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Abstract. We consider the mathematical model of a rigid ball moving in a viscous incompressible fluid occupying a bounded domain Ω, with an external force acting on the ball. We investigate in particular the case when the external force is what would be produced by a spring and a damper connecting the center of the ball h to a fixed point $h_1 \in \Omega$. If the initial fluid velocity is sufficiently small, and the initial h is sufficiently close to h_1, then we prove the existence and uniqueness of global (in time) solutions for the model. Moreover, in this case, we show that h converges to h_1, and all the velocities (of the fluid and of the ball) converge to zero. Based on this result, we derive a control law that will bring the ball asymptotically to the desired position h_1 even if the initial value of h is far from h_1, and the path leading to h_1 is winding and complicated. Now, the idea is to use the force as described above, with one end of the spring and damper at h, while other end is jumping between a finite number of points in Ω, that depend on h (a switching feedback law).

Key words. fluid-structure interactions, Navier-Stokes equations, PD controller, global solutions, asymptotic stability, switching feedback.

AMS Subject Classification: 35Q35,35D05, 35Q30, 35Q72, 76D03

1. Introduction and main results

We consider a coupled system described by nonlinear partial and ordinary differential equations modelling the motion of a rigid body inside a viscous incompressible

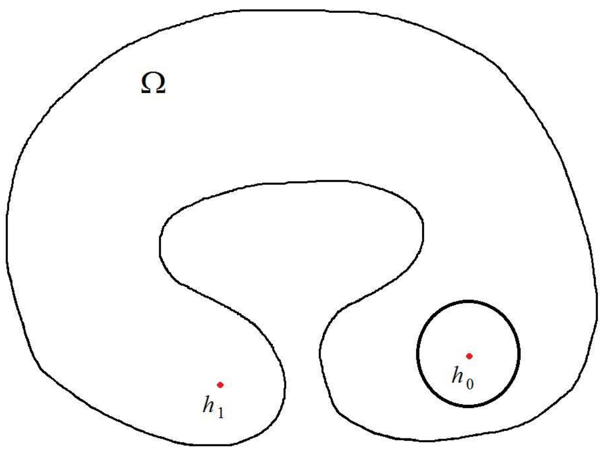

Figure 1: A domain Ω with the ball shown with the center at its initial position h_0. The feedback (1.12) is trying to bring the ball to the desired position h_1. This is equivalent to having connected a spring and a damper between the center of the ball and h_1, so that the spring is pulling the ball towards h_1. In the situation shown here, if the initial velocities are small, then the ball will not reach h_1. The more sophisticated feedback (1.19) will do the job.

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fluid in a bounded domain Ω. The fluid flow is described by the classical *Navier-Stokes equations* (see (1.1)–(1.2) below), whereas the motion of the ball-shaped rigid body is governed by the *Newton laws* (see (1.6)-(1.7) below), including an external control force denoted by u acting on the ball.

The domain occupied by the fluid and the rigid ball is $\Omega \subset \mathbb{R}^3$, a connected open bounded set with C^2 boundary. The rigid ball has radius 1 and its center is located at the (variable) point h which is at a distance > 1 from the boundary $\partial\Omega$. We denote by $\mathcal{B}(h)$ the closed set occupied by the ball. The fluid is homogeneous with *density* $\rho > 0$ and *viscosity* $\nu > 0$ and it occupies the domain

$$\mathcal{F}(h) = \Omega \setminus \mathcal{B}(h).$$

The full system of equations modelling the system, for $t \geqslant 0$, is

$$\rho\dot{v} - \nu\Delta v + \rho(v\cdot\nabla)v + \nabla p = 0, \qquad x \in \mathcal{F}(h(t)), \tag{1.1}$$
$$\operatorname{div} v = 0, \qquad x \in \mathcal{F}(h(t)), \tag{1.2}$$
$$v = 0, \qquad x \in \partial\Omega, \tag{1.3}$$
$$\dot{h} = g, \tag{1.4}$$
$$v = g(t) + \omega(t) \times (x - h(t)), \qquad x \in \partial\mathcal{B}(h(t)), \tag{1.5}$$

$$m\dot{g} = -\int_{\partial\mathcal{B}(h)} \sigma(v,p)n \,\mathrm{d}\Gamma + u, \tag{1.6}$$

$$J\dot{\omega} = -\int_{\partial\mathcal{B}(h)} (x-h) \times \sigma(v,p)n \,\mathrm{d}\Gamma, \tag{1.7}$$

$$h(0) = h_0, \quad \dot{h}(0) = g_0, \quad \omega(0) = \omega_0, \tag{1.8}$$

$$v(x,0) = v_0(x), \qquad x \in \mathcal{F}(h_0). \tag{1.9}$$

In the above system the *state variables* are $v(x,t)$ (the Eulerian velocity field of the fluid), $h(t)$ (the position of the center of the rigid ball), its time derivative $g(t)$, and $\omega(t)$ (the angular velocity of the ball). The function $p(x,t)$ is the pressure of the fluid, which is not a state variable, because at any time instant it can be computed from v at the same instant, up to an additive constant. We have denoted by $n(x,t)$ the unit normal to $\partial\mathcal{B}(h(t))$ at the point $x \in \partial\mathcal{B}(h(t))$, directed to the interior of the ball, and by m and J the mass and the moment of inertia of the rigid ball. (If we would take $\nu = 0$, then (1.1)–(1.2) would be called *Euler's equations*, but then the other equations and the nature of the system would change.) We have denoted by $\sigma(v,p)$ the tensor defined by

$$\sigma_{ij}(v,p) = -p\delta_{ij} + \nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (i,\, j \in \{1,2,3\}). \tag{1.10}$$

This is the *stress tensor* in the fluid, and $\mathrm{d}\Gamma$ is the surface measure on $\partial\mathcal{B}(h)$. We also need a notation for the set of points where the center of the ball can be:

$$\Omega^\circ = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 1\}, \tag{1.11}$$

and we assume this set to be connected.

The main difficulty in the analysis of (1.1)–(1.9) is that the Navier-Stokes equations are valid in a non-cylindrical space-time domain. This domain depends on the solution, so that we have here a free boundary problem. Early references addressing these difficulties are Conca, San Martín and Tucsnak [1], Desjardins and Esteban [3] and Hoffman and Starovoitov [10]. The case $\Omega = \mathbb{R}^3$ has been considered in Galdi and Silvestre [7] and by Cumsille and Takahashi [2]. The global existence and uniqueness of strong solutions has been proved for sufficiently small v_0 in Takahashi [17]. Local existence of strong solutions in the L^p context has been proved in Geissert, Götze and Hieber [8]. The existence of global weak solutions for $u = 0$ (with possible contacts between the rigid body and $\partial\Omega$) has been proved in San Martín, Starovoitov and Tucsnak [16] and in Feireisl [5]. Most the above references considered the case in which u in (1.6) is a given function of time (often identically equal to zero). The literature on this subject is large and, inevitably, we have left out some relevant references. For instance, there are several works studying a fluid with a rigid body moving according to a prescribed trajectory (that is independent of the fluid).

One of the contributions of our work is that we prove the existence and uniqueness of global (in time) strong solutions of (1.1)–(1.9) when u is given by a feedback law of the form

$$u(t) = k_p[h_1 - h(t)] - k_d\dot{h}(t), \tag{1.12}$$

with a given $h_1 \in \Omega^\circ$ and $k_p > 0$, $k_d \geqslant 0$. This feedback may be regarded as a proportional-derivative (PD) controller, as is often used in control engineering. Another interpretation is that the force u from (1.12) is generated by a spring (with constant k_p) and a mechanical damper (with constant k_d) connected between $h(t)$ and a fixed anchor point h_1. This result (Theorem 1.1 below) assumes that the initial velocity field v_0 as well as the initial data g_0, ω_0 and $h_1 - h_0$ are sufficiently small in a suitable sense. In addition to existence and uniqueness of global strong solutions of (1.1)–(1.9) with (1.12), we show that these solutions satisfy

$$\lim_{t\to\infty} h(t) = h_1, \qquad \lim_{t\to\infty} g(t) = 0, \qquad \lim_{t\to\infty} \omega(t) = 0, \tag{1.13}$$

$$\lim_{t\to\infty} \|v(\cdot,t)\|_{\mathcal{H}^1(\mathcal{F}(h(t)))} = 0. \tag{1.14}$$

The last formula of course implies that $\lim_{t\to\infty} \|v(\cdot,t)\|_{L^2(\mathcal{F}(h(t)))} = 0$, a fact that will be proved before proving (1.14), using energy estimates.

We need more notation. If W is a Hilbert space and $q \in \mathbb{N}$, we denote by $\mathcal{H}^q_{\rm loc}(0,\infty;W)$ the space of those $v : (0,\infty) \to W$ for which the restriction $v|_{(0,T)}$ is in $\mathcal{H}^q(0,T;W)$, for every $T > 0$. If M is a subset of W then $\mathcal{H}^q_{\rm loc}(0,\infty;M)$ is the set of those functions in $\mathcal{H}^q_{\rm loc}(0,\infty;W)$ that have their range in M. If x is a vector in a finite-dimensional normed space, then we denote its norm by $|x|$.

We now introduce sets of vector-valued functions with four components $\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix}$ which include possible state trajectories of the system (1.1)–(1.9) with an arbitrary control input u. Thus, v is the velocity field, but extended to all of Ω, by considering also the velocity field of the rigid ball. This function is required to satisfy $\operatorname{div} v = 0$, in accordance with (1.2). The function h represents the position of the center of the rigid ball (which must be with values in Ω°), $g = \dot{h}$ is the velocity of the center of the ball, while ω is its angular velocity. For compatibility, we impose that at any moment $t > 0$, the restriction of v to $\mathcal{B}(h(t))$ must be equal to the velocity field of the rigid ball, as determined by h, g and ω:

$$\mathcal{T}L^2_{\rm loc}([0,\infty);L^2(\Omega)) = \left\{ \begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \begin{matrix} L^2_{\rm loc}([0,\infty);L^2(\Omega)) \\ \mathcal{H}^2_{\rm loc}(0,\infty;\Omega^\circ) \\ \mathcal{H}^1_{\rm loc}(0,\infty;\mathbb{R}^3) \\ \mathcal{H}^1_{\rm loc}(0,\infty;\mathbb{R}^3) \end{matrix} \;\middle|\; \begin{matrix} v(x,t) = g(t) + \\ \omega(t) \times (x - h(t)) \\ \forall t > 0,\ x \in \mathcal{B}(h(t)), \\ \operatorname{div} v = 0 \end{matrix} \right\}.$$

The subset $\mathcal{T}C([0,\infty);\mathcal{H}^1(\Omega))$ is defined similarly, but with $C([0,\infty);\mathcal{H}^1(\Omega))$ in place of $L^2_{\rm loc}([0,\infty);L^2(\Omega))$ in the top line. Another subset $\mathcal{T}\mathcal{H}^1_{\rm loc}(0,\infty;L^2(\Omega))$ is defined similarly, but with $\mathcal{H}^1_{\rm loc}(0,\infty;L^2(\Omega))$ in place of $L^2_{\rm loc}([0,\infty);L^2(\Omega))$ in the top line. Finally, the more complicated set

$$\mathcal{T}L^2_{\rm loc}([0,\infty);\mathcal{H}^2(\mathcal{F}(h))) \subset \mathcal{T}C([0,\infty);\mathcal{H}^1(\Omega))$$

consists of those $\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \mathcal{T}C([0,\infty);\mathcal{H}^1(\Omega))$ for which $v(\cdot,t)|_{\mathcal{F}(h(t))} \in \mathcal{H}^2(\mathcal{F}(h(t))$ for almost every $t>0$ and the scalar function $t \to \|v(\cdot,t)|_{\mathcal{F}(h(t))}\|_{\mathcal{H}^2}$ is in $L^2_{\rm loc}[0,\infty)$.

We need to introduce also suitable sets of functions for the pressure p in (1.1). For any given $h \in \mathcal{H}^2_{\rm loc}(0,\infty;\Omega^\circ)$ we set

$$L^2_{\rm loc}([0,\infty);\mathcal{H}^1(\mathcal{F}(h))) = \left\{ p \in L^2_{\rm loc}([0,\infty);L^2(\Omega)) \;\middle|\; \begin{array}{c} p|_{\mathcal{F}(h(t))} \in \mathcal{H}^1(\mathcal{F}(h(t)) \text{ and} \\ p|_{\mathcal{B}(h(t))} = 0 \text{ for a.e. } t \geqslant 0, \\ \int_0^T \|p\|^2_{\mathcal{H}^1(\mathcal{F}(h(t))}\mathrm{d}t < \infty \\ \text{for all } T>0 \end{array} \right\}.$$

Two important intermediate results can be stated as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be an open, connected and bounded set with $\partial\Omega$ of class C^2 and let $h_1 \in \Omega^\circ$. Then for each $k_p > 0$ and $k_d \geqslant 0$ there exists $\delta > 0$, depending only on Ω, k_p, k_d and on the distance* $\mathrm{dist}(h_1,\partial\Omega)$*, such that for every $h_0 \in \Omega^\circ$, $v_0 \in \mathcal{H}^1(\mathcal{F}(h_0);\mathbb{R}^3)$ and every $g_0, \omega_0 \in \mathbb{R}^3$ satisfying*

$$\begin{cases} \operatorname{div} v_0 = 0, & \text{in } \mathcal{F}(h_0), \\ v_0(x) = 0, & \text{for } x \in \partial\Omega, \\ v_0(x) = g_0 + \omega_0 \times (x-h_0), & \text{for } x \in \partial\mathcal{B}(h_0), \\ \|v_0\|_{\mathcal{H}^1(\mathcal{F}(h_0))} + |g_0| + |\omega_0| + |h_1 - h_0| \leqslant \delta, \end{cases} \tag{1.15}$$

there exists a strong solution of (1.1)–(1.9) *with the feedback* (1.12)*, on the time interval* $[0,\infty)$*. This solution satisfies*

$$\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \mathcal{T}L^2_{\rm loc}([0,\infty);\mathcal{H}^2(\mathcal{F}(h))) \cap \mathcal{T}\mathcal{H}^1_{\rm loc}(0,\infty;L^2(\Omega)), \tag{1.16}$$

$$p \in L^2_{\rm loc}([0,\infty);\mathcal{H}^1(\mathcal{F}(h))), \tag{1.17}$$

$$v \in L^\infty([0,\infty);\mathcal{H}^1(\Omega)). \tag{1.18}$$

The above solution is unique up to an additive perturbation of p that depends only on time.

