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# Tests of isotropy for rough textures of trended images.

Frédéric J.P. RICHARD

## *Abstract:*

In this paper, we propose a statistical methodology to test whether the texture of an image is isotropic or not. This methodology is based on the well-known quadratic variations defined as averages of square image increments. Specific to our approach, these variations are computed in different directions using grid-preserving image rotations. We study asymptotically these variations in a framework of intrinsic random fields allowing us to take into account the presence of polynomial trends in images. We establish a convergence result linking variation and scale logarithms through an asymptotic Gaussian linear model. This model involves direction-dependent intercepts which are equal when textures are isotropic. Hence, we test the texture isotropy using Fisher tests that check the validity of the assumption of the intercept equality. These tests are validated using 6000 realizations of anisotropic fractional Brownian fields simulated on a grid of size  $100 \times 100$ . Results show that more than 70% of anisotropic cases can be detected with less than 1% of misclassified isotropic cases.

*Key words and phrases:* texture analysis, image processing, isotropy, anisotropy, intrinsic random field, statistical test, fractional Brownian field, quadratic variations.

## 1 Introduction

The notion of Texture usually refers to an image aspect which is essential for processing images (see [27, 60] and references therein for an introduction to Texture). From a study to another, the definition of textures varies depending mainly on analysis approaches, those approaches being based on statistical models [11, 26, 34, 44, 54], variational image decompositions [4, 5, 52], patches [31, 55, 62], image filters [29, 41, 56, 61, 64], textons [30, 32, 38], etc. In this work, we deal with rough textures which appear on irregular images (this irregularity will be given a Holderian meaning in Section 2.2). From this viewpoint,

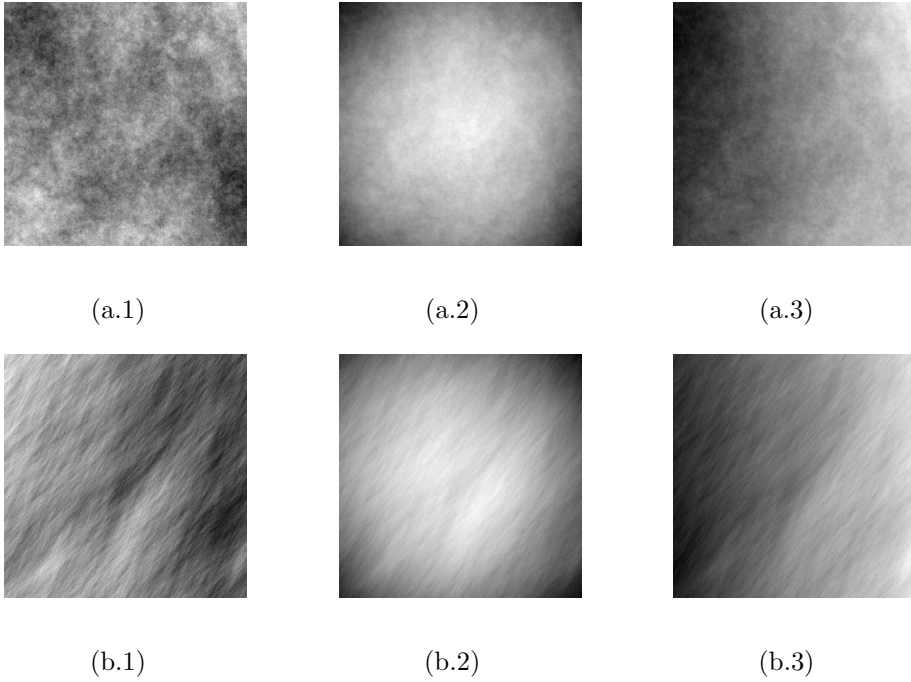


Figure 1: Isotropy and anisotropy of textures of images with trends. (a.1) Realization  $X_a$  of a (isotropic) fractional Brownian field with constant Hurst function  $\beta \equiv 0.22$ , (b.1) realization  $X_b$  of a anisotropic fractional Brownian field with a Hurst function ranging in  $(0.13, 0.76)$ , (a.2)  $X_a + P_1$ , where  $P_1$  is an isotropic polynomial of degree 2, (b.2)  $X_b + P_1$ , (a.3)  $X_a + P_2$ , where  $P_2$  is an anisotropic polynomial of degree 2, (b.3)  $X_b + P_2$ .

textures are associated to high frequency components of images, and not related to their possible trends and other low frequency components. This is illustrated in Figure 1 where some images having textures of interest are shown with and without trend.

In these examples, we can further distinguish two types of textures: the isotropic and anisotropic ones. Isotropic textures (first row) are uniform in all directions whereas the anisotropic ones (second row) are oriented. Anisotropy is one of the main features for the analysis of textures. It was early recognized as an important feature for pre-attentive and attentive vision [43, 57]. It is also relevant for the characterization of material properties in Material Science [35], the diagnostic or prognostic of diseases in Medicine [13, 16, 20, 48, 59], the fingerprint

authentication in Biometry [40, 42], the analysis of chemical movements through aquifers in Hydro-Geology [10], etc. In this paper, our goal is to construct statistical tests to decide whether the texture of an irregular image is isotropic or not. Let us outline that those tests are focused on the texture of the image, and does not concern its trend. In particular, images of the first row of Figure 1 should all be considered as isotropic, even though the third one has an anisotropic trend.

In [59], the issue of testing the texture isotropy was addressed in the framework of anisotropic fractional Brownian fields (AFBF). Introduced in [18], AFBF are some Gaussian random fields that are spatial extensions on  $\mathbb{R}^d$  ( $d > 1$ ) of the fractional Brownian motion [50]. Having zero-mean and stationary increments, they are characterized by a semi-variogram  $v_Z$  whose harmonizable representation

$$\forall x, y \in \mathbb{R}^d, v_Z(x) = \frac{1}{2} \mathbb{E}((Z(x+y) - Z(y))^2) = \int_{\mathbb{R}^d} |e^{i\langle w, x \rangle} - 1|^2 f(w) dw \quad (1)$$

is specified by a spectral density  $f$  of the form

$$f(w) = \tau \left( \frac{w}{|w|} \right) |w|^{-2\beta \left( \frac{w}{|w|} \right) - d}. \quad (2)$$

In this expression, the so-called topothesy function  $\tau$  and Hurst function  $\beta$  are two even, positive, and bounded functions defined on the unit sphere  $S^{d-1} = \{w \in \mathbb{R}^d, |w| = 1\}$  of  $\mathbb{R}^d$ . They are two directional functions that make the model anisotropic. In particular cases when these functions are constant, the model becomes isotropic. The Hurst function is further assumed to range in an interval  $(H, \overline{H}) \subset (0, 1)$  for almost points of  $S^{d-1}$ . An AFBF has irregular realizations (see Section 2.2 for details). Hence, due to its properties, AFBF are well-suited to model anisotropic textures of irregular images. Some realizations of AFBF (in isotropic and anisotropic cases) are shown in Figure 1. In [59], an isotropy test was built upon the statistical validation of an hypothesis of the form  $\beta(s_1) = \beta(s_2) = H$  for a couple  $(s_1, s_2)$  of directions. This hypothesis only gives a necessary condition for the field isotropy, and is focused on two chosen directions. Hence, the test is likely to fail detecting a texture anisotropy which checks this condition. Moreover, the test does not cover anisotropies induced by the topothesy function.

