 v^2 q N^2} \beta_{(q-1)N}.$$

Finally, noting that $\mathbb{P}(|\Im(S_{\mathbf{B}_{n,m}}(z)) - \mathbb{E} \Im(S_{\mathbf{B}_{n,m}}(z))| > 2x)$ also admits the same upper bound and choosing $q = \lceil n/(\log n)^\alpha \rceil + 1$, (4.10) follows ending the proof of Theorem 2.1.

5 Proof of Theorem 2.2

To prove Theorem 2.2 it suffices to show that for any $z \in \mathbb{C}^+$,

$$S_{\mathbf{B}_n}(z) \rightarrow S(z) \quad \text{a.s.} \quad (5.1)$$

where $S(z)$ satisfies equation (2.7). However, by Theorem 2.1, it becomes sufficient to prove that, for any $z \in \mathbb{C}^+$,

$$\lim_{n \rightarrow \infty} \mathbb{E}(S_{\mathbf{B}_n}(z)) = S(z). \quad (5.2)$$

Now, let $(Z_k)_{k \in \mathbb{Z}}$ be a centered Gaussian process with real values, whose covariance function is given for any $k, \ell \in \mathbb{Z}$ by

$$\text{Cov}(Z_k, Z_\ell) = \text{Cov}(X_k, X_\ell). \quad (5.3)$$

For a positive integer n , we consider n independent copies of the Gaussian process $(Z_k)_{k \in \mathbb{Z}}$ that are also independent of $(X_k)_{k \in \mathbb{Z}}$. We shall denote these copies by $(Z_k^{(i)})_{k \in \mathbb{Z}}$ for $i = 1, \dots, n$. Now, consider the $N \times n$ matrix $\mathcal{Z}_n = ((\mathcal{Z}_n)_{u,v})_{uv} = (Z_u^{(v)})_{uv}$ and note that its columns are independent copies of the vector $(Z_1, \dots, Z_N)^T$. Finally, consider its associated Gram matrix

$$\mathbf{G}_n = \frac{1}{n} \mathcal{Z}_n \mathcal{Z}_n^T. \quad (5.4)$$

Banna and Merlevède proved in Section 4.4 of [3] that

$$\lim_{n \rightarrow \infty} \mathbb{E}(S_{\mathbf{G}_n}(z)) = S(z). \quad (5.5)$$

This was accomplished with the help of Theorem 1.1 of Silverstein [14] combined with arguments developed in the proof of Theorem 1 of Yao [16]. Thus, the proof of (5.2) is reduced to prove, for any $z \in \mathbb{C}^+$,

$$\lim_{n \rightarrow \infty} |\mathbb{E}(S_{\mathbf{B}_n}(z)) - \mathbb{E}(S_{\mathbf{G}_n}(z))| = 0. \quad (5.6)$$

The proof of (5.6), being technical, will be divided into three major steps (Sections 5.1 to 5.3).

5.1 A First Approximation

By (2.6), there exists a sequence $(a_n)_{n \geq 1}$ of positive integers such that

$$\lim_{n \rightarrow +\infty} a_n = +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} a_n n \|X_0 - \mathbb{E}(X_0 | \mathcal{F}_{-n}^n)\|_2^2 = 0. \quad (5.7)$$

Let $p := p(m) = a_m m$ and define $k_N = \left\lceil \frac{N}{p+3m} \right\rceil$. We write now the subset $\{1, \dots, Nn\}$ as a union of disjoint subsets of \mathbb{N} as follows:

$$[1, Nn] \cap \mathbb{N} = \bigcup_{i=1}^n [(i-1)N + 1, iN] \cap \mathbb{N} = \bigcup_{i=1}^n \bigcup_{\ell=1}^{k_N+1} I_\ell^i \cup J_\ell^i,$$

where, for $i \in \{1, \dots, n\}$ and $\ell \in \{1, \dots, k_N\}$,

$$\begin{aligned} I_\ell^i &:= [(i-1)N + (\ell-1)(p+3m) + 1, (i-1)N + (\ell-1)(p+3m) + p] \cap \mathbb{N}, \\ J_\ell^i &:= [(i-1)N + (\ell-1)(p+3m) + p + 1, (i-1)N + \ell(p+3m)] \cap \mathbb{N}, \end{aligned}$$

