
Test of isotropy for rough textures of trended images.

Frédéric J.P. Richard

Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373.

Supplementary Material

This supplementary material contains proofs for Theorem 3.4 and some propositions stated in the main paper (Sections S1 and S2). It also includes a description of a method used to estimate the covariance of quadratic variations (Section S3).

S1 Convergence study

S1.1 A multivariate Breuer Major Theorem

The original Breuer-Major theorem was shown for stationary processes in Breuer and Major (1983), and extended to multivariate fields by Arcones (1994). Another formulation of the Breuer-Major Theorem is demonstrated by Biermé et al. (2011, Theorem 3.2) using the Malliavin calculus. We state a specific version of this theorem which is sufficient for the proof of Theorem 3.4.

Theorem 1 (Breuer-Major theorem). *Let $d, l \in \mathbb{N}^*$, and $X_N = (X_N[k])_{k \in \mathbb{Z}^d}$ be centered Gaussian stationary fields with values in \mathbb{R}^l . Assume that there exist functions $g_{a,b}^N$ in $L^2([0, 2\pi]^d)$ (spectral densities) such that, for all $a, b \in \llbracket 1, l \rrbracket$ and $k \in \mathbb{Z}^d$,*

$$\text{Cov}(X_a^N[k], X_b^N[0]) = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} e^{i\langle w, k \rangle} g_{a,b}^N(w) dw.$$

Further assume that, for all $a, b \in \llbracket 1, l \rrbracket$, $g_{a,b}^N$ converges in $L^2([0, 2\pi]^d)$ to a function $g_{a,b}$ as N tends to $+\infty$. Define

$$\forall k \in \mathbb{Z}^d, r_{a,b}[k] = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} e^{i\langle w, k \rangle} g_{a,a}(w) dw,$$

and assume that $r_{a,b}[0] = 1$. Then,

$$\frac{1}{N^{d/2}} \sum_{k \in \llbracket 1, N \rrbracket^d} ((X^N[k])^2 - \mathbf{1}) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \Sigma),$$

where $\mathbf{1}$ is the unit vector of size l and Σ is a $l \times l$ -matrix having terms

$$\Gamma_{a,b} = 2 \sum_{k \in \mathbb{Z}^d} (r_{a,b}[k])^2 = \frac{2}{(2\pi)^d} \int_{[0, 2\pi]^d} |g_{a,b}(w)|^2 dw.$$

S1.2 Proof of Theorem 3.4

In a first part, we prove the asymptotic normality of the random vector

$$U^N = \left(\frac{W_a^N}{\mathbb{E}(W_a^N)} \right)_{a \in \mathcal{F}} \quad (\text{S1.1})$$

defined with quadratic variations W_a^N of Equation (23) (main paper). Then, we deduce the asymptotic normality (26) (main paper) of $Y^N = (Y_a^N)_{a \in \mathcal{F}}$. In a second part, we further specify terms of this convergence.

Part 1. For establishing the asymptotic normality of U^N , we use a multivariate version of the Breuer-Major theorem recalled above. For that, let us first notice that

$$N^{\frac{d}{2}}(U_a^N - 1) = \frac{N^{\frac{d}{2}}}{N_e} \sum_{m \in \mathcal{E}_N} \left((X_a^N[m])^2 - 1 \right) \underset{N \rightarrow +\infty}{\sim} \frac{1}{N^{\frac{d}{2}}} \sum_{k \in [1, N]^d} \left((X_a^N[k])^2 - 1 \right),$$

with $X_a^N[m] = V_a^N[m]/\sqrt{\mathbb{E}((V_a^N[m])^2)}$. So, if the Breuer-Major theorem could be applied to the vector-valued random field $X^N = ((X_a^N[m])_{a \in \mathcal{F}}, m \in \mathbb{Z}^d)$, it would follow that

$$N^{\frac{d}{2}}(U^N - \mathbf{1}) \underset{N \rightarrow +\infty}{\xrightarrow{d}} \mathcal{N}(0, \Sigma), \quad (\text{S1.2})$$

