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To cite this version:

Sebastian Donoso, Fabien Durand, Alejandro Maass, Samuel Petite. On automorphism groups of low complexity minimal subshifts. Ergodic Theory and Dynamical Systems, 2016. hal-01076427

HAL Id: hal-01076427

https://hal.science/hal-01076427v1

Submitted on 22 Oct 2014

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ON AUTOMORPHISM GROUPS OF LOW COMPLEXITY MINIMAL SUBSHIFTS

SEBASTIÁN DONOSO, FABIEN DURAND, ALEJANDRO MAASS, AND SAMUEL PETITE

Abstract. In this article we study the automorphism group $\mathrm{Aut}(X,\sigma)$ of a minimal subshift (X,σ) of low *word* complexity. In particular, we prove that $\mathrm{Aut}(X,\sigma)$ is virtually $\mathbb{Z}$ for aperiodic minimal subshifts with affine complexity on a subsequence, more precisely, the quotient of this group by the one generated by the shift map is a finite group. In addition, we provide examples to show that any finite group can be obtained in this way. The class considered includes minimal substitutions, linearly recurrent subshifts and even some minimal subshifts with polynomial complexity. In the case of polynomial complexity, first we prove that for minimal subshifts with polynomial recurrence any finitely generated subgroup of $\mathrm{Aut}(X,\sigma)$ is virtually nilpotent. Then, we describe several subshifts of polynomial complexity to illustrate that the group of automorphisms can still be virtually $\mathbb{Z}$. This behavior seems to be very frequent for low complexity minimal subshifts. The main technique in this article relies on the study of classical relations among points used in topological dynamics, in particular asymptotic pairs.

1. Introduction

An automorphism of a topological dynamical system (X,T), where $T\colon X\to X$ is a homeomorphism of the compact metric space X, is a homeomorphism from X to itself which commutes with T. We call $\mathrm{Aut}(X,T)$ the group of automorphisms of (X,T). There is a similar definition for measurable automorphisms when we consider an invariant measure μ for the system (X,T) or a general measure preserving system. The group of measurable automorphisms is historically denoted by $C(T)$ that stands for the *centralizer group* of (X,T,μ).

The study of automorphism groups is a classical and widely considered subject in ergodic theory. The group $C(T)$ has been intensively studied for mixing systems of finite rank. We refer to [17] for an interesting survey. Let us mention some key theorems. D. Ornstein proved in [28] that a mixing rank one dynamical system (X,T,μ) has a trivial (measurable) automorphism group: it consists in powers of T. Later, A. del Junco [11] showed that the famous weakly mixing (but not mixing) rank one Chacon subshift also shares this property. Finally, for mixing systems of finite rank J. King and J.-P. Thouvenot proved in [22] that $C(T)$ is virtually $\mathbb{Z}$. That is, its quotient by the subgroup $\langle T\rangle$ generated by T is a finite group.

Date: October 20, 2014.

2010 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37B10.

Key words and phrases. Minimal subshifts, automorphism group, complexity.

This research was partially supported by grants Basal-CMM & Fondap 15090007, CONICYT Doctoral fellowship 21110300, ANR SubTile and the cooperation project MathAmSud DYSTIL. The first and third authors thanks University of Picardie Jules Verne where this research was finished.

In the non weakly mixing case, B. Host and F. Parreau [21] proved, for a family of constant length substitution subshifts, that $C(\sigma)$ is also virtually $\mathbb{Z}$ and equals to $\mathrm{Aut}(X,\sigma)$, where σ is the shift map. Concomitantly, M. Lemańczyk and M. Mentzen [24] realized any finite group as the quotient of $C(\sigma)$ by $\langle\sigma\rangle$ with constant length substitution subshifts.

Priorly to these results, in the topological setting G. A. Hedlund in [18] described the automorphism groups for a family of binary substitutions including the Thue-Morse subshift. Precisely, he proved that $\mathrm{Aut}(X,\sigma)$ is generated by the shift and a flip map (a map which interchanges the letters). In the positive entropy situation, M. Boyle, D. Lind and D. Rudolph [5] obtained that the group of automorphisms of mixing subshifts of finite type contains various subgroups, so this group is large in relation to previous examples.

In this paper we focus on the group of automorphisms $\mathrm{Aut}(X,\sigma)$ of minimal subshifts of subaffine complexity and, more generally, on zero entropy subshifts without assuming any mixing condition. All evidence described before in the measurable and topological context shows that we must expect that *low complexity* systems have a simple automorphism group. This is one of the main questions we want to address in this paper. Here, by complexity we mean the increasing function $p_X\colon \mathbb{N}\to\mathbb{N}$ that for $n\in\mathbb{N}$ counts the number of *words* of length n appearing in points of the subshift.

Recently V. Salo and I. Törmä in [33] proved that for subshifts generated by constant length or primitive Pisot substitutions the group of automorphisms is virtually $\mathbb{Z}$. This generalizes a result of E. Coven for constant length substitutions on two letters [9]. In [33] is asked whether the same result holds for any primitive substitution or more generally for linearly recurrent subshifts. In this paper we answer positively to this question, proving the following more general theorem whose proof is given in Section 3.

Theorem 1.1. *Let (X,σ) be an aperiodic minimal subshift. If*

$$\liminf_{n\in\mathbb{N}} \frac{p_X(n)}{n} < \infty$$

then $\mathrm{Aut}(X,\sigma)$ is virtually $\mathbb{Z}$.

The class of systems satisfying the condition of Theorem 1.1 includes primitive substitutions, linearly recurrent subshifts [14] and more generally subaffine complexity subshifts or even some families with polynomial complexity (see Section 4). In addition, we illustrate this by realizing any finite group as the quotient group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$, where (X,σ) is a substitutive subshift. We observe that this result can be obtained combining the main results of [21] and [24] but we prefer to present here a different and straightforward proof.

Extending Theorem 1.1 for subshifts of polynomial complexity seems to be more intriguing. Nevertheless, several classes of examples still show that $\mathrm{Aut}(X,\sigma)$ has small growth rate. Indeed, in Sections 3 and 4 we give classes of minimal subshifts with polynomial complexity where $\mathrm{Aut}(X,\sigma)$ is virtually nilpotent (Theorem 3.8) and in most cases the finite group is Abelian. Also, very recently, V. Cyr and B. Kra [10] proved the fact that for transitive subshifts with subquadratic complexity $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is periodic, meaning that any element in this group has finite order. Their proof translates the question into a coloring problem of $\mathbb{Z}^2$ and uses a deep combinatorial result of A. Quas and L. Zamboni [31]. Our results arise from

obstructions related to some classical and some less classical equivalence relations associated to fibers of special topological factors. This idea was already used by J. Olli in [27] to prove that $\mathrm{Aut}(X,\sigma)$ of Sturmian subshifts consists only in powers of the shift by studying the irrational rotation defining the subshift. Here, we consider the maximal nilfactor ([20],[35]) of a minimal subshift to find a class of examples with arbitrarily big polynomial complexity whose group of automorphisms is virtually $\mathbb{Z}$.

2. Preliminaries, notation and background

2.1. **Topological dynamical systems.** A topological dynamical system or just a system is a homeomorphism $T\colon X \to X$ where X is a compact metric space. It is classically denoted by (X,T). Let dist denote a distance on X and for $x \in X$ $\mathrm{Orb}(x)$ denotes its orbit $\{T^n x; n \in \mathbb{Z}\}$. A topological dynamical system is minimal if the orbit of any point is dense in X and is transitive if at least one orbit is dense in X.

An *automorphism* of the topological dynamical system (X,T) is a homeomorphism ϕ of the space X such that $\phi \circ T = T \circ \phi$. We let $\mathrm{Aut}(X,T)$ denote the group of automorphisms of (X,T).

Let (X,T) be a topological dynamical system. We say that $x, y \in X$ are *proximal* if there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that

$$\lim_{i\to+\infty} \mathrm{dist}(T^{n_i}x, T^{n_i}y) = 0.$$

A stronger condition than proximality is asymptoticity. Two points $x, y \in X$ are said to be *asymptotic* if

$$\lim_{n\to+\infty} \mathrm{dist}(T^n x, T^n y) = 0.$$

Non trivial asymptotic pairs may not exist in an arbitrary topological dynamical system but it is well known that a non-empty aperiodic subshift always admits one [2].

A factor map between the topological dynamical systems (X,T) and (Y,S) is a continuous onto map $\pi\colon X \to Y$ such that $\pi \circ T = S \circ \pi$. We also say that (Y,S) is a factor of (X,T) or that (X,T) is an extension of (Y,S). We also use the notation $\pi\colon (X,T) \to (Y,S)$ to indicate a factor map. If $\pi\colon X \to Y$ is a factor map, we say that (X,T) is a *proximal extension* of (Y,S) if for $x, x' \in X$ the condition $\pi(x) = \pi(x')$ implies that x, x' are proximal. For minimal systems, the extension (X,T) is called an *almost one-to-one* extension if there exists a point $y \in Y$ with at most one pre-image for the map π.

The relation between these two notions is given by the following folklore lemma.

Lemma 2.1. *Let $\pi\colon (X,T) \to (Y,S)$ be an almost one-to-one extension of minimal systems. Then, (X,T) is a proximal extension of (Y,S)*

Proof. Let $y_0 \in Y$ be a point with one pre-image by π. For any points $x, x' \in X$ with the same image $\pi(x) = \pi(x')$, there is a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $T^{n_i}\pi(x)$ converges to y_0 as i goes to infinity. Since $\pi(x) = \pi(x')$, the sequence $(T^{n_i}\pi(x'))_{i\in\mathbb{N}}$ also converges to y_0. Taking a subsequence if needed, the sequences $(T^{n_i}x)_{i\in\mathbb{N}}$ and $(T^{n_i}x')_{i\in\mathbb{N}}$ converge to the same unique point in the pre-image of y_0 by π. This shows that points x and x' are proximal. $\square$

2.2. **Subshifts.** Throughout this paper the *shift map* is denoted by $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ where $\mathcal{A}$ is some finite alphabet and $\mathcal{A}^{\mathbb{Z}}$ is the set of bi-infinite sequences $(x_n)_{n\in\mathbb{Z}}$ on this alphabet. It is defined by $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. The elements of $\mathcal{A}$ are called *letters*. A *subshift* is a topological dynamical system (X,σ) where X is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ (we consider the product topology on $\mathcal{A}^{\mathbb{Z}}$). Observe that even if the alphabet may change we continue to denote the shift map by σ. For convenience, when we state general results about topological dynamical systems we use the notation (X,T), and we use (X,σ) to state particular results about subshifts. Let $\phi\colon (X,\sigma) \to (Y,\sigma)$ be a *factor map*. By the Curtis-Hedlund-Lyndon Theorem, ϕ is determined by a local map $\hat{\phi}\colon \mathcal{A}^{2\mathbf{r}+1} \to \mathcal{A}$ in such way that $\phi(x)_n = \hat{\phi}(x_{n-\mathbf{r}} \dots x_n \dots x_{n+\mathbf{r}})$ for any $n \in \mathbb{Z}$ and $x \in X$, where $\mathbf{r} \in \mathbb{N}$ is called a *radius* of ϕ. The local map $\hat{\phi}$ naturally extends to the set of words of length at least $2\mathbf{r}+1$, and we denote this map also by $\hat{\phi}$.

A *word* $w = w_0 \dots w_{\ell-1}$ with symbols w_i's on the alphabet $\mathcal{A}$ is a element of the free monoid $\mathcal{A}^*$ endowed with the concatenation. The *length* of w is ℓ and is denoted by $|w|$. For a subshift (X,σ) we call *language* of X to the set $\mathcal{L}(X)$ of all words $x_n x_{n+1} \dots x_{n+\ell-1}$, $x_n \in \mathcal{A}$, *appearing* in some sequence $x \in X$. We denote by $\mathcal{L}_n(X)$ the set of words of $\mathcal{L}(X)$ of length n. The map $p_X\colon \mathbb{N} \to \mathbb{N}$ defined by $p_X(n) = \sharp\mathcal{L}_n(X)$ is called the *complexity function* of (X,σ).