and, for $\ell = k_N + 1$, $I_{k_N+1}^i = \emptyset$ and

$$J_{k_N+1}^i = [(i-1)N + k_N(p+3m) + 1, iN] \cap \mathbb{N}.$$

Note that for all $i \in \{1, \dots, n\}$, $J_{k_N+1}^i = \emptyset$ if $k_N(p+3m) = N$. Now, let M be a fixed positive number not depending on (n, m) and let φ_M be the function defined by $\varphi_M(x) = (x \wedge M) \vee (-M)$. Setting

$$B_{i,\ell} = (\varepsilon_{(i-1)N+(\ell-1)(p+3m)+1-m}, \dots, \varepsilon_{(i-1)N+(\ell-1)(p+3m)+p+m}), \quad (5.8)$$

we define the sequences $(\tilde{X}_{k,m,M})_{k \geq 1}$ and $(\bar{X}_{k,m,M})_{k \geq 1}$ as follows:

$$\tilde{X}_{k,m,M} = \begin{cases} \mathbb{E}(\varphi_M(X_k) | B_{i,\ell}) & \text{if } k \in I_\ell^i \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{X}_{k,m,M} = \tilde{X}_{k,m,M} - \mathbb{E}(\tilde{X}_{k,m,M}). \quad (5.9)$$

To soothe the notations, we shall write $\tilde{X}_{k,m}$ and $\bar{X}_{k,m}$ instead of $\tilde{X}_{k,m,M}$ and $\bar{X}_{k,m,M}$ respectively. Note that for any $k \geq 1$,

$$\|\bar{X}_{k,m}\|_2 \leq 2\|\tilde{X}_{k,m}\|_2 = 2\|\mathbb{E}(\varphi_M(X_k) | B_{i,\ell})\|_2 \leq 2\|\varphi_M(X_k)\|_2 \leq 2\|X_k\|_2 = 2\|X_0\|_2, \quad (5.10)$$

and

$$\|\bar{X}_{k,m}\|_\infty \leq 2\|\tilde{X}_{k,m}\|_\infty \leq 2M, \quad (5.11)$$

where the last equality in (5.10) follows from the stationarity of $(X_k)_k$. As $\bar{X}_{k,m}$ is $\sigma(B_{i,\ell})$ -measurable then it can be written as a measurable function h_k of $B_{i,\ell}$, i.e.

$$\bar{X}_{k,m} = h_k(B_{i,\ell}). \quad (5.12)$$

Finally, let $\bar{\mathcal{X}}_{n,m} = ((\bar{\mathcal{X}}_{n,m})_{ij})_{i,j} = (\bar{X}_{(j-1)N+i,m})_{i,j}$ and set

$$\bar{\mathbf{B}}_{n,m} := \frac{1}{n} \bar{\mathcal{X}}_{n,m} \bar{\mathcal{X}}_{n,m}^T. \quad (5.13)$$

We shall approximate \mathbf{B}_n by $\bar{\mathbf{B}}_{n,m}$ by the following proposition:

Proposition 5.1 *Let \mathbf{B}_n and $\bar{\mathbf{B}}_{n,m}$ be the matrices defined in (2.3) and (5.13) respectively then if $\lim_{n \rightarrow \infty} N/n = c \in (0, \infty)$, we have for any $z \in \mathbb{C}^+$,*

$$\lim_{m \rightarrow +\infty} \limsup_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |\mathbb{E}(S_{\mathbf{B}_n}(z)) - \mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}}(z))| = 0. \quad (5.14)$$

Proof. By (4.5) and Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned} & |\mathbb{E}(S_{\mathbf{B}_n}(z)) - \mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}}(z))| \\ & \leq \frac{\sqrt{2}}{v^2} \left\| \frac{1}{N} \text{Tr}(\mathbf{B}_n + \bar{\mathbf{B}}_{n,m}) \right\|_1^{1/2} \left\| \frac{1}{Nn} \text{Tr}(\mathcal{X}_n - \bar{\mathcal{X}}_{n,m})(\mathcal{X}_n - \bar{\mathcal{X}}_{n,m})^T \right\|_1^{1/2} \end{aligned} \quad (5.15)$$