where $\mathbf{1}$ is the unit vector of the same size as U^N , and Σ is a covariance matrix. But, using Proposition 3.3 (main paper), the spectral density of X^N can be specified as

$$g_{a,b}^N(w) = \frac{f_{a,b}^N(w)}{\sqrt{\mathbb{E}((V_a^N[m])^2)}\sqrt{\mathbb{E}((V_b^N[m])^2)}},$$

where $\mathbb{E}((V_a^N[m])^2) = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} f_{a,a}^N(w) dw$. Thus, it suffices to show the convergence in $L^2([0, 2\pi]^d)$ of $g_{a,b}^N$. This convergence results from the next lemma whose proof is postponed at the end of the section.

Lemma 1. *Take the same conditions as in Theorem 3.4 (main paper). Consider the multivariate spectral density $f_{a,b}^N$ of V^N given by Equation (22) of Proposition 3.3 (main paper). Then, for any $a, b \in \mathcal{F}$, as N tends to $+\infty$, $N^{2H} f_{a,b}^N$ converges in $L^2([0, 2\pi]^d)$ to the function $f_{a,b}$ defined by Equation (28) (main paper).*

Due to Lemma 1, $N^{2H} f_{a,b}^N$ tends to $f_{a,b}$ in $L^2([0, 2\pi]^d)$ and, *a fortiori* in $L^1([0, 2\pi]^d)$. Hence, for $a \in \mathcal{F}$, $N^{2H} \mathbb{E}((V_a^N[m])^2)$ converges to

$$C_a = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} f_{a,a}(w) dw. \quad (\text{S1.3})$$

Therefore, $g_{a,b}^N$ tends in $L^2([0, 2\pi]^d)$ to $g_{a,b} = f_{a,b}/\sqrt{C_a C_b}$.

Consequently, the Breuer-Major theorem yields the asymptotic normality (26) (main paper) for a covariance matrix Σ whose terms are defined by Equation (27) (main paper).

S1. CONVERGENCE STUDY

Now, let G be the differentiable function mapping $(\mathbb{R}_*^+)^{n_f}$ into \mathbb{R}^{n_f} defined by $G(y)_a = \log(y_a)$ for $a \in \mathcal{F}$. Since $N^{d/2}(U^N - \mathbf{1}) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \Sigma)$, we have

$$N^{d/2}(G(U^N) - G(\mathbf{1})) \xrightarrow[N \rightarrow +\infty]{d} \nabla G(\mathbf{1})' \mathcal{N}(0, \Sigma).$$

using the multivariate Δ -method. But, for $a \in \mathcal{F}$,

$$G(U^N)_a - G(\mathbf{1})_a = Y_a^N + \log(N^{2H}) - \log(C_a) + R_a^N,$$

where $R_a^N = \log(C_a) - \log(N^{2H} \mathbb{E}(W_a^N))$ and C_a is defined by Equation (S1.3). Moreover, due to Lemma 1, $\lim_{N \rightarrow +\infty} R_a^N = 0$. Hence, the asymptotic normality (26) (main paper) follows for $\zeta^N = (\log(C_a) - \log(N^{2H}))_{a \in \mathcal{F}}$ and Σ defined by Equation (27) (main paper).

Part 2. Let us notice that

$$C_a = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} \int_{\mathbb{R}^d} |\hat{v}(T'_a(w+z))|^2 \delta\left(\frac{w+z}{|w+z|}\right) |w+z|^{-2H-d} d\Delta(z) dw,$$

where $d\Delta(z) = \sum_{k \in \mathbb{Z}^d} \delta_{2k\pi}(z)$ is a counting measure on \mathbb{R}^d . But, $C_a < +\infty$. Hence, by application of the Lebesgue-Fubini theorem, we obtain

$$C_a = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \int_{[0, 2\pi]^d} |\hat{v}(T'_a(w+2k\pi))|^2 \delta\left(\frac{w+2k\pi}{|w+2k\pi|}\right) |w+2k\pi|^{-2H-d} dw.$$