In the proof of Theorem 1.1 we will need the following well known notion that is intimately related to asymptotic pairs. A word $w \in \mathcal{L}(X)$ is said to be *left special* if there exist at least two distinct letters a and b such that aw and bw both belong to $\mathcal{L}(X)$. In the same way we define *right special* words.

2.3. **Equicontinuous systems.** We recall that a topological dynamical system (X,T) is equicontinuous if the family of transformations $\{T^n; n \in \mathbb{Z}\}$ is equicontinuous. It is well known that for an equicontinuous minimal system (X,T), the closure of the group $\langle T\rangle$ in the set of homeomorphisms of X, for the uniform topology, is a compact Abelian group G acting transitively on X (see [2]).

A typical example is when X is topologically a Cantor set. The dynamical system (X,T) is then called an *odometer*. One can show that X is then a pro-finite group. More precisely, there exists a nested sequence of finite index subgroups $\dots \subset \Gamma_{n+1} \subset \Gamma_n \subset \dots \subset \Gamma_0 \subset \mathbb{Z}$, with $\bigcap_{n\geq 0}\Gamma_n$ trivial, such that X is isomorphic to the inverse limit

$$\lim_{\leftarrow n}(\mathbb{Z}/\Gamma_n, \pi_n) = \{(x_n)_{n\in\mathbb{N}}; x_n \in \mathbb{Z}/\Gamma_n,\ x_n = \pi_n(x_{n+1})\ \forall n \geq 0\},$$

where $\pi_n\colon \mathbb{Z}/\Gamma_{n+1} \to \mathbb{Z}/\Gamma_n$ denotes the canonical projection. The addition in this group is thus given for $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \in \lim_{\leftarrow n}(\mathbb{Z}/\Gamma_n, \pi_n)$, by

$$(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (x_n +_n y_n)_{n\in\mathbb{N}},$$

with $+_n$ stands for the addition in $\mathbb{Z}/\Gamma_n$. The group $\mathbb{Z}$ is a dense subgroup through the injection $i\colon k \mapsto (k \mod \Gamma_n)_{n\in\mathbb{N}}$. The action T is then given by the addition by $i(1)$ in the group X. It is a minimal and uniquely ergodic action on X.

2.4. **Nilsystems.** We introduce a special class of systems which allows us to compute the automorphism group in some interesting examples and that generalizes equicontinuous systems.

Let G be a group. For $g,h \in G$ we write $[g,h] = ghg^{-1}h^{-1}$ for the commutator of g and h and for $A,B \subseteq G$ we write $[A,B]$ for the subgroup spanned by $\{[a,b]; a \in$

$A, b \in B\}$. The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $d \geq 1$ be an integer. We say that G is *d-step nilpotent* if G_{d+1} is the trivial subgroup. Notice that a subgroup of a d-step nilpotent group is also d-step nilpotent, and any Abelian group is 1-step nilpotent.

Let G be a d-step nilpotent Lie group and Γ a discrete cocompact subgroup of G. The compact manifold $X = G/\Gamma$ is called a *d-step nilmanifold*. The group G acts on X by left translations and we write this action as $(g, x) \mapsto gx$. Let $\tau \in G$ and T be the transformation $x \mapsto \tau x$. Then (X, T) or (X, τ) is called a *d-step nilsystem*. Nilsystems are distal systems, meaning that there are no proximal pairs and they are uniquely ergodic when the action is minimal. See [3] and [23] for general references.

An important subclass of nilsystems is the class of *affine nilsystems*. Let $d \in \mathbb{N}$ and let A be a $d \times d$ integer matrix such that $(A - Id)^d = 0$ (such a matrix is called *unipotent*). Let $\vec{\alpha} \in \mathbb{T}^d$ and consider the transformation $T\colon \mathbb{T}^d \to \mathbb{T}^d$, $x \mapsto Ax + \vec{\alpha}$. Let G be the group spanned by A and all the translations of $\mathbb{T}^d$. Since A is unipotent one can check that G is a d-step nilpotent Lie group. The stabilizer of 0 is the subgroup Γ spanned by A thus we can identify $\mathbb{T}^d$ with G/Γ. The topological dynamical system $(\mathbb{T}^d, T) = (G/\Gamma, T)$ is called a *d-step affine nilsystem* and it is proved in [29] that this system is minimal if the projection of $\vec{\alpha}$ on $\mathbb{T}^d/ker(A - Id)$ defines a minimal rotation.

2.5. **Maximal nilfactor.** Let (X, T) be a minimal topological dynamical system. We say that $x, y \in X$ are regionally proximal of order d if for any $\delta > 0$ there exists $x', y' \in X$ and $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $\mathrm{dist}(x, x') < \delta$, $\mathrm{dist}(y, y') < \delta$ and

$$\mathrm{dist}(T^{\vec{n}\cdot\vec{\epsilon}}x', T^{\vec{n}\cdot\epsilon}y') < \delta$$

for every $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d \setminus \{(0, \ldots, 0)\}$, where $\vec{n} \cdot \epsilon = \sum_{i=1}^d n_i \cdot \epsilon_i$.

Let $\mathbf{RP}^{[d]}(X)$ denote the set of regionally proximal points of order d. It is proved that $\mathbf{RP}^{[d]}(X)$ is an equivalence relation, first in [20] in the distal case and then in [35] in the general minimal case. Of course the map T preserves each class, so that it induces a map T_d on the quotient space $Z_d(X) := X/\mathbf{RP}^{[d]}(X)$.

Theorem 2.2 ([20],[35])**.** *Let (X, T) be a minimal topological dynamical system. Then $(Z_d(X), T_d)$ is isomorphic to an inverse limit of minimal d-step nilsystems. Moreover, it is the* maximal factor *of (X, T) with this property or* maximal d-step nilsystem*, meaning that any other factor which is an inverse limit of minimal d-step nilsystems is a factor of $(Z_d(X), T_d)$.*

In particular, $(Z_1(X), T_1)$ is the maximal equicontinuous factor of (X, T) [2].

2.6. **General results on automorphisms through factor maps.**

Definition 2.3. *Let $\pi\colon (X, T) \to (Y, S)$ be a factor map between the minimal systems (X, T) and (Y, S), and let ϕ be an automorphism of (X, T). We say that π is* compatible *with ϕ if $\pi(x) = \pi(x')$ implies $\pi(\phi(x)) = \pi(\phi(x'))$ for any $x, x' \in X$. We say that π is* compatible *with* $\mathrm{Aut}(X, T)$ *if π is compatible with any $\phi \in \mathrm{Aut}(X, T)$.*

If the factor map $\pi\colon (X, T) \to (Y, S)$ is compatible with $\mathrm{Aut}(X, T)$ we can define the projection $\widehat{\pi}(\phi) \in \mathrm{Aut}(Y, S)$ as the map defined by the equation $\pi(\phi(x)) = \widehat{\pi}(\phi)(\pi(x))$ for every $x \in X$. We have that $\widehat{\pi}$ is a group morphism.

Notice that the map $\widehat{\pi}\colon \mathrm{Aut}(X,T) \to \mathrm{Aut}(Y,S)$ may not be onto or injective. For an irrational rotation on the circle, the group of automorphisms is the whole circle but for its Sturmian extension the group of automorphisms is $\mathbb{Z}$ [27]. We will show in Lemma 2.12 that this extension is compatible, hence $\widehat{\pi}$ is not onto. On the other hand, the projection on the trivial system $\widehat{\pi}$ cannot be injective.

By considering compatible factor maps in this section we prove that the automorphism group of a d-step nilsystem and any proximal extension is d-step nilpotent. Before considering factors we need two general properties of automorphism groups.

Lemma 2.4. *Let (X,T) be a minimal topological dynamical system. Then the action of $\mathrm{Aut}(X,T)$ on X is free, meaning that every non-trivial element in $\mathrm{Aut}(X,T)$ has no fixed points.*

Proof. Let $\phi \in \mathrm{Aut}(X,T)$ and let $x \in X$ such that $\phi(x) = x$. Since ϕ commutes with T and is continuous, then by minimality ϕ is the identity. □

Lemma 2.5. *Let (X,T) be a minimal topological dynamical system. Let $x \in X$ and $\phi \in \mathrm{Aut}(X,T)$. Then x and $\phi(x)$ are proximal if and only if ϕ is the identity.*

Proof. We prove the non-trivial direction. Let $x \in X$ and $\phi \in \mathrm{Aut}(X,T)$ such that $(x,\phi(x))$ is a proximal pair. There exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $\lim_{i\to+\infty} \mathrm{dist}(T^{n_i}x, T^{n_i}\phi(x)) = 0$. We can assume that $T^{n_i}x$ converges to $y \in X$ and therefore $\mathrm{dist}(y,\phi(y)) = 0$. By Lemma 2.4 ϕ is the identity. □

Then considering a proximal extension between minimal systems we get that,

Lemma 2.6. *Let $\pi\colon (X,T) \to (Y,S)$ be a proximal extension between minimal systems and suppose that π is compatible with $\mathrm{Aut}(X,T)$. Then $\widehat{\pi}\colon \mathrm{Aut}(X,T) \to \mathrm{Aut}(Y,S)$ is an injection.*

Proof. It suffices to prove that $\widehat{\pi}(\phi) = id_Y$, where id_Y is the identity map on Y, implies that $\phi = id_X$. Let ϕ be an automorphism with $\widehat{\pi}(\phi) = id_Y$. For $x \in X$ we have that $\pi(\phi(x)) = \widehat{\pi}(\phi)\pi(x) = \pi(x)$. Since π is proximal, points x and $\phi(x)$ are proximal. By Lemma 2.5 we get the conclusion. □

Now we study the automorphism group of d-step nilsystems.

Lemma 2.7. *Let $\pi\colon (X,T) \to (Y,S)$ be an almost one-to-one extension between minimal systems. Then, for any integer $d \geq 1$, their maximal d-step nilsystems $(Z_d(X),T_d)$ and $(Z_d(Y),S_d)$ are topologically conjugated.*

Proof. Let $\pi_d\colon X \to Z_d(X)$ and $\widetilde{\pi}_d\colon Y \to Z_d(Y)$ denote the quotient maps and let $p\colon Z_d(X) \to Z_d(Y)$ be the natural factor map (it exists by maximality of $Z_d(X)$). Let $x \in X$ be such that $\pi^{-1}\{\pi(x)\} = \{x\}$. We claim that $p^{-1}\{p(\pi_d(x))\} = \{\pi_d(x)\}$. Let x' be such that $p(\pi_d(x)) = p(\pi_d(x'))$. Then $\widetilde{\pi}_d(\pi(x)) = \widetilde{\pi}_d(\pi(x'))$ and thus $(\widetilde{\pi}(x),\widetilde{\pi}(x')) \in \mathbf{RP}^{[d]}(Y)$. Particularly, there exists a sequence $(\vec{n}_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}^d$ such that $S^{\vec{n}_i\cdot\epsilon}\pi(x')$ converges to $\pi(x)$ for every $\epsilon \in \{0,1\}^d \setminus \{(0,\dots,0)\}$. Taking a subsequence we can assume that $T^{\vec{n}_i\cdot\epsilon}x'$ converges to some point in $\pi^{-1}\{\pi(x)\}$. By our assumption, $T^{\vec{n}_i\cdot\epsilon}x'$ can only converge to x for every $\epsilon \in \{0,1\}^d \setminus \{(0,\dots,0)\}$ and thus $(x,x') \in \mathbf{RP}^{[d]}(X)$, which means that $\pi_d(x) = \pi_d(x')$. It follows that p is an almost one-to-one map between distal systems and thus it is a topological conjugacy. □

We deduce that,

Corollary 2.8. *Let $\pi\colon (X,T) \to (Y,S)$ be an almost one-to-one extension between minimal systems. If (Y,S) is a nilsystem of order d, then it is the maximal d-step nilsystem of (X,T).*

For instance, since any Sturmian subshift is an almost one-to-one extension of a rotation on the circle [14], this rotation is its maximal 1-step nilsystem or more classically its maximal equicontinuous factor. Similarly, Toeplitz subshifts are the symbolic almost one-to-one extensions of odometers [13]. These odometers are hence their maximal 1-step nilsystems.