By the definition of \mathbf{B}_n , $N^{-1} \mathbb{E}|\text{Tr}(\mathbf{B}_n)| = \|X_0\|_2^2$. Similarly and due to the fact that $pk_N \leq N$ and (5.10),

$$\frac{1}{N} \mathbb{E}|\text{Tr}(\bar{\mathbf{B}}_{n,m})| = \frac{1}{Nn} \sum_{i=1}^n \sum_{\ell=1}^{k_N} \sum_{k \in I_\ell^i} \|\bar{X}_{k,m}\|_2^2 \leq 4\|X_0\|_2^2. \quad (5.16)$$

Moreover, by the construction of \mathcal{X}_n and $\bar{\mathcal{X}}_{n,m}$, we have

$$\frac{1}{Nn} \mathbb{E}|\text{Tr}(\mathcal{X}_n - \bar{\mathcal{X}}_{n,m})(\mathcal{X}_n - \bar{\mathcal{X}}_{n,m})^T| = \frac{1}{Nn} \sum_{i=1}^n \sum_{\ell=1}^{k_N} \sum_{k \in I_\ell^i} \|X_k - \bar{X}_{k,m}\|_2^2 + \frac{1}{Nn} \sum_{i=1}^n \sum_{\ell=1}^{k_N+1} \sum_{k \in J_\ell^i} \|X_k\|_2^2$$

Now, since X_k is centered, we write for $k \in I_\ell^i$,

$$\begin{aligned} \|X_k - \bar{X}_{k,m}\|_2 &= \|X_k - \tilde{X}_{k,m} - \mathbb{E}(X_k - \tilde{X}_{k,m})\|_2 \leq 2\|X_k - \tilde{X}_{k,m}\|_2 \\ &\leq 2\|X_k - \mathbb{E}(X_k|B_{i,\ell})\|_2 + 2\|\tilde{X}_{k,m} - \mathbb{E}(X_k|B_{i,\ell})\|_2 \end{aligned}$$

Analyzing the second term of the last inequality, we get

$$\|\tilde{X}_{k,m} - \mathbb{E}(X_k|B_{i,\ell})\|_2 = \|\mathbb{E}(X_k - \varphi_M(X_k)|B_{i,\ell})\|_2 \leq \|X_k - \varphi_M(X_k)\|_2 = \|(|X_0| - M)_+\|_2. \quad (5.17)$$

As X_0 belongs to \mathbb{L}^2 , then $\lim_{M \rightarrow +\infty} \|(|X_0| - M)_+\|_2 = 0$. Now, we note that for $k \in I_\ell^i$, $\sigma(\varepsilon_{k-m}, \dots, \varepsilon_{k+m}) \subset \sigma(B_{i,\ell})$ which implies that

$$\begin{aligned} \|X_k - \mathbb{E}(X_k|B_{i,\ell})\|_2 &\leq \|X_k - \mathbb{E}(X_k|\varepsilon_{k-m}, \dots, \varepsilon_{k+m})\|_2 \\ &= \|X_0 - \mathbb{E}(X_0|\varepsilon_{-m}, \dots, \varepsilon_m)\|_2 = \|X_0 - X_{0,m}\|_2. \end{aligned} \quad (5.18)$$

where the first equality is due to the stationarity. Therefore, by (5.17), (5.18), the fact that $pk_N \leq N$ and

$$\text{Card}\left(\bigcup_{i=1}^n \bigcup_{\ell=1}^{k_N+1} J_\ell^i\right) \leq Nn - npk_N,$$

we infer that

$$\begin{aligned} \frac{1}{Nn} \mathbb{E}|\text{Tr}(\mathcal{X}_n - \bar{\mathcal{X}}_{n,m})(\mathcal{X}_n - \bar{\mathcal{X}}_{n,m})^T| &\leq 8\|X_0 - X_{0,m}\|_2^2 + 8\|(|X_0| - M)_+\|_2^2 \\ &\quad + (3(a_m + 3)^{-1} + a_m m N^{-1})\|X_0\|_2^2. \end{aligned} \quad (5.19)$$

Thus starting from (5.15), considering the upper bounds (5.16) and (5.19), we derive that there exists a positive constant C not depending on (n, m, M) such that

$$\limsup_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} |\mathbb{E}(S_{\mathbf{B}_n}(z)) - \mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}}(z))| \leq \frac{C}{v^2} \left(\|X_0 - X_{0,m}\|_2^2 + \frac{3}{a_m} \right).$$

Taking the limit on m , Proposition 5.1 follows by the martingale convergence theorem in \mathbb{L}^2 .