Theorem 1.2. *With the notation and assumptions of Theorem* 1.1*, the solution (v,p,h,g,ω) of* (1.1)–(1.9) *satisfies* (1.13) *and* (1.14)*.*

The above theorem tells us that we can move the ball from any initial point $h_0 \in \Omega^\circ$ asymptotically to another point $h_1 \in \Omega^\circ$, if h_0 and h_1 are sufficiently close to each other, by connecting a spring and a damper to the ball and pulling the ball towards h_1. One reason why we need h_0 and h_1 to be close to each other is to ensure that on the way, the ball will not hit the boundary $\partial\Omega$.

We would like to have a result that tells us something similar to the above theorem, but without the requirement that h_0 is close to h_1. We achieve this by imposing a more complicated control law, in which the anchor point of the spring and damper is not fixed at h_1, but instead it jumps between a finite number of possible points. In this way, it is possible to navigate the ball along a curved path, as may be necessary due to the shape of Ω, see Figure 1.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be an open, connected and bounded set with $\partial\Omega$ of class C^2, with Ω° connected. Then for each h_0, $h_1 \in \Omega^\circ$ and $k_p > 0, k_d \geqslant 0$ there exists $\delta > 0$ such that if $v_0 \in \mathcal{H}^1(\mathcal{F}(h_0))$, $g_0, \omega_0 \in \mathbb{R}^3$ satisfy*

$$\begin{cases} \operatorname{div} v_0 = 0, & \text{in } \mathcal{F}(h_0), \\ v_0(x) = 0, & \text{for } x \in \partial\Omega, \\ v_0(x) = g_0 + \omega_0 \times (x - h_0), & \text{for } x \in \partial\mathcal{B}_0, \\ \|v_0\|_{\mathcal{H}^1(\mathcal{F}(h_0))} + |g_0| + |\omega_0| \leqslant \delta, \end{cases}$$

then there exists a piecewise constant function $s : [0, \infty) \to \Omega^\circ$ such that the strong solution of (1.1)–(1.9) *with*

$$u(t) = k_p[s(t) - h(t)] - k_d\dot{h}(t) \qquad (t \geqslant 0), \tag{1.19}$$

satisfies the stability properties (1.13) *and* (1.14)*.*

As pointed above, one of the main difficulties to study the system (1.1)–(1.9) comes from the fact that the domain of the fluid is moving. To overcome this difficulty, we introduce in Section 2 a change of variables in order to rewrite the system in a cylindrical domain. Using this change of variables, we prove in Section 3 the local in time existence and uniqueness of a solution. The energy estimates and $\mathcal{H}^1$ estimates established respectively in Section 4 and in Section 5 allow to deduce the global in time existence of solutions under a smallness assumption of the initial data. Then, using the feedback (1.12), we show in Section 6 that the $\mathcal{H}^1$ norm of the solutions tends to zero and that the position of the center of the ball tends to h_1. This result is the keystone to deduce the proof of the main results in Section 7. Finally, in Section 8, we state similar results for the bidimensional case. In that case, we can skip the smallness condition that is necessary in dimension 3.

2. Changing variables to a fixed domain

In this section we recall the construction of a change of variables which, when applied to the system (1.1)–(1.9), transforms equation (1.1) in a PDE valid, for every $t \geq 0$, in the fixed domain $\mathcal{F}(h(0))$. This change of variables has been widely used in the study of fluid-structure interactions so that most of the results in this section are stated without proofs. We refer to [17] and to San Martín and Tucsnak [13] for the detailed proofs.

Let $T > 0$ and let $h \in \mathcal{H}^2(0, T; \Omega^\circ)$ (this means that h is of class $\mathcal{H}^2$ on the interval $(0, T)$ and its range is in Ω° defined in (1.11)). As in the previous section, we denote by $\mathcal{B}(h(t))$ the closed ball of radius 1 centered at $h(t)$ and we set $\mathcal{F}(h(t)) = \Omega \setminus \mathcal{B}(h(t))$. We define the function $w : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ by

$$w(x,t) = \frac{1}{2}\dot{h}(t) \times x\,.$$

It is easily seen that

$$\operatorname{rot} w(x,t) = \dot{h}(t) \qquad \text{for } \ t \geqslant 0,\ x \in \mathbb{R}^3.$$

Using the Sobolev embedding theorem, a short argument shows that there exists $\varepsilon > 0$ such that $h \in C([0,T]; \overline{\Omega_{1+\varepsilon}})$, where Ω_α the set of points in Ω that are at a distance larger than α from $\partial\Omega$. In particular,

$$|x - h(t)| \geqslant 1 + \varepsilon \qquad \forall\, x \in \partial\Omega,\ t \in (0,T). \tag{2.1}$$

Let $\xi \in C^\infty(\mathbb{R}^3)$ be a function with compact support contained in $\Omega_{\frac{\varepsilon}{2}}$ and with $\xi \equiv 1$ on $\overline{\Omega}_\varepsilon$. We define the vector field $\Lambda : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ by

$$\Lambda(x,t) = \operatorname{rot}(\xi w)(x,t) \qquad (x \in \mathbb{R}^3,\ t \geqslant 0). \tag{2.2}$$

It is not difficult to check that for every $t \in [0,T]$ we have

$$\Lambda(x,t) = \begin{cases} \dot{h}(t) & \text{if} \quad x \in \overline{\Omega}_\varepsilon \supset \mathcal{B}(h(t)), \\ 0 & \text{if} \quad \qquad\qquad x \notin \Omega_{\frac{\varepsilon}{2}}. \end{cases}$$

Next, consider the time dependent vector field $X(\cdot,t)$ satisfying

$$\begin{cases} \dfrac{\partial X}{\partial t}(y,t) = \Lambda(X(y,t),t), & \text{for } \ y \in \mathbb{R}^3,\ t > 0, \\ X(y,0) = y & \text{for } \ y \in \mathbb{R}^3. \end{cases} \tag{2.3}$$

The first properties of the map X are summarized in the following two lemmas. As mentioned above, we refer to [17] and to [13] for the detailed proofs.

Lemma 2.1. *With the above assumptions, for all $y \in \Omega$, the initial-value problem (2.3) has a unique solution $X(y,\cdot) : [0,\infty) \to \Omega$ and for every $t \geqslant 0$, we have that the mapping $y \mapsto X(y,t)$ is a C^∞-diffeomorphism of Ω and from $\mathcal{F}(h(0))$ onto $\mathcal{F}(h(t))$. Moreover, X satisfies the following conditions:*

1. *The restriction of X to $\mathcal{B}(h(0))$ is a translation, i.e.,*

$$X(y,t) = y + h(t) - h(0) \qquad \textit{for } \ y \in \mathcal{B}(h(0)),\ t \geqslant 0.$$

2. *The Jacobian matrix J_X of X satisfies*

$$\det J_X(y,t) = 1 \qquad \textit{for } \ y \in \Omega,\ t \geqslant 0.$$

3. *The map $y \mapsto X(y,t)$ is invertible for every $t \geqslant 0$ and its inverse map Y satisfies*

$$\begin{aligned} \frac{\partial Y}{\partial t}(x,t) \;&=\; -\,(\Lambda(x,t)\cdot\nabla)Y(x,t), \quad \textit{for}\ \ x\in\Omega,\ t>0,\\ Y(x,0) \;&=\; x, \qquad\qquad\qquad\qquad\qquad\ \textit{for}\ \ x\in\Omega. \end{aligned} \tag{2.4}$$

We mention that the map $x \mapsto Y(x,\tau)$ (for a fixed $\tau > 0$) can also be found by solving (2.3) backwards in time, i.e., solving the final time problem

$$\begin{aligned} \frac{\partial \tilde{Y}}{\partial t}(x,t) \;&=\; \Lambda(\tilde{Y}(x,t),t), \quad \text{for}\ \ x\in\Omega,\ t\in[0,\tau],\\ \tilde{Y}(x,\tau) \;&=\; x, \qquad\qquad\qquad \text{for}\ \ x\in\Omega \end{aligned}$$

and then setting $Y(x,\tau) = \tilde{Y}(x,0)$.

The result below gives some information on the "distance" from $X(y,t)$ to y for positive t and it shows that this distance is "controlled" by $h(t)$ and its derivatives.

Lemma 2.2. *Let $T > 0$ and assume that $h \in \mathcal{H}^2(0,T;\Omega^\circ)$ and that $\varepsilon > 0$ satisfies* (2.1)*. Then there exists a positive constant K that depends only on ε and Ω such that the function X defined by* (2.3) *satisfies:*

$$\|X - \mathrm{id}_\Omega\|_{C(\Omega\times[0,T])} \leqslant K\|\dot{h}\|_{L^1([0,T];\mathbb{R}^3)}, \tag{2.5}$$

$$\|\nabla X - I_3\|_{C(\Omega\times[0,T])} \leqslant K\|\dot{h}\|_{L^1([0,T];\mathbb{R}^3)} \exp\left(K\|\dot{h}\|_{L^1([0,T];\mathbb{R}^3)}\right), \tag{2.6}$$

$$\left\|\nabla^2 X\right\|_{C(\Omega\times[0,T])} + \left\|\nabla^3 X\right\|_{C(\Omega\times[0,T])} \leqslant K\|\dot{h}\|_{L^1([0,T];\mathbb{R}^3)} \exp\left(K\|\dot{h}\|_{L^1([0,T];\mathbb{R}^3)}\right). \tag{2.7}$$

Moreover, the above estimates are still valid if we replace $\nabla_y^\alpha X$ with $\nabla_x^\alpha Y$, where $0 \leqslant \alpha \leqslant 3$.

In the above result, we use the supremum norm on $C(\Omega \times [0,T])$ and we write

$$\nabla^\alpha X = \left(\frac{\partial^\alpha X_i}{\partial y_1^{\beta_1}\partial y_2^{\beta_2}\partial y_3^{\beta_3}}\right)_{\beta\in\mathbb{N}^3,\ \beta_1+\beta_2+\beta_3=\alpha}.$$

In order to transform (1.1)–(1.9) into a system written in a cylindrical domain, we define, following Inoue and Wakimoto [11], the vector field $V : \mathcal{F}(h(0))\times[0,T] \to \mathbb{R}^3$ and the scalar field $P : \mathcal{F}(h(0)) \times [0,T] \to \mathbb{R}$ by

$$V(y,t) = J_Y(X(y,t),t)v(X(y,t),t) \qquad (y \in \mathcal{F}(h(0)),\ \ t\in[0,T]), \tag{2.8}$$

$$P(y,t) = p(X(y,t),t) \qquad (y\in\mathcal{F}(h(0)),\ \ t \geqslant 0), \tag{2.9}$$

where J_Y is the Jacobian of the inverse map Y of X, introduced in Lemma 2.1.

In order to write the equations satisfied by $V(y,t)$ and $P(y,t)$ we define for each $i \in \{1,2,3\}$ the differential operators

$$(LV)_i = \sum_{j,k=1}^{3} \frac{\partial}{\partial y_j}\left(g^{jk}\frac{\partial V_i}{\partial y_k}\right) + 2\sum_{j,k,l=1}^{3} g^{kl}\Gamma_{jk}^{i}\frac{\partial V_j}{\partial y_l}$$
$$+ \sum_{j,k,l=1}^{3}\left\{\frac{\partial}{\partial y_k}(g^{kl}\Gamma_{jl}^{i}) + \sum_{m=1}^{n} g^{kl}\Gamma_{jl}^{m}\Gamma_{km}^{i}\right\}V_j, \quad (2.10)$$

$$(NV)_i = \sum_{j=1}^{3} V_j\frac{\partial V_i}{\partial y_j} + \sum_{j,k=1}^{3}\Gamma_{jk}^{i}V_jV_k, \quad (2.11)$$

$$(MV)_i = \sum_{j=1}^{3}\frac{\partial Y_j}{\partial t}\frac{\partial V_i}{\partial y_j} + \sum_{j,k=1}^{3}\left\{\Gamma_{jk}^{i}\frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y_j}\right\}V_j, \quad (2.12)$$

$$(GP)_i = \sum_{j=1}^{3} g^{ij}\frac{\partial P}{\partial y_j}. \quad (2.13)$$

We have denoted, for each i, j, $k \in \{1,2,3\}$, (see, for instance, [4])

$$g^{ij}(y,t) = \sum_{k=1}^{3}\frac{\partial Y_i}{\partial x_k}(X(y,t),t)\frac{\partial Y_j}{\partial x_k}(X(y,t),t) \qquad \text{(metric contravariant tensor)},$$
$$g_{ij}(y,t) = \sum_{k=1}^{3}\frac{\partial X_k}{\partial y_i}(y,t)\frac{\partial X_k}{\partial y_j}(y,t) \qquad \text{(metric covariant tensor)},$$

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{3} g^{kl}\left\{\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\right\} \quad \text{(Christoffel's symbol)}. \quad (2.14)$$

Indeed, this follows in an elementary way from (2.6), (2.7) and (2.14).