Besides, AFBF model relies upon some stationarity assumptions which restrict their application to images without any trend. To overcome this limitation,

we propose to work in a general framework of Gaussian intrinsic random fields (IRF), which is already very popular in Spatial Statistic [23, 25, 51]. These fields generalize fields with stationary increments. They are characterized by a so-called generalized covariance which can be represented in an harmonizable form extending the one in Equation (1) (see [33, 51] and Section 2.1 for details). Using this representation, we focus on IRF having a spectral density  $f$  which satisfies

$$|w| > A \Rightarrow f(w) - \tau\left(\frac{w}{|w|}\right) |w|^{-2\beta\left(\frac{w}{|w|}\right) - d} \leq C|w|^{-2H-d-\gamma}. \quad (3)$$

for some positive constants  $A, C$  and  $\gamma$ , and some topothesy and Hurst functions  $\tau$  and  $\beta$ . Such a condition implies that the field texture is similar to the one of an AFBF. However, this condition only constrains high frequencies of the field. At low frequency, the spectral density only fulfills general requirements of an IRF density, allowing the field to have a polynomial trend of an arbitrary order. In this work, images are considered as realizations of Gaussian IRF satisfying Condition (3). The texture of such an image is then called isotropic whenever the condition

$$\tau(s) = \tau_0 \in \mathbb{R}^+, \beta(s) = H \in (0, 1) \quad (4)$$

holds almost everywhere on  $S^{d-1}$ . We construct statistical tests to validate the hypothesis that a given image is the realization of an isotropic IRF satisfying Condition (4). As opposed to the one in [59], these tests are designed to detect texture anisotropies whatever their direction and type.

The construction of our test statistics is based on quadratic variations. These variations are averages of square increments computed on an observation grid (see Section 2.1 for a definition of increments). In the literature, they are commonly used to estimate parameters characterizing a field irregularity, *e.g.* the Hurst index of fields related to the fractional Brownian motion in [22, 28, 46, 65], or the local Hurst function of the multifractional Brownian fields and their generalizations in [6, 7, 8, 9, 24, 39]. In [17, 59], such variations were specifically used in combination to a Radon transform for the estimation of the Hurst function of an AFBF. But, due to a discretization issue of the Radon transform, their application was limited to the estimation of function values in vertical and horizontal directions. Variations we use have some original features. First, they are computed on rotated and rescaled images so as to provide both direction and

scale information. Image rotations and rescalings are carefully chosen to preserve the observation grid and avoid data interpolations. Furthermore, the order of increments is adapted to the order of the polynomial trend so as to annihilate its effect on the estimation. We establish a convergence result linking variation and scale logarithms through an asymptotic Gaussian linear model. This model includes some direction-dependent intercepts which are equal when the texture is isotropic. Hence, we propose to test the texture isotropy using some Fisher tests which verify the assumption of the equality of these intercepts.

## 2 Image and Texture Modeling

In this work, an image will be considered as a realization of a random field. Let  $d > 1$  be the dimension of the image (usually,  $d = 2$  or  $d = 3$ ), and  $Z$  be a random field defined on  $\mathbb{R}^d$ , *i.e.* a set of random variables defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , and indexed in  $\mathbb{R}^d$ . For  $N \in \mathbb{N}^*$ ,  $k \in \mathbb{Z}^d$  and  $x \in \mathbb{R}^d$ ,  $Z(x)$  and  $Z^N[k]$  will denote random variables of  $Z$  at positions  $x$  and  $k/N$ , respectively. An image will be defined as a realization of random variables  $Z_N[k]$  at points  $k$  of the grid  $\mathcal{G}_N = \llbracket 1, N \rrbracket^d$ . We will assume that  $Z$  is mean square continuous at any point of  $\mathbb{R}^d$ .

In what follows, we recall some elements about intrinsic random fields (IRF) which are required to set our framework for the modeling of random fields (interested readers may refer to [12, 23, 25, 51, 63] for more comprehensive presentations of IRF).

### 2.1 Intrinsic random fields

In the sequel, we will use some shorthands for manipulating multivariate polynomials: Given a multi-index  $l = (l_1, \dots, l_d) \in \mathbb{N}^d$ , we denote by  $x^l$  the monomial  $x_1^{l_1} \cdots x_d^{l_d}$  defined for  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ , and by  $|l| = \sum_{j=1}^d l_j$  its order.

**Definition 2.1.** *Let  $Z$  be a random field on  $\mathbb{R}^d$  and  $M \in \mathbb{N}$ . An increment of order  $M$  (or  $M$ -increment) of  $Z$  is a linear combination  $Z_{\lambda,x}$  of random variables from  $Z$*

$$Z_{\lambda,x} = \sum_{i=1}^m \lambda_i Z(x_i), \quad (5)$$

defined with a set  $\lambda = (\lambda_i)_{i=1}^m$  of scalar values and a set  $x = (x_i)_{i=1}^m$  of points in  $\mathbb{R}^d$  satisfying the condition

$$\sum_{i=1}^m \lambda_i x_i^l = 0, \forall l \in \mathbb{N}^d, |l| \leq M. \quad (6)$$

Due to Condition (6), the linear combination defined by  $\lambda$  and  $x$  removes from  $Z$  any polynomial trend of order less or equal to  $M$ . Random variables  $Z(x_i) - Z(x_j)$ , defined with arbitrary  $x_i \neq x_j$ , are examples of 0-increments. In Section 3.1, we will construct increments of arbitrary orders on a grid.

**Definition 2.2.** *Let  $M \in \mathbb{N}$ . A random field  $Z$  is an intrinsic random field of order  $M$  (or  $M$ -IRF) if, for any sets  $\lambda = (\lambda_i)_{i=1}^m \in \mathbb{R}^m$  and  $x = (x_i)_{i=1}^m$  of points in  $\mathbb{R}^d$  satisfying Condition (6), the random field defined for all  $y \in \mathbb{R}^d$  by*

$$V_{\lambda,x}(y) = \sum_{i=1}^m \lambda_i Z(x_i + y),$$

has a zero mean, and is second-order stationary, i.e.

$$\begin{aligned} \mathbb{E}(V_{\lambda,x}(y)) &= 0, \forall y \in \mathbb{R}^d, \\ \mathbb{E}(V_{\lambda,x}(y)V_{\lambda,x}(z)) &= K_{\lambda,x}(y - z), \forall y, z \in \mathbb{R}^d. \end{aligned}$$

Field models previously used in [14, 22, 59, 65] are continuous 0-IRF. Since their increments have a zero mean, their expectation is only a constant. Consequently, they cannot be used to describe fields having a polynomial trend of degree higher than 0. Such a description becomes possible with IRF of higher orders.