5.2 Approximation by a Gram Matrix with Independent Blocks

By Berbee's classical coupling lemma [4], one can construct by induction a sequence of random variables $(\varepsilon_k^*)_{k \geq 1}$ such that:

- For any $1 \leq i \leq n$ and $1 \leq \ell \leq k_N$,

$$B_{i,\ell}^* = (\varepsilon_{(i-1)N+(\ell-1)(p+3m)+1-m}^*, \dots, \varepsilon_{(i-1)N+(\ell-1)(p+3m)+p+m}^*)$$

has the same distribution as $B_{i,\ell}$ defined in (5.8).

- The array $(B_{i,\ell}^*)_{1 \leq i \leq n, 1 \leq \ell \leq k_N}$ is i.i.d.
- For any $1 \leq i \leq n$ and $1 \leq \ell \leq k_N$, $\mathbb{P}(B_{i,\ell} \neq B_{i,\ell}^*) \leq \beta_m$.

(see page 484 of [15] for more details concerning the construction of the array $(B_{i,\ell}^*)_{i,\ell \geq 1}$). We define now the sequence $(\bar{X}_{k,m}^*)_{k \geq 1}$ as follows:

$$\bar{X}_{k,m}^* = h_k(B_{i,\ell}^*) \quad \text{if } k \in I_\ell^i, \quad (5.20)$$

where the functions h_k are defined in (5.12).

We construct the $N \times n$ random matrix $\bar{\mathcal{X}}_{n,m}^* = ((\bar{\mathcal{X}}_{n,m}^*)_{ij})_{i,j} = (\bar{X}_{(j-1)N+i,m}^*)_{i,j}$. Note that the block of entries $(\bar{X}_{k,m}^*, k \in I_\ell^i)$ is independent of $(\bar{X}_{k,m}^*, k \in I_{\ell'}^{i'})$ if $(i, \ell) \neq (i', \ell')$. Thus, $\bar{\mathcal{X}}_{n,m}^*$ has independent blocks of dimension p separated by null blocks whose dimension is at least $3m$. Setting

$$\bar{\mathbf{B}}_{n,m}^* := \frac{1}{n} \bar{\mathcal{X}}_{n,m}^* \bar{\mathcal{X}}_{n,m}^{*T}, \quad (5.21)$$

we approximate $\bar{\mathbf{B}}_{n,m}$ by the Gram matrix $\bar{\mathbf{B}}_{n,m}^*$ as shown in the following proposition.

Proposition 5.2 *Let $\bar{\mathbf{B}}_{n,m}$ and $\bar{\mathbf{B}}_{n,m}^*$ be defined in (5.13) and (5.21) respectively. Assuming that $\lim_{n \rightarrow \infty} N/n = c \in (0, \infty)$ and $\lim_{n \rightarrow \infty} \beta_n = 0$ then for any $z \in \mathbb{C}^+$,*

$$\lim_{m \rightarrow +\infty} \limsup_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |\mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}}(z)) - \mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}^*}(z))| = 0. \quad (5.22)$$

Proof. By (4.5) and Cauchy-Schwarz's inequality, it follows that

$$\begin{aligned} & |\mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}}(z)) - \mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}^*}(z))| \\ & \leq \frac{\sqrt{2}}{v^2} \left\| \frac{1}{N} \text{Tr}(\bar{\mathbf{B}}_{n,m} + \bar{\mathbf{B}}_{n,m}^*) \right\|_1^{1/2} \left\| \frac{1}{Nn} \text{Tr}(\bar{\mathcal{X}}_{n,m} - \bar{\mathcal{X}}_{n,m}^*)(\bar{\mathcal{X}}_{n,m} - \bar{\mathcal{X}}_{n,m}^*)^T \right\|_1^{1/2} \end{aligned} \quad (5.23)$$