We need a version of the space $\mathcal{T}L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ introduced in Section 1, but this time on a finite time interval. For any $T>0$ we set

$$\mathcal{T}L^2([0,T];L^2(\Omega)) = \left\{ \begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \begin{matrix} L^2([0,T];L^2(\Omega)) \\ \mathcal{H}^2(0,T;\Omega^\circ) \\ \mathcal{H}^1(0,T;\mathbb{R}^3) \\ \mathcal{H}^1(0,T;\mathbb{R}^3) \end{matrix} \;\middle|\; \begin{matrix} v(x,t) = g(t) + \\ \omega(t)\times(x-h(t)) \\ \forall\, t\in(0,T),\ x\in\mathcal{B}(h(t)), \\ \operatorname{div} v = 0, \quad g = \dot{h} \end{matrix} \right\}.$$

The subset $\mathcal{T}C([0,T];\mathcal{H}^1(\Omega))$ is defined similarly, but with $C([0,T];\mathcal{H}^1(\Omega))$ in place of $L^2([0,T];L^2(\Omega))$ in the top line. Another subset $\mathcal{T}\mathcal{H}^1(0,T;L^2(\Omega))$ is defined similarly, but with $\mathcal{H}^1(0,T;L^2(\Omega))$ in place of $L^2([0,T];L^2(\Omega))$ in the top line. The more complicated set

$$\mathcal{T}L^2([0,T];\mathcal{H}^2(\mathcal{F}(h))) \subset \mathcal{T}C([0,T];\mathcal{H}^1(\Omega))$$

consists of those $\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \mathcal{T}C([0,T];\mathcal{H}^1(\Omega))$ for which $v(\cdot,t)|_{\mathcal{F}(h(t))} \in \mathcal{H}^2(\mathcal{F}(h(t))$ holds for almost every $t \in [0,T]$ and the scalar function $t \to \|v(\cdot,t)|_{\mathcal{F}(h(t))}\|_{\mathcal{H}^2}$ is in $L^2[0,T]$. For any given $h \in \mathcal{H}^2(0,T;\Omega^\circ)$ we set

$$L^2([0,T];\mathcal{H}^1(\mathcal{F}(h))) = \left\{ p \in L^2([0,T];L^2(\Omega)) \;\middle|\; \begin{array}{c} p|_{\mathcal{F}(h(t))} \in \mathcal{H}^1(\mathcal{F}(h(t)) \text{ and} \\ p|_{\mathcal{B}(h(t))} = 0 \text{ for a.e. } t \in [0,T], \\ \int_0^T \|p\|^2_{\mathcal{H}^1(\mathcal{F}(h(t))}\mathrm{d}t < \infty \end{array} \right\}.$$

With this notation we have the following result, which is proved in [17]:

Proposition 2.3. *Let $h \in \mathcal{H}^2(0,T;\Omega^\circ)$ and let $\varepsilon > 0$ be such that* (2.1) *holds. Denote $h_0 = h(0)$, $\mathcal{F}_0 = \mathcal{F}(h_0)$ and $\mathcal{B}_0 = \mathcal{B}(h_0)$. Let $h_1 \in \Omega^\circ$ and let v, g, ω, p be functions such that*

$$\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \mathcal{T}L^2([0,T];\mathcal{H}^2(\mathcal{F}(h))) \cap \mathcal{T}\mathcal{H}^1(0,T;L^2(\Omega)), \tag{2.15}$$

$$p \in L^2([0,T];\mathcal{H}^1(\mathcal{F}(h))). \tag{2.16}$$

These functions satisfy (1.1)–(1.9) *with* (1.12) *if and only if the functions V, P defined by* (2.8)–(2.9) *satisfy the regularity conditions*

$$V \in L^2([0,T];\mathcal{H}^2(\mathcal{F}_0;\mathbb{R}^3)) \cap C([0,T];\mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3)) \cap \mathcal{H}^1(0,T;L^2(\mathcal{F}_0;\mathbb{R}^3)), \tag{2.17}$$

$$P \in L^2([0,T];\mathcal{H}^1(\mathcal{F}_0)), \tag{2.18}$$

and together with g and ω they satisfy the equations

$$\rho\frac{\partial V}{\partial t} - \nu(LV) + \rho(MV) + \rho(NV) + (GP) = 0 \quad \text{in } \; \mathcal{F}_0 \times (0,T), \tag{2.19}$$

$$\operatorname{div} V = 0 \quad \text{in } \; \mathcal{F}_0 \times (0,T), \tag{2.20}$$

$$V = 0 \quad \text{on } \; \partial\Omega, \tag{2.21}$$

$$\dot{h} = g, \tag{2.22}$$

$$V(y,t) = g(t) + \omega(t) \times (y - h_0) \quad \text{on } \; \partial\mathcal{B}_0 \times (0,T), \tag{2.23}$$

$$m\dot{g}(t) = -\int_{\partial\mathcal{B}_0} \sigma(V,P)n\,\mathrm{d}\Gamma + k_p[h_1 - h(t)] - k_d g(t) \quad \textit{for } \; t \in (0,T), \tag{2.24}$$

$$J\dot{\omega}(t) = -\int_{\partial\mathcal{B}_0} (y - h_0) \times \sigma(V,P)n\mathrm{d}\Gamma \qquad \textit{for } \; t \in (0,T), \tag{2.25}$$

$$V(y,0) = v_0(y) \qquad \textit{for } \; y \in \mathcal{F}_0, \tag{2.26}$$

$$g(0) = g_0, \qquad \omega(0) = \omega_0, \tag{2.27}$$

where, denoting by I the identity matrix and using D from (4.1)*,*

$$\sigma(V,P) = -PI + 2\nu D(V).$$

The next result asserts that, for T small enough, the operators L and G defined in (2.10), (2.13) are close to the operators Δ and ∇, respectively, and that the operator M defined in (2.12) is small in an appropriate sense. This lemma is taken from [13, Lemma 2.3.4]. We continue to use the notation $\mathcal{F}_0 = \mathcal{F}(h(0))$.

Lemma 2.4. *Let $\varepsilon > 0$ and let $h \in \mathcal{H}^2(0,T;\Omega^\circ)$ be such that (2.1) holds. Let V and P be functions as in (2.17) and (2.18). Denote*

$$R_1 = \|V\|_{L^2(0,T;\mathcal{H}^2(\mathcal{F}_0;\mathbb{R}^3))} + \|V\|_{C([0,T];\mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3))} + \|\nabla P\|_{L^2(0,T;L^2(\mathcal{F}_0;\mathbb{R}^3))} \tag{2.28}$$

$$\eta = \|\dot{h}\|_{\mathcal{H}^1(0,T;\mathbb{R}^3)} + \|\dot{h}\|_{L^\infty(0,T;\mathbb{R}^3)}. \tag{2.29}$$

Then there exist a positive constant K, depending only on Ω, ε and the physical parameters (ν, ρ), such that

1. $\|\nu(L-\Delta)V\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \leqslant KTR_1\eta\exp(K\eta T)$,
2. $\|\rho MV\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \leqslant KT^{\frac{1}{2}}R_1\eta\exp(K\eta T)$,
3. $\|(G-\nabla)P\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \leqslant KTR_1\eta\exp(K\eta T)$.

We end this section by an estimate for the nonlinear operator N defined in (2.11). To accomplish this, we first recall the following result:

Lemma 2.5. *Let $T > 0$ and denote $\mathcal{I} = [0,T]$. Let V and W be functions as in (2.17). Then $(W\cdot\nabla)V \in L^{5/2}(\mathcal{I};L^2(\mathcal{F}_0))$, and for all $i,j \in \{1,2,3\}$, we have that $W_iV_j \in C(\mathcal{I};L^2(\mathcal{F}_0))$. Moreover, there exist $C_1, C_2 > 0$, depending only on Ω, such that:*

$$\|(W\cdot\nabla)V\|_{L^{5/2}(\mathcal{I};L^2(\mathcal{F}_0))} \leqslant C_1\|W\|_{C(\mathcal{I};\mathcal{H}^1(\mathcal{F}_0))}\|V\|^{1/5}_{C(\mathcal{I};\mathcal{H}^1(\mathcal{F}_0))}\|V\|^{4/5}_{L^2(\mathcal{I};\mathcal{H}^2(\mathcal{F}_0))}, \tag{2.30}$$

$$\|W_iV_j\|_{C(\mathcal{I};L^2(\mathcal{F}_0))} \leqslant C_2\|W\|_{C(\mathcal{I};\mathcal{H}^1(\mathcal{F}_0))}\|V\|_{C(\mathcal{I};\mathcal{H}^1(\mathcal{F}_0))}. \tag{2.31}$$

The first estimate appears as Lemma 5.2 in [18], for a two-dimensional domain. However, thanks to the continuous Sobolev embedding $\mathcal{H}^1(\Omega) \subset L^q(\Omega)$, valid for any bounded domain $\Omega \subset \mathbb{R}^3$ and for $1 \leqslant q \leqslant 6$, the same proof works also for a three-dimensional domain. The second estimate is a direct consequence of the same Sobolev embedding (with $q = 4$) and the simple estimate $\|ab\|_{L^2} \leqslant \|a\|_{L^4}\|b\|_{L^4}$. Using the above lemma, one can prove the following result.

Lemma 2.6. *With the notation and assumptions in Lemma 2.4, there exists a positive constant K, depending only on Ω and ε, such that for all $T \leqslant 1$,*

$$\|\rho NV\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \leqslant KR_1^2T^{\frac{1}{10}}\exp(K\eta). \tag{2.32}$$

3. Local existence of solutions

In this section we show that the method introduced in [17] and [18] in order to prove local in time existence and uniqueness of solutions of (1.1)–(1.9) for $u = 0$ can be adapted for u given in feedback form by (1.12). Consider the C^2 domain Ω and the points $h_0 \in \Omega^\circ$ (initial position of the center of the ball) and $h_1 \in \Omega^\circ$ (anchor point of the spring and damper) to be fixed, with $\operatorname{dist}(h_0, \partial\Omega) > 1 + \varepsilon$ for some $\varepsilon > 0$. Recall the notation $\mathcal{F}_0 = \mathcal{F}(h_0)$ and introduce

$$\widehat{\mathcal{H}}^1(\mathcal{F}_0) = \left\{ q \in \mathcal{H}^1(\mathcal{F}_0),\ \int_{\mathcal{F}_0} q(x)\,\mathrm{d}x = 0 \right\}.$$

A basic ingredient in tackling the above local existence and uniqueness problem consists in studying the linearized system

$$\frac{\partial V}{\partial t} - \nu \Delta V + \nabla P = f_1, \quad \text{in } \mathcal{F}_0 \times [0,T], \tag{3.1}$$

$$\operatorname{div} V = 0, \quad \text{in } \mathcal{F}_0 \times [0,T], \tag{3.2}$$

$$V = 0, \quad \text{on } \partial\Omega, \tag{3.3}$$

$$V(y,t) = g(t) + \omega(t) \times (y - h_0), \quad y \in \partial\mathcal{B}_0,\ t \in [0,T], \tag{3.4}$$

$$m\dot{g}(t) = -\int_{\partial\mathcal{B}(h_0)} \sigma(V,P) n \mathrm{d}\Gamma - k_d g(t) + f_2(t), \quad t \in [0,T], \tag{3.5}$$

$$J\dot{\omega}(t) = -\int_{\partial\mathcal{B}_0} (y - h_0) \times \sigma(V,P) n \,\mathrm{d}y, \quad t \in [0,T], \tag{3.6}$$

$$V(x,0) = v_0(x), \quad x \in \mathcal{F}_0, \tag{3.7}$$

$$g(0) = g_0, \quad \omega(0) = \omega_0, \tag{3.8}$$

where $T > 0$, combined with the application of a fixed point method. These equations have been obtained from (2.19)–(2.27) by replacing the terms LV and GP with their linear approximations ΔV and with ∇P, respectively, and "throwing out" the other nonlinear terms (any term that depends on h is considered nonlinear). Note that the function h does not appear in the above equations. The new functions f_1 and f_2 will account for all the modifications that we did.

Proposition 3.1. *With the above notation, let*

$$f_1 \in L^2((0,T); L^2(\mathcal{F}_0; \mathbb{R}^3)), \quad f_2 \in L^2((0,T); \mathbb{R}^3), \quad v_0 \in \mathcal{H}^1(\mathcal{F}_0; \mathbb{R}^3)$$

and $g_0, \omega_0 \in \mathbb{R}^3$ *be such that*

$$\operatorname{div} v_0 = 0 \quad \text{in } \mathcal{F}_0, \qquad v_0 = 0 \quad \text{on } \partial\Omega,$$
$$v_0(y) = g_0 + \omega_0 \times (y - h_0), \quad y \in \partial\mathcal{B}_0.$$

Then the system (3.1)–(3.8) *admits a unique solution* (V, P, g, ω) *with*

$$V \in L^2((0,T); \mathcal{H}^2(\mathcal{F}_0; \mathbb{R}^3)) \cap C([0,T]; \mathcal{H}^1(\mathcal{F}_0; \mathbb{R}^3)) \cap \mathcal{H}^1((0,T); L^2(\mathcal{F}_0; \mathbb{R}^3)),$$

$$P \in L^2((0,T);\widehat{\mathcal{H}}^1(\mathcal{F}_0)/\mathbb{R}), \quad g,\,\omega \in \mathcal{H}^1(\mathcal{I};\mathbb{R}^3).$$

Moreover, there exists a $K_\varepsilon > 0$, depending only on Ω and on ε, such that

$$\begin{aligned}\|V\|_{L^2((0,T);\mathcal{H}^2(\mathcal{F}_0;\mathbb{R}^3))} + \|V\|_{C([0,T];\mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3))} + \|V\|_{\mathcal{H}^1((0,T);L^2(\mathcal{F}_0;\mathbb{R}^3))} \\ + \|\nabla P\|_{L^2((0,T);L^2(\mathcal{F}_0;\mathbb{R}^3))} + \|g\|_{\mathcal{H}^1(\mathcal{I};\mathbb{R}^3)} + \|\omega\|_{\mathcal{H}^1(\mathcal{I};\mathbb{R}^3)} \\ \leqslant K_\varepsilon \left[\|v_0\|_{\mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3)} + |g_0| + |\omega_0| + \left(\|f_1\|^2_{L^2(\mathcal{I};L^2(\mathcal{F}_0;\mathbb{R}^3))} + \|f_2\|^2_{L^2(\mathcal{I};\mathbb{R}^3)}\right)^{\frac{1}{2}}\right]. \qquad (3.9)\end{aligned}$$

This follows from Takahashi [17, Corollary 5.4]. Actually the proof in [17] is for a two-dimensional domain, but (as explained in [17, Section 9]) the same proof works also in three dimensions.