**Example 2.1.** *For instance, letting  $W$  be a continuous zero-mean 0-IRF, and  $(\alpha_l)_l$  a set of square integrable random variables, the random field*

$$Z(x) = \sum_{l \in \mathbb{N}^d, |l| \leq M} \alpha_l x^l + W(x),$$

is a continuous  $M$ -IRF with a polynomial trend of degree  $M$ .

The correlation structure of a continuous  $M$ -IRF  $Z$  is characterized by a generalized covariance, which is defined as a function  $K_Z$  satisfying

$$\mathbb{E}(Z_{\lambda,x} Z_{\mu,y}) = \sum_{i=1}^m \sum_{j=1}^n \lambda_i \mu_j K_Z(x_i - y_j) \quad (7)$$

for any pair of  $M$ -increments  $Z_{\lambda,x}$  and  $Z_{\mu,y}$  of  $Z$ . Generalized covariances extend ordinary covariances of stationary fields, for which Equation (7) holds for any linear combination of  $Z$ . Besides, when  $Z$  is a 0-IRF with a generalized covariance  $K_Z$ , its semi-variogram  $v_Z(h) = -K_Z(h)$ . A generalized covariance plays the same characterization role for a  $M$ -IRF as the semi-variogram for a 0-IRF.

For any continuous  $M$ -IRF  $Z$ , there exists a continuous generalized covariance  $K_Z$ , which is unique up to an even polynomial of order  $2M$  [33, 51]. This function is symmetric and  $M$ -conditionally positive definite, *i.e.* it satisfies

$$\sum_{i,j} \lambda_i \lambda_j K_Z(x_i - x_j) \geq 0,$$

for any sets  $(\lambda_i)_i$  and  $(x_i)_i$  satisfying Condition (6). Reciprocally, any continuous, symmetric and  $M$ -conditionally positive definite is a generalized covariance of a continuous  $M$ -IRF.

Besides,  $M$ -conditionally positive definite functions can be characterized using a spectral theorem established by Vilenkin and Gelfand for generalized random functions [33], and developed specifically for ordinary random fields by Matheron [51]. Next, we recall a corollary of this theorem which gives a sufficient condition for a function to be a generalized covariance.

**Theorem 2.3.** *A function  $K$  on  $\mathbb{R}^d$  is a generalized covariance of an  $M$ -IRF if it is of the form*

$$K(h) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (\cos(\langle w, h \rangle) - \mathbf{1}_B(w) P_M(\langle w, h \rangle)) f(w) dw + Q(h), \quad (8)$$

where  $P_M(t) = 1 - \frac{t^2}{2} + \dots + \frac{(-1)^M}{(2M)!} t^{2M}$ ,  $\mathbf{1}_B(w)$  is the indicator function of an arbitrary neighborhood of 0,  $Q$  an arbitrary even polynomial of degree  $\leq 2M$ , and  $f$  is an even and positive function satisfying the integrability conditions

$$\forall A > 0, \int_{|w| < A} |w|^{2M+2} f(w) dw < \infty \quad \text{and} \quad \int_{|w| > A} f(w) dw < \infty. \quad (9)$$

The function  $f$  is called the spectral density of the field. Conditions (9) concern the spectral density at low and high frequencies, respectively. In particular, the first condition becomes weaker as  $M$  increases. This enables to include densities having a larger power at low frequencies, and to account for fields having polynomial trends of larger order.



**Example 2.2.** *The anisotropic fractional Brownian fields, whose spectral density has the form (2), are examples of 0-IRF. Such fields can be easily extended to arbitrary IRF by letting*

$$\bar{H} = \text{ess sup}\{\beta(s), s \in S^{d-1}, \tau(s) \neq 0\} \quad (10)$$

*vary in  $(0, +\infty)$  instead of  $(0, 1)$ . The obtained IRF is then of order  $M \geq \bar{H} - 1$ .*

**Definition 2.4.** *We say that an  $M$ -IRF is Gaussian if any uplet of its  $M$ -increments is a Gaussian random vector.*

## 2.2 Hölder irregularity

As mentioned in Introduction, images we consider in this work are irregular. We define this irregularity in a Hölder sense [1, 12].

**Definition 2.5.** *A field  $Z$  satisfies a uniform stochastic Hölder condition of order  $\alpha \in (0, 1)$  if, for any compact set  $C \subset \mathbb{R}^d$ , there exists an almost surely finite, positive random variable  $A_C$  such that the Hölder condition*

$$|Z(x) - Z(y)| \leq A_C |x - y|^\alpha. \quad (11)$$

*holds for any  $x, y \in C$ , with probability one.*

*If there exists  $H \in (0, 1)$  for which Condition (11) holds for any  $\alpha < H$  but not for  $\alpha > H$ , then we say that  $Z$  admits  $H$  as critical Hölder exponent, or that  $Z$  is  $H$ -Hölder.*

The critical Hölder exponent  $H$  of a field  $Z$  quantifies its irregularity. It is close to 0 for the most irregular fields and to 1 for the least ones. Visually, field textures look rougher and rougher as  $H$  decreases to 0.

The Hölder irregularity of a Gaussian  $M$ -IRF with spectral density can be characterized from the asymptotic behavior of its spectral density at high frequency (as  $|w| \rightarrow +\infty$ ). To be more specific, we state a proposition which is borrowed from [12] (Proposition 2.1.6 and Proposition 2.1.7).

**Proposition 2.6.** *Let  $Z$  be a continuous Gaussian IRF of an arbitrary order with a spectral density  $f$ , and  $H \in (0, 1)$ .  $Z$  is  $H$ -Hölder if and only if, for any*

$0 < \alpha < H < \beta < 1$ , there exist positive constants  $A$ ,  $B_1$ ,  $B_2$ , and a positive measure subset  $E$  of the unit sphere  $S^{d-1}$  of  $\mathbb{R}^d$  such that for almost all  $w \in \mathbb{R}^d$

$$|w| \geq A \Rightarrow f(w) \leq B_2 |w|^{-2\alpha-d}, \quad (12)$$

$$|w| \geq A \text{ and } \frac{w}{|w|} \in E \Rightarrow f(w) \geq B_1 |w|^{-2\beta-d}. \quad (13)$$

Condition (12) (resp. condition (13)) ensures that the critical Hölder exponent of a Gaussian IRF is above (resp. below)  $H$ .

**Example 2.3.** *In the particular case of an anisotropic fractional Brownian field of density (2), Conditions (12) and (13) are fulfilled for*

$$H = \text{ess inf} \{ \beta(s), s \in S^{d-1}, \tau(s) \neq 0 \}. \quad (14)$$

*It is the same for an IRF whose spectral density fulfills Condition (3). Hence, both fields are  $H$ -Hölder.*

### 3 Test construction

In this section, we present into details the construction of isotropy tests.