The next result is a characterization of the automorphisms of an equicontinuous system. In particular, we get that its group of automorphisms is Abelian.

Lemma 2.9. *Let (X,T) be an equicontinuous minimal system. Then $\mathrm{Aut}(X,T)$ is the closure of the group $\langle T\rangle$ in the set of homeomorphisms of X for the topology of uniform convergence. Moreover $\mathrm{Aut}(X,T)$ is homeomorphic to X.*

We deduce the automorphism group of an odometer is the whole odometer group.

Proof. Let x be a point in X and let ϕ be an automorphism in $\mathrm{Aut}(X,T)$. Let G denote the closure of the group $\langle T\rangle$. By Ascoli's Theorem it is a compact Abelian group. Let us consider a sequence of integers $(n_i)_{i\in\mathbb{N}}$ such that the sequence $(T^{n_i}x)_{i\in\mathbb{N}}$ converges to $\phi(x)$. Taking a subsequence if needed, we can assume that the sequence of maps $(T^{n_i})_{i\in\mathbb{N}}$ converges to a homeomorphism g in G. So we get that $g^{-1}\circ\phi(x) = x$, and then ϕ belongs to G by Lemma 2.4. This lemma also ensures that the map $g \in G \mapsto g(x)$ is a homeomorphism onto its image. The latter being T invariant, it is equal to X. □

If $(Y,S) = (G/\Gamma,\tau)$ is a d-step nilsystem, we can generalize previous results.

Theorem 2.10. *Let $(G/\Gamma,\tau)$ be a minimal d-step nilsystem. Then $\mathrm{Aut}(G/\Gamma,\tau)$ is a d-step nilpotent group.*

Proof. Let ϕ be an automorphism of $(G/\Gamma,\tau)$ and let μ be the unique ergodic invariant measure of $(G/\Gamma,\tau)$. Since ϕ commutes with τ we have that $\tau\phi\mu = \phi\tau\mu = \phi\mu$ and thus $\tau\phi\mu = \phi\mu$. Since μ is the unique τ-invariant measure we have that $\phi\mu = \mu$. Hence ϕ is also an automorphism of the system $(G/\Gamma,\tau,\mu)$ in the measurable setting (a measure preserving transformation that commutes with τ). It is proved in [19] that G/Γ is isomorphic to $\mathcal{G}/\Gamma'$ where $\mathcal{G}$ is a nilpotent Lie group of measure preserving transformations of G/Γ. By Lemma 5.5 of [19], ϕ belongs to $\mathcal{G}$ and thus $\mathrm{Aut}(G/\Gamma,\tau)$ is d-step nilpotent. □

Theorem 2.11. *Let (X,T) be an inverse limit of d-step nilsystems. Then, its automorphism group $\mathrm{Aut}(X,T)$ is a d-step nilpotent group.*

On the other hand, by definition of the regionally proximal relation of order d we have that,

Lemma 2.12. *Let (X,T) be a minimal topological dynamical system. Let $\phi \in \mathrm{Aut}(X,T)$. Then $(x,y) \in \mathbf{RP}^{[d]}(X)$ if and only if $(\phi(x),\phi(y)) \in \mathbf{RP}^{[d]}(X)$. Consequently, the projection $\pi_d\colon X \to Z_d(X)$ from X to its maximal d-step nilsystem is compatible with $\mathrm{Aut}(X,T)$.*

Combining Theorem 2.11, Lemma 2.12 and Lemma 2.6 we get,

Corollary 2.13. *Let (X,T) be a proximal extension of a minimal d-step nilsystem. Then,* $\mathrm{Aut}(X,T)$ *is a d-step nilpotent group.*

Since Sturmian and Toeplitz subshifts are almost one-to-one extensions of their maximal equicontinuous factors, they are also proximal extensions (Lemma 2.1). We obtain as a corollary that their automorphism groups are Abelian. More precisely, Lemma 2.12 and Lemma 2.6 imply that their automorphism groups are subgroups of the automorphism group of their maximal equicontinuous factors, characterized in Lemma 2.9. In addition, it is not difficult to construct minimal symbolic almost one-to-one extensions of d-tep nilsystems by considering codings on well chosen partitions. An example will be developed in Section 4.

3. On the automorphisms of subshifts with polynomial complexity

In this section we prove the main results of this paper. We start by proving Theorem 1.1 and in a second part we give new proofs of byproduct results from [21, 24]. Namely, a characterization of the automorphisms of bijective constant length substitutions and the realization of any finite group as the quotient $\mathrm{Aut}(X,T)/\langle T\rangle$. We end this section by presenting a tentative generalization of Theorem 1.1 to polynomial complexity by using a result on the growth rate of groups.

For the sequel, we recall that a group G satisfies *virtually* a property $\mathtt{P}$ (*e.g.*, nilpotent, solvable, isomorphic to a given group, ...) if there is a finite index subgroup $H \subset G$ satisfying the property $\mathtt{P}$.

3.1. **Proof of Theorem 1.1.** Let (X,T) be a topological dynamical system. It is clear from the definition that for any proximal (asymptotic) pair $(x,y) \in X \times X$ and for any $\phi \in \mathrm{Aut}(X,T)$ we have that $(\phi(x),\phi(y))$ is a proximal (asymptotic) pair. We say that the asymptotic pairs (x,y) and (x',y') belong to the same class if they are in the same orbit, meaning that there exists $n \in \mathbb{Z}$ such that $(x',y') = (T^n x, T^n y)$. A class of asymptotic pairs is a (non closed) $T \times T$-invariant subset of $X \times X$. We denote by $[(x,y)]$ the class of the asymptotic pair (x,y). We say that two classes $[(x,y)]$, $[(x',y')]$ are *equivalent* if there is an asymptotic pair $(x_1',y_1') \in [(x',y')]$ such that $x = x_1'$ or x and x_1' are asymptotic. This defines an equivalence relation and any class is called an *asymptotic component*. We denote by $\mathcal{AS}_{[(x,y)]}$ the asymptotic component of the class $[(x,y)]$ and by $\mathcal{AS}$ the collection of asymptotic components.

It is also plain to check for $\phi \in \mathrm{Aut}(X,T)$ and two equivalent asymptotic classes $[(x,y)]$ and $[(x',y')]$, that classes $[(\phi(x),\phi(y))]$ and $[(\phi(x'),\phi(y'))]$ are also equivalent. So the automorphism ϕ induces a permutation $j(\phi)$ on the collection $\mathcal{AS}$ of asymptotic components of (X,T). By denoting $\mathrm{Per}\mathcal{AS}$ the set of such permutations, formally we have the group morphism

$$\begin{aligned} j\colon \mathrm{Aut}(X,T) &\to \mathrm{Per}\mathcal{AS} \\ \phi &\mapsto \left(\mathcal{AS}_{[x,y]} \mapsto \mathcal{AS}_{[(\phi(x),\phi(y))]}\right). \end{aligned} \tag{1}$$

In the case of subshifts, the following lemma is a key observation which relates the complexity of the subshift with asymptotic classes. The proof relies in classical ideas from [32].

Lemma 3.1. *Let (X,σ) be a subshift. If (X,σ) has a sublinear complexity, then there is a finite number of asymptotic classes. More generally, if the complexity*

$p_X(n)$ satisfies

$$\liminf_{n\to+\infty} \frac{p_X(n)}{n} < +\infty \ ,$$

then there is a finite number of asymptotic classes.

In particular, this lemma provides a sufficient condition to bound the number of asymptotic components.

Proof. For the first statement see [32] Lemma V. 22. For the second claim we proceed as follows. The hypothesis implies the existence of a constant κ and an increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $p_X(n_i+1)-p_X(n_i)\leq\kappa$. Indeed, if not, for any $A>0$ and for any integer n large enough we have $p_X(n+1)-p_X(n)\geq A$. It follows that $p_X(n)-p_X(m)=\sum_{i=m}^{n-1} p_X(i+1)-p_X(i)\geq (n-m)A$ for any $n\geq m$ enough large. From here we get that $\liminf_{n\to+\infty}\frac{p_X(n)}{n}\geq A$ which is a contradiction since A is arbitrary.

Hence, the number of left special words of length n_i (see Section 2.2 for the definition) is bounded by κ. Any asymptotic pair defines a sequence with arbitrarily long special words, so there are at most κ asymptotic classes. □

A second main ingredient for proving Theorem 1.1 is the following direct corollary of Lemma 2.5. We recall that an asymptotic pair is proximal and that the map j used in the following corollary has been defined in (1).

Corollary 3.2. *Let (X,T) be a minimal topological dynamical system with at least one asymptotic pair. We have the following exact sequence*

$$1 \longrightarrow \langle T\rangle \xrightarrow{\mathrm{Id}} \mathrm{Aut}(X,T) \xrightarrow{j} \mathrm{Per}\mathcal{AS},$$

where $\mathrm{Per}\mathcal{AS}$ denotes the set of permutations on the collection of asymptotic components of (X,T). Moreover, for any automorphism ϕ, the permutation $j(\phi)$ has a fixed point if and only if ϕ is a power of T.

As a byproduct of this result and Lemma 3.1 we get Theorem 1.1 that we recall and extend here.

Theorem. *Let (X,σ) be a minimal aperiodic subshift with $\liminf_{n\to+\infty}\frac{p_X(n)}{n}<+\infty$. Then,*

(1) $\mathrm{Aut}(X,\sigma)$ *is virtually isomorphic to $\mathbb{Z}$.*
(2) The quotient group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is isomorphic to a finite subgroup of permutations without fixed points. In particular, $\sharp\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ divides the number of asymptotic components.

Proof. Only the second part of statement (2) is not straightforward from Corollary 3.2. The group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ acts freely on the finite set of asymptotic component $\mathcal{AS}$: the stabilizer of any point is trivial. Thus, $\mathcal{AS}$ is decomposed into disjoint $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$-orbits, and any such orbit has the same cardinality as $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$. □

Statement (2) of the theorem enables us to perform explicit computations of the automorphism group for easy cases. A first example comes from Sturmian subshifts. It is well-known that this system admits just one asymptotic component, so any automorphism is a power of the shift map. A bit more general case is when

the number of asymptotic components is a prime p (*e.g.*, 2 for the Thue-Morse subshift), then the group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is a subgroup of $\mathbb{Z}/p\mathbb{Z}$: either the trivial one or $\mathbb{Z}/p\mathbb{Z}$. In particular, since the Thue-Morse subshift admits an automorphism which is not the power of the shift map, then the quotient automorphism group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

One could ask whether the automorphism group is computable algorithmically, at least for substitution subshifts, or explicitly by theoretical arguments for some families of subshifts. This will be achieved in [16] for substitutive and linearly recurrent subshifts.

Statement (2) is not a real restriction. Given any finite group G, it acts on itself by left multiplication $L_g(h) = g \cdot h$ for $g, h \in G$. The map L_g defines then a permutation on the finite set G without fixed points. So G is a subgroup of elements of the permutation group on $\sharp G$ elements which verifies statement (2) in the theorem. Thus, it is natural to ask whether we can realize any finite group as $\mathrm{Aut}(X,T)/\langle T\rangle$ or if we can characterize those finite groups. This is done in the next subsection.

Finally, notice that the complexity condition of Theorem 1.1 is compatible with $\limsup_{n\to+\infty} p_X(n)/n = +\infty$. In Section 4.4 we construct a minimal subshift with subexponential complexity satisfying $\liminf_{n\to+\infty} p_X(n)/n < +\infty$ and $\limsup_{n\to+\infty} p_X(n)/n^d = +\infty$ for every $d > 1$. Thus, in this case, the automorphism group is virtually $\mathbb{Z}$ by Theorem 1.1.