Remark 3.2. Let us denote

$$\mathcal{H}^1_{\mathrm{div},\partial\Omega}(\mathcal{F}_0) = \left\{V \in \mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3) \mid \operatorname{div} V = 0 \text{ in } \mathcal{F}_0,\ V = 0 \text{ on } \partial\Omega\right\}.$$

From a system theoretic point of view, the equations (3.1)–(3.8) determine a well-posed linear control system with input $\tilde{u} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2(\mathcal{F}_0;\mathbb{R}^3) \times \mathbb{R}^3$ and state $z = \begin{bmatrix} V \\ g \\ \omega \end{bmatrix} \in \mathcal{H}^1_{\mathrm{div},\partial\Omega}(\mathcal{F}_0) \times \mathbb{R}^3 \times \mathbb{R}^3$, in the sense of [24] (see also [21, Section 4.5]). Thus, the equations can be reformulated in the form $\dot{z} = Az + B\tilde{u}$, where A generates a strongly continuous semigroup on the state space X. This state space X consists of those triples $\begin{bmatrix} V \\ g \\ \omega \end{bmatrix}$ as above that satisfy the compatibility condition $V(x) = g + \omega \times (x - h_0)$ for all $x \in \partial\mathcal{B}_0$. The control operator B is unbounded since the vector $B\tilde{u} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}$ need not be in X. The semigroup generated by A is analytic.

We have the following local in time existence and uniqueness result:

Theorem 3.3. *Suppose that $\partial\Omega$ is of class C^2 and let $\varepsilon, \delta > 0$. Let $h_0, h_1 \in \Omega$, with* $\operatorname{dist}(h_0, \partial\Omega) > 1 + \varepsilon$, *let $g_0, \omega_0 \in \mathbb{R}^3$ and let $v_0 \in \mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3)$. Moreover, assume that*

$$\begin{cases} \operatorname{div} v_0 = 0, & \text{in } \mathcal{F}_0, \\ v_0 = 0, & \text{on } \partial\Omega, \\ v_0(x) = g_0 + \omega_0 \times (x - h_0), & x \in \partial\mathcal{B}(h_0), \\ \|v_0\|_{\mathcal{H}^1(\mathcal{F}_0)} + |g_0| + |\omega_0| + |h_1 - h_0| \leqslant \delta. \end{cases}$$

Then there exists $T_{max} > 0$, depending only on ε and on δ, such that the equations (1.1)–(1.9) *with* (1.12) *admit, for every $T \in [0, T_{max})$, a solution with*

$$\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \mathcal{T}L^2([0,T];L^2(\Omega)) \cap \mathcal{T}\mathcal{H}^1(0,T;L^2(\Omega)),$$

$$p \in L^2([0,T];\mathcal{H}^1(\mathcal{F}(h))).$$

This solution is unique up to an additive perturbation of p that depends only on t.

Proof. **The first step:** For each $T > 0$ we define

$$E_T = L^2([0,T]; L^2(\mathcal{F}_0; \mathbb{R}^3)) \times L^2([0,T]; \mathbb{R}^3),$$

which is the Hilbert space with the product norm. This is the space in which we choose $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$ in the system (3.1)–(3.8). For any $R > 0$ we denote by $B_T(R)$ the closed ball centered at the origin and of radius R in the Hilbert space E_T. For any $R > 0$ we shall define a complicated input-output mapping $\mathcal{Z}_T$ for the system (3.1)–(3.8), defined on a short time interval $[0,T]$ with $T < 1$, and for all $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] \in B_T(R)$. This time horizon T depends on Ω, ε, v_0, g_0, ω_0 and on R, in a way that will be specified later. The map

$$\mathcal{Z}_T : B_T(R) \to E_T\,, \qquad \mathcal{Z}_T \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3.10}$$

will be defined in three stages. Throughout, we assume that $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] \in E_1$, because even if f_1, f_2 are defined on a shorter interval $[0,T]$, we can extend them to $[0,1]$ by taking them to be zero for $t > T$. It will be convenient to denote

$$\lambda_0 = \|v_0\|_{\mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3)} + |g_0| + |\omega_0|\,. \tag{3.11}$$

In the *first stage*, we solve the linear equations (3.1)–(3.8) with the given v_0, g_0 and ω_0 on the time interval $[0,1]$, which is possible according to Proposition 3.1. Moreover, if $\begin{bmatrix} V \\ g \\ \omega \end{bmatrix}$ and P are the solution, then we know that these functions satisfy the smoothness conditions and the estimates stated in Proposition 3.1. In particular, $g \in \mathcal{H}^1(0,1;\mathbb{R}^3)$ and for every $t \in [0,1]$ we have (from (3.9))

$$\begin{aligned} |g(t)| \leqslant |g_0| + \int_0^t |\dot{g}(\theta)| \mathrm{d}\theta \ &\leqslant\ |g_0| + \|g\|_{\mathcal{H}^1(0,1;\mathbb{R}^3)} \\ &\leqslant\ |g_0| + K_\varepsilon \lambda_0 + K_\varepsilon \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{E_t}. \end{aligned} \tag{3.12}$$

In the *second stage*, we define the function $h : [0,1] \to \mathbb{R}^3$ by

$$h(t) = h_0 + \int_0^t g(\theta) \mathrm{d}\theta\,.$$

Clearly $h \in \mathcal{H}^2(0,1;\mathbb{R}^3)$ and since $\mathrm{dist}(h(0), \partial\Omega) > 1 + \varepsilon$, due to (3.12) there exists a $T > 0$ such that (2.1) holds for any $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] \in B_T(R)$. Indeed, this T must satisfy

$$T \cdot [|g_0| + K_\varepsilon \lambda_0 + K_\varepsilon R] \ \leqslant\ \mathrm{dist}(h_0, \partial\Omega) - 1 - \varepsilon\,.$$

Thus, T depends only on the initial conditions and on R. Using h we define a t-dependent transformation X on Ω as in Lemma 2.1, and its inverse Y.

In the *third stage* we define the operators L, N, M, G from (2.10)–(2.13), for $t \in [0,T]$, based on the transformations X and Y constructed in the second stage. Finally, we define the functions y_1 and y_2 by

$$y_1 = \nu(L-\Delta)V - \rho MV + (\nabla - G)P - \rho NV\,, \tag{3.13}$$

$$y_2 = k_p\,(h_1 - h)\,. \tag{3.14}$$

We see from Lemmas 2.4 and 2.6 and from Proposition 3.1 that indeed $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in E_T$, so that $\mathcal{Z}_T$ is now well defined.

The second step: For any fixed $R > 0$, we have seen in the first step that $\mathcal{Z}_T$ can be defined on $B_T(R)$, if $T > 0$ is sufficiently small. Now we show that if T is sufficiently small (possibly it needs to be smaller than in the first step), then $\mathcal{Z}_T$ leaves $B_T(R)$ invariant: $\mathcal{Z}_T B_T(R) \subset B_T(R)$. We take $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] \in B_T(R)$.

To estimate $\|y_1\|$, first we note, using (3.12), that

$$\|\dot{h}\|_{L^1([0,T];\mathbb{R}^3)} = \|g\|_{L^1([0,T];\mathbb{R}^3)} \leqslant T\left(|g_0| + K_\varepsilon\lambda_0 + K_\varepsilon \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{E_T}\right).$$

Using Lemma 2.4 and Lemma 2.6, we deduce that

$$\|y_1\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \leqslant K(\eta^2 + R_1^2)T^{\frac{1}{10}}\exp(K\eta),$$

with (from Proposition 3.1),

$$\eta \leqslant K_\varepsilon(\lambda_0 + R), \quad R_1 \leqslant K_\varepsilon(\lambda_0 + R).$$

As a consequence there exists a constant K depending on the geometry, the physical parameters and ε such that

$$\|y_1\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \leqslant K(\lambda_0 + R)^2 T^{\frac{1}{10}}\exp(K(\lambda_0 + R)).$$

To estimate $\|y_2\|$, we observe that for all $t \in [0,T]$ we have $|h_1 - h(t)| < D_\Omega$, where D_Ω is the diameter of Ω. Hence

$$\|y_2\|_{L^2([0,T];\mathbb{R}^3)} \leqslant k_p D_\Omega T^{\frac{1}{2}}.$$

Looking at the last two estimates, it is now clear that for $T > 0$ sufficiently small, $\|\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]\| \leqslant R$, i.e., $B_T(R)$ is invariant under $\mathcal{Z}_T$.

The third step: We show that for any $R > 0$, $\mathcal{Z}_T$ has a fixed point in $B_T(R)$, if $T > 0$ is sufficiently small. Throughout we assume that $T \in (0,1)$ is sufficiently small so that $B_T(R)$ is invariant under $\mathcal{Z}_T$, but this is not enough to guarantee a fixed point, we may need to take T even smaller than in the second step.

We take $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{smallmatrix}\right] \in B_T(R)$. Let V, P, g, ω, respectively $\tilde{V}, \tilde{P}, \tilde{g}, \tilde{\omega}$ be the corresponding solutions of (3.1)–(3.8). We then set

$$h(t) = h_0 + \int_0^t g(\theta)\,\mathrm{d}\theta, \qquad \tilde{h}(t) = h_0 + \int_0^t \tilde{g}(\theta)\,\mathrm{d}\theta,$$

and we denote by X and Y, respectively $\tilde{X}$ and $\tilde{Y}$ the transformations constructed from h, respectively $\tilde{h}$, according to the procedure described in Lemma 2.1. Let L, N, M, G, respectively $\tilde{L}, \tilde{N}, \tilde{M}, \tilde{G}$ be the corresponding operators obtained according to (2.10)–(2.13), for $t \in [0,T]$, based on the transformations. Finally, let $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$, respectively $\left[\begin{smallmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{smallmatrix}\right]$ be the corresponding outputs defined by (3.13)–(3.14).

According to (3.13) we have

$$
\begin{aligned}
y_1 - \tilde{y}_1 = \nu(L-\tilde{L})V - \rho(M-\tilde{M})V + (\tilde{G}-G)P - \rho(NV - \tilde{N}\tilde{V}) \\
+ \nu(\tilde{L}-\Delta)(V-\tilde{V}) - \rho\tilde{M}(V-\tilde{V}) + (\nabla - \tilde{G})(P-\tilde{P}). \quad (3.15)
\end{aligned}
$$

Therefore, the last three terms in (3.15) can be estimated by using Lemma 2.4 with $V-\tilde{V}$ in place of V, $P-\tilde{P}$ in place of P, and $\tilde{L}, \tilde{M}, \tilde{G}$ in place of L, M, G. We obtain, using the notation K_ε introduced in the second step,

$$
\begin{aligned}
&\left\| \nu(\tilde{L}-\Delta)(V-\tilde{V}) - \rho\tilde{M}(V-\tilde{V}) + (\nabla - \tilde{G})(P-\tilde{P}) \right\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \\
&\quad \leqslant K_\varepsilon T^{\frac{1}{2}} \left(\|\dot{\tilde{h}}\|_{\mathcal{H}^1(0,T)} + \|\dot{\tilde{h}}\|_{L^\infty(0,T)} \right) \exp\left(K_\varepsilon \left(\|\dot{\tilde{h}}\|_{\mathcal{H}^1(0,T)} + \|\dot{\tilde{h}}\|_{L^\infty(0,T)} \right) \right) \\
&\qquad \times \left(\|V-\tilde{V}\|_{L^2([0,T];\mathcal{H}^2(\mathcal{F}_0;\mathbb{R}^3))} + \|V-\tilde{V}\|_{C([0,T];\mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3))} \right. \\
&\qquad\qquad \left. + \|\nabla(P-\tilde{P})\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \right).
\end{aligned}
$$

Now we estimate the norms on the right-hand side above using (3.9), which leads to

$$
\begin{aligned}
&\left\| \nu(\tilde{L}-\Delta)(V-\tilde{V}) - \rho\tilde{M}(V-\tilde{V}) + (\nabla - \tilde{G})(P-\tilde{P}) \right\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \\
&\qquad \leqslant K T^{\frac{1}{2}} (\lambda_0 + R) \exp\left(K(\lambda_0 + R) \right) \cdot |||f - \tilde{f}|||, \quad (3.16)
\end{aligned}
$$

where we have used the notation

$$
|||f-\tilde{f}||| = \left(\|f_1 - \tilde{f}_1\|^2_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} + \|f_2 - \tilde{f}_2\|^2_{L^2([0,T];\mathbb{R}^3)} \right)^{\frac{1}{2}}. \quad (3.17)
$$

To estimate the first three terms on the right-hand side of (3.15), we note that Proposition 6.14 in [17] implies, modulo some slight adaptations, that there exists a constant $K > 0$, depending only on the geometry, the physical arameters and ε such that

$$
\begin{aligned}
&\left\| \nu(L-\tilde{L})V - \rho(M-\tilde{M})V + (\tilde{G}-G)P \right\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \\
&\qquad \leqslant K T^{\frac{1}{2}} Q \left(\|\dot{h} - \dot{\tilde{h}}\|_{L^\infty([0,T];\mathbb{R}^3)} \right) \exp\left(K \|\dot{h}\|_{L^1(0,T)} \right),
\end{aligned}
$$

where

$$
Q = \|V\|_{L^2([0,T];\mathcal{H}^2(\mathcal{F}_0;\mathbb{R}^3))} + \|V\|_{C([0,T];\mathcal{H}^1(\mathcal{F}_0;\mathbb{R}^3))} + \|\nabla P\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))}.
$$

Since, by Proposition 3.1 we have $Q + \|\dot{h}\|_{L^1(0,T)} \leqslant K_\varepsilon(\lambda_0 + R)$, it follows that

$$\left\|\nu(L-\tilde{L})V - \rho(M-\tilde{M})V + (\tilde{G}-G)P\right\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))}$$
$$\leqslant KT^{\frac{1}{2}}(\lambda_0+R)\exp\left(K(\lambda_0+R)\right)|||f-\tilde{f}|||. \tag{3.18}$$