After a variable change $\zeta = w + 2k\pi$ in each integral, we further get

$$C_a = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{v}(T'_a \zeta)|^2 \delta\left(\frac{\zeta}{|\zeta|}\right) |\zeta|^{-2H-d} d\zeta.$$

Next, using the variable change $w = |u_a| \zeta$, we have $C_a = |u_a|^{2H} C_H(\arg(u_a), v)$, where $\arg(u_a)$ is the angle of the rotation $\frac{T'_a}{|u_a|}$ and $C_H(\arg(u_a), v)$ is defined by Equation (31) (main paper). Then, the expression of ζ given by Equations (29) and (30) of the main paper follows.

Furthermore, let us notice that, when the texture of Z is isotropic, the function $\delta \equiv \tau_0 \in \mathbb{R}_*^+$. Hence, in this case, we obtain

$$C_H(\arg(u_a), v) = \frac{\tau_0}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{v}(w)|^2 |w|^{-2H-d} d\zeta.$$

by applying the variable change $w = \frac{T'_a}{|u_a|} \zeta$. Therefore, in this case, $C_H(\arg(u_a), v)$ only depends on v .

Proof of Lemma 1. Let us define

$$G^N(w) = N^{2H+d} f(Nw) - \delta \left(\frac{w}{|w|} \right) |w|^{-2H-d},$$

and, for $L \in \mathbb{N}^*$, consider $S_L^N(w) = \sum_{0 < |k| \leq L} G^N(w + 2k\pi)$. Since f satisfies Condition (3) (main paper), we have

$$f(N(w + 2k\pi)) \leq C_N |k|^{-2H-d},$$

for $k \neq 0$, $w \in [0, 2\pi]^d$, and N sufficiently large. Hence, $\sum_{0 < |k|} f(N(w + 2k\pi))$ is normally convergent, and, as L tends to $+\infty$, S_L^N tends uniformly to

$$S^N = N^{2H+d} \sum_{0 < |k|} f(N(w + 2k\pi)) - \sum_{0 < |k|} \delta \left(\frac{w + 2k\pi}{|w + 2k\pi|} \right) |w + 2k\pi|^{-2H-d}.$$

Let us now consider

$$U^N = \int_{[0, 2\pi]^d} |\hat{v}(T'_a w) \hat{v}(T'_b w)|^2 \left(S^N(w) + G^N(w) \right)^2 dw,$$

and show that U^N tends to 0 as N tends to $+\infty$.

First, let us quote that, for all $w \in [0, 2\pi]^d$, $S^N(w)$ is bounded by

$$I^N = \int_{\mathbb{R}^d \setminus B(0, A)} |G^N(w)| dw = \int_{S^{d-1}} \int_A^{+\infty} |G^N(\rho s)| \rho^{d-1} d\rho ds,$$

where $B(0, A)$ denotes a ball of \mathbb{R}^d centered at 0 of radius $0 < A \leq 2\pi$. Further notice that

$$|G^N(\rho s)| \leq G_1^N(\rho s) + G_2^N(\rho s), \quad (\text{S1.4})$$

$$\text{where } G_1^N(\rho s) = N^{2H+d} \left| f(N\rho s) - \tau(s)(N\rho)^{-2\beta(s)-d} \right|, \quad (\text{S1.5})$$

$$\text{and } G_2^N(\rho s) = \left| N^{2(H-\beta(s))} \tau(s) \rho^{-2\beta(s)-d} - \delta(s) \rho^{-2H-d} \right|. \quad (\text{S1.6})$$

Hence, $I^N \leq I_1^N + I_2^N$ with $I_j^N = \int_{S^{d-1}} \int_A^{+\infty} G_j^N(\rho s) \rho^{d-1} d\rho ds$.