3.2. **A characterization of $\mathbf{Aut}(X,\sigma)/\langle\sigma\rangle$ for constant length substitutions.** In this section, by using the results of Section 3.1, we provide a characterization of the automorphism group for subshifts given by a constant length substitution $\tau : \mathcal{A} \to \mathcal{A}^*$ on a finite alphabet $\mathcal{A}$. Our characterization follows from the one of asymptotic components. We deduce then new and direct proofs of two already known results. The first one is due to B. Host and F. Parreau [21] on the characterization of the automorphism group of bijective constant length substitutions. The second one is a combination of results in [24] and [21], giving an explicit example of a substitutive minimal subshift (X,σ) such that $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is isomorphic to an arbitrary finite group G. Notice that in [24] the authors have a similar statement but in the measurable setting.

We recall that a substitution $\tau\colon \mathcal{A} \to \mathcal{A}^*$ is of constant length $\ell > 0$ if any word $\tau(a)$ for the letter $a \in \mathcal{A}$ is of length ℓ. A substitution of constant length is *bijective* if the corresponding letters at position $i \in \{0,\dots,\ell-1\}$ of all $\tau(a)$'s are pairwise distinct. We denote by X_τ the subshift

$$X_\tau = \{x \in \mathcal{A}^{\mathbb{Z}};\ \text{any word of } x \text{ appears in } \tau^n(a) \text{ for some } n \geq 0 \text{ and } a \in \mathcal{A}\}.$$

For constant length substitution, it is well known (*e.g.* see [32]) that the subshift (X_τ,τ) is minimal if and only if the substitution τ is *primitive*, that is, for some power $p \geq 0$ and any letter $a \in \mathcal{A}$, the word $\tau^p(a)$ contains all the letters of the alphabet. Recall that the substitution τ is *aperiodic* if and only if X_τ is infinite.

Lemma 3.3. *Let τ be a primitive aperiodic bijective constant length substitution. Let $(x,y) = ((x_n)_{n\in\mathbb{Z}},(y_n)_{n\in\mathbb{Z}}) \in X_\tau^2$ be an asymptotic pair with $x_n = y_n$ for any $n \geq 0$ and $x_{-1} \neq y_{-1}$. Then, there exists an asymptotic pair $((x'_n)_{n\in\mathbb{Z}},(y'_n)_{n\in\mathbb{Z}}) \in X_\tau^2$ with $x'_n = y'_n$ for any $n \geq 0$ and $x'_{-1} \neq y'_{-1}$, such that*

$$\tau((x'_n)_{n\in\mathbb{Z}}) = (x_n)_{n\in\mathbb{Z}} \text{ and } \tau((y'_n)_{n\in\mathbb{Z}}) = (y_n)_{n\in\mathbb{Z}}.$$

Proof. Let ℓ be the length of the substitution τ. By the classical result of B. Mossé [25, 26] on recognizability, the substitution $\tau\colon X_\tau \to \tau(X_\tau)$ is one-to-one. Moreover, the collection $\{\sigma^k\tau(X_\tau)\colon k = 0, \dots, \ell-1\}$ is a clopen partition of X_τ. So, there are $x' = (x'_n)_{n\in\mathbb{Z}}, y' = (y'_n)_{n\in\mathbb{Z}} \in X_\tau$ and $0 \le k_x, k_y < \ell$ such that $\sigma^{k_x}\tau(x') = x$ and $\sigma^{k_y}\tau(y') = y$.

We claim that we have $k_x = k_y = 0$. Since the sequences x and y are asymptotic, there are integers $n \ge 0$, $k' \in \{0, \dots, \ell-1\}$ such that $\sigma^n(x), \sigma^n(y) \in \sigma^{k'}(\tau(X_\tau))$. The substitution τ is of constant length ℓ, so we have $\sigma^\ell \circ \tau = \tau \circ \sigma$. Therefore, we get x and y are in the same clopen set $\sigma^k(\tau(X_\tau))$ for some $k \in \{0, \dots, \ell-1\}$. Let us assume that $k \ge 1$. The words $x_{-1}x_0 \dots x_{k-1}$, $y_{-1}y_0 \dots y_{k-1}$ are then prefixes of the words $\tau(x'_{-1})$ and $\tau(y'_{-1})$ respectively. Since the substitution τ is bijective and $x_0 = y_0$, we have $x'_{-1} = y'_{-1}$. In particular, we get $x_{-1} = y_{-1}$: a contradiction.

To finish the proof, notice that the substitution τ is injective on the letters, so we obtain $x'_n = y'_n$ for any $n \ge 0$ and $x'_{-1} \ne y'_{-1}$. □

Lemma 3.4. *Let τ be a primitive aperiodic bijective constant length substitution. Then, there exists an integer $p \ge 0$ such that for any asymptotic pair $((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}) \in X_\tau^2$ the one-sided infinite sequences*

$$(x_{n+n_0})_{n\ge 0}, (y_{n+n_0})_{n\ge 0} \text{ are equal for some } n_0 \in \mathbb{Z} \text{ and fixed by } \tau^p.$$

Proof. Shifting the indices if needed by some σ^{n_0}, we can assume that for the asymptotic pair $(x, y) = ((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}})$ we have $x_n = y_n$ for any integer $n \ge 0$ and $x_{-1} \ne y_{-1}$. Let $p \ge 0$ be an integer such that for any letter $a \in \mathcal{A}$, any word in $\{\tau^{pn}(a)\}_{n\ge 1}$ starts with the same letter. Hence, the sequence of sequences $(\tau^{pn}(aa\cdots))_{n\ge 0}$ converges to a one-sided infinite word fixed by τ^p.

Applying inductively Lemma 3.3 to the substitution τ^p, we get a sequence of asymptotic pairs $((x^{(i)}, y^{(i)}))_{i\ge 0}$ verifying the conclusions of the lemma and such that $\tau^p(x^{(i+1)}) = x^{(i)}$, $\tau^p(y^{(i+1)}) = y^{(i)}$, $x^{(0)} = x$ and $y^{(0)} = y$. By the definition of p, all sequences $x^{(i)}$ and also $y^{(i)}$, $i \ge 0$, share the same letter a at index 0. The conclusion of the lemma follows straightforwardly since we assume that $\tau^{pn}(a\cdots)$ converges to a one-sided infinite word fixed by τ^p. □

Thanks to this lemma we can obtain another proof of the following result due to B. Host and F. Parreau.

Theorem 3.5. [21] *Let τ be a primitive bijective constant length substitution. Then, any automorphism of the subshift X_τ is the composition of some power of the shift with an automorphism ϕ of radius 0. Moreover, its local rule $\hat{\phi}\colon \mathcal{A} \to \mathcal{A}$ satisfies*

$$\tau \circ \hat{\phi} = \hat{\phi} \circ \tau. \tag{2}$$

Conversely, notice that a local map satisfying (2) defines an automorphism of the subshift. Hence we obtain an algorithm to determine in this case the group of automorphisms since there is just a finite number of local rules of radius 0.

Proof. Notice first that when X_τ is finite, it is reduced to a finite orbit. Hence any automorphism is a power of the shift map by Lemma 2.9.

Let us assume now that the substitution τ is aperiodic and let $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}} \in X_\tau$ be two asymptotic sequences. Lemma 3.4 provides a power $p \ge 0$ such that, shifting the sequences if needed by some σ^{n_0}, we can assume that $(x_n)_{n\ge 0}$ and $(y_n)_{n\ge 0}$ coincide and are fixed by τ^p.

Let ϕ be an automorphism of the subshift (X_τ, σ). The pair $(\phi(x), \phi(y))$ is also an asymptotic pair. Again, Lemma 3.4 ensures that for some integer $n_1 \in \mathbb{Z}$, the sequences $(\phi(x)_{n+n_1})_{n\geq 0}$ and $(\phi(y)_{n+n_1})_{n\geq 0}$ coincide and are also fixed by τ^p (observe as stated in Lemma 3.4, we can use the same power p for any couple of asymptotic pairs). In the following, we will consider the automorphism $\phi' = \sigma^{n_1} \circ \phi$, thus by definition, the sequence $(\phi'(x)_n)_{n\geq 0}$ is also fixed by τ^p.

Let $\mathbf{r}$ and $\hat{\phi}'$ denote the radius and the local map of ϕ' respectively. Taking a power of τ^p if needed, we can assume that the length ℓ of τ^p is greater than $2\mathbf{r}+1$. Suppose now that $x_n = x_m$ for some $n, m \geq 0$. We have $\phi'(x)_{m\ell+\mathbf{r}} = \hat{\phi}'(x_{m\ell} \dots x_{m\ell+2\mathbf{r}}) = \hat{\phi}'(\tau^p(x_m)_{[0,2\mathbf{r}]}) = \hat{\phi}'(\tau^p(x_n)_{[0,2\mathbf{r}]}) = \phi'(x)_{n\ell+\mathbf{r}}$, where for a word $u = u_0 \dots u_{\ell-1}$, $u_{[0,2\mathbf{r}]}$ stands for the prefix $u_0 \dots u_{2\mathbf{r}}$. Since $\phi'(x)_{n\ell+\mathbf{r}}$ and $\phi'(x)_{m\ell+\mathbf{r}}$ are the $r+1^{\text{th}}$ letters of the words $\tau^p(\phi'(x)_n)$ and $\tau^p(\phi'(x)_m)$ respectively, and the substitution τ is bijective, we obtain that $\phi'(x)_n = \phi'(x)_m$.

Hence, we can define the local map $\hat{\psi}\colon \mathcal{A} \to \mathcal{A}$ by $\hat{\psi}(x_n) = \phi'(x)_n$ for any $n \geq 0$. This provides a shift commuting map $\psi\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that for any word w in the language $\mathcal{L}(X_\tau)$, we have that $\psi(\tau^p(w)) = \tau^p(\psi(w))$. Thus $\psi(X_\tau) \subset X_\tau$. Since the substitution τ is bijective we also get relation (2). In the same way, using ϕ'^{-1} instead of ϕ' we obtain that ψ is invertible. By construction, we have that $\psi^{-1}\phi'(x)$ is asymptotic to x, so by Lemma 2.5, $\psi = \phi' = \sigma^{n_1} \circ \phi$. □

A second consequence of Lemma 3.4 is the realization of any finite group as the group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ for a substitutive subshift of constant length.

Proposition 3.6. *Given a finite group G, there is a substitutive minimal subshift (X, σ) such that $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is isomorphic to G.*

Proof. The Fibonacci subshift is both a substitutive and a Sturmian subshift, then by previous discussion the quotient group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is trivial. Then, let us assume that the finite group G is not trivial. We choose an enumeration of its elements $G = \{g_0, g_1, \dots, g_{p-1}\}$ with $p \geq 2$ where g_0 denotes the neutral element.

For an element $h \in G$, we denote by $L_h\colon G \to G$ the bijection $g \mapsto hg$. We consider the alphabet G, viewed as a finite set, and define the substitution τ from the set of letters G into the set of words G^*, by

$$\tau\colon g \mapsto L_g(g_0)L_g(g_1)\cdots L_g(g_{p-1}).$$

Since the map L_g is a bijection on G, the substitution τ of constant length is primitive and bijective. Thus the associated subshift (X_τ, σ).

Moreover observe that for any letter $g \in G$, the word $\tau(g)$ starts by the letter g, so any sequence $(\tau^n(gg\cdots))_{n\geq 1}$ converges to a τ-invariant infinite word.