The fourth term on the right-hand side of (3.15) can be estimated using Proposition 6.15 in [17] (again, with slight adaptations): there exists a constant $K > 0$, depending only on the geometry, the physical parameters and ε such that

$$\begin{aligned}\left\|\rho(NV-\tilde{N}\tilde{V})\right\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} &\\ \leqslant KT^{\frac{1}{10}}\left(\|V\|+\|\tilde{V}\|\right)&\exp\left(K(\|\dot{h}\|_{L^\infty(0,T)}+\|\dot{\tilde{h}}\|_{L^\infty(0,T)})\right)\|V-\tilde{V}\| \\ &+ KT\exp\left(K(\|\dot{h}\|_{L^\infty(0,T)}+\|\dot{\tilde{h}}\|_{L^\infty(0,T)})\right)\|\tilde{V}\|^2\|\dot{h}-\dot{\tilde{h}}\|_{L^\infty(0,T)},\end{aligned}$$

where all the norms without subscript are taken in $\mathcal{U}(0,T;\mathcal{F}_0)$. Combining the above with (3.9) gives

$$\left\|\rho(NV-\tilde{N}\tilde{V})\right\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))}$$
$$\leqslant KT^{1/10}(\lambda_0+R)\exp\left(K(\lambda_0+R)\right)|||f-\tilde{f}|||. \tag{3.19}$$

Combining (3.15), (3.16), (3.18) and (3.19), we obtain that there exists a constant $K > 0$, depending only on the geometry, the physical parameters and ε such that

$$\|y_1-\tilde{y}_1\|_{L^2([0,T];L^2(\mathcal{F}_0;\mathbb{R}^3))} \leqslant KT^{1/10}(\lambda_0+R)\exp\left(K(\lambda_0+R)\right)\left\|\begin{bmatrix}f_1\\f_2\end{bmatrix}-\begin{bmatrix}\tilde{f}_1\\\tilde{f}_2\end{bmatrix}\right\|_{E_T}. \tag{3.20}$$

Now we estimate $y_2 - \tilde{y}_2$, using (3.14):

$$\begin{aligned}\|y_2-\tilde{y}_2\|_{L^2([0,T];\mathbb{R}^3)} &= k_p\|h-\tilde{h}\|_{L^2([0,T];\mathbb{R}^3)} \leqslant k_pT^{\frac{1}{2}}\sup_{t\in[0,T]}|h(t)-\tilde{h}(t)| \\ &\leqslant k_pT^{\frac{1}{2}}\int_0^T|g(t)-\tilde{g}(t)|\mathrm{d}t \leqslant k_pT\|g-\tilde{g}\|_{L^2([0.T];\mathbb{R}^3)}.\end{aligned}$$

Using (3.9) we get

$$\|y_2-\tilde{y}_2\|_{L^2([0,T];\mathbb{R}^3)} \leqslant k_pK_\varepsilon T\left\|\begin{bmatrix}f_1\\f_2\end{bmatrix}-\begin{bmatrix}\tilde{f}_1\\\tilde{f}_2\end{bmatrix}\right\|_{E_T}.$$

Combining this last estimate with (3.20), we obtain that there is a constant $K > 0$, depending only on the geometry, the physical parameters and ε such that

$$\left\|\begin{bmatrix}y_1\\y_2\end{bmatrix}-\begin{bmatrix}\tilde{y}_1\\\tilde{y}_2\end{bmatrix}\right\|_{E_T} \leqslant KT^{1/10}\left[(\lambda_0+R)\exp\left(K(\lambda_0+R)\right)+k_p\right]\left\|\begin{bmatrix}f_1\\f_2\end{bmatrix}-\begin{bmatrix}\tilde{f}_1\\\tilde{f}_2\end{bmatrix}\right\|_{E_T}.$$

Now we see that for T small enough, $\mathcal{Z}_T$ is a strict contraction, so that $\mathcal{Z}_T$ has a fixed point in $B_T(R)$. $\square$

We remark that, together with the outputs y_1 and y_2, the well-posed control system from Remark 3.2 could be called a non-linear well-posed system, in the sense that on any finite time interval, we have continuous mappings from the initial state and the input function to the final state and the output function.

4. Energy estimates

In this section we derive some energy estimates for the strong solutions of (1.1)–(1.9). We use the notation introduced in Section 1 and we introduce the *strain rate tensor* $D(v)$ by

$$D_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \text{for} \quad i, j \in \{1,2,3\}. \tag{4.1}$$

Proposition 4.1. *Let $T > 0$ and let $u \in L^2_{\mathrm{loc}}([0,T];\mathbb{R}^3)$ and let $\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix}$ be a solution of* (1.1)–(1.9) *for $t \in [0,T]$, with a corresponding pressure field p, satisfying the regularity assumptions* (2.15)–(2.16)*. Then for all $\tau \in [0,T]$,*

$$\frac{\rho}{2}\int_{\mathcal{F}(h(\tau))} |v(x,\tau)|^2 \,\mathrm{d}x + \frac{m}{2}|g(\tau)|^2 + \frac{J}{2}|\omega(\tau)|^2 - \frac{\rho}{2}\int_{\mathcal{F}(h_0)} |v_0(x)|^2 \,\mathrm{d}x - \frac{m}{2}|g_0|^2 - \frac{J}{2}|\omega_0|^2$$
$$= \int_0^\tau \left[-2\nu \int_{\mathcal{F}(h(t))} |D(v)(x,t)|^2 \,\mathrm{d}x + u(t)\cdot g(t)\right] \mathrm{d}t. \tag{4.2}$$

Proof. We first note that (1.1) can be rewritten as Cauchy's equation

$$\rho[\dot v + (v\cdot\nabla)v] - \operatorname{div}\sigma(v,p) = 0 \quad \text{in} \quad L^2_{\mathrm{loc}}([0,\infty); L^2(\mathcal{F}(h))),$$

where σ has been defined in (1.10) and $(\operatorname{div}\sigma)_i = \sum_{j=1}^3 \frac{\partial \sigma_{ij}}{\partial x_j}$. If we take the pointwise inner product of both sides of the last formula with v (which, according to (1.16) is a continuous function of time, with values in $L^2(\Omega;\mathbb{R}^3)$) and we integrate over $\mathcal{F}(h(t))$, it follows that we have, for almost every $t \geqslant 0$,

$$\frac{\rho}{2}\int_{\mathcal{F}(h(t))} \frac{\partial}{\partial t}|v|^2 \,\mathrm{d}x + \rho \int_{\mathcal{F}(h(t))} [(v\cdot\nabla)v]\cdot v \,\mathrm{d}x = \int_{\mathcal{F}(h(t))} (\operatorname{div}\sigma(v,p))\cdot v \,\mathrm{d}x. \tag{4.3}$$

Applying the Reynolds transport formula (see Gurtin [9, Section III.10]) to the first term above, this first term can be rewritten as follows:

$$\frac{\rho}{2}\int_{\mathcal{F}(h(t))} \frac{\partial}{\partial t}|v|^2 \,\mathrm{d}x = \frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(h(t))} |v|^2 \,\mathrm{d}x - \frac{\rho}{2}\int_{\partial\mathcal{B}(h(t))} |v|^2 \,(v\cdot n)\,\mathrm{d}\Gamma.$$

Integrating by parts the second term in (4.3), it follows that

$$\rho\int_{\mathcal{F}(h(t))} [(v\cdot\nabla)v]\cdot v \,\mathrm{d}x = \frac{\rho}{2}\int_{\partial\mathcal{B}(h(t))} |v|^2 \,(v\cdot n)\,\mathrm{d}\Gamma.$$

Integrating by parts the right-hand side of (4.3) and using (1.2), it follows that

$$\int_{\mathcal{F}(h(t))} \operatorname{div} \sigma(v, p) \cdot v \, \mathrm{d}x = \int_{\partial\mathcal{B}(h(t))} \sigma(v, p) n \cdot v \, \mathrm{d}\Gamma - 2\nu \int_{\mathcal{F}(h(t))} |D(v)|^2 \, \mathrm{d}x . \qquad (4.4)$$

Putting together (4.3)–(4.4), it follows that

$$\frac{\rho}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}(h(t))} |v|^2 \, \mathrm{d}x = -2\nu \int_{\mathcal{F}(h(t))} |D(v)|^2 \, \mathrm{d}x + \int_{\partial\mathcal{B}(h(t))} \sigma(v, p) n \cdot v \, \mathrm{d}\Gamma . \qquad (4.5)$$

On the other hand, taking the inner products of (1.6) with $\dot{h}(t)$, of (1.7) with $\omega(t)$ and using (1.5), it follows that

$$\frac{m}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\dot{h}(t)|^2 + \frac{J}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\omega(t)|^2 = -\int_{\partial\mathcal{B}(h(t))} \sigma(v, p) n \cdot v \, \mathrm{d}\Gamma + u(t) \cdot \dot{h}(t). \qquad (4.6)$$

Adding (4.5) and (4.6) we obtain on the left-hand side the time derivative of the energy function

$$E(t) = \frac{\rho}{2} \int_{\mathcal{F}(h(t))} |v(x, t)|^2 \, \mathrm{d}x + \frac{m}{2} |g(t)|^2 + \frac{J}{2} |\omega(t)|^2 .$$

It is easy to see from (1.16) that $E \in \mathcal{H}^1_{\mathrm{loc}}(0, \infty)$. Therefore, by integration we obtain the conclusion (4.2). $\square$

The above proposition suggests a candidate to a feedback control such that the corresponding state trajectory satisfies (1.13) and (1.14). We take the PD controller applied to the position error $h_1 - h(t)$, given in (1.12). As already explained in Section 1, this feedback is equivalent to a spring and a damper between the points $h(t)$ and h_1. Intuitively, we expect the resulting feedback system to satisfy the stability requirements (1.13) and (1.14), if the ball could go in a straight line from h_0 to h_1. In fact, we expect this feedback control to work even in some other cases, but not always: see Figure 1 for a geometry where the control (1.12) will not bring the ball into the desired final position.

Proposition 4.2. *With the notation and assumption of Proposition* 4.1*, let* $h_1 \in \Omega^\circ$ *and assume that* u *satisfies the feedback law* (1.12)*. Then*

$$\frac{\rho}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}(h(t))} |v(x, t)|^2 \, \mathrm{d}x + \frac{m}{2} \frac{\mathrm{d}}{\mathrm{d}t} |g(t)|^2 + \frac{J}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\omega(t)|^2 + \frac{k_p}{2} \frac{\mathrm{d}}{\mathrm{d}t} |h(t) - h_1|^2$$
$$= -2\nu \int_{\mathcal{F}(h(t))} |D(v)(x, t)|^2 \, \mathrm{d}x - k_d |g(t)|^2 . \qquad (4.7)$$

Proof. If we combine the obvious formula

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |h(t) - h_1|^2 = (h(t) - h_1) \cdot g(t),$$

with (4.2), it follows that

$$\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(h(t))}|v(x,t)|^2\,\mathrm{d}x+\frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}|g(t)|^2+\frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\omega(t)|^2+\frac{k_p}{2}\frac{\mathrm{d}}{\mathrm{d}t}|h(t)-h_1|^2$$
$$=\ -2\nu\int_{\mathcal{F}(h(t))}|D(v)(x,t)|^2\,\mathrm{d}x+[u(t)+k_p(h(t)-h_1)]\cdot g(t)\,.$$

The above formula, combined with (1.12), yields the conclusion (4.7). □

Corollary 4.3. *With the notation and assumption of Proposition* 4.1*, let $h_1 \in \Omega^\circ$ and assume that u satisfies the feedback law* (1.12)*. Then, for every $t \in [0, T]$*

$$\rho\int_{\mathcal{F}(h(t))}|v(x,t)|^2\,\mathrm{d}x+m|g(t)|^2+|\omega(t)|^2+k_p|h(t)-h_1|^2$$
$$\leqslant \rho\int_{\mathcal{F}(h_0)}|v_0(x)|^2\,\mathrm{d}x+m|g_0|^2+|\omega_0|^2+k_p|h_0-h_1|^2. \quad (4.8)$$

Moreover, there exists $\delta, \varepsilon > 0$, depending only on Ω, k_p and h_1, such that if

$$\rho\int_{\mathcal{F}(h_0)}|v_0(x)|^2\,\mathrm{d}x+m|g_0|^2+|\omega_0|^2+k_p|h_0-h_1|^2\ \leqslant \delta^2,$$

then

$$\mathrm{dist}(h(t),\partial\Omega)\ \geqslant 1+\varepsilon \qquad \forall\, t\in[0,T].$$

Proof. The estimate (4.8) is a direct consequence of (4.7). In order to prove the last statement in the corollary, using the fact that $h_1 \in \Omega^\circ$, we can choose $\varepsilon > 0$ such that

$$\mathrm{dist}(h_1,\partial\Omega)\ \geqslant 1+2\varepsilon.$$

From (4.8) we deduce

$$|h(t)-h_1|^2\leqslant|h_0-h_1|^2+\frac{1}{k_p}\left(\rho\int_{\mathcal{F}(h_0)}|v_0(x)|^2\,\mathrm{d}x+m|g_0|^2+|\omega_0|^2\right).$$

Consequently, if

$$|h_0-h_1|^2+\frac{1}{k_p}\left(\rho\int_{\mathcal{F}(h_0)}|v_0(x)|^2\,\mathrm{d}x+m|g_0|^2+|\omega_0|^2\right)\ <\varepsilon^2,$$

then

$$\mathrm{dist}(h(t),\partial\Omega)\ \geqslant 1+\varepsilon \qquad \forall\, t\in[0,T]. \qquad \square$$

5. Proof of Theorem 1.1

In order to prove the global existence result in Theorem 1.1 we need the following result, which can be proved by slightly adapting the proof of Lemma 4.3 in [2].