#### 3.1 Quadratic variations

First, we describe increments (see Definition 2.1) which compose quadratic variations. These increments are computed by application to the discrete field  $Z^N$  of a linear filter

$$\forall m \in \mathbb{Z}^d, V^N[m] = \sum_{k \in \llbracket 0, L \rrbracket^d} v[k] Z^N[m - k], \quad (15)$$

determined by a convolution kernel  $v$  of a finite support  $\llbracket 0, L \rrbracket^d$  with  $L \in \mathbb{N}^d$ . Let  $\mathcal{Q}_v$  be the characteristic polynomial associated to  $v$

$$\forall z \in \mathbb{R}^d, \mathcal{Q}_v(z) = \sum_{k \in \llbracket 0, L \rrbracket^d} v[k] z^k. \quad (16)$$

We now state a necessary and sufficient condition on partial derivatives of  $\mathcal{Q}$  for the random variable  $V^N[m]$  to be a  $K$ -increment of  $Z$  for any  $m \in \mathbb{Z}^d$  (the proof is given in Appendix C).

**Proposition 3.1.** *Let  $K \in \mathbb{N}$ . For all  $m \in \mathbb{Z}^d$ , random variables  $V^N[m]$  are  $K$ -increments of  $Z$  if and only if*

$$\forall l \in \llbracket 0, K \rrbracket^d, |l| \leq K, \frac{\partial^{|l|} \mathcal{Q}_v}{\partial z_1^{l_1} \dots \partial z_d^{l_d}}(1, \dots, 1) = 0. \quad (17)$$

**Example 3.1.** *Let  $L \in \mathbb{N}^d$ ,  $|L| > 0$ , and  $v$  be the kernel associated to the characteristic polynomial  $\mathcal{Q}_v(z) = (z_1 - 1)^{L_1} \dots (z_d - 1)^{L_d}$ . From Proposition 3.1, it follows that  $v$  induces increments of order  $K = |L| - 1$ . Its terms are*

$$v[l] = (-1)^{|l|} \binom{L_1}{l_1} \dots \binom{L_d}{l_d}, \text{ if } l \in \llbracket 0, L \rrbracket^d, \text{ } v[l] = 0 \text{ otherwise,} \quad (18)$$

where  $\binom{n}{k}$  stands for the binomial coefficient. For  $0 < |L| \leq 2$  and  $d = 2$ , such kernels yield the main increments used in [14, 22, 65] for the estimation of Hölder irregularity.

Besides, our statistical test involves computing increments at different scales and orientations. To do so, the observation field  $Z^N$  is transformed into discrete fields  $Z_T^N$  by applying combinations  $T$  of rescaling and rotation which maps  $\mathbb{Z}^d$  into itself:

$$\forall k \in \mathbb{Z}^d, Z_T^N[k] = Z\left(\frac{Tk}{N}\right) = Z^N[Tk]. \quad (19)$$

When  $d = 2$ , such transforms have the form

$$T_u = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} = |u|^2 \begin{pmatrix} \cos(\arg(u)) & -\sin(\arg(u)) \\ \sin(\arg(u)) & \cos(\arg(u)) \end{pmatrix}, \text{ for } u \in \mathbb{Z}^2 \setminus \{(0, 0)\},$$

corresponding to a rescaling of factor  $|u|^2$  and a rotation of angle  $\arg(u)$ . When  $d = 3$ , examples of transforms are

$$T_u = \begin{pmatrix} u_1 & -u_2 & 0 \\ u_2 & u_1 & 0 \\ 0 & 0 & |u|^2 \end{pmatrix}, \begin{pmatrix} u_1 & 0 & -u_2 \\ 0 & |u|^2 & 0 \\ u_2 & 0 & u_1 \end{pmatrix} \text{ or } \begin{pmatrix} |u|^2 & 0 & 0 \\ 0 & u_1 & -u_2 \\ 0 & u_2 & u_1 \end{pmatrix}.$$

In both cases, transforms can be identified by a single vector  $u$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . The application of a filter  $v$  to a transform field  $Z_T^N$  leads to random variables

$$\forall m \in \mathbb{Z}^d, V_T^N[m] = \sum_k v[k] Z\left(\frac{m - Tk}{N}\right) = \sum_k v[k] Z^N[m - Tk]. \quad (20)$$

**Proposition 3.2.** *Random variables  $V_T^N[m]$  are  $K$ -increments of  $Z$  if and only if the characteristic polynomial of  $v$  fulfills Condition (17).*

The proof of this Proposition is postponed in Appendix C.

Next, we select a set of transforms. We will use a multi-index  $a = (i, j)$  to classify transforms according to their rotation angle: the transform  $T_a$  will be identified by a vector  $u_a$ . It will correspond to a  $j$ th transform among those that have a same rotation angle  $\arg u_{i1}$ . The rescaling factor of this transform is  $|u_a|^2$ . The index  $a$  will vary in a set

$$\mathcal{F} = \{i \in \llbracket 1, n_b \rrbracket, j \in \llbracket 1, p_i \rrbracket\},$$

where  $n_b$  is the number of different rotation angles and  $p_i \in \mathbb{N}^*$  depends on  $i$ . We denote by  $n_f$  the cardinal of  $\mathcal{F}$ . We have an extra transform identified by  $u_{00}$  which will serve for a normalization purpose (see below). We write  $\tilde{\mathcal{F}} = \mathcal{F} \cup \{(0, 0)\}$ .

Applying a chosen filter  $v$  to transformed fields  $T_a$ , we obtain a set of increments denoted  $V_a^N[m]$ , for  $m \in \mathbb{Z}^d$ . We define the vector-valued random field  $V^N = ((V_a^N[m])_{a \in \tilde{\mathcal{F}}}, m \in \mathbb{Z}^d)$ , and state some of its properties under the assumption that  $Z$  is an IRF (see Appendix C for the proof).

**Proposition 3.3.** *For some  $M \in \mathbb{N}$ , let  $Z$  be a continuous  $M$ -IRF with a spectral density  $f$ . Assume that the vector-valued field  $V^N$  is constructed using increments of order  $K \geq M$ . Then,  $V^N$  is stationary with zero mean. Moreover, for any  $a, b \in \tilde{\mathcal{F}}$  and  $m, n \in \mathbb{Z}^d$ ,*

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} f_{a,b}^N(w) e^{i\langle m-n, w \rangle} dw, \quad (21)$$

where  $f_{a,b}^N$ , the multivariate spectral density of  $V^N$ , is a function of  $L^1([0, 2\pi]^d)$  given by

$$f_{a,b}^N(w) = N^d \hat{v}(T'_a w) \overline{\hat{v}(T'_b w)} \sum_{k \in \mathbb{Z}^d} f(N(w + 2k\pi)), \quad (22)$$

with  $T'$  the transpose of  $T$ ,  $\hat{v}(w) = \sum_{k \in \mathbb{Z}^d} v[k] e^{-i\langle k, w \rangle}$  the discrete Fourier transform of  $v$ , and  $\overline{\hat{v}}$  its conjugate.