We claim that the subshift (X_τ, σ) is not periodic, *i.e.*, not reduced to a periodic orbit. To show this it suffices to give an example of an asymptotic pair. The word g_0g_1 belongs to the language $\mathcal{L}(X_\tau)$ of the subshift X_τ. Hence the words $\tau(g_0)\tau(g_1)$ and its sub-word $g_{p-1}g_1$ (which is different from the word g_0g_1) also belong to $\mathcal{L}(X_\tau)$. It follows for any integer $n \geq 0$ that the words $\tau^n(g_0).\tau^n(g_1)$ and $\tau^n(g_{p-1}).\tau^n(g_1)$ are also in the language. Taking a subsequence if needed, these words converge as n goes to infinity to two sequences x and $y \in X_\tau$ that are, by construction, asymptotic.

Given an element $g \in G$ we extend the definition of the map L_g to G^* by defining for a word $w = h_1 \dots h_n$, $L_g(w) := L_g(h_1) \dots L_g(h_n)$. By concatenation, it defines

a left continuous G-action on $G^{\mathbb{Z}}$. It is important to note that we have the relation for any $g, h \in G$

$$L_g(\tau(h)) = \tau(L_g(h)). \tag{3}$$

Hence any map L_g preserves the subshift X_τ and we have a left action of G on X_τ. It is plain to check that $L\colon g \mapsto L_g$ defines an injection of G into $\mathrm{Aut}(X_\tau, \sigma)$. Actually, we claim that we have a converse which allows to finish the proof.

Lemma 3.7. *For the subshift X_τ defined above the map*

$$\begin{array}{rcl} \varphi\colon \mathbb{Z} \times G & \to & \mathrm{Aut}(X_\tau, \sigma) \\ (n, g) & \mapsto & \sigma^n \circ L_g \end{array}$$

is a group isomorphism.

Proof of Lemma 3.7. To show the injectivity of the map φ let us assume there are $g \in G$ and an integer k such that $L_g(x) = \sigma^k(x)$ for any $x \in X$. Necessarily $k = 0$, otherwise the infinite sequence $L_g\big(\lim_{n\to+\infty} \tau^n(g^{-1}g^{-1}\cdots)\big)$, which is equal to $\lim_n \tau^n(g_0 g_0 \cdots)$ by formula (3), is ultimately periodic. This is impossible since the subshift X_τ is not periodic. The injectivity of the map L implies finally that the map φ is injective.

To show it is also onto, it is enough to prove that any automorphism $\phi \in \mathrm{Aut}(X_\tau, \sigma)$ may be written as a power of the shift composed with a map of the kind L_g. Let (x, y) be an asymptotic pair. By Lemma 3.4 up to shift x, y and compose ϕ with a power of the shift map, there exist $g_1, g_2 \in G$ such that the sequences x, y are positively asymptotic to $\lim_{n\to+\infty} \tau^n(g_1)$, and $\phi(x), \phi(y)$ are positively asymptotic to $\lim_{n\to+\infty} \tau^n(g_2 g_2 \cdots)$. It follows from (3) that the points x and $L_{g_1(g_2^{-1})} \circ \phi(x)$ are asymptotic. So, by Lemma 2.5 the maps ϕ and $(L_{g_1(g_2^{-1})})^{-1} = L_{g_2(g_1^{-1})}$ coincide. □

□

3.3. **Recurrence and growth rate of groups.** We try to extend Theorem 1.1 to subshifts with higher complexity. For this, we need to introduce a stronger condition. We define, for a topologically transitive subshift (X, σ) and an integer $n \geq 1$, a *local recurrence time*:

$$N_X(n) := \inf\{|w|;\ w \in \mathcal{L}(X) \text{ contains any word of } X \text{ of length } n\}.$$

Clearly, this value is well defined and satisfies $N_X(n) \geq p_X(n) + n$. For instance, it is well-known that any primitive substitutive subshift is linearly repetitive meaning that $\sup_{n\geq 1} \frac{N_X(n)}{n} < +\infty$. We obtain the following result.

Theorem 3.8. *Let (X, σ) be a transitive subshift such that $\sup_{n\geq 1} \frac{N_X(n)}{n^d} < +\infty$ for some $d \geq 1$. Then, there is a constant C depending only on d, such that any finitely generated subgroup of $\mathrm{Aut}(X, \sigma)$ is virtually nilpotent of step at most C.*

Proof. Let $\mathcal{S} = \langle \phi_1, \ldots, \phi_\ell \rangle \subset \mathrm{Aut}(X, \sigma)$ be a finitely generated group. Let $\mathbf{r}$ be an upper bound of the radii of the local maps associated to all generators ϕ_i of $\mathcal{S}$ and their inverses. For $n \in \mathbb{N}$ consider

$$B_n(\mathcal{S}) = \{\phi_{i_1}^{s_1} \cdots \phi_{i_m}^{s_m}; 1 \leq m \leq n,\ i_1, \ldots, i_m \in \{1, \ldots, \ell\},\ s_1, \ldots, s_m \in \{1, -1\}\}\ .$$

Let w be a word of length $N_X(2n\mathbf{r}+1)$ containing any word of length $(2n\mathbf{r}+1)$ of X. If $\phi, \phi' \in B_n(\mathcal{S})$ are different then $\phi(w) \neq \phi'(w)$. Then, $B_n(\mathcal{S})$ can be injected

into the set of words of length $N_X(2n\mathbf{r}+1)-2\mathbf{r}$ (the injection is just the valuation of ϕ on w). This implies that $\sharp B_n(\mathcal{S}) \leq p_X(N_X(2n\mathbf{r}+1)-2\mathbf{r})$. We deduce from the hypothesis on N_X that $\sharp B_n(\mathcal{S}) \leq n^{d^2+1}$ for all large enough integers $n \in \mathbb{N}$. Therefore, by the quantitative result of Y. Shalom and T. Tao in [34] generalizing Gromov's classical result on the growth rate of groups, we get the conclusion. □

Notice that the constant C may be given explicitly in the result of [34]. It is clear that a subshift of polynomial local recurrence complexity has a polynomial complexity. The converse is not clear, but an additional possible condition is that the subshift has bounded repetitions of words. The natural question here is whether the automorphism group of a minimal subshift of polynomial local recurrence complexity, or just polynomial complexity, is finitely generated.

4. Gallery of examples

We present here examples of subshifts with various complexities. The first two examples are substitutive subshifts with superlinear complexity. Even if we can not apply straightforwardly the main results of the paper (*e.g.*, the substitutions are not primitive), we study their asymptotic components to prove their automorphism groups are isomorphic to $\mathbb{Z}$. Next, we define a coding of a nil-translation with a polynomial complexity of arbitrary high degree but having an automorphism group which is virtually $\mathbb{Z}$. To enlarge the zoology of automorphism groups we provide a subshift whose automorphism group is isomorphic to $\mathbb{Z}^d$. We end with a subshift whose complexity is, for infinitely many integers, subaffine and superpolynomial. Theorem 1.1 applies in this case.

4.1. **Substitutions with superlinear complexity.** Recall that substitutive subshifts have a prescribed complexity: with growth bounded or equivalent to n, $n\log\log n$, $n\log n$, or to n^2 (see [30]). Below we give two examples having a unique asymptotic component. This is enough to conclude that their automorphism groups are isomorphic to $\mathbb{Z}$.

4.1.1. *A $n\log\log n$ complexity substitutive subshift.* Let $\mathcal{A}=\{a,b\}$ and consider the substitution $\tau_1\colon \mathcal{A}\to\mathcal{A}^*$ defined by

$$\tau_1(a)=aba \text{ and } \tau_1(b)=bb.$$

We set

$$X_{\tau_1}=\{x\in\{a,b\}^{\mathbb{Z}};\ \text{ any word of } x \text{ appears in some } \tau_1^n(c), n\geq 0, c\in\{a,b\}\}.$$

It can be checked that (X_{τ_1},σ) is a non minimal transitive subshift. Moreover, it is proven in [6] that its complexity is equivalent to $n\log_2\log_2 n$.

In the sequel we need some specific notations. For a sequence $x\in\{a,b\}^{\mathbb{Z}}$ we set $x^-=\cdots x_{-2}x_{-1}$, $x^+=x_0x_1\cdots$ and $x=x^-.x^+$. Let $b^{+\infty}=bbbbb\ldots\in\mathcal{A}^{\mathbb{N}}$ and $b^{-\infty}=\ldots bbbbb\in\mathcal{A}^{\mathbb{Z}_{<0}}$, where $\mathbb{Z}_{<0}$ is the set of negative integers. Thus the sequence $x=\ldots bb.bb\ldots\in\{a,b\}^{\mathbb{Z}}$ can be written $b^{-\infty}.b^{+\infty}$. In the same spirit we put $\tau_1^{+\infty}(c)$ for $\lim_{n\to+\infty}\tau_1^n(cc\ldots)$, when it exists in $\{a,b\}^{\mathbb{N}}$, and, $\tau_1^{-\infty}(c)$ for $\lim_{n\to+\infty}\tau_1^n(\cdots cc)$, when it exists in $\{a,b\}^{\mathbb{Z}_{<0}}$.

Let us ckeck (X_{τ_1},σ) has a unique asymptotic component. We show that asymptotic points should end with $b^{+\infty}$.

Let (x, y) be an asymptotic pair. We can suppose, shifting if needed, that

$$x = x^- a.x^+ = \cdots x^-_{-4}x^-_{-3}x^-_{-2}a.x^+_0x^+_1x^+_2\cdots$$
$$y = y^- b.x^+ = \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.x^+_0x^+_1x^+_2\cdots.$$

Observe that $x^+_0 = b$ because aa does not belong to $\mathcal{L}(X_{\tau_1})$:

$$x = \cdots x^-_{-4}x^-_{-3}x^-_{-2}a.bx^+_1x^+_2x^+_3\cdots$$
$$y = \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.bx^+_1x^+_2x^+_3\cdots.$$

Suppose $x^+_1 = a$. Then, we should have $x^+_2x^+_3 = bb$ because aba is necessarily followed by bb. Thus, $bbabb$ should appear in some element of x which is not the case. Therefore $x^+_1 = b$:

$$x = \cdots x^-_{-4}x^-_{-3}x^-_{-2}a.bbx^+_2x^+_3x^+_4\cdots$$
$$y = \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.bbx^+_2x^+_3x^+_4\cdots.$$

Suppose x^+ begins with $b^{2n+1}a$ for some $n \geq 1$. Then, $abab^{2n+1}aba$ should belong to the language of X_{τ_1}. But it should appear in some $\tau_1(u)$ and then we must have $abab^{2n+1}aba = \tau_1(ava)$ for some word $v \in \mathcal{L}(X_{\tau_1})$, hence $b^{2n+1} = \tau_1(v)$, which is not possible. Thus, x^+ begins with $b^{2n}a$ for some $n \geq 1$ or it is equal to $b^{+\infty}$. Suppose we are in the first situation:

$$x = \cdots x^-_{-4}aba.b^{2n}abax^+_{2n+3}\cdots$$
$$y = \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.b^{2n}abax^+_{2n+3}\cdots.$$

It can be checked that τ_1 is one-to-one on X_{τ_1}. Consequently, there are two unique sequences

$$x^{(1)} = x^{(1-)}a.b^nabax^{(1+)} \text{ and } y^{(1)} = y^{(1-)}b.b^nabax^{(1+)} \tag{4}$$

belonging to X_{τ_1} such that

$$x = \tau_1(x^{(1-)}a).\tau(b^nabax^{(1+)}) \text{ and } y = \tau_1(y^{(1-)}b).\tau(b^nabax^{(1+)}).$$

Thus, $(x^{(1)}, y^{(1)})$ is also an asymptotic pair. From the observation made before, n should be even and we can obtain a new asymptotic pair $(x^{(2)}, y^{(2)})$ having the shape given by (4). Of course n is decreasing at each step and we can continue until $n = 1$: we get an asymptotic pair $(x^{(k)}, y^{(k)})$ such that

$$x^{(k)} = \cdots a.baba\cdots$$
$$y^{(k)} = \cdots b.baba\cdots.$$

But $ababa$ does not belong to $\mathcal{L}(X_{\tau_1})$. Consequently $x^+ = b^{+\infty}$ and (X_{τ_1}, σ) has a unique asymptotic component.