Lemma 5.1. *Let* $h \in \mathcal{H}^2_{\text{loc}}((0,\infty);\Omega^\circ)$ *and assume that* $u \in \mathcal{T}L^2_{\text{loc}}([0,\infty);\mathcal{H}^2(\mathcal{F}(h)))$ *and* $q \in L^2_{\text{loc}}([0,\infty);\mathcal{H}^1(\mathcal{F}(h)))$ *with* $u = \ell + k \times (x-h)$ *on* $\partial\mathcal{B}(h)$. *Then, for almost every* $t \in [0,\infty)$ *we have, using the symbol* : *to denote the Hilbert-Schmidt inner product for* 3×3 *matrices,*

$$\nu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}(h)} |Du|^2 \,\mathrm{d}x = -\int_{\mathcal{F}(h)} \operatorname{div} \sigma(u,q) \cdot (\dot{u} + (\Lambda \cdot \nabla) u - (u \cdot \nabla)\Lambda) \,\mathrm{d}x$$
$$+ \int_{\partial\mathcal{B}(h)} \sigma(u,q) n \cdot \Big(\dot{\ell} + \dot{k} \times (x-h) - \omega \times \ell + (k \times \omega) \times (x-h) \Big) \,\mathrm{d}\Gamma$$
$$+ \int_{\mathcal{F}(h)} 2\nu Du : \left[D\left((u \cdot \nabla)\Lambda \right) - (\nabla u)(\nabla \Lambda) \right] \mathrm{d}x. \quad (5.1)$$

We also need the following result, which is a direct consequence of the Poincaré and of the Korn inequalities:

Lemma 5.2. *Let* $h \in \Omega^\circ$ *and assume that* $v \in \mathcal{H}^1_0(\Omega)$, *with* $D(v) \equiv 0$ *in* $\mathcal{S}(h)$. *Then there exists a positive constant* C *independent of* v *and* h *such that*

$$\int_{\mathcal{F}(h)} |v|^2 \, dx + m|g|^2 + J|\omega|^2 + \int_{\mathcal{F}(h)} |\nabla v|^2 \mathrm{d}x \leqslant C \int_{\mathcal{F}(h)} |Dv|^2 \mathrm{d}x. \quad (5.2)$$

Proof of Theorem 1.1. By classical arguments, the local in time solution (v,p,h,g,ω) constructed in Theorem 3.3 can be extended to a maximal solution defined on the time interval $[0,T_*)$ with $T_* \in (0,\infty]$. Moreover, if $T_* < \infty$, then we have either

$$\operatorname{dist}(\mathcal{B}(h(t)), \partial\Omega) \to 0 \quad \text{as} \quad t \to T_* \quad (5.3)$$

or

$$\|v(\cdot,t)\|_{\mathcal{H}^1(\mathcal{F}(h(t)))} \to \infty \quad \text{as} \quad t \to T_*. \quad (5.4)$$

From (4.3), we already know that there exists δ such that if the last condition in (1.15) holds, then (5.3) never occurs. We now prove that for δ possibly smaller, (5.4) is also false. This will imply that $T_* = \infty$.

For $t \in [0,T_*)$ we take the inner product of (1.1) with $\dot{v} + (\Lambda \cdot \nabla)v - (v \cdot \nabla)\Lambda$ and we integrate on $\mathcal{F}(h(t))$, to obtain

$$\int_{\mathcal{F}(h(t))} \rho \dot{v} \cdot [\dot{v} + (\Lambda \cdot \nabla)v - (v \cdot \nabla)\Lambda] \,\mathrm{d}x$$
$$- \int_{\mathcal{F}(h(t))} \operatorname{div} \sigma(v,p) \cdot [\dot{v} + (\Lambda \cdot \nabla)v - (v \cdot \nabla)\Lambda] \,\mathrm{d}x$$
$$+ \int_{\mathcal{F}(h(t))} \rho (v \cdot \nabla) v \cdot [\dot{v} + (\Lambda \cdot \nabla)v - (v \cdot \nabla)\Lambda] \,\mathrm{d}x = 0.$$

Using Lemma 5.1, we deduce from the above relation that

$$\int_{\mathcal{F}(h(t))} \rho\dot{v}\cdot[\dot{v}+(\Lambda\cdot\nabla)v-(v\cdot\nabla)\Lambda]\,\mathrm{d}x$$
$$+\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(h)}|Dv|^2\,\mathrm{d}x-\int_{\partial\mathcal{B}(h)}\sigma(v,p)n\cdot[\dot{g}+\dot{\omega}\times(x-h)-\omega\times g]\,\mathrm{d}\Gamma$$
$$-\int_{\mathcal{F}(h)}2\nu Dv:[D\left((v\cdot\nabla)\Lambda\right)-(\nabla v)(\nabla\Lambda)]\,\mathrm{d}x$$
$$+\int_{\mathcal{F}(h(t))}\rho(v\cdot\nabla)v\cdot[\dot{v}+(\Lambda\cdot\nabla)v-(v\cdot\nabla)\Lambda]\,\mathrm{d}x\,=\,0.$$

Inserting (1.6), (1.12) and (1.7) into the above equation, we obtain

$$\int_{\mathcal{F}(h(t))} \rho\dot{v}\cdot[\dot{v}+(\Lambda\cdot\nabla)v-(v\cdot\nabla)\Lambda]\,\mathrm{d}x+\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(h)}|Dv|^2\,\mathrm{d}x$$
$$+[m\dot{g}+k_d g+k_p(h-h_1)]\cdot(\dot{g}-\omega\times g)+J|\dot{\omega}|^2$$
$$-\int_{\mathcal{F}(h)}2\nu Dv:[D\left((v\cdot\nabla)\Lambda\right)-(\nabla v)(\nabla\Lambda)]\,\mathrm{d}x$$
$$+\int_{\mathcal{F}(h(t))}\rho(v\cdot\nabla)v\cdot[\dot{v}+(\Lambda\cdot\nabla)v-(v\cdot\nabla)\Lambda]\,\mathrm{d}x\,=\,0.$$

We can write the above equation as

$$\int_{\mathcal{F}(h(t))}\rho|\dot{v}|^2\,\mathrm{d}x+m|\dot{g}|^2+J|\dot{\omega}|^2+\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(h)}|Dv|^2\,\mathrm{d}x+\frac{1}{2}k_d\frac{\mathrm{d}}{\mathrm{d}t}|g|^2$$
$$=\int_{\mathcal{F}(h)}2\nu Dv:[D\left((v\cdot\nabla)\Lambda\right)-(\nabla v)(\nabla\Lambda)]\,\mathrm{d}x$$
$$-\int_{\mathcal{F}(h(t))}\rho(v\cdot\nabla)v\cdot[\dot{v}+(\Lambda\cdot\nabla)v-(v\cdot\nabla)\Lambda]\,\mathrm{d}x-\int_{\mathcal{F}(h(t))}\rho\dot{v}\cdot[(\Lambda\cdot\nabla)v-(v\cdot\nabla)\Lambda]\,\mathrm{d}x$$
$$-\left(k_p(h-h_1)\right)\cdot(\dot{g}-\omega\times g)+m\dot{g}\cdot(\omega\times g). \quad (5.5)$$

Applying Lemma 5.2 and performing a standard computation, we deduce from equation (5.5) and from the definition of Λ in (2.2) that

$$\int_{\mathcal{F}(h(t))}\rho|\dot{v}|^2\,\mathrm{d}x+m|\dot{g}|^2+J|\dot{\omega}|^2+\nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(h)}|Dv|^2\,\mathrm{d}x+\frac{1}{2}k_d\frac{\mathrm{d}}{\mathrm{d}t}|g|^2$$
$$\leqslant C(|g|+|g|^2)\int_{\mathcal{F}(h)}|Dv|^2\,\mathrm{d}x+C\int_{\mathcal{F}(h(t))}\rho|(v\cdot\nabla)v|^2\,\mathrm{d}x. \quad (5.6)$$

Using the change of variables introduced in Section 2, we have

$$[(v\cdot\nabla)v]\circ X\,=\,(\nabla X)(NV),$$

where V is defined by (2.8) and NV is defined by (2.11). Recall the notation $\mathcal{F}_0 = \mathcal{F}(h_0)$. We can check that

$$\int_{\mathcal{F}(h)} \rho |(v \cdot \nabla) v|^2 \,\mathrm{d}x \leqslant C \|\nabla X\|^2_{L^\infty(\Omega)} \int_{\mathcal{F}_0} |(V \cdot \nabla) V|^2 \,\mathrm{d}x \\ + C \|\nabla X\|^2_{L^\infty(\Omega)} \|\Gamma\|^2_{L^\infty(\Omega)} \int_{\mathcal{F}_0} |V|^4 \,\mathrm{d}x. \quad (5.7)$$

Next, by applying the Hölder inequality combined with the continuous embeddings $\mathcal{H}^{\frac{1}{2}}(\mathcal{F}_0)$ in $L^3(\mathcal{F}_0)$ and $\mathcal{H}^{\frac{3}{4}}(\mathcal{F}_0)$ in $L^4(\mathcal{F}_0)$ and with an interpolation inequality, we obtain that there exists a positive constant C depending on Ω and on h such that

$$\begin{aligned} \int_{\mathcal{F}(h)} \rho |(v \cdot \nabla) v|^2 \,\mathrm{d}x &\leqslant C \|\nabla X\|^2_{L^\infty(\Omega)} \|V\|^2_{L^6(\mathcal{F}_0)} \|\nabla V\|^2_{L^3(\mathcal{F}_0)} \\ &\quad + C \|\nabla X\|^2_{L^\infty(\Omega)} \|\Gamma\|^2_{L^\infty(\Omega)} \|V\|^4_{L^4(\mathcal{F}_0)} \\ &\leqslant C \|\nabla X\|^2_{L^\infty(\Omega)} \|\nabla V\|^3_{L^2(\mathcal{F}_0)} \|\nabla V\|_{H^1(\mathcal{F}_0)} \\ &\quad + C \|\nabla X\|^2_{L^\infty(\Omega)} \|\Gamma\|^2_{L^\infty(\Omega)} \|V\|_{L^2(\mathcal{F}_0)} \|V\|^3_{\mathcal{H}^1(\mathcal{F}_0)}. \end{aligned} \quad (5.8)$$

Applying Lemma 5.2 and Lemma 2.2, we deduce from the above inequality that

$$\int_{\mathcal{F}(h)} \rho |(v \cdot \nabla) v|^2 \,\mathrm{d}x \leqslant C \|V\|_{L^2(\mathcal{F}_0)} \|DV\|^3_{L^2(\mathcal{F}_0)} + C \|DV\|^3_{L^2(\mathcal{F}_0)} \|\nabla^2 V\|_{L^2(\mathcal{F}_0)}. \quad (5.9)$$

To estimate $\|\nabla^2 V\|_{L^2(\mathcal{F}_0)}$, we recall that

$$-\nu \Delta v + \nabla p = -\rho \dot{v} - \rho (v \cdot \nabla) v, \qquad x \in \mathcal{F}(h(t)), \quad (5.10)$$

$$\operatorname{div} v = 0, \qquad x \in \mathcal{F}(h(t)), \quad (5.11)$$

$$v = 0, \qquad x \in \partial\Omega, \quad (5.12)$$

$$v = g(t) - \omega(t) \times n, \qquad x \in \partial\mathcal{B}(h(t)). \quad (5.13)$$

Thus as long as $\operatorname{dist}(h, \partial\Omega) > 1 + \varepsilon$, there exists a positive constant such that

$$\|\nabla^2 v\|_{L^2(\mathcal{F}(h))} \leqslant C \left(\|\dot{v}\|_{L^2(\mathcal{F}(h))} + \|(v \cdot \nabla) v\|_{L^2(\mathcal{F}(h))} + |g| + |\omega| \right). \quad (5.14)$$

On the other hand, the change of variables of Section 2 and some tedious calculation yield

$$\|\nabla^2 V\|_{L^2(\mathcal{F}_0)} \leqslant C \left(\|\nabla^2 v\|_{L^2(\mathcal{F}(h))} + \|Dv\|_{L^2(\mathcal{F}(h))} \right). \quad (5.15)$$

Combining (5.14) and (5.15), we obtain

$$\|\nabla^2 V\|_{L^2(\mathcal{F}_0)} \leqslant C \left(\|\dot{v}\|_{L^2(\mathcal{F}(h))} + \|(v \cdot \nabla) v\|_{L^2(\mathcal{F}(h))} + \|Dv\|_{L^2(\mathcal{F}(h))} \right). \quad (5.16)$$

Inserting the above inequality in (5.9), we deduce

$$\int_{\mathcal{F}(h)} \rho |(v \cdot \nabla) v|^2 \,\mathrm{d}x \leqslant C \|v\|_{L^2(\mathcal{F}(h))} \|Dv\|^3_{L^2(\mathcal{F}(h))} \\ + C \|Dv\|^6_{L^2(\mathcal{F}(h))} + \eta \|\dot{v}\|^2_{L^2(\mathcal{F}(h))}, \quad (5.17)$$

where η is a positive constant that can be chosen arbitrarily small. More precisely, taking η small enough, and gathering (5.6) and (5.17), we obtain

$$\frac{1}{2}\int_{\mathcal{F}(h)} \rho|\dot{v}|^2\,\mathrm{d}x + m|\dot{g}|^2 + J|\dot{\omega}|^2 + \nu\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}(h)} |Dv|^2\,\mathrm{d}x + \frac{1}{2}k_d\frac{\mathrm{d}}{\mathrm{d}t}|g|^2$$
$$\leqslant C(|g|+|g|^2)\int_{\mathcal{F}(h)} |Dv|^2\,\mathrm{d}x + C\|v\|_{L^2(\mathcal{F}(h))}\|Dv\|^3_{L^2(\mathcal{F}(h))}$$
$$+ C\|Dv\|^6_{L^2(\mathcal{F}(h))}. \quad (5.18)$$