Notice that $\|\bar{X}_{k,m}^*\|_2 = \|h_k(B_{i,\ell}^*)\|_2 = \|h_k(B_{i,\ell})\|_2 = \|\bar{X}_{k,m}\|_2 \leq 2\|X_0\|_2$, where the second equality follows from the fact that $B_{i,\ell}^*$ is distributed as $B_{i,\ell}$ whereas the last inequality follows from (5.10). Thus, we get by the definition of $\bar{\mathbf{B}}_{n,m}^*$ and the fact that $pk_N \leq N$,

$$\frac{1}{N} \mathbb{E} |\text{Tr}(\bar{\mathbf{B}}_{n,m}^*)| = \frac{1}{Nn} \sum_{i=1}^n \sum_{\ell=1}^{k_N} \sum_{k \in I_\ell^i} \|\bar{X}_{k,m}^*\|_2^2 \leq 4\|X_0\|_2^2. \quad (5.24)$$

Considering (5.23), (5.16) and (5.24), we infer that Proposition 5.2 follows once we prove that

$$\lim_{m \rightarrow +\infty} \limsup_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \frac{1}{Nn} \mathbb{E} \left| \text{Tr}(\bar{\mathcal{X}}_{n,m} - \bar{\mathcal{X}}_{n,m}^*)(\bar{\mathcal{X}}_{n,m} - \bar{\mathcal{X}}_{n,m}^*)^T \right| = 0. \quad (5.25)$$

By the construction of $\bar{\mathcal{X}}_{n,m}$ and $\bar{\mathcal{X}}_{n,m}^*$, we write

$$\frac{1}{Nn} \mathbb{E} \left| \text{Tr}(\bar{\mathcal{X}}_{n,m} - \bar{\mathcal{X}}_{n,m}^*)(\bar{\mathcal{X}}_{n,m} - \bar{\mathcal{X}}_{n,m}^*)^T \right| = \frac{1}{Nn} \sum_{i=1}^n \sum_{\ell=1}^{k_N} \sum_{k \in I_\ell^i} \|\bar{X}_{k,m} - \bar{X}_{k,m}^*\|_2^2. \quad (5.26)$$

Now, let L be a fixed positive real number strictly less than M and not depending on (n, m, M) . To control the term $\|\bar{X}_{k,m} - \bar{X}_{k,m}^*\|_2^2$, we write for $k \in I_\ell^i$,

$$\begin{aligned} \|\bar{X}_{k,m} - \bar{X}_{k,m}^*\|_2^2 &= \|(h_k(B_{i,\ell}) - h_k(B_{i,\ell}^*)) \mathbf{1}_{B_{i,\ell} \neq B_{i,\ell}^*}\|_2^2 \\ &\leq 4 \|h_k(B_{i,\ell}) \mathbf{1}_{B_{i,\ell} \neq B_{i,\ell}^*}\|_2^2 = 4 \|\bar{X}_{k,m} \mathbf{1}_{B_{i,\ell} \neq B_{i,\ell}^*}\|_2^2 \\ &\leq 12 \|\bar{X}_{k,m} - \mathbb{E}(X_k | B_{i,\ell})\|_2^2 + 12 \|\mathbb{E}(X_k | B_{i,\ell}) - \mathbb{E}(\varphi_L(X_k) | B_{i,\ell})\|_2^2 \\ &\quad + 12 \|\mathbb{E}(\varphi_L(X_k) | B_{i,\ell}) \mathbf{1}_{B_{i,\ell} \neq B_{i,\ell}^*}\|_2^2. \end{aligned}$$

Since $\mathbb{P}(B_{i,\ell} \neq B_{i,\ell}^*) \leq \beta_m$ and $\varphi_L(X_k)$ is bounded by L , we get

$$\|\mathbb{E}(\varphi_L(X_k) | B_{i,\ell}) \mathbf{1}_{B_{i,\ell} \neq B_{i,\ell}^*}\|_2^2 \leq L^2 \beta_m.$$

Moreover, it follows by the fact that X_k is centered and (5.17) that

$$\|\bar{X}_{k,m} - \mathbb{E}(X_k|B_{i,\ell})\|_2^2 \leq 4\|\tilde{X}_{k,m} - \mathbb{E}(X_k|B_{i,\ell})\|_2^2 \leq 4\|(|X_0| - M)_+\|_2^2$$

and

$$\|\mathbb{E}(X_k|B_{i,\ell}) - \mathbb{E}(\varphi_L(X_k)|B_{i,\ell})\|_2^2 \leq \|X_k - \varphi_L(X_k)\|_2^2 = \|(|X_0| - L)_+\|_2^2.$$

Hence gathering the above upper bounds and noting that $pk_N \leq N$, we infer that

$$\frac{1}{Nn} \sum_{i=1}^n \sum_{\ell=1}^{k_N} \sum_{k \in I_\ell^i} \|\bar{X}_{k,m} - \bar{X}_{k,m}^*\|_2^2 \leq 48\|(|X_0| - M)_+\|_2^2 + 12\|(|X_0| - L)_+\|_2^2 + 12L^2\beta_m.$$

Letting first M , then m and finally L tend to infinity, the right-hand-side converges to zero which considering (5.26) implies (5.25) and thus the Proposition.