Furthermore, it can be checked, using already used arguments, that $z^-.b^{+\infty}$ is in $X_{\tau_1} \setminus \{b^{-\infty}.b^{+\infty}\}$ if and only if $z^- = \tau_1^{-\infty}(a)b^n$ for some non-negative integer n. Hence, if (x, y) is an asymptotic pair then x and y belong to

$$\{b^{-\infty}.b^{+\infty}, \sigma^i(\tau^{-\infty}(a).b^{+\infty}); i \in \mathbb{Z}\}.$$

4.1.2. *A n^2 complexity substitutive subshift.* Below we use the notation of the previous section. Consider the substitution $\tau_2 : \mathcal{A} \to \mathcal{A}^*$ defined by

$$\tau_2(a) = aab \text{ and } \tau_2(b) = b.$$

It is easy to check that the subshift (X_{τ_2}, σ) is transitive but not minimal. Moreover, from [30] its complexity is of the order n^2. Before showing it has a unique asymptotic component, let us introduce some key concepts for the treatment of this example.

Let x be a sequence of $\mathcal{B}^{\mathbb{N}}$, where $\mathcal{B}$ is an alphabet. We denote by $\mathcal{L}(x)$ the set of words having an occurrence in x. A *return word* to $u \in \mathcal{L}(x)$ is a word $w \in \mathcal{L}(x)$ such that wu belongs to $\mathcal{L}(x)$, contains exactly two occurrences of u and has u as a prefix. We denote by $\mathcal{R}_x(u)$ the set of return words to u.

In [7] is defined the notion of *sparse sequence* on the alphabet $\mathcal{B}$. It is an element x of $\mathcal{B}^{\mathbb{N}}$ satisfying:

$$\exists b \in A,\ \forall n \in \mathbb{N}, b^n \in \mathcal{L}(x) \text{ and } \#\mathcal{R}_x(b^n) = 2.$$

It is proven that $p_x(n)$ (the number of words of length n appearing in x) is less than or equal to $(n^2 + n + 2)/2$ whenever x is sparse. In Example 4.7.67 of [7] it is claimed that $x = \tau_2^{+\infty}(a)$ is sparse. Using Lemma 4.5.15 in [7] one can deduce that for all $n \geq 1$,

$$\mathcal{R}_x(b^n) = \{b, b^n u\}, \text{ where } \tau_2^n(a) = ub^n. \tag{5}$$

We show (X_{τ_2}, σ) has a unique asymptotic component. Let (x, y) be an asymptotic pair. It suffices to prove that x and y end with $b^{+\infty}$. We can suppose that $x = x^-.x_0^+ax^+$ and $y = y^-.y_0^+bx^+$. We set $x^+ = x_2^+x_3^+\cdots$.

Suppose that $x_2^+ = a$. Then the only possibility to have bax_3^+ in $\mathcal{L}(X_{\tau_2})$ is $x_3^+ = a$. Consequently, aaa would belong to $\mathcal{L}(X_{\tau_2})$, which is not the case. Therefore, $x_2^+ = b$ and necessarily $x_{-1}^-x_0^- = ba$:

$$x = \cdots x_{-1}^-b.aabx_3^+x_4^+x_4^+\cdots$$
$$y = \cdots y_{-2}^-y_{-1}^-.y_0^+bbx_3^+x_4^+x_5^+\cdots.$$

Suppose we are in the following situation:

$$x = \cdots x_{-1}^-b.aab^naax_{n+4}^+\cdots$$
$$y = \cdots y_{-2}^-y_{-1}^-.y_0^+bb^naax_{n+4}^+\cdots.$$

From (5) one gets that

$$x = \cdots x_{-1}^-b.aab^n\tau_2^{n+1}(a)\cdots$$
$$y = \cdots y_{-2}^-y_{-1}^-.y_0^+bb^n\tau_2^{n+1}(a)\cdots.$$

Then, using (5) again, x would have an occurrence of $w = \tau_2^n(a)\tau_2^n(a)\tau_2^n(a)$, but w does not belong to $\mathcal{L}(X_{\tau_2})$. Indeed, if it was the case, by a finite recurrence we prove that aaa should belong to $\mathcal{L}(X_{\tau_2})$, which is not the case. Hence, $x_3^+x_4^+x_4^+\cdots = b^{+\infty}$ and (X_{τ_2}, σ) has a unique asymptotic component.

Observe that $(\sigma^{-n}(\tau_2^n(a^{-\infty})).b^{+\infty})$ converges in X_{τ_2}. Let z denote its limit. We can check that if (x, y) is an asymptotic pair then x and y belong to

$$\{b^{-\infty}.b^{+\infty}, \sigma^i(z); i \in \mathbb{Z}\}.$$

We finish this section by proving that in both examples (X_{τ_1}, σ) and (X_{τ_2}, σ) the group of automorphisms is isomorphic to $\mathbb{Z}$.

Lemma 4.1. *Let τ denote either the substitution τ_1 or τ_2. Then, the group* $\mathrm{Aut}(X_\tau, \sigma)$ *is generated by the shift map σ.*

Observe that the main result of [10] gives only that the group $\mathrm{Aut}(X_{\tau_1}, \sigma)$ is a periodic group.

Proof. Let us first recall that for any asymptotic pair (x, y) of (X_τ, σ), x and y belong to

$$\{b^{-\infty}.b^{+\infty}, \sigma^i(z); i \in \mathbb{Z}\},$$

for some $z \in X_\tau$.

Notice that (X_τ, σ) has a unique minimal subsystem, namely $(\{b^{-\infty}.b^{+\infty}\}, \sigma)$. Moreover, it is clear that an automorphism ϕ of the subshift (X_τ, σ) maps any minimal subsystem onto a minimal subsystem, so ϕ fixes the sequence $b^{-\infty}.b^{+\infty}$. The morphism ϕ mapping asymptotic pairs onto asymptotic pairs, $\sigma^i(z)$ should be mapped to some $\sigma^j(z)$. The orbit $\{\sigma^k(z); k \in \mathbb{Z}\}$ being dense in X_τ one deduces that $\phi \circ \sigma^i = \sigma^j$. Thus, ϕ is a power of the shift map. □

4.2. **Coding a nil-translation.** We introduce a class of examples of symbolic systems with polynomial complexity of arbitrarily high degree and with a group of automorphisms which is virtually $\mathbb{Z}$. We build these systems as symbolic extensions of minimal nilsystems.

We start by stating some general results we need and then review some generalities about the coded systems.

Let (X, T) be a minimal topological dynamical system and let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite collection of subsets of X. We say that $\mathcal{U}$ covers X if $\bigcup_{i=1}^m U_i = X$. For two covers $\mathcal{U} = \{U_1, \dots, U_m\}$ and $\mathcal{V} = \{V_1, \dots, V_p\}$ of X we let $U \vee V$ denote the cover given by $\{U_i \cap V_j; i = 1, \dots m,\ j = 1, \dots p\}$.

Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite cover of X and let $\mathcal{A}$ denote the set $\{1, \dots, m\}$. We say that $\omega = (w_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ is a $\mathcal{U}$-name of x if $x \in \bigcap_{i \in \mathbb{Z}} T^{-i} U_{w_i}$. Let $X_{\mathcal{U}}$ denote the set

$$\{\omega \in A^{\mathbb{Z}}; \bigcap_{i \in \mathbb{Z}} T^{-i} U_{w_i} \neq \emptyset\} \subseteq \mathcal{A}^{\mathbb{Z}}.$$

It is easy to check that $X_{\mathcal{U}}$ is closed when each U_i is closed and if we let $\overline{\mathcal{U}}$ denote the collection $\{\overline{U}_1, \dots, \overline{U}_m\}$ we have that $\overline{X_{\mathcal{U}}} \subset X_{\overline{\mathcal{U}}}$. For $N \in \mathbb{N}$, let

$$\mathcal{U}_N = \bigvee_{i=-N}^{N} T^{-i} \mathcal{U}.$$

We say that the cover $\mathcal{U}$ separates points if every $\omega \in X_{\overline{\mathcal{U}}}$ is a name of exactly one $x \in X$. If $\mathcal{U}$ separates points in X, we can build a factor map π between $(\overline{X_{\mathcal{U}}}, \sigma)$ and (X, T) where $\pi(\omega)$ is defined as the unique point in $\bigcap_{i \in \mathbb{Z}} T^{-i} \overline{U_{w_i}}$.

Lemma 4.2. *Let (X, T) be a minimal topological dynamical system and let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a partition which covers and separates points in X. Let suppose that for every $N \in \mathbb{N}$ every atom of $\mathcal{U}_N$ has non-empty interior, then $(\overline{X_{\mathcal{U}}}, \sigma)$ is a minimal system.*

Proof. Let $\omega, \omega' \in \overline{X_{\mathcal{U}}}$ and let $N \in \mathbb{N}$. We denote $x = \pi(\omega)$ and $x' = \pi(\omega')$. By definition we have that $\bigcap_{-N}^{N} T^{-i} U_{w_i} \neq \emptyset$ and therefore it has non-empty interior. Since (X, T) is minimal, there exists $n \in \mathbb{Z}$ such that $T^n x' \in \operatorname{int}(\bigcap_{-N}^{N} T^{-i} U_{w_i})$. This implies that $w'_{[n-N,n+N]} = w_{[-N,N]}$ and the proof is finished. □

Now we compute the automorphism groups of symbolic extensions of some nilsystems. First we recall the construction of the systems studied in [1]. Let us consider the infinite matrix $A = (a_{i,j})_{i,j\in\mathbb{N}}$ where $a_{i,j} = \binom{j}{i}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 3 & 4 & \cdots \\ & & 1 & 3 & 6 & \cdots \\ & & & 1 & 4 & \cdots \\ & & & & 1 & \cdots \\ & & & & \cdots & \cdots \end{pmatrix}.$$

It is proven in Section 4 of [1] that for all $i \in \mathbb{N}$, A^i is well defined and

$$A^i = \begin{pmatrix} 1 & i & i^2 & i^3 & i^4 & \cdots \\ & 1 & 2i & 3i^2 & 4i^3 & \cdots \\ & & 1 & 3i & 6i^2 & \cdots \\ & & & 1 & 4i & \cdots \\ & & & & 1 & \cdots \\ & & & & \cdots & \cdots \end{pmatrix}.$$

Let $\alpha \in [0, 1]$ be an irrational number. For any $d \in \mathbb{N}$, consider A_{d+1} the restriction of A to $(d + 1) \times (d + 1)$ coordinates. We let $T_d \colon \mathbb{T}^d \to \mathbb{T}^d$ denote the application that maps $(x_0, \dots x_d)$ to the d first coordinates of $A_{d+1}(x_0, \dots, x_d, \alpha)^t$. For example, T_2 is the application $(x_0, x_1) \mapsto (x_0 + x_1 + \alpha, x_1 + 2\alpha)$ and T_3 is the application $(x_0, x_1, x_2) \mapsto (x_0 + x_1 + x_2 + \alpha, x_1 + 2x_2 + 3\alpha, x_2 + 3\alpha)$.

We can represent the transformation T_d as $T_d(x) = A_d x + \vec{\alpha}$ where $\vec{\alpha}$ is the restriction to the first d-coordinates of the last column of A_{d+1} multiplied by α. This is the classical presentation of an affine nilsystem.

Fix $d \in \mathbb{N}$ and for $i, n \in \mathbb{Z}$ let $H_{i,n}$ be the plane given by the equation $\sum_{k=0}^{d-1} i^k x_k + i^d \alpha = n$. It can be proven that $H_{i,n} = T_d^{-i} H_{0,n}$ and for a fixed value of i, the planes $H_{i,n}$ are projected in $\mathbb{T}^d$ to the same plane $\widehat{H}_i$. We remark that

$$\widehat{H}_0 = \{(0, x_1, \dots, x_{d-1}); (x_1, \dots, x_{d-1}) \in \mathbb{T}^{d-1}\}.$$

We refer to Section 4 of [1] for further details.