We deduce from (5.18) and from Lemma 5.2 the existence of a positive constant C_1 such that

$$\nu\|Dv(t)\|^2_{L^2(\mathcal{F}(h(t)))} + \frac{1}{2}k_d|g(t)|^2 \leqslant \nu\|Dv_0\|^2_{L^2(\mathcal{F}_0)} + \frac{1}{2}k_d|g_0|^2$$
$$+ C_1\nu\int_0^t \left(\|Dv(s)\|^3_{L^2(\mathcal{F}(h(s)))} + \|Dv(s)\|^6_{L^2(\mathcal{F}(h(s)))}\right)\mathrm{d}s. \quad (5.19)$$

Assume that the constant δ in the last estimate of (1.15) is chosen such that

$$\nu\|Dv_0\|^2_{L^2(\mathcal{F}_0)} + \frac{1}{2}k_d|g_0|^2 + \frac{C_1}{2}\left[\int_{\mathcal{F}_0} \rho|v_0|^2\,\mathrm{d}x + m|g_0|^2 + J|\omega_0|^2\right] < \nu. \quad (5.20)$$

Then combining (5.19) and (4.7), we deduce that for every $t \in [0, T_*)$,

$$\|Dv(t)\|_{L^2(\mathcal{F}(h(t)))} < 1. \quad (5.21)$$

This contradicts (5.4) so that indeed $T_* = +\infty$. Moreover, it is now clear that (5.21) holds for every $t \in [0,\infty)$, hence we have (1.18). $\square$

6. Proof of Theorem 1.2

To prove Theorem 1.2, we need to show (1.13) and (1.14). We work with the notation and assumptions of Section 1. Let $\mathbb{X} = L^2(\Omega) \times \Omega^\circ \times \mathbb{R}^3 \times \mathbb{R}^3$ and define $W_1, W_2 : \mathbb{X} \to \mathbb{R}$ by

$$W_1\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} = \frac{1}{2}\left(\rho\int_{\mathcal{F}(h)} |v_0(x)|^2\,\mathrm{d}x + m|g_0|^2 + J|\omega_0|^2\right) \qquad \left(\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \in \mathbb{X}\right), \quad (6.1)$$

$$W_2\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} = \frac{k_p}{2}|h_0 - h_1|^2 \qquad \left(\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \in \mathbb{X}\right). \quad (6.2)$$

Let $\mathbb{X}_1 = \mathcal{H}_0^1(\Omega) \times \Omega^\circ \times \mathbb{R}^3 \times \mathbb{R}^3$ and define $W_3 : \mathbb{X}_1 \to \mathbb{R}$ by

$$W_3 \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} = 2\nu \int_{\mathcal{F}(h)} |D(v_0)(x)|^2 \,\mathrm{d}x + k_d |g_0|^2 \qquad \left(\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \in \mathbb{X}_1 \right). \tag{6.3}$$

Let $\delta > 0$ be the constant in Theorem 1.1 and let

$$\mathbb{V} = \left\{ \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \in \mathbb{X}_1 \;\middle|\; \begin{array}{c} v_0(x) = g_0 + \omega_0 \times (x - h_0) \quad x \in \mathcal{B}(h_0) \\ \operatorname{div} v_0 = 0 \quad x \in \Omega \\ \|v_0\|_{\mathcal{H}^1(\mathcal{F}(h_0))} + |g_0| + |\omega_0| + |h_1 - h_0| \leqslant \delta \end{array} \right\}.$$

For $t \geqslant 0$ we set

$$S(t) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} = \begin{bmatrix} v(t,\cdot) \\ h(t) \\ g(t) \\ \omega(t) \end{bmatrix} \qquad \left(\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \in \mathbb{V} \right),$$

where $\begin{bmatrix} v(t,\cdot) \\ h(t) \\ g(t) \\ \omega(t) \end{bmatrix}$ is the corresponding solution of (1.1)–(1.9) constructed in Theorem 1.1. With this notation, estimate (4.7) writes, for every $t_1,\ t_2 \geqslant 0$,

$$W_1 \left(S(t_1) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \right) + W_2 \left(S(t_1) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \right) - W_1 \left(S(t_2) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \right) - W_2 \left(S(t_2) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \right)$$
$$= \int_{t_1}^{t_2} W_3 \left(S(\sigma) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \right) \mathrm{d}\sigma \qquad \left(\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \in \mathbb{V} \right). \tag{6.4}$$

We first prove that the kinetic energy W_1 of the fluid-rigid system tends to zero when $t \to \infty$.

Proposition 6.1. *Under the assumptions of Theorem 1.1 we have*

$$\lim_{t\to\infty} W_1 \left(S(t) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \right) = 0. \tag{6.5}$$

Proof. Within this proof we denote, for the sake of simplicity

$$W_k \left(S(t) \begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix} \right) = W_k(t) \qquad (k \in \{1,2,3\}, \quad t \geqslant 0).$$

From (6.4), we deduce that for every $t_2 > t_1$,

$$W_1(t_2) - W_1(t_1) \leqslant |W_2(t_1) - W_2(t_2)| + \int_{t_1}^{t_2} W_3(s)\,\mathrm{d}s.$$

From (4.7) and (5.21), we deduce that there exists $M > 0$ such that

$$W_1(t_2) - W_1(t_1) \leqslant M(t_2 - t_1) \qquad (t_2 > t_1 \geqslant 0). \tag{6.6}$$

On the other hand, (4.7) and Lemma 5.2 imply that $W_1 \in L^1[0,\infty)$. Taking $f = W_1$, all the assumptions in Lemma 10.1 are satisfied, so that we obtain (6.5). $\square$

The next proposition complements the above proposition and states that

$$W_2\left(S(t)\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix}\right) \to 0.$$

For this, we have to define weak solutions for the problem (1.1)–(1.9) with the feedback (1.12). First we define, for every $h \in \Omega^\circ$, the space

$$\mathcal{H}^1_R(h) := \left\{\varphi \in \mathcal{H}^1_0(\Omega) \mid \operatorname{div}\varphi = 0 \text{ in } \Omega, \quad D(\varphi) = 0 \text{ in } \mathcal{B}(h)\right\}. \tag{6.7}$$

We notice that if the fluid velocity in (1.1)–(1.9) satisfies $v(t,\cdot) \in \mathcal{H}^1_0(\mathcal{F}(h(t)))$, then, by extending it by

$$v(t,x) = g(t) + \omega(t) \times (x - h(t)) \qquad (t \geqslant 0,\ x \in \mathcal{B}(h(t)), \tag{6.8}$$

we obtain $v(t,\cdot) \in \mathcal{H}^1_R(h(t))$ for every $t \geqslant 0$. More precisely, according to Lemma 1.1 in [19, p. 23], for any $\varphi \in \mathcal{H}^1_R(h)$, there exists g_φ and ω_φ such that

$$\varphi(x) = g_\varphi + \omega_\varphi \times (x - h) \qquad (x \in \mathcal{B}(h)). \tag{6.9}$$

We can also extend the density ρ of the fluid by setting

$$\rho(t,x) = \frac{3m}{4\pi} \qquad (t \geqslant 0,\ x \in \mathcal{B}(h(t)).$$

Definition 6.2. *Assume $T > 0$. A quadruple* $\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix}$ *is a* weak solution *of* (1.1)–(1.9) *with the feedback* (1.12) *on* $(0,T)$ *if*

$$v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathcal{H}^1_0(\Omega)), \qquad h \in W^{1,\infty}(0,T), \qquad g,\omega \in L^\infty(0,T),$$

$$h(t) \in \Omega^\circ, \qquad \dot{h} = g,$$

$$v(t,\cdot) \in \mathcal{H}^1_R(h(t)) \quad \textit{in } (0,T), \quad \textit{with } (6.8)$$

and (using again the symbol : for the Hilbert-Schmidt inner product)

$$\int_{(0,T)\times\Omega} (-\rho v \cdot \varphi + 2\nu D(v) : D(\varphi))\, \mathrm{d}x\mathrm{d}t = \int_{\Omega} \rho(0,x)v(0,x)\cdot\varphi(0,x)\,\mathrm{d}x + \int_0^T (k_p[h-h_1] + k_d g)\cdot g_\varphi\,\mathrm{d}t, \quad (6.10)$$

for all $\varphi \in C^1([0,T];\mathcal{H}_0^1(\Omega))$ *with* $\varphi(t,\cdot) \in \mathcal{H}_R^1(h(t))$ *for every* $t \in [0,T]$ *and satisfying* (6.9)*, together with* $\varphi(T,\cdot) = 0$.

The above set of weak solutions is closed, with respect to the appropriate weak topology, as stated below (see, for instance, [20] or [16] for the proof).

Proposition 6.3. *Let* $h_0 \in \Omega_{1+\varepsilon}$ *and* $v_0 \in \mathcal{H}_R^1(h_0)$*. Let* $(h_{0n})_{n\geqslant 1}$*,* $(v_{0n})_{n\geqslant 1}$ *be two sequences such that* $h_{0n} \in \Omega_{1+\varepsilon}$ *and* $v_{0n} \in \mathcal{H}_R^1(h_n)$ *for every* $n \geqslant 1$*. Let* $\left(\begin{bmatrix} v_n \\ h_n \\ g_n \\ \omega_n \end{bmatrix}\right)_{n\geqslant 1}$ *be a sequence of weak solutions of* (1.1)*–*(1.9) *on* $(0,T)$*, with the feedback* (1.12) *and, for each* $n \geqslant 1$*, with the initial conditions* $\left(\begin{bmatrix} v_{0n} \\ h_{0n} \\ g_{0n} \\ \omega_{0n} \end{bmatrix}\right)$*. Assume that*

$$h_{0n} \to h_0, \qquad v_{0n} \to v_0 \quad \text{in } L^2(\Omega),$$

and that there exist v*,* h*,* g*,* ω *such that*

$$h_n \rightharpoonup h,\ g_n \rightharpoonup g,\ \omega_n \rightharpoonup \omega \qquad \text{in} \quad L^\infty(0,T) \quad \text{weak star},$$

$$v_n \rightharpoonup v \quad \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathcal{H}_0^1(\Omega)) \quad \text{weak star.}$$

Then $\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix}$ *is a weak solution of* (1.1)*–*(1.9) *on* $(0,T)$*, with the feedback* (1.12) *and with the initial conditions* $\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix}$*, where* g_0, ω_0 *are determined by* v_0 *via* (6.8).

Proposition 6.4. *Under the assumptions of Theorem* 1.1 *we have*

$$\lim_{t\to\infty} (W_1 + W_2)\left(S(t)\begin{bmatrix} v_0 \\ h_0 \\ g_0 \\ \omega_0 \end{bmatrix}\right) = 0. \qquad (6.11)$$

Proof. We know from Proposition 6.1 that

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{L^2(\mathcal{F}(h(t)))} = 0, \qquad \lim_{t\to\infty} \dot{h}(t) = 0. \tag{6.12}$$

Moreover, since $h(t) \in \Omega_{1+\varepsilon}$ for every $t \geqslant 0$ we have that the set $(h(t))_{t\geqslant 0}$ is relatively compact in $\mathbb{R}^3$ and all its limit points lie in Ω°. Let $(t_n)_{n\geqslant 0}$ be a sequence of positive numbers such that

$$t_n \to \infty, \qquad \lim_{n\to\infty} h(t_n) = h^* \in \Omega^\circ. \tag{6.13}$$

We also know from (6.4) that the map $t \mapsto W_3\left(S(t)[v_0\ h_0\ g_0\ \omega_0]^\top\right)$ is in $L^1[0,\infty)$ so that, given $T > 0$, we have

$$\lim_{n\to\infty} \int_{t_n}^{T+t_n} W_3\left(S(t)[v_0\ h_0\ g_0\ \omega_0]^\top\right) \mathrm{d}t = 0.$$

A change of variables and the semigroup property of the family $(S(t))_{t\geqslant 0}$ imply that

$$\lim_{n\to\infty} \int_0^T W_3\left(S(\tau)S(t_n)[v_0\ h_0\ g_0\ \omega_0]^\top\right) \mathrm{d}\tau = 0. \tag{6.14}$$

On the other hand, we know from (6.12) and (6.13) that

$$\lim_{n\to\infty} S(t_n)[v_0\ h_0\ g_0\ \omega_0]^\top = [0\ h^*\ 0\ 0]^\top \qquad \text{in } \mathbb{X}. \tag{6.15}$$

Denote

$$[v_{0n}\ h_{0n}\ g_{0n}\ \omega_{0n}]^\top = S(t_n)[v_0\ h_0\ g_0\ \omega_0]^\top$$

and

$$[v_n(\tau,\cdot)\ h_n(\tau)\ g_n(\tau)\ \omega_n(\tau)]^\top = S(\tau)[v_{0n}(\cdot)\ h_{0n}\ g_{0n}\ \omega_{0n}]^\top \qquad (\tau \in [0,T]).$$

Then $[v_n\ h_n\ g_n\ \omega_n]^\top$ is the strong solution of (1.1)–(1.9) with the feedback (1.12) in $(0,T)$ associated to the initial condition $[v_{0n}\ h_{0n}\ g_{0n}\ \omega_{0n}]^\top$ and thus it is a weak solution for the same problem.

Using the energy estimate (6.4) it follows that there exists (h,v) such that, up to the extraction of a subsequence, the sequences (h_n) and (v_n) satisfy the assumptions in Proposition 6.3. Consequently, $[v\ h\ g\ \omega]^\top$ is a weak solution of (1.1)–(1.9) on $(0,T)$, with initial conditions $[0\ h^*\ 0\ 0]^\top$.