5.3 Approximation by a Gram Matrix Associated with a Gaussian Process

In order to prove (5.6), it suffices, in view of (5.14) and (5.22), to prove the following convergence: for any $z \in \mathbb{C}^+$,

$$\lim_{m \rightarrow +\infty} \limsup_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |\mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}^*}(z)) - \mathbb{E}(S_{\mathbf{G}_n}(z))| = 0. \quad (5.27)$$

With this aim, we shall first approximate \mathbf{G}_n by another sample Gaussian covariance matrix $\tilde{\mathbf{G}}_n$ having the same structure as $\bar{\mathbf{B}}_{n,m}^*$. So let $\tilde{Z}_k^{(i)}$ be defined for any $1 \leq i \leq n$ and $1 \leq k \leq N$ by

$$\tilde{Z}_k^{(i)} = \begin{cases} Z_k^{(i)} & \text{if } k \in I_\ell^1 \text{ for some } \ell \in \{1, \dots, k_N\} \\ 0 & \text{otherwise,} \end{cases}$$

where the $Z_k^{(i)}$'s are independent copies of the Gaussian process defined in (5.3). We define now the $N \times n$ matrix $\tilde{\mathbf{Z}}_n = ((\tilde{Z}_n)_{u,v})_{uv} = (\tilde{Z}_u^{(v)})_{uv}$ and finally set

$$\tilde{\mathbf{G}}_n = \frac{1}{n} \tilde{\mathbf{Z}}_n \tilde{\mathbf{Z}}_n^T. \quad (5.28)$$

Provided that $\lim_{n \rightarrow \infty} n/N = c \in (0, \infty)$, we have by Proposition 4.2 of [3] that for any $z \in \mathbb{C}^+$,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} |\mathbb{E}(S_{\mathbf{G}_n}(z)) - \mathbb{E}(S_{\tilde{\mathbf{G}}_n}(z))| = 0.$$

Therefore, the proof of (5.27) is reduced to prove for any $z \in \mathbb{C}^+$,

$$\lim_{m \rightarrow +\infty} \limsup_{M \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |\mathbb{E}(S_{\bar{\mathbf{B}}_{n,m}^*}(z)) - \mathbb{E}(S_{\tilde{\mathbf{G}}_n}(z))| = 0. \quad (5.29)$$

Proposition 5.3 *Provided that $\lim N/n = c \in (0, \infty)$ and assuming that (2.5) and (2.6), the convergence (5.29) holds for $\bar{\mathbf{B}}_{n,m}^*$ and $\tilde{\mathbf{G}}_n$ defined in (5.21) and (5.28) respectively.*

Proof. We don't give a full proof of this proposition because the computation involved has been almost exactly done in the proof of Proposition 4.3 of [3]. Similarly as $\bar{\mathcal{X}}_{n,m}^*$, the matrix $\bar{\mathcal{X}}_n$ considered in [3] has independent blocks separated by blocks of zero entries. The dimensions of blocks considered differ but this does not affect the proof. For instance, $p = m^2$ in [3], however here $p = a_m m$ with a_m being defined in (5.7).

Therefore, following the lines of the proof of Proposition 4.3 of [3], we get for any $z = u + iv \in \mathbb{C}^+$,

$$\begin{aligned} & \left| \mathbb{E}(S_{\mathbf{B}_{n,m}^*}(z)) - \mathbb{E}(S_{\tilde{\mathbf{G}}_n}(z)) \right| \\ & \leq \frac{C(1+M^5)N^{1/2}p^2}{v^3(1 \wedge v)n} + \frac{C(1+M^2)(1+c^2(n))p}{nv^3(1 \wedge v^2)} + \frac{C(1+c^2(n))}{v^2(1 \wedge v)} \sum_{k \geq m+1} |\text{Cov}(X_0, X_k)| \\ & \quad + \frac{C}{nv^2(1 \wedge v)(N \wedge n)} \sum_{i=1}^n \sum_{s=1}^{k_N} \sum_{k, \ell \in I_s^i} |\text{Cov}(\bar{X}_{k,m}^*, \bar{X}_{\ell,m}^*) - \text{Cov}(X_k, X_\ell)|, \end{aligned} \quad (5.30)$$