Since f satisfies Condition (3) (main paper) and τ is bounded, we have

$$I_1^N \leq c_1 N^{-\gamma} A^{-2H-\gamma},$$

for some $c_1 > 0$, and large N . So, $\lim_{N \rightarrow +\infty} I_1^N = 0$.

Besides, for $\eta > 0$, let us define sets

$$E_\eta = \{s \in S^{d-1}, H < \beta(s) < H + \eta\} \text{ and } F_\eta = \{s \in S^{d-1}, \beta(s) \geq H + \eta\}. \quad (\text{S1.7})$$

When $s \in E_\eta \cup F_\eta$, $\delta(s) = 0$, so that $G_2^N(\rho s) \leq cN^{2(H-\beta(s))} \rho^{-2\beta(s)-d}$. When $s \in E_0$, $\tau(s) = \delta(s)$ and $\beta(s) = H$, so that $G_2^N(\rho s) = 0$. Hence,

$$I_2^N \leq c_2 \int_{E_\eta \cup F_\eta} N^{2(H-\beta(s))} A^{-2\beta(s)} ds.$$

Thus, $I_2^N \leq \tilde{c}_2(\mu(E_\eta) + N^{-2\eta})$, where $\mu(E_\eta)$ is the Lebesgue measure of E_η over the sphere S^{d-1} . Let us show that $\lim_{\eta \rightarrow 0^+} \mu(E_\eta) = 0$. Assume it is not the case. Then, there exists $c_0 > 0$ and a decreasing sequence $(\eta_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} \eta_n = 0$ and $\mu(E_{\eta_n}) > c_0$. Since $E_\eta \subset E_{\eta'}$ when $\eta < \eta'$, the sequence $\mu(E_\eta)$ decreases, and admits a positive limit as η decreases to 0. This implies that $\mu(\bigcap_{\eta < \eta_0} E_\eta) > 0$. So, take $s \in \bigcap_{\eta < \eta_0} E_\eta$. It satisfies $H < \beta(s) < H + \eta$ for all

S1. CONVERGENCE STUDY

$\eta > 0$. This yields $\beta(s) = H$, which is contradictory.

Now, for $0 < \alpha < 1$, let us set $\eta_N = \log(N)^{-\alpha}/2$. We then obtain

$$I_2^N \leq \tilde{c}_2(\mu(E_{\log(N)^{-\alpha}}) + e^{-\log(N)^{1-\alpha}}), \quad (\text{S1.8})$$

and $\lim_{N \rightarrow +\infty} I_2^N = 0$. Therefore, $\lim_{N \rightarrow +\infty} I^N = 0$. Thus, on $[0, 2\pi]^d$, S^N converges uniformly to 0, as N tends to 0.

Consequently, the integral U^N is bounded by $c_1(\sup_{w \in [0, 2\pi]^d} |S^N(w)|)^2 + c_2 J^N$, where

$$J^N = \int_{[0, 2\pi]^d} |\hat{v}(T'_a w) \hat{v}(T'_b w)|^2 ((G^N(w))^2 + G^N(w)) dw.$$

Let us then decompose J^N into a sum of two integrals $J_1^N = \int_{B(0, A)} \cdots dw$ and $J_2^N = \int_{[0, 2\pi]^d \setminus B(0, A)} \cdots dw$, and study separately these integrals.

Notice that $\hat{v}(y) = Q_v(e^{iy_1}, \dots, e^{iy_d})$ where Q_v is the characteristic polynomial of v . Hence, using Proposition 3.2 and Taylor expansions of Q_v in the neighborhood of 0, we obtain $|\hat{v}(y)|^2 \leq C|y|^{2K+2}$ for some $C > 0$. Therefore, J_1^N is bounded by

$$\begin{aligned} & c_3 N^{4(H-K-1)+d} \int_{B(0, \frac{A}{N})} |w|^{4(K+1)} (f^2(w) + f(w)|w|^{-2H-d} + |w|^{-4H-2d}) dw \\ & + c_4 N^{2(H-2K-2)} \int_{B(0, \frac{A}{N})} |w|^{4(K+1)} (f(w) + |w|^{-2H-d}) dw, \end{aligned}$$