By combining (6.15) and Proposition 6.3, it follows that there exists a weak solution $[v\ h\ g\ \omega]^\top$ associated with the initial conditions $[0\ h^*\ 0\ 0]^\top$ such that

$$D(v_n) \rightharpoonup D(v) \quad \text{in } L^2(0,T;L^2(\Omega)).$$

In particular, the above convergence and (6.14) yield

$$\|D(v)\|_{L^2((0,T)\times\Omega)} \leqslant \liminf_n \|D(v_n)\|_{L^2((0,T)\times\Omega)} = 0.$$

We deduce that $v = 0$ in $(0,T)\times\Omega$ and in particular $g = \omega = 0$. Writing that $[0\ h^*\ 0\ 0]^\top$ satisfies (6.10) for all $\varphi \in C^1([0,T];\mathcal{H}_0^1(\Omega))$, $\varphi(t,\cdot) \in \mathcal{H}_R^1(h(t))$ with (6.9), and $\varphi(T,\cdot) = 0$, we obtain that $h^* = h_1$, which ends the proof. □

We can now end the proof of Theorem 1.2.

Proof of Theorem 1.2. It remains to prove (1.14). We already know that

$$\int_{\mathcal{F}(h(t))} |v(t,x)|^2 \,\mathrm{d}x + m|g(t)|^2 + J|\omega(t)|^2 + \frac{k_p}{2}|h(t) - h_1|^2 \to 0 \quad \text{if} \quad t \to \infty \tag{6.16}$$

and (5.21) holds. Thus, from (1.18) and (5.18) we deduce that there exists $\eta > 0$ such that

$$\nu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{F}(h)} |Dv|^2 \,\mathrm{d}x + \frac{1}{2} k_d \frac{\mathrm{d}}{\mathrm{d}t} |g|^2 \leqslant \eta \qquad \text{(for a.e. } t \geqslant 0\text{)}. \tag{6.17}$$

We set

$$W_4(t) := \int_{\mathcal{F}(h(t))} |Dv(t)|^2 \,\mathrm{d}x .$$

From (6.17) it follows that

$$W_4(t) - W_4(s) \leqslant \eta(t-s) \qquad (t > s \geqslant 0).$$

Moreover, we know from (4.7) that $W_4 \in L^1[0, \infty)$. By applying Lemma 10.1 in the appendix the conclusion follows. □

7. Proof of Theorem 1.3

Proof of Theorem 1.3. Assume $h_0, h_1 \in \Omega^\circ$. Since Ω° is open and connected, it is path-connected and thus there exists a continuous curve $\gamma : [0, L] \to \Omega^\circ$ such that $\gamma(0) = h_0$ and $\gamma(L) = h_1$. Since $\mathrm{Im}\,(\gamma)$ is compact, there exists $\varepsilon > 0$ such that

$$\mathrm{dist}(\mathrm{Im}\,(\gamma), \partial\Omega) \geqslant 1 + 2\varepsilon.$$

From Theorem 1.1 and Theorem 1.2, there exists $\delta > 0$, depending only on Ω, k_p, k_d and ε, such that for evry $h^* \in \mathrm{Im}\,(\gamma)$ satisfying

$$\|v_0\|_{\mathcal{H}^1(\mathcal{F}(h_0))} + |g_0| + |\omega_0| + |h^* - h_0| \leqslant 2\delta ,$$

the solution of (1.1)–(1.9), with

$$u(t) = k_p[h^* - h(t)] - k_d \dot{h}(t) ,$$

is global in time and, as $t \to \infty$, we have

$$\|v(\cdot, t)\|_{\mathcal{H}^1(\mathcal{F}(h(t)))} + |g(t)| + |\omega(t)| + |h^* - h(t)| \to 0 .$$

The construction of the control function u is based on the above facts and on the choice of a family $(\widetilde{h}_k)$ of points of $\mathrm{Im}\,(\gamma)$, such that for every $k \in \{0 \ldots, K\}$ we have

$$\widetilde{h}_0 = h_0, \quad \widetilde{h}_K = h_1, \quad |\widetilde{h}_k - \widetilde{h}_{k+1}| < \delta.$$

We next construct a corresponding partition $0 = t_0^* < t_1^* < \cdots < t_K^* = +\infty$ of $[0, \infty)$ as follows: assuming that $k \in \{0 \ldots, K\}$ and $t_k^* \geqslant 0$ is such that

$$\|v(\cdot, t_k^*)\|_{\mathcal{H}^1(\mathcal{F}(h(t_k^*)))} + |g(t_k^*)| + |\omega(t_k^*)| + |\widetilde{h}_k - h(t_k^*)| \leqslant 2\delta,$$

we set

$$u_k(t) = k_p(\widetilde{h}_{k+1} - h(t)) - k_d \dot{h}(t)) \qquad (t \geqslant t_k^*).$$

According to Theorem 1.1 and Theorem 1.2, there exists $t_{k+1}^* > t_k^*$ such that

$$\|v(\cdot, t_{k+1}^*)\|_{\mathcal{H}^1(\mathcal{F}(h(t_{k+1}^*)))} + |g(t_{k+1}^*)| + |\omega(t_{k+1}^*)| + |\widetilde{h}_{k+1} - h(t_{k+1}^*)| \leqslant 2\delta.$$

Using iteratively the above argument it follows that we have indeed (1.13), (1.14) with

$$s(t) = \widetilde{h}_j \qquad (j \in \{1 \ldots, K\},\ t \in [t_{j-1}, t_j)).$$

□

8. The bidimensional case

In this section we consider the two-dimensional case, i.e., we assume that Ω is a bounded open set in $\mathbb{R}^2$ and the rigid ball is replaced by a disk of radius 1 in $\mathbb{R}^2$. More precisely, we consider the system described (for every $t \geqslant 0$) by

$$\rho\dot{v} - \nu\Delta v + \rho(v \cdot \nabla)v + \nabla p = 0, \qquad x \in \mathcal{F}(h(t)), \tag{8.1}$$

$$\mathrm{div}\ v = 0, \qquad x \in \mathcal{F}(h(t)), \tag{8.2}$$

$$v = 0, \qquad x \in \partial\Omega, \tag{8.3}$$

$$\dot{h} = g, \tag{8.4}$$

$$v = g(t) + \omega(t)(x - h)^\perp, \qquad x \in \partial\mathcal{B}(h(t)), \tag{8.5}$$

$$m\dot{g} = -\int_{\partial\mathcal{B}(h)} \sigma(v, p)n \,\mathrm{d}\Gamma + u, \tag{8.6}$$

$$J\dot{\omega} = -\int_{\partial\mathcal{B}(h)} (x - h)^\perp \cdot \sigma(v, p)n \,\mathrm{d}\Gamma, \tag{8.7}$$

$$h(0) = h_0, \quad \dot{h}(0) = g_0, \quad \omega(0) = \omega_0, \tag{8.8}$$

$$v(x, 0) = v_0(x), \qquad x \in \mathcal{F}(h_0). \tag{8.9}$$

In the above system, we have used the notation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^\perp = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

We note that $\omega(t) \in \mathbb{R}$. We use again a feedback law of the form

$$u(t) = k_p[h_1 - h(t)] - k_d\dot{h}(t),$$

with a given $h_1 \in \Omega^\circ$ and $k_p > 0$, $k_d \geqslant 0$.

Similarly as for the usual Navier-Stokes system, we can derive stronger versions of the results obtained in three dimensions. More precisely, the two dimensional versions of Theorems 1.1, 1.2 and 1.3 are:

Theorem 8.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, connected and bounded set with $\partial\Omega$ of class C^2 and let $h_1 \in \Omega^\circ$.*

Then for every $h_0 \in \Omega^\circ$, $v_0 \in \mathcal{H}^1(\mathcal{F}(h_0); \mathbb{R}^2)$ and every $(g_0, \omega_0) \in \mathbb{R}^3$ satisfying

$$\begin{cases} \operatorname{div} v_0 = 0, & \text{in } \mathcal{F}(h_0), \\ v_0(x) = 0, & \text{for } x \in \partial\Omega, \\ v_0(x) = g_0 + \omega_0(x - h_0)^\perp, & \text{for } x \in \partial\mathcal{B}(h_0), \end{cases}$$

there exists $k_p > 0$, depending only on Ω, $\|v\|_{\mathcal{H}^1(\mathcal{F}(h_0))}$, g_0, ω_0, h_0 and h_1, such that for all $k_d \geqslant 0$, there exists a strong solution of (8.1)–(8.9) *with the feedback* (1.12), *on the time interval* $[0, \infty)$. *This solution satisfies*

$$\begin{bmatrix} v \\ h \\ g \\ \omega \end{bmatrix} \in \mathcal{T} L^2_{\mathrm{loc}}([0, \infty); \mathcal{H}^2(\mathcal{F}(h))) \cap \mathcal{T}\mathcal{H}^1_{\mathrm{loc}}(0, \infty; L^2(\Omega)), \tag{8.10}$$

$$p \in L^2_{\mathrm{loc}}([0, \infty); \mathcal{H}^1(\mathcal{F}(h))). \tag{8.11}$$

The above solution is unique up to an additive perturbation of p that depends only on time.

Theorem 8.2. *With the notation and assumptions in Theorem 8.1, the constructed solution (v, p, h, g, ω) of* (8.1)–(8.9) *satisfies* (6.11).

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^2$ be an open, connected and bounded set with $\partial\Omega$ of class C^2 and let $h_0,\ h_1 \in \Omega^\circ$. Assume $v_0 \in \mathcal{H}^1(\mathcal{F}(h_0); \mathbb{R}^2)$ and $g_0 \in \mathbb{R}^2, \omega_0 \in \mathbb{R}$ satisfy*

$$\begin{cases} \operatorname{div} v_0 = 0, & \text{in } \mathcal{F}(h_0), \\ v_0(x) = 0, & \text{for } x \in \partial\Omega, \\ v_0(x) = g_0 + \omega_0(x - h_0)^\perp, & \text{for } x \in \partial\mathcal{B}_0. \end{cases}$$

Then there exists $k_p > 0$ such that for all $k_d \geqslant 0$ there exists a piecewise constant function $s : [0, \infty) \to \Omega^\circ$ such that the system (8.1)–(8.9), *with*

$$u(t) = k_p[s(t) - h(t)] - k_d\dot{h}(t). \tag{8.12}$$

admits a strong solution, i.e., this solution has the regularity properties (8.10)–(8.11). *Moreover, this solution is unique up to an additive perturbation of p that depends only on time and the stability property* (6.11) *holds.*

We do not provide the proofs of the above three theorems since they are completely similar to the proofs of Theorems 1.1, 1.2 and 1.3. The only difference is that in the proof of Theorem 8.1 it is not difficult to show, by estimates similar to those in the proof of Theorem 1.1, that the $\mathcal{H}^1$ norm of the velocity field does not blow up in finite time (without any smallness assumption).

9. Concluding remarks

We have studied an infinite-dimensional nonlinear dynamical system coupling the Navier-Stokes equations with the rigid body dynamics, in the presence of a free boundary. We have proposed a PD-type controller which asymptotically steers the rigid body to a prescribed final position, while the velocities of the fluid and of the rigid body tend to zero. The stabilizing mechanism is the viscosity of the fluid which, due to the assumptions on the initial data and to the coupling at the interface, suffices to stabilize both the fluid and the solid. The coupling of Navier-Stokes equations with rigid body dynamics has been investigated in an "inverted" context in Mazzone *et al.* [14, 6], where the rigid body has a cavity filled with a Navier-Stokes fluid. In the situation studied in [14, 6] there is no free boundary but, as in our case, it is shown that the viscosity of a fluid stabilizes both the fluid and the solid to one of a finite number of equilibrium states.

An interesting open question is the extension of our stability analysis (in the three dimensional case) to weak solutions which satisfy a strong energy inequality. In order to perform this extension, a first step should be proving the existence of weak solutions for fluid-rigid problems, which is an open question. Another interesting open question is the large-time behavior of our system when the initial data are small and the control u vanishes. Our methods can be easily adapted to show that the velocities of the fluid and of the rigid body tend to zero when $t \to \infty$, but it seems difficult to say something precise about the asymptotic behavior of the position $h(t)$ when $t \to \infty$. This question seems difficult, even in the simplified model, in which the fluid-rigid system fills the whole space, as considered in Vázquez and Zuazua [22, 23] or in Munnier and Zuazua [15].

10. Appendix: A Barbălat type lemma

A tool which is frequently used to deduce asymptotic stability of nonlinear systems using Lyapunov-like approaches is Barbălat's lemma. One of the version of this result says that if $f \in L^1[0,\infty)$ is uniformly continuous on $[0,\infty)$ then $\lim_{t\to\infty} f(t) = 0$, see, for instance, Logemann and Ryan [12, p. 177]. Our version below shows that for positive functions f the uniform continuity property can be slightly relaxed.

Lemma 10.1. *Assume that $f \in L^1[0,\infty)$ is a non negative continuous function such that, the following "right uniform continuity" property holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$f(t) - f(s) \leqslant \varepsilon \qquad\qquad (t,\ s \in [0,\infty),\ s \leqslant t \leqslant s + \delta).$$

Then $\lim_{t\to\infty} f(t) = 0$.

Proof. Let us assume, by contradiction, that f does not converge to zero for $t \to \infty$. Since f is non negative, this means that there exists $\varepsilon > 0$ and a sequence (t_n) of positive numbers such that $t_n \to \infty$ and

$$f(t_n) \geqslant \varepsilon \qquad\qquad (n \in \mathbb{N}).$$

Denote

$$\delta_n = \max\left\{\delta > 0 \ \mid \ f(t_n - \delta) \leqslant \frac{\varepsilon}{2}\right\} \qquad (n \in \mathbb{N}),$$

so that

$$f(t_n) - f(t_n - \delta_n) \geqslant \frac{\varepsilon}{2} \qquad\qquad (n \in \mathbb{N}). \tag{10.1}$$

Since $f(t) \geqslant \frac{\varepsilon}{2}$ for $t \in [t_n - \delta_n, t_n]$ and $f \in L^1[0, \infty)$, it follows that

$$\sum_{n \in \mathbb{N}} \delta_n < \infty, \quad \text{hence} \quad \lim_{n \to \infty} \delta_n = 0.$$

Using the above convergence and the uniform right-continuity of f, it follows that

$$\lim_{n \to \infty} [f(t_n) - f(t_n - \delta_n)] = 0,$$

which contradicts (10.1). $\square$

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