We now consider the set  $\mathcal{E}_N$  of cardinal  $N_e$  including all points  $m$  of  $\mathbb{Z}^d$  for which all increments  $V_a^N[m]$  can be computed using exclusively observations on

the grid  $\mathcal{G}_N$ . Then, we define the so-called quadratic variations on  $\mathcal{E}_N$  as

$$\forall a \in \tilde{\mathcal{F}}, W_a^N = \frac{1}{N_e} \sum_{m \in \mathcal{E}_N} (V_a^N[m])^2. \quad (23)$$

These variations account for those of [14, 22], which can be obtained for  $d = 2$  with  $u \propto (1, 0)$  (*i.e.* without any field rotation). They also contains variations defined in [65] using some field rotations. However, they are more generic as they include variations obtained with arbitrary grid-preserving rotations and  $K$ -order filters.

For  $a \in \mathcal{F}$ , we also define normalized log-variations  $Y_a^N$  and log-scales  $x_a$  as

$$Y_a^N = \log \left( \frac{W_a^N}{W_{00}^N} \right) \quad \text{and} \quad x_a = \log \left( \frac{|u_a|^2}{|u_{00}|^2} \right), \text{ respectively.} \quad (24)$$

### 3.2 Gaussian linear model

The construction of the statistical test is based on asymptotic properties of the random vector  $Y^N = (Y_a^N)_{a \in \mathcal{F}}$ . These properties are stated in the following theorem.

**Theorem 3.4.** *Assume that  $Z$  is a continuous Gaussian  $M$ -IRF with a spectral density  $f$  satisfying Condition (3). Let*

$$H = \text{ess inf} \{ \beta(s), s \in S^{d-1}, \tau(s) \neq 0 \}. \quad (25)$$

*Assume that  $H \in (0, 1)$ , and the set  $E_0 = \{s \in S^{d-1}, \beta(s) = H, \tau(s) \neq 0\}$  is of positive measure on  $S^{d-1}$ . Consider a log-variation vector  $Y^N$  constructed using increments of orders  $K \geq M + 1$ , and  $K \geq M/2 + d/4$  if  $d > 4$ .*

*For  $a, b \in \tilde{\mathcal{F}}$ , let  $f_{a,b}$  be a function defined for almost  $w \in [0, 2\pi]^d$  by*

$$f_{a,b}(w) = \hat{v}(T'_a w) \bar{\hat{v}}(T'_b w) \sum_{k \in \mathbb{Z}^d} \delta \left( \frac{w + 2k\pi}{|w + 2k\pi|} \right) |w + 2k\pi|^{-2H-d}, \quad (26)$$

*where  $\delta$  is defined, for almost  $s \in S^{d-1}$ , by  $\delta(s) = \lim_{\rho \rightarrow +\infty} f(s\rho)\rho^{2H+d}$ .*

*Then,*

$$N^{\frac{d}{2}}(Y^N - \zeta) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \Sigma), \quad (27)$$

*for a vector  $\zeta \in \mathbb{R}^{n_f}$  and a  $n_f \times n_f$ -covariance matrix  $\Sigma$ , both specified next.*

The expectation term  $\zeta$  is of the form

$$\forall a = (i, j) \in \mathcal{F}, \zeta_{ij} = x_{ij}H + \beta_i, \quad (28)$$

where  $\beta_i$  depends only on the filter  $v$  and the rotation angle  $\arg(u_{i1})$ . This term is given by

$$\beta_i = \log \left( \frac{C_H(\arg(u_{i1}), v)}{C_H(\arg(u_{00}), v)} \right), \quad (29)$$

where

$$C_H(\arg(u_a), v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{v} \left( \frac{T'_a}{|u_a|} w \right) \right|^2 \delta \left( \frac{w}{|w|} \right) |w|^{-2H-d} dw. \quad (30)$$

The matrix  $\Sigma = (\Sigma_{a,b})_{a,b \in \mathcal{F}}$  has terms  $\Sigma_{a,b} = \Gamma_{a,b} - \Gamma_{a,00} - \Gamma_{00,b} - \Gamma_{00,00}$ , where

$$\Gamma_{\tilde{a}, \tilde{b}} = \frac{2(2\pi)^d \int_{[0, 2\pi]^d} |f_{\tilde{a}, \tilde{b}}(w)|^2 dw}{\int_{[0, 2\pi]^d} f_{\tilde{a}, \tilde{a}}(w) dw \int_{[0, 2\pi]^d} f_{\tilde{b}, \tilde{b}}(w) dw}, \forall \tilde{a}, \tilde{b} \in \tilde{\mathcal{F}}. \quad (31)$$

Moreover, when  $Z$  has an isotropic texture,  $\beta_i = 0$  for all  $i \in \llbracket 1, n_b \rrbracket$ .

This theorem is proved in Section 4. It brings out a linear asymptotic dependency between variations and parameters related to field irregularity and anisotropy. This dependency can be formalized in terms of a Gaussian linear model

$$\forall (i, j) \in \mathcal{F}, Y_{ij}^N = x_{ij}H + \beta_i + \epsilon_{ij}^N, \quad (32)$$

where  $\epsilon_{ij}^N$  are correlated Gaussian variables. This model can be interpreted as a generalized model of analysis of covariance, in which the  $x_{ij}$  are regression variables, and the rotation angle (index  $j$ ) used to form  $Y_{ij}^N$  is a qualitative factor. As mentioned in Theorem 3.4, parameters  $\beta_i$  vary according to the rotation index, and become constant when the texture is isotropic. Hence, for testing the texture isotropy, we propose to check statistically the null hypothesis

$$\mathcal{H}_0 : \forall i \in \llbracket 1, n_b \rrbracket, \beta_i = \tilde{\beta}, \quad (\text{isotropy}) \quad (33)$$

against its alternative hypothesis  $\mathcal{H}_1 : \exists i, j, \beta_i \neq \beta_j$  (anisotropy).

### 3.3 Parameter estimation

Model (32) can also be formulated in a matricial form. Let  $Y^N$  and  $\epsilon^N$  be the two random vectors of size  $n_f$  formed by terms  $Y_a^N$  and  $\epsilon_a^N$ , respectively.

Define a parameter vector  $\theta$  of size  $n_b + 1$ . Set a design matrix  $X$ , of size  $n_f \times (n_b + 1)$ , having terms  $X_{ij,0} = x_{ij}$  on the  $(i, j)^{th}$  row and the first column and,  $X_{ij,m} = \delta_{i=m}$  on  $(i, j)$ th row of the  $m$ th column. Then, Model (32) is equivalent to

$$Y^N = X\theta + \epsilon^N, \epsilon^N \sim \mathcal{N}(0, \Sigma). \quad (34)$$

Using a generalized least square criterion, parameters  $\theta$  of this model can be estimated by

$$\tilde{\theta}^N = P(\Sigma)Y^N \quad \text{with} \quad P(\Sigma) = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}. \quad (35)$$

In particular, the field irregularity  $H$  is estimated by  $\tilde{H}^N = \hat{\theta}_0^N = P_0(\Sigma)Y^N$ ,  $P_0(\Sigma)$  being the first row of the matrix  $P(\Sigma)$ . Similarly, for  $i \in \llbracket 1, n_b \rrbracket$ , parameters  $\beta_i^N$  are estimated by  $\tilde{\theta}_i^N$ .