for some $c_3, c_4 > 0$. In this upper bound, integrals of the form $\int_{B(0, \epsilon)} |w|^u f(w) dw$ are finite for both $u = 4(K+1) - 2H - d$ and $u = 4(K+1)$, since Z is a M -IRF and $K \geq M/2 + d/4$. Moreover, $\sup_{w \in B(0, \epsilon)} |w|^{2K+3} f(w) \leq c < +\infty$, since Z is a M -IRF and $K_a \geq M$. Therefore,

$$\int_{B(0, \epsilon)} |w|^{4(K+1)} f^2(w) dw \leq c \int_{B(0, \epsilon)} |w|^{2K+1} f(w) dw < +\infty,$$

since $K \geq M+1$. Besides, integrals of the form $\int_{B(0, \epsilon)} |w|^u dw$ are finite for $u = 4(K+1) - 2H - d$ and $u = 4(K+1) - 4H - 2d$, since $K \geq d/4$. Consequently,

$$J_1^N \leq \tilde{c}_3 N^{4(H-K-1)+d} + \tilde{c}_4 N^{2(H-2K-2)},$$

and $\lim_{N \rightarrow +\infty} J_1^N = 0$, since $K \geq d/4$.

Besides, using the bound (S1.4), we obtain

$$J_2^N \leq \int_{S^{d-1}} \int_A^{+\infty} (G_1^N(\rho s) + G_2^N(\rho s))^2 \rho^{d-1} d\rho ds.$$

Then, using previous bounds on G_1^N and G_2^N , we get

$$J_2^N \leq c_5 (N^{-2\gamma} + N^{-2\eta} + \mu(E_{\eta})) \Gamma_H,$$

with $c_5 > 0$ and $\Gamma_H = \int_A^{+\infty} \rho^{-4H-d-1} d\rho < +\infty$. Setting again $\eta_N = \log(N)^{-\alpha}/2$ with $0 < \alpha < 1$, we obtain $\lim_{N \rightarrow +\infty} J_2^N = 0$. Therefore, $\lim_{N \rightarrow +\infty} J^N = 0$.

Consequently, $\lim_{N \rightarrow +\infty} U^N = 0$. This implies the convergence of $N^{2H} f_{a,b}^N$ to $f_{a,b}$ in $L^2([0, 2\pi]^d)$.

S2 Proofs of Propositions

Proof of Proposition 3.1. By definition, the kernel v leads to K -order increments of Z if and only if $\sum_{k \in \llbracket 0, L \rrbracket^d} v[k](m - k)^l = 0, \forall m \in \mathbb{Z}^d, \forall l \in \llbracket 0, K \rrbracket^d, |l| \leq K$, which is also equivalent to

$$\sum_{k \in L} v[k]k^l = 0, \forall l \in \llbracket 0, K \rrbracket^d, |l| \leq K.$$

Besides, we have

$$\frac{\partial^{|l|} \mathcal{Q}_v}{\partial z^l}(z) = \sum_{k_1=l_1}^{L_1} \cdots \sum_{k_d=l_d}^{L_d} v[k] \prod_{j_1=0}^{l_1-1} (k_1 - j_1) \cdots \prod_{j_d=0}^{l_d-1} (k_d - j_d) z^{k-l},$$

using the convention that $\prod_{j_i=0}^{l_i-1} (k_i - j_i) = 1$ if $l_i = 0$. From that, we deduce the recurrence equations

$$\frac{\partial^{|l|} \mathcal{Q}_v}{\partial z^l}(z) = \sum_{k \in \llbracket 0, L \rrbracket^d} v[k]k^l z^{k-l} - \sum_{j \in \llbracket 1, d \rrbracket, l_j \geq 1} \frac{l_j - 1}{z_j} \frac{\partial^{|l|-1} \mathcal{Q}_v}{\partial z^{l-e_j}}(z),$$

where e_j is the j th vector of the canonical basis of \mathbb{R}^d . In particular,

$$\forall j, l, \frac{\partial^{|l|} \mathcal{Q}_v}{\partial z^l}(1, \dots, 1) + \sum_{j \in \llbracket 1, d \rrbracket, l_j \geq 1} (l_j - 1) \frac{\partial^{|l|-1} \mathcal{Q}_v}{\partial z^{l-e_j}}(1, \dots, 1) = \sum_{k \in \llbracket 0, L \rrbracket^d} v[k]k^l.$$