where C is a constant not depending on (n, m, M) and $c(n) = N/n$. Since $c(n) \rightarrow c \in (0, \infty)$, it follows that

$$\lim_{n \rightarrow \infty} \frac{(1+M^5)N^{1/2}p^2}{n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{(1+M^2)p}{n} = 0.$$

On the other hand, we get by (2.5) that $\lim_{m \rightarrow \infty} \sum_{k \geq m+1} |\text{Cov}(X_0, X_k)| = 0$. Therefore, Proposition 5.3 follows if we prove that, for any $z \in \mathbb{C}^+$,

$$\lim_{m \rightarrow \infty} \limsup_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n(N \wedge n)} \sum_{i=1}^n \sum_{s=1}^{k_N} \sum_{k, \ell \in I_s^i} |\text{Cov}(\bar{X}_{k,m}^*, \bar{X}_{\ell,m}^*) - \text{Cov}(X_k, X_\ell)| = 0. \quad (5.31)$$

In fact, as $B_{i,s}^*$ is distributed as $B_{i,s}$ then by (5.9) and (5.20) we have for $k \in I_s^i$,

$$\mathbb{E}(\bar{X}_{k,m}^*) = \mathbb{E}(h_k(B_{i,s}^*)) = \mathbb{E}(h_k(B_{i,s})) = \mathbb{E}(\bar{X}_{k,m}) = 0.$$

For the same reasons, we have for $k, \ell \in I_s^i$,

$$\text{Cov}(\bar{X}_{k,m}^*, \bar{X}_{\ell,m}^*) = \mathbb{E}(\bar{X}_{k,m}^* \bar{X}_{\ell,m}^*) = \mathbb{E}(h_k(B_{i,s}^*) h_\ell(B_{i,s}^*)) = \mathbb{E}(h_k(B_{i,s}) h_\ell(B_{i,s})) = \text{Cov}(\bar{X}_{k,m}, \bar{X}_{\ell,m})$$

Moreover, letting for $k \in I_s^i$, $\hat{X}_{k,m} = \mathbb{E}(X_k | B_{i,s})$ and noting that $\mathbb{E}(\hat{X}_{k,m}) = 0$, we write

$$\begin{aligned} & \sum_{k \in I_s^i} \sum_{\ell \in I_s^i} |\text{Cov}(\bar{X}_{k,m}^*, \bar{X}_{\ell,m}^*) - \text{Cov}(X_k, X_\ell)| = \sum_{k \in I_s^i} \sum_{\ell \in I_s^i} |\text{Cov}(\bar{X}_{k,m}, \bar{X}_{\ell,m}) - \text{Cov}(X_k, X_\ell)| \\ & \leq \sum_{k \in I_s^i} \sum_{\ell \in I_s^i} |\text{Cov}(\bar{X}_{k,m}, \bar{X}_{\ell,m}) - \text{Cov}(\hat{X}_{k,m}, \hat{X}_{\ell,m})| + \sum_{k \in I_s^i} \sum_{\ell \in I_s^i} |\text{Cov}(\hat{X}_{k,m}, \hat{X}_{\ell,m}) - \text{Cov}(X_k, X_\ell)|. \end{aligned} \quad (5.32)$$

Now, by the stationarity of $(X_k)_{k \in \mathbb{Z}}$, the fact that the random variables are centered, (5.11) and Cauchy-Schwarz's inequality, we get