### 3.4 Fisher test

To test hypothesis  $\mathcal{H}_0$  against  $\mathcal{H}_1$ , we construct a Fisher test. Under  $\mathcal{H}_0$ , Model (32) reduces to the model

$$Y^N = \tilde{X}\alpha + \epsilon^N, \epsilon^N \sim \mathcal{N}(0, \Sigma), \quad (36)$$

where  $\alpha$  is a parameter vector of size 2, and  $\tilde{X} = (x_a)_{a \in \mathcal{F}}$  is the design matrix of size  $n_f \times 2$  having terms  $\tilde{X}_{ij,0} = x_{ij}$  on the  $(i, j)^{th}$  row and the first column and, 1 on rows of the second column. The GLS estimator of parameters of this model is

$$\hat{\alpha}^N = \tilde{P}(\Sigma)Y^N \quad \text{with} \quad \tilde{P}(\Sigma) = (\tilde{X}'\Sigma^{-1}\tilde{X})^{-1}\tilde{X}'\Sigma^{-1}, \quad (37)$$

Then, we define the test statistic

$$F^N(\Sigma) = \frac{n_f - n_b - 1}{n_b - 1} \frac{|X\hat{\beta}^N - \tilde{X}\hat{\alpha}^N|^2}{|Y - X\hat{\beta}^N|^2}. \quad (38)$$

Under  $\mathcal{H}_0$ ,  $F^N(\Sigma)$  has a Fisher law distribution with  $(n_b - 1, n_f - n_b - 1)$  as degrees of freedom. Let  $\alpha \in (0, 1)$ , and set  $s_\alpha$  to be the quantile of order  $1 - \alpha$  of a  $(n_b - 1, n_f - n_b - 1)$ -Fisher law distribution. Then, the rejection interval  $R_\alpha = \{F^N > s_\alpha\}$  defines an isotropy test with a confidence level  $\alpha$ .

The expression of the covariance matrix  $\Sigma$  depends on the spectral density of  $Z$ , which is unknown. Hence, to implement the test, we compute  $F^N(\tilde{\Sigma}^N)$  instead

of  $F^N(\tilde{\Sigma})$  using an estimate  $\tilde{\Sigma}^N$  of the covariance matrix of the log-variations (see Appendix B for more details).

## 4 Convergence study

This section is devoted to the proof of Theorem 3.4. In a first part, we prove the asymptotic normality of the random vector

$$U^N = \left( \frac{W_a^N}{\mathbb{E}(W_a^N)} \right)_{a \in \tilde{\mathcal{F}}} \quad (39)$$

defined with quadratic variations  $W_a^N$  of Equation (23). Then, we deduce the asymptotic normality (27) of  $Y^N = (Y_a^N)_{a \in \mathcal{F}}$ . In a second part, we further specify terms of this convergence.

**Part 1.** For establishing the asymptotic normality of  $U^N$ , we use a multivariate version of the Breuer-Major theorem which is recalled in Appendix A. For that, let us first notice that

$$N^{\frac{d}{2}}(T_a^N - \mathbf{1}) = \frac{N^{\frac{d}{2}}}{N_e} \sum_{m \in \mathcal{E}_N} ((X_a^N[k])^2 - 1) \underset{N \rightarrow +\infty}{\sim} \frac{1}{N^{\frac{d}{2}}} \sum_{k \in [1, N]^d} ((X_a^N[k])^2 - 1),$$

with  $X_a^N[m] = V_a^N[m] / \sqrt{\mathbb{E}((V_a^N[m])^2)}$ . So, if the Breuer-Major theorem could be applied to the vector-valued random field  $X^N = ((X_a^N[m])_{a \in \tilde{\mathcal{F}}}, m \in \mathbb{Z}^d)$ , it would follow that

$$N^{\frac{d}{2}}(T_N - \mathbf{1}) \underset{N \rightarrow +\infty}{\xrightarrow{d}} \mathcal{N}(0, \Sigma), \quad (40)$$

where  $\mathbf{1}$  is the unit vector of the same size as  $U^N$ , and  $\Gamma$  is a covariance matrix. But, using Proposition 3.3, the spectral density of  $X^N$  can be specified as

$$g_{a,b}^N(w) = \frac{f_{a,b}^N(w)}{\sqrt{\mathbb{E}((V_a^N[m])^2)} \sqrt{\mathbb{E}((V_b^N[m])^2)}},$$

where  $\mathbb{E}((V_a^N[m])^2) = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} f_{a,a}^N(w) dw$ . Thus, it suffices to show the convergence in  $L^2([0, 2\pi]^d)$  of  $g_{a,b}^N$ . This convergence results from the next lemma whose proof is postponed at the end of the section.



**Lemma 4.1.** *Take the same conditions as in Theorem 3.4. Consider the multivariate spectral density  $f_{a,b}^N$  of  $V^N$  given by Equation (22) of Proposition 3.3. Then, for any  $a, b \in \tilde{\mathcal{F}}$ , as  $N$  tends to  $+\infty$ ,  $N^{2H} f_{a,b}^N$  converges in  $L^2([0, 2\pi]^d)$  to the function  $f_{a,b}$  defined by Equation (26).*

Due to Lemma 4.1,  $N^{2H} f_{a,b}^N$  tends to  $f_{a,b}$  in  $L^2([0, 2\pi]^d)$  and, *a fortiori* in  $L^1([0, 2\pi]^d)$ . Hence, for  $a \in \tilde{\mathcal{F}}$ ,  $N^{2H} \mathbb{E}((V_a^N[m])^2)$  converges to

$$C_a = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} f_{a,a}(w) dw. \quad (41)$$

Therefore,  $g_{a,b}^N$  tends in  $L^2([0, 2\pi]^d)$  to  $g_{a,b} = f_{a,b}/\sqrt{C_a C_b}$ .

Consequently, the Breuer-Major theorem yields the asymptotic normality (40) for a covariance matrix  $\Gamma$  whose terms are defined by Equation (31).