We conclude the proof by recurrence on the order of the partial derivatives of \mathcal{Q} . \square

Proof of Proposition 3.2. Let P be a polynomial of degree $l \leq K$. We notice

$$\sum_k v[k]P\left(\frac{m - T_u k}{N}\right) = \sum_k v[k]P \circ T_u\left(\frac{m' - k}{N}\right), \text{ with } m' = T_u^{-1}m.$$

But, any rescaling or rotation $P \circ T_u$ of a polynomial P remains a polynomial of the same degree. Hence, $\sum_k v[k]P\left(\frac{m - T_u k}{N}\right) = 0$ for all P of degree $l \leq K$ if and only if $\sum_k v[k]P\left(\frac{m' - k}{N}\right) = 0$ for all P of degree $l \leq K$. According to Proposition 3.1, this only holds if an and only if Condition (17) (main paper) is satisfied. \square

Proof of Proposition 3.3. Since, for $a \in \mathcal{F}$ and $m \in \mathbb{Z}^d$, $V_a^N[m]$ is an increment of Z of order $\geq M$, it has zero mean. Moreover, for any $a, b \in \mathcal{F}$ and $m, n \in \mathbb{Z}^d$, Theorem 2.3 yields

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \sum_{k, l \in \mathbb{Z}^d} v[k]v_b[l]K_Z\left(\frac{m - T_a k}{N} - \frac{n - T_b l}{N}\right) < +\infty,$$

where K_Z is a generalized covariance of the form (8). But, for any even polynomial P of degree $2M$, we can write $P(x - y) = \sum_{|l|=0}^M q_l(y)x^l + \sum_{|l|=0}^M q_l(x)y^l$ where q_l are polynomials of degree up to $2M$. Hence, since $V_a^N[m]$ and $V_b^N[n]$ are increments of order $K \geq M$,

$$\sum_{k, l \in \mathbb{Z}^d} v[k]v[l]P\left(\frac{m - T_a k}{N} - \frac{n - T_b l}{N}\right) = 0,$$

S3. COVARIANCE ESTIMATION

for any even polynomial P of degree $2M$, including P_M and Q of Equation (8). Therefore,

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sum_{k,l \in \mathbb{Z}^d} v[k]v[l] \cos \left(\left\langle \frac{m - T_a k}{N} - \frac{n - T_b l}{N}, w \right\rangle \right) f(w) dw,$$

and, since f is even,

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sum_{k,l \in \mathbb{Z}^d} v[k]v[l] e^{i \langle \frac{m-n}{N} - \frac{T_a k}{N} + \frac{T_b l}{N}, w \rangle} f(w) dw.$$

From this, we deduce

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{v} \left(\frac{T'_a w}{N} \right) \bar{\hat{v}}_b \left(\frac{T'_b w}{N} \right) e^{i \langle \frac{m-n}{N}, w \rangle} f(w) dw.$$

Since this expression exclusively depends on $m-n$, and not on m and n , V^N is stationary. Using a variable change $\zeta = w/N$, and a decomposition of the integral domain, we further obtain

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \int_{[0, 2\pi]^d + 2k\pi} \hat{v}(T'_a \zeta) \bar{\hat{v}}_b(T'_b \zeta) e^{i 2\pi \langle m-n, \zeta \rangle} f(N\zeta) N^d d\zeta,$$