We consider the partition $\mathcal{U}$ given by the cells generated by the planes $\widehat{H}_0, \dots, \widehat{H}_{d-1}$. The partition $\bigvee_{i=0}^{n+d-1} T_d^{-i} \mathcal{U}$ coincides with the cells generated by the planes $\widehat{H}_0, \dots, \widehat{H}_{n+d-1}$ (see Section 6 of [1]). Let $(x_0, \dots, x_{d-1})$ and $(y_0, \dots, y_{d-1})$ be different points in $\mathbb{T}^d$ and let $\overline{k} = \max\{k; x_k \neq y_k\}$. Then the difference (in $\mathbb{R}$) between $\sum_{k=0}^{d-1} i^k x_k + i^d \alpha$ and $\sum_{k=0}^{d-1} i^k y_k + i^d \alpha$ grows to infinity as i goes to infinity since this difference behaves like $i^{\overline{k}}(x_{\overline{k}} - y_{\overline{k}})$. This implies that for big enough N, $(x_0, \dots, x_{d-1})$ and $(y_0, \dots, y_{d-1})$ lie on different cells of $\bigvee_{i=-N}^{N} T_d^{-i} \mathcal{U}$ since for big enough i these points are separated by the cells generated by $\widehat{H}_i$.

We recall that $(\overline{X_{\mathcal{U}}}, \sigma)$ is the subshift associated to $\mathcal{U}$. By Lemma 4.2, one can see that $(\overline{X_{\mathcal{U}}}, \sigma)$ is a minimal system and it is an extension of $(\mathbb{T}^d, T_d)$ since $\mathcal{U}$ separates points. Moreover, the complexity function of $(\overline{X_{\mathcal{U}}}, \sigma)$ is given by

$$p(n) = \frac{1}{V(0,1,\dots,d-1)} \sum_{0\le k_1<k_2<\cdots<k_d\le n+d-1} V(k_1,k_2,\cdots,k_d)$$

where $V(k_1,k_2,\cdots,k_d) = \prod_{1\le i<j\le d}(k_j-k_i)$ is a Vandermonde determinant. We remark that varying $d\in\mathbb{N}$ we get an arbitrarily large complexity with a polynomial growth.

By construction and Corollary 2.8 we also get:

Lemma 4.3. *The maximal d-step nilfactor of* $(\overline{X_{\mathcal{U}}},\sigma)$ *is the nilsystem* $(\mathbb{T}^d,T_d)$.

Proposition 4.4. *The group* $\mathrm{Aut}(\overline{X_{\mathcal{U}}},\sigma)$ *is virtually* $\mathbb{Z}$.

Proof. Let ϕ be an automorphism of $(\overline{X_{\mathcal{U}}},\sigma)$ and let $\pi\colon \overline{X_{\mathcal{U}}}\to\mathbb{T}^d$ be the natural factor map. Let $W=\{\omega\in X; \#\pi^{-1}\{\pi(\omega)\}\ge 2\}$ be the set of points where π is not one-to-one. Since ϕ preserves the regionally proximal pairs of order d, we have that W is invariant under ϕ. We remark that the projection of W under π are the points which fall in $F:=\widehat{H}_0\cup\widehat{H}_1\cup\cdots\cup\widehat{H}_{d-1}$ under some power of T, which is nothing but $\bigcup_{j\in\mathbb{Z}}T^jF=\bigcup_{j\in\mathbb{Z}}T^j\widehat{H}_0$. We have that the projection $\widehat{\pi}(\phi)$ is an automorphism that commutes with the affine ergodic transformation T which has eigenvalues equal to 1. By Theorem 2 and Corollary 1 in [36] we have that $\widehat{\pi}(\phi)$ has the form $(x_0,\dots,x_{d-1})^t\mapsto B(x_0,\dots,x_{d-1})^t+\beta$ where B is an integer matrix and $\beta\in\mathbb{T}^d$. Since W is invariant under ϕ we get that the projection $\widehat{\pi}(\phi)$ leaves invariant $\bigcup_{j\in\mathbb{Z}}T^j\widehat{H}_0$. Particularly, since $\widehat{H}_0$ is the restriction of a plane to $\mathbb{T}^d$, so is its image under $\widehat{\pi}(\phi)$ and therefore there exists $j\in\mathbb{Z}$ such that $\widehat{\pi}(\phi)\widehat{H}_0=T^j\widehat{H}_0$. Hence, the automorphism $T^{-j}\widehat{\pi}(\phi)$ leaves invariant $\widehat{H}_0$. So we are left to study the automorphisms of $(\mathbb{T},T_d)$ which leave invariant $\widehat{H}_0$. Let φ be such an automorphism. By [36] we can assume that φ has the form

$$\varphi\begin{pmatrix} x_0\\ x_1\\ \vdots\\ x_{d-1}\end{pmatrix} = B\begin{pmatrix} x_0\\ x_1\\ \vdots\\ x_{d-1}\end{pmatrix}+\begin{pmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_{d-1}\end{pmatrix}$$

where the matrix $B=(b_{j,k})_{j,k=1\dots,d}$ has integer entries and $\vec{\beta}=(\beta_0,\dots,\beta_{d-1})^t\in\mathbb{T}^d$. Since φ commutes with T we have that B commutes with A_d (as real matrices) and $(B-Id)\vec{\alpha}=(A_d-Id)\vec{\beta}$ in $\mathbb{T}^d$.

Since $\varphi(0,x_1,\dots,x_{d-1})\in\widehat{H}_0$, for any $(x_1,\dots,x_{d-1})\in\mathbb{T}^{d-1}$ we deduce that $b_{1,2}=\cdots=b_{1,d}=0=\beta_0$. Since $A_d^iB=BA_d^i$ for any $i\in\mathbb{N}$, by looking at the first row of these matrices we deduce that for any $j=1,\dots,d$ and any $i\in\mathbb{N}$

$$\sum_{k=1,k\neq j}^{d}(b_{j,k})i^{k-1}+(b_{j,j}-b_{1,1})i^{j-1}=0.$$

Since the vectors $(1,i,i^2,\dots,i^{d-1})$ are linearly independent for different values of i we deduce that $B=b_{1,1}I_d$. Therefore, $(A_d-Id)\beta=(B-Id)\vec{\alpha}=(b_{1,1}-1)\vec{\alpha}$. Since A_d is upper triangular with ones in the diagonal, this condition implies that $(b_{1,1}-1)\alpha\in\mathbb{Q}$ and thus $b_{1,1}=1$. We conclude that B is the identity matrix and then φ is the rotation by $\vec{\beta}:=(0,\beta_1,\dots,\beta_{d-1})^t$ and $(A_d-Id)\vec{\beta}\in\mathbb{Z}^d$. We can write this system as

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ & 0 & 2 & 3 & 4 & \cdots \\ & & 0 & 3 & 6 & \cdots \\ & & & 0 & 4 & \cdots \\ & & & & 0 & d \\ & & & & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \beta_1 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{pmatrix} \in \mathbb{Z}.$$

This implies that $d\beta_{d-1} \in \mathbb{Z}$ and this is possible for finitely many $\beta_{d-1} \in \mathbb{T}$. Inductively, we deduce that there are finitely many (and rational) solutions $\vec{\beta} = (0, \beta_1, \ldots, \beta_{d-1})^t$ in $\mathbb{T}^d$. This means that the group of automorphisms that leaves invariant $\widehat{H}_0$ is a finite group of rational rotations. Therefore, $\widehat{\pi}(\mathrm{Aut}(\overline{X_{\mathcal{U}}}, \sigma))$ is spanned by T_d and a finite group. Since the map $\widehat{\pi}\colon \mathrm{Aut}(\overline{X_{\mathcal{U}}}, \sigma) \to \mathrm{Aut}(\mathbb{T}, T_d)$ is an injection we have that $\mathrm{Aut}(\overline{X_{\mathcal{U}}}, \sigma)$ is spanned by σ and a finite group. The result follows. □

4.3. **Example of a larger automorphism group.** We remark that the statement of Theorem 1.1 is no longer valid for an arbitrary polynomial complexity, as the following shows.

Proposition 4.5. *For any $d \in \mathbb{N}$, there exists a minimal subshift (X, σ) with complexity satisfying $\lim_{n\to+\infty} p_X(n)/n^{d+1} = 0$ and where $\mathrm{Aut}(X, \sigma)$ is isomorphic to $\mathbb{Z}^d$.*

Proof. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{R} \setminus \mathbb{Q}$ be rationally independent numbers. For every $i = 1, \ldots, d$, let (X_i, σ_i) be the Sturmian extension of the rotation R_{α_i} by the angle α_i on the circle $\mathbb{S}^1$, and let $X = X_1 \times X_2 \cdots \times X_d$ and $\sigma = \sigma_1 \times \sigma_2 \cdots \times \sigma_d$. We remark that for any $i = 1, \ldots, d$, on (X_i, σ_i) the proximal relation and the regionally proximal relation coincide and thus the proximal relation is an equivalence relation. Since the maximal equicontinuous factor of (X_i, σ_i) is $(\mathbb{S}^1, R_{\alpha_i})$ via the factor map π_i, by [2], Chapter 11, theorems 7 and 9, we have that (X, σ) is a minimal system and the product system $((\mathbb{S}^1)^d, R_{\alpha_1} \times \cdots \times R_{\alpha_d})$ is its maximal equicontinuous factor.

The complexity function of any (X_i, σ_i) is $n + 1$, so we get that the complexity function of (X, σ) is $(n+1)^d$. On the other hand, we observe that $\phi_1 \times \cdots \times \phi_d$ belongs to $\mathrm{Aut}(X, \sigma)$ for any choice of $\phi_i \in \mathrm{Aut}(X_i, \sigma_i)$. Since for every i, $\mathrm{Aut}(X_i, \sigma_i)$ is $\mathbb{Z}$, we conclude that $\mathbb{Z}^d$ can be embedded as a subgroup of $\mathrm{Aut}(X, \sigma)$.

We claim this embedding is actually an isomorphism. To prove this, recall that the Sturmian subshift X_i is an almost one-to-one extension of a rotation on the circle via an onto map $\pi_i\colon X_i \to \mathbb{S}^1$ that it is injective except on the orbit of the unit $\mathrm{Orb}_{R_{\alpha_i}}(1)$, where any point has two pre-images (*e.g.*, see [14]). By Lemma 2.12, for any automorphism $\phi \in \mathrm{Aut}(X, \sigma)$, the automorphism $\hat{\pi}(\phi)$ preserves the set of points in $(\mathbb{S}^1)^d$ having a maximum number (namely 2^d) of pre-images for the factor map $\pi = \pi_1 \times \cdots \times \pi_d$. This set is the product set $\mathrm{Orb}_{R_{\alpha_1}}(1) \times \cdots \times \mathrm{Orb}_{R_{\alpha_d}}(1)$. Clearly, the group of automorphisms of the form $\hat{\pi}(\sigma_1^{n_1} \times \cdots \times \sigma_d^{n_d})$, $n_1, \ldots, n_d \in \mathbb{Z}$, acts transitively on this set. Since the group $\mathrm{Aut}((\mathbb{S}^1)^d, R_{\alpha_1} \times \cdots \times R_{\alpha_d})$ acts freely and the morphism $\hat{\pi}$ is injective (Lemma 2.6), we get that any automorphism $\phi \in \mathrm{Aut}(X, \sigma)$ may be written as a product of automorphisms in $\mathrm{Aut}(X_i, \sigma_i)$. □

4.4. **Subshift with subexponential complexity.** In this section we give an example of a minimal subshift (X, σ) generated by a uniformly recurrent sequence $x \in \{0,1\}^{\mathbb{Z}}$ such that:

- There exists C such that for infinitely many n's one has $p_X(n) \leq Cn$.
- For any subexponential function ϕ there are infinitely many n's such that $p_X(n) \geq \phi(n)$, where subexponential means that $\lim_{n\to+\infty} \phi(n)/\alpha^n = 0$ for all $\alpha \in \mathbb{R}$.