Now, let  $G$  be the differentiable function mapping  $(\mathbb{R}_*^+)^{n_f+1}$  into  $\mathbb{R}^{n_f}$  defined by  $G(u)_a = \log(u_a/u_{00})$  for  $a \in \mathcal{F}$ . Since  $N^{d/2}(U^N - \mathbf{1}) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \Sigma)$ , we have

$$N^{d/2}(G(U^N) - G(\mathbf{1})) \xrightarrow[N \rightarrow +\infty]{d} \nabla G(\mathbf{1})' \mathcal{N}(0, \Sigma).$$

using the multivariate  $\Delta$ -method. But, for  $a \in \mathcal{F}$ ,

$$G(U^N)_a - G(\mathbf{1})_a = Y_a^N - \log\left(\frac{C_a}{C_{00}}\right) + R_a^N,$$

where  $R_a^N = \log(C_a/C_{01}) - \log(\mathbb{E}(W_a^N)/\mathbb{E}(W_{01}^N))$  and  $C_a$  is defined by Equation (41). Moreover, due to Lemma 4.1,  $\lim_{N \rightarrow +\infty} R_a^N = 0$ . Hence, the asymptotic normality (27) follows for  $\zeta = (\log(C_a/C_{00}))_{a \in \mathcal{F}}$  and the covariance matrix  $\Sigma$  defined in the theorem.

**Part 2.** Let us notice that

$$C_a = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} \int_{\mathbb{R}^d} |\hat{v}(T'_a(w+z))|^2 \delta\left(\frac{(w+z)}{|w+z|}\right) |w+z|^{-2H-d} d\Delta(z) dw,$$

where  $d\Delta(z) = \sum_{k \in \mathbb{Z}^d} \delta_{2k\pi}(z)$  is a counting measure on  $\mathbb{R}^d$ . But,  $C_a < +\infty$ . Hence, by application of the Lebesgue-Fubini theorem, we obtain

$$C_a = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \int_{[0, 2\pi]^d} |\hat{v}(T'_a(w+2k\pi))|^2 \delta\left(\frac{w+2k\pi}{|w+2k\pi|}\right) |w+2k\pi|^{-2H-d} dw.$$

After a variable change  $\zeta = w + 2k\pi$  in each integral, we further get

$$C_a = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{v}(T'_a \zeta)|^2 \delta \left( \frac{\zeta}{|\zeta|} \right) |\zeta|^{-2H-d} d\zeta.$$

Next, using the variable change  $w = |u_a| \zeta$ , we have  $C_a = |u_a|^{2H} C_H(\arg(u_a), v)$ , where  $\arg(u_a)$  is the angle of the rotation  $\frac{T'_a}{|u_a|}$  and  $C_H(\arg(u_a), v)$  is defined by Equation (30). Then, the expression of  $\zeta$  given by Equations (28) and (29) follows.

Furthermore, let us notice that, when the texture of  $Z$  is isotropic, the function  $\delta \equiv \tau_0 \in \mathbb{R}_*^+$ . Hence, in this case, we obtain

$$C_H(\arg(u_a), v) = \frac{\tau_0}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{v}(w)|^2 |w|^{-2H-d} d\zeta.$$

by applying the variable change  $w = \frac{T'_a}{|u_a|} \zeta$ . Therefore, in this case,  $C_H(\arg(u_a), v)$  only depends on  $v$ .

**Proof of Lemma 4.1.** Let us define

$$G^N(w) = N^{2H+d} f(Nw) - \delta \left( \frac{w}{|w|} \right) |w|^{-2H-d},$$

and, for  $L \in \mathbb{N}^*$ , consider  $S_L^N(w) = \sum_{0 < |k| \leq L} G^N(w + 2k\pi)$ . Since  $f$  satisfies Condition (3), we have

$$f(N(w + 2k\pi)) \leq C_N |k|^{-2H-d},$$

for  $k \neq 0$ ,  $w \in [0, 2\pi]^d$ , and  $N$  sufficiently large. Hence,  $\sum_{0 < |k|} f(N(w + 2k\pi))$  is normally convergent, and, as  $L$  tends to  $+\infty$ ,  $S_L^N$  tends uniformly to

$$S^N = N^{2H+d} \sum_{0 < |k|} f(N(w + 2k\pi)) - \sum_{0 < |k|} \delta \left( \frac{w + 2k\pi}{|w + 2k\pi|} \right) |w + 2k\pi|^{-2H-d}.$$

Let us now consider

$$U^N = \int_{[0, 2\pi]^d} |\hat{v}(T'_a w) \hat{v}(T'_b w)|^2 (S^N(w) + G^N(w))^2 dw,$$

and show that  $U^N$  tends to 0 as  $N$  tends to  $+\infty$ .

First, let us quote that, for all  $w \in [0, 2\pi]^d$ ,  $S^N(w)$  is bounded by

$$I^N = \int_{\mathbb{R}^d \setminus B(0, A)} |G^N(w)| dw = \int_{S^{d-1}} \int_A^{+\infty} |G^N(\rho s)| \rho^{d-1} d\rho ds,$$

where  $B(0, A)$  denotes a ball of  $\mathbb{R}^d$  centered at 0 of radius  $0 < A \leq 2\pi$ . Further notice that

$$|G^N(\rho s)| \leq G_1^N(\rho s) + G_2^N(\rho s), \quad (42)$$

$$\text{where } G_1^N(\rho s) = N^{2H+d} \left| f(N\rho s) - \tau(s)(N\rho)^{-2\beta(s)-d} \right|, \quad (43)$$

$$\text{and } G_2^N(\rho s) = \left| N^{2(H-\beta(s))} \tau(s) \rho^{-2\beta(s)-d} - \delta(s) \rho^{-2H-d} \right|. \quad (44)$$

Hence,  $I^N \leq I_1^N + I_2^N$  with  $I_j^N = \int_{S^{d-1}} \int_A^{+\infty} G_j^N(\rho s) \rho^{d-1} d\rho ds$ .

Since  $f$  satisfies Condition (3) and  $\tau$  is bounded, we have

$$I_1^N \leq c_1 N^{-\gamma} A^{-2H-\gamma},$$

for some  $c_1 > 0$ , and large  $N$ . So,  $\lim_{N \rightarrow +\infty} I_1^N = 0$ .

Besides, for  $\eta > 0$ , let us define sets

$$E_\eta = \{s \in S^{d-1}, H < \beta(s) < H + \eta\} \text{ and } F_\eta = \{s \in S^{d-1}, \beta(s) \geq H + \eta\}. \quad (45)$$

When  $s \in E_\eta \cup F_\eta$ ,  $\delta(s) = 0$ , so that  $G_2^N(\rho s) \leq cN^{2(H-\beta(s))} \rho^{-2\beta(s)-d}$ . When  $s \in E_0$ ,  $\tau(s) = \delta(s)$  and  $\beta(s) = H$ , so that  $G_2^N(\rho s) = 0$ . Hence,

$$I_2^N \leq c_2 \int_{E_\eta \cup F_\eta} N^{2(H-\beta(s))} A^{-2\beta(s)} ds.$$

Thus,  $I_2^N \leq \tilde{c}_2(\mu(E_\eta) + N^{-2\eta})$ , where  $\mu(E_\eta)$  is the Lebesgue measure of  $E_\eta$  over the sphere  $S^{d-1}$ . Let us show that  $\lim_{\eta \rightarrow 0^+} \mu(E_\eta) = 0$ . Assume it is not the case. Then, there exists  $c_0 > 0$  and a decreasing sequence  $(\eta_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} \eta_n = 0$  and  $\mu(E_{\eta_n}) > c_0$ . Since  $E_\eta \subset E_{\eta'}$  when  $\eta < \eta'$ , the sequence  $\mu(E_\eta)$  decreases, and admits a positive limit as  $\eta$  decreases to 0. This implies that  $\mu(\bigcap_{\eta < \eta_0} E_\eta) > 0$ . So, take  $s \in \bigcap_{\eta < \eta_0} E_\eta$ . It satisfies  $H < \beta(s) < H + \eta$  for all  $\eta > 0$ . This yields  $\beta(s) = H$ , which is contradictory.