After a variable change $\zeta = w + 2k\pi$ in each integral, we then get

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{[0, 2\pi]^d} g_{a,b}^N(w, z) dw d\Delta(z),$$

where $d\Delta(u) = \sum_{k \in \mathbb{Z}^d} \delta_{2k\pi}(u)$ is a counting measure on \mathbb{R}^d and $g_{a,b}^N(w, z) = N^d \hat{v}(T'_a w) \bar{\hat{v}}_b(T'_b w) e^{i \langle m-n, w \rangle} f(N(w+z))$. Since $\mathbb{E}(V_a^N[m]V_b^N[n]) < +\infty$, the Lebesgue Fubini theorem implies that $f_{a,b}^N(w) = \int_{\mathbb{R}^d} g_{a,b}^N(w, z) d\Delta(z)$ is almost everywhere defined, and that $\int_{[0, 2\pi]^d} f_{a,b}^N(w) dw < +\infty$. \square

S3 Covariance estimation

In this section, we construct an estimate of the covariance matrix Σ^N of log-variations Y^N involved in the linear model (29) (main paper). According to the proof in Section S1, the random vector Y^N has the same asymptotical covariance Σ as the random vector U^N defined by Equation (S1.1). Hence, we approximate Σ^N by an estimate of the covariance matrix of U^N .

We have

$$\mathbb{E}(U_a^N U_b^N) = \frac{1}{N_e^2 \mathbb{E}(W_a^N) \mathbb{E}(W_b^N)} \sum_{p,q \in \mathcal{E}_N} \mathbb{E}((V_a^N[p])^2 (V_a^N[q])^2).$$

But $(V_a^N[p], V_a^N[q])$ are centered Gaussian vectors. Thus, $E((V_a^N[p])^2 (V_a^N[q])^2) = 2(E(V_a^N[p] V_a^N[q]))^2$.

Moreover,

$$E(V_a^N[p] V_a^N[q]) = \sum_{k,l} v[k]v[l] K_Z \left(\frac{p - T_a k - q + T_b l}{N} \right),$$

where K_Z is the generalized covariance of the IRF Z . Let \tilde{H} be an estimate of the Hölder irregularity of Z (e.g. an OLS estimate of H in the linear model (29) (main paper)). Approximating

the generalized covariance K_Z by the one of a fractional Brownian field of order \tilde{H} , it follows that

$$E(V_a^N[p]V_a^N[q]) \simeq C_{\tilde{H}} \sum_{k,l} v[k]v[l]|p - T_a k - q + T_b l|^{2\tilde{H}}.$$

Using the same approximation, we also have

$$\mathbb{E}(W_a^N) = \mathbb{E}((V_a^N[0])^2) \simeq \sum_{k,l} v[k]v[l]|T_a(l - k)|^{2\tilde{H}}.$$

Hence, we get

$$\Sigma_{a,b}^N \simeq \sum_{\delta \in \Delta \mathcal{E}_N} \frac{N_\delta \left(\sum_{k,l} v[k]v[l]|\delta - T_a k + T_b l|^{2\tilde{H}} \right)^2}{N_e^2 \left(\sum_{k,l} v[k]v[l]|T_a(l - k)|^{2\tilde{H}} \right) \left(\sum_{k,l} v[k]v[l]|T_b(l - k)|^{2\tilde{H}} \right)},$$

where $\Delta \mathcal{E}_N = \{\delta = p - q, p, q \in \mathcal{E}_N\}$ and N_δ is the number of couples $(p, q) \in \mathcal{E}_N^2$ for which $\delta = p - q$.

References

- M.A. Arcones (1994). Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors. *Ann. Probab.*, 22(4):2242–2274.
- H. Biermé, A. Bonami, and J. R. León (2011). Central limit theorems and quadratic variations in terms of spectral density. *Electron. J. Probab.*, 16(13):362–395.
- P. Breuer and P. Major (1983). Central limit theorems for non-linear functionals of Gaussian fields. *J. Multivariate Anal.*, 13:425–441.