As for subshifts $p_z(n)$ will stand for the number of words of length n occurring in the sequence $z \in \{0,1\}^{\mathbb{Z}}$ or $z \in \{0,1\}^{\mathbb{N}}$.

The proofs of the two following lemmas are left to the reader.

Lemma 4.6. *Let ξ be a substitution on $\{0,1\}^*$ of constant length L and τ be an endomorphism of $\{0,1\}^*$ having all words of length 2 in its images. Let $x \in \{0,1\}^{\mathbb{N}}$. Then, for any $y \in \{0,1\}^{\mathbb{N}}$ having occurrences of all words of length 2 and $0 \leq l \leq L$ we have*

$$p_{\xi\circ\tau(x)}(l) = p_{\xi(y)}(l).$$

Below ρ stands for the Morse substitution: $\rho(0) = 01$ and $\rho(1) = 10$.

Lemma 4.7. *Let ξ be a substitution on $\{0,1\}^*$ of constant length L. Let $x \in \{0,1\}^{\mathbb{N}}$. We have*

$$p_{\xi\circ\rho^3(x)}(2L) \leq 6L.$$

Below, when a substitution τ is of constant length L we set $|\tau| = L$. Let us construct inductively the sequence x. In fact, we will construct two increasing sequences of integers $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$, and a sequence of morphisms $(\tau_i)_{i\geq 1}$ such that

(1) $x = \lim_{i\to\infty} \rho^3\tau_1 \ldots \rho^3\tau_i(0^\infty)$, where $0^\infty = 00\cdots$,
(2) $a_1 < b_1 < a_2 < b_2 < \ldots$,
(3) $p_x(a_i) \leq 3a_i$, $i \in \mathbb{N}$ and
(4) $p_x(b_i) \geq \phi(b_i)$, $i \in \mathbb{N}$.

We start fixing $a_1 = 2$. Let $x^{(1)} = \rho^3(0^\infty)$. Then, $p_{x^{(1)}}(a_1) = 4$, which is less than $3a_1$.

Let k_1 be such that $2^{k_1} \geq \phi(k_1|\rho^3|)$ (observe it is always possible because ϕ has a subexponential growth) and τ_1 be a substitution of $\{0,1\}^*$ of length $L_1 = 2^{m_1}$ such that $\tau_1(0)$ starts with 0 and the number of words of length k_1 in $\tau_1(0)$ and $\tau_1(1)$ is 2^{k_1}. We set

$$b_1 = k_1|\rho^3| \text{ and } y^{(1)} = \rho^3\tau_1(0^\infty).$$

One gets

$$p_{y^{(1)}}(b_1) \geq \phi(b_1).$$

Moreover, notice that from Lemma 4.6 one has that

$$p_{y^{(1)}}(l) = p_{x^{(1)}}(l)$$

for all $l \leq |\rho^3|$. Now consider $x^{(2)} = \rho^3\tau_1\rho^3(0^\infty)$. Then from Lemma 4.7

$$p(2|\rho^3\tau_1|) \leq 6|\rho^3\tau_1|.$$

Setting $a_2 = 2|\rho^3\tau_1|$, one gets $p_{x^{(2)}}(a_2) \leq 3a_2$.

Let $k_2 \geq k_1$ be such that $2^{k_2} \geq \phi(k_2|\rho^3\tau_1\rho^3|)$ and τ_2 be a substitution of $\{0,1\}^*$ of length $L_2 = 2^{m_2}$ such that the number of words of length k_2 in $\tau_2(0)$ and $\tau_2(1)$ is 2^{k_2}. We set

$$b_2 = k_2|\rho^3\tau_1\rho^3| \text{ and } y^{(2)} = \rho^3\tau_1\rho^3\tau_2(0^\infty).$$

One gets that $p_{y^{(2)}}(b_2)$ is greater than $\phi(b_2)$. Moreover, notice that from Lemma 4.6 one has that

$$\begin{aligned} p_{y^{(2)}}(l) &= p_{x^{(2)}}(l) \quad \forall l \leq |\rho^3\tau_1\rho^3|, \\ p_{x^{(2)}}(l) &= p_{y^{(1)}}(l) \quad \forall l \leq |\rho^3\tau_1|, \\ p_{y^{(1)}}(l) &= p_{x^{(1)}}(l) \quad \forall l \leq |\rho^3|. \end{aligned}$$

Thus, $p_{y^{(2)}}(a_1) \leq 3a_1$, $p_{y^{(2)}}(b_1) \geq \phi(b_1)$ and $p_{y^{(2)}}(a_2) \leq 3a_2$.

Now suppose we have constructed:

(1) morphisms τ_i of constant length such that $\tau_i(0)$ starts with 0, $1 \leq i \leq n$,
(2) $x^{(i)} = \rho^3\tau_1 \dots \tau_{i-1}\rho^3(0^\infty)$,
(3) $y^{(i)} = \rho^3\tau_1 \dots \tau_{i-1}\rho^3\tau_i(0^\infty)$, $1 \leq i \leq n$,
(4) $a_1 < b_1 < a_2 < \dots < a_n < b_n$, such that
  (a) $a_i = 2|\rho^3\tau_1 \dots \rho^3\tau_{i-1}|$,
  (b) $b_i \geq |\rho^3\tau_1 \dots \rho^3\tau_{i-1}\rho^3|$,
  (c) $x^{(i)}[0, |\rho^3\tau_1 \dots \tau_{i-1}\rho^3|]$ is a prefix of $y^{(i)}$,
  (d) $y^{(i)}[0, |\rho^3\tau_1 \dots \tau_{i-1}\rho^3\tau_i|]$ is a prefix of $x^{(i+1)}$,
  (e) $p_{y^{(i)}}(l) = p_{x^{(i)}}(l)$ for all $l \leq |\rho^3\tau_1 \dots \tau_{i-1}\rho^3|$,
  (f) $p_{y^{(n)}}(a_i) \leq 3a_i$, $1 \leq i \leq n$, and,
  (g) $p_{y^{(n)}}(b_i) \geq \phi(b_i)$, $1 \leq i \leq n-1$.

We have seen this construction is realizable for $n = 2$. Proceeding as we did for the first cases, it is not difficult to see that it can be achieved for every $n \geq 1$.

To conclude, it suffices to observe that $(y^{(n)})_{n\geq 1}$ converges to the sequence x we are looking for. Indeed, the convergence follows from (4c) and (4d). Also observe that $y^{(n)}$ is a prefix of x. It is a classical exercise to show that x is uniformly recurrent. From (4g) we get that $p_x(b_i) \geq \phi(b_i)$ for all $i \in \mathbb{N}$. For the last point, $p_x(a_i) \leq 3a_i$ for all $i \in \mathbb{N}$, it comes from (4f) because it is true for all $n \geq 1$.

5. Comments and open questions

A standard question related to automorphisms is to determine if the transformation T has a root. That is, does it exist a transformation U such that $U^p = T$ for some integer $p \geq 0$. A classical way to deal with this problem is to notice that a root is an automorphism.

The automorphism group is also related to the collection of conjugacy maps between two systems. If π_1 and π_2 are two conjugacy maps between the same systems, then $\pi_1 \circ \pi_2^{-1}$ is an automorphism. Hence, a characterization of when the automorphism group is trivial, *i.e.* $\mathrm{Aut}(X, T)$ is generated by T, implies rigidity results in both problems.

5.1. **Automorphisms and nilfactors.** We have shown that a large family of minimal subshifts, either with sublinear or other type of polynomial complexity, have automorphisms groups that are virtually $\mathbb{Z}$. Even in the case of minimal subshifts obtained as extensions of minimal systems whose automorphism group is much complex (the case of extensions of nilsystems). So a natural question is whether this behaviour is generally true just because the fibres over particular topological factors are constrained.

5.2. **Eigenvalues, roots of T and automorphisms.** We obtain, in the good cases, that the group of automorphisms is a subgroup of the corresponding one of a maximal nilfactor. This proves that there are connections between automorphisms and continuous eigenvalues. To study these relations we can focus on rational eigenvalues. So it is natural to ask: does a Cantor minimal system (X, T) admit a non trivial automorphism with finite order or have some roots, are there constraints on the rational continuous eigenvalues of (X, T) ?

Classical examples of Toeplitz sequences with a unique asymptotic component (so $\mathrm{Aut}(X, \sigma)$ is generated by σ) show that the converse is false: a system may have rational eigenvalues and no automorphisms of finite order.

5.3. **Complexity versus group of automorphisms.** The results of [10] and of this paper show a relation between complexity and the growth rate of the groups. Is it possible to be more precise ? For instance, is it true that for a transitive subshift with a subquadratic complexity the group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is finite ? The result in [10] states it is periodic, but we do not know any subshift with this complexity and with an infinite group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ (see Section 4.1.2).
Inspired by the main theorem of this paper and examples in Section 4.3 we ask,

Question. Let (X, σ) be a minimal subshift such that

$$d = \inf\{\delta \in \mathbb{N}; 0 < \liminf p_X(n)/n^\delta < +\infty\} > 0.$$

Is it true that it is virtually $\mathbb{Z}^d$? Can we extend these results to the category of transitive subshifts ?

5.4. **Measurable versus continuous automorphisms.** The main result in [21] shows a rigidity result, any measurable automorphism is almost everywhere continuous for bijective constant length substitutions. Is it possible to enlarge this class of subshifts with the same rigidity property ? A first answer is negative: This is not true for substitution of non-constant length and even for Pisot substitution on the alphabet $\{0, 1\}$. Consider the two substitutions τ and ξ defined by $\tau(0) = 010$, $\tau(1) = 01$, $\xi(0) = 001$ and $\xi(1) = 10$. Let (X_τ, σ) and (X_ξ, σ) be the subshift they generate. It can be shown that they are both measure theoretically isomorphic to $(\mathbb{S}^1, R_\alpha)$ (see [4]), where R_α is the rotation of angle $\alpha = (1 + \sqrt{5})/2$, and, thus (X_τ, σ) and (X_ξ, σ) are measure theoretically isomorphic. But they cannot be topologically isomorphic because their dimension groups are not isomorphic (see [15] for their computations).

5.5. **Realization of automorphism groups.** By the Curtis-Hedlund-Lyndon theorem, the collection of automorphisms of a subshift is countable. We leave open the realization of any countable group as an automorphism group. More precisely,

Question. Given a countable group G (not necessarily finitely generated). Does it exist a minimal subshift (X, σ) such that $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is isomorphic to G ?

Notice that Toeplitz sequences can also be realized on residually finite groups [8]. A priori, they may provide interesting solutions in this class. But, as stated in the remark below Corollary 2.13, their automorphism group is Abelian. This kills any non commutative group realization by this way.

A second question that was left open in the text is when the automorphism group of polynomial complexity subshifts is finitely generated. Again, the intuition

is based on the fact that asymptotic classes in such systems are few, and prevent the growth of the automorphism group.

If we restrict to some families of subshifts (*e.g.* Sturmian or Toeplitz sub shifts), we prove that their automorphism groups are subgroups of their maximal equicontinuous factors. Can we characterize these groups for the Sturmian and Toeplitz cases ?

5.6. **Extension to higher dimensional subshifts.** A natural question is to study the generalization to $\mathbb{Z}^d$-subshifts with $d \geq 1$, or to the geometrical notion of tilings and Delone sets of $\mathbb{R}^d$. More intrinsically, the method used in [10] illustrates how properties on complexity of multidimensional subshifts give information on one dimensional subshifts. Unfortunately, this uses results related to the famous Nivat's conjecture, that are known to be true only in dimension two. Nevertheless, we can hope a similar strategy may provide interesting information on multidimensional subshifts. We believe that the study of asymptotic components or the similar notion of non-expansive directions in higher dimensions may also provide useful results. For instance in [12], such a way leads to prove that the automorphism group of the minimal component of the Robinson subshift of finite type is trivial.

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