$$\begin{aligned} & |\text{Cov}(\bar{X}_{k,m}, \bar{X}_{\ell,m}) - \text{Cov}(\hat{X}_{k,m}, \hat{X}_{\ell,m})| \\ & = |\text{Cov}(\bar{X}_{k,m}, \bar{X}_{\ell,m} - \hat{X}_{\ell,m}) - \text{Cov}(\hat{X}_{k,m} - \bar{X}_{k,m}, \hat{X}_{\ell,m} - \bar{X}_{\ell,m}) - \text{Cov}(\hat{X}_{k,m} - \bar{X}_{k,m}, \bar{X}_{\ell,m})| \\ & \leq 2M \|\bar{X}_{\ell,m} - \hat{X}_{\ell,m}\|_1 + \|\bar{X}_{k,m} - \hat{X}_{k,m}\|_2 \|\bar{X}_{\ell,m} - \hat{X}_{\ell,m}\|_2 + 2M \|\bar{X}_{k,m} - \hat{X}_{k,m}\|_1 \\ & \leq 8M \|(|X_0| - M)_+\|_1 + 4 \|(|X_0| - M)_+\|_2^2, \end{aligned} \quad (5.33)$$

where the last inequality follows from (5.17). Moreover, $(|x| - M)_+ \leq 2|x| \mathbf{1}_{|x| \geq M}$ which in turn implies that $M(|x| - M)_+ \leq 2|x|^2 \mathbf{1}_{|x| \geq M}$. So, overall we get

$$\sum_{k \in I_s^i} \sum_{\ell \in I_s^i} |\text{Cov}(\bar{X}_{k,m}, \bar{X}_{\ell,m}) - \text{Cov}(\hat{X}_{k,m}, \hat{X}_{\ell,m})| \leq 32p^2 \mathbb{E}(X_0^2 \mathbf{1}_{|X_0| \geq M}). \quad (5.34)$$

Again, by the stationarity of $(X_k)_{k \in \mathbb{Z}}$, the fact that the random variables are centered, (5.10), Cauchy-Schwarz's inequality and (5.18), we get

$$\begin{aligned} |\text{Cov}(\widehat{X}_{k,m}, \widehat{X}_{\ell,m}) - \text{Cov}(X_k, X_\ell)| &= |\mathbb{E}(X_k X_\ell) - \mathbb{E}(\mathbb{E}(X_k|B_{i,s})\mathbb{E}(X_\ell|B_{i,s}))| \\ &= |\mathbb{E}(X_k(X_\ell - \mathbb{E}(X_\ell|B_{i,s})))| = \left| \mathbb{E}\left((X_k - \mathbb{E}(X_k|B_{i,s}))(X_\ell - \mathbb{E}(X_\ell|B_{i,s}))\right) \right| \\ &\leq \|X_k - \mathbb{E}(X_k|B_{i,s})\|_2 \|X_\ell - \mathbb{E}(X_\ell|B_{i,s})\|_2 \leq \|X_0 - \mathbb{E}(X_0|\varepsilon_{-m}, \dots, \varepsilon_m)\|_2^2. \end{aligned} \quad (5.35)$$

Therefore,

$$\sum_{k \in I_s^i} \sum_{\ell \in I_s^i} |\text{Cov}(\widehat{X}_{k,m}, \widehat{X}_{\ell,m}) - \text{Cov}(X_k, X_\ell)| \leq 2p^2 \|X_0 - \mathbb{E}(X_0|\varepsilon_{-m}, \dots, \varepsilon_m)\|_2^2. \quad (5.36)$$

Therefore, starting from (5.32) together with (5.34) and (5.36), it follows that there exists a constant C not depending on (n, m, M) such that

$$\begin{aligned} \frac{1}{n(N \wedge n)} \sum_{i=1}^n \sum_{s=1}^{k_N} \sum_{k \in I_s^i} \sum_{\ell \in I_s^i} |\text{Cov}(\bar{X}_{k,m}^*, \bar{X}_{\ell,m}^*) - \text{Cov}(X_k, X_\ell)| \\ \leq \frac{CN}{N \wedge n} (a_m m \mathbb{E}(X_0^2 \mathbf{1}_{|X_0| \geq M}) + a_m m \|X_0 - \mathbb{E}(X_0|\varepsilon_{-m}, \dots, \varepsilon_m)\|_2^2). \end{aligned} \quad (5.37)$$

Letting n tend to infinity, $N/(N \wedge n)$ converges to $1 \vee c$ since $\lim_{n \rightarrow +\infty} N/n = c$. Then letting M tend to infinity, the first term on the right hand side converges to zero since X_0 belongs to \mathbb{L}^2 . Finally, letting m tend to infinity, the last term converges to zero by (5.7), which ends the proof of Proposition 5.3 and thus of Theorem 2.2.

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