Now, for  $0 < \alpha < 1$ , let us set  $\eta_N = \log(N)^{-\alpha}/2$ . We then obtain

$$I_2^N \leq \tilde{c}_2(\mu(E_{\log(N)^{-\alpha}}) + e^{-\log(N)^{1-\alpha}}), \quad (46)$$

and  $\lim_{N \rightarrow +\infty} I_2^N = 0$ . Therefore,  $\lim_{N \rightarrow +\infty} I^N = 0$ . Thus, on  $[0, 2\pi]^d$ ,  $S^N$  converges uniformly to 0, as  $N$  tends to 0.

Consequently, the integral  $U^N$  is bounded by  $c_1(\sup_{w \in [0, 2\pi]^d} |S^N(w)|)^2 + c_2 J^N$ , where

$$J^N = \int_{[0, 2\pi]^d} |\hat{v}(T'_a w) \hat{v}(T'_b w)|^2 ((G^N(w))^2 + G^N(w)) dw.$$

Let us then decompose  $J^N$  into a sum of two integrals  $J_1^N = \int_{B(0, A)} \cdots dw$  and  $J_2^N = \int_{[0, 2\pi]^d \setminus B(0, A)} \cdots dw$ , and study separately these integrals.

Notice that  $\hat{v}(y) = Q_v(e^{iy_1}, \dots, e^{iy_d})$  where  $Q_v$  is the characteristic polynomial of  $v$ . Hence, using Proposition 3.2 and Taylor expansions of  $Q_v$  in the neighborhood of 0, we obtain  $|\hat{v}(y)|^2 \leq C|y|^{2K+2}$  for some  $C > 0$ . Therefore,  $J_1^N$  is bounded by

$$c_3 N^{4(H-K-1)+d} \int_{B(0, \frac{A}{N})} |w|^{4(K+1)} (f^2(w) + f(w)|w|^{-2H-d} + |w|^{-4H-2d}) dw \\ + c_4 N^{2(H-2K-2)} \int_{B(0, \frac{A}{N})} |w|^{4(K+1)} (f(w) + |w|^{-2H-d}) dw,$$

for some  $c_3, c_4 > 0$ . In this upper bound, integrals of the form  $\int_{B(0, \epsilon)} |w|^u f(w) dw$  are finite for both  $u = 4(K+1) - 2H - d$  and  $u = 4(K+1)$ , since  $Z$  is a  $M$ -IRF and  $K \geq M/2 + d/4$ . Moreover,  $\sup_{w \in B(0, \epsilon)} |w|^{2K+3} f(w) \leq c < +\infty$ , since  $Z$  is a  $M$ -IRF and  $K_a \geq M$ . Therefore,

$$\int_{B(0, \epsilon)} |w|^{4(K+1)} f^2(w) dw \leq c \int_{B(0, \epsilon)} |w|^{2K+1} f(w) dw < +\infty,$$

since  $K \geq M + 1$ . Besides, integrals of the form  $\int_{B(0, \epsilon)} |w|^u dw$  are finite for  $u = 4(K+1) - 2H - d$  and  $u = 4(K+1) - 4H - 2d$ , since  $K \geq d/4$ . Consequently,

$$J_1^N \leq \tilde{c}_3 N^{4(H-K-1)+d} + \tilde{c}_4 N^{2(H-2K-2)},$$

and  $\lim_{N \rightarrow +\infty} J_1^N = 0$ , since  $K \geq d/4$ .

Besides, using the bound (42), we obtain

$$J_2^N \leq \int_{S^{d-1}} \int_A^{+\infty} (G_1^N(\rho s) + G_2^N(\rho s))^2 \rho^{d-1} d\rho ds.$$

Then, using previous bounds on  $G_1^N$  and  $G_2^N$ , we get

$$J_2^N \leq c_5 (N^{-2\gamma} + N^{-2\eta} + \mu(E_\eta)) \Gamma_H,$$

with  $c_5 > 0$  and  $\Gamma_H = \int_A^{+\infty} \rho^{-4H-d-1} d\rho < +\infty$ . Setting again  $\eta_N = \log(N)^{-\alpha}/2$  with  $0 < \alpha < 1$ , we obtain  $\lim_{N \rightarrow +\infty} J_2^N = 0$ . Therefore,  $\lim_{N \rightarrow +\infty} J^N = 0$ .

Consequently,  $\lim_{N \rightarrow +\infty} U^N = 0$ . This implies the convergence of  $N^{2H} f_{a,b}^N$  to  $f_{a,b}$  in  $L^2([0, 2\pi]^d)$ .

## 5 Numerical results

We conducted some experiments on AFBF (see definition in Equation (2)) simulated using the turning-band method developed in [15]. This method was applied with approximately 500 bands to simulate field realizations on a grid of size  $100 \times 100$ . We simulated fields by triplets consisting of an isotropic fractional Brownian field (*i.e.* AFBF with constant Hurst and topothesy functions), an AFBF with a non-constant Hurst function and a constant topothesy (*i.e.* with a Hurst-induced anisotropy), and an AFBF with a constant Hurst function and a non-constant topothesy function (*i.e.* with a topothesy-induced anisotropy). For anisotropic fields, the Hurst and topothesy functions were specified using a function  $g_{a_1, a_2, \delta_1, \delta_2, \varphi}$  depending on a few parameters  $a_1 > 0$ ,  $a_2 > 0$ ,  $0 < \delta_1 \leq \pi/2$ ,  $0 < \delta_2 < \pi/2 - \delta_1$ ,  $\varphi \in (0, \pi)$ . This function was defined as the  $\pi$ -periodic and  $\mathcal{C}^2$  function which is equal to  $a_1$  on a set of intervals  $\cup_{k \in \mathbb{Z}} (\varphi - \delta_1 + k\pi, \varphi + \delta_1 + k\pi)$ ,  $a_2$  in another disjoint set  $\cup_{k \in \mathbb{Z}} (\varphi + \pi/2 - \delta_2 + k\pi, \varphi + \pi/2 + \delta_2 + k\pi)$ , and polynomial of order 2 outside those two sets. Such a function leads to a field whose the anisotropy is oriented in a privileged direction  $\varphi$  (see Examples in Figure 5). For each field triplet, we set parameters  $\delta_1$ ,  $\delta_2$ ,  $\varphi$  by sampling from uniform distributions on  $(0, \pi/2)$ ,  $(0, \pi/2 - \delta_1)$ , and  $(0, \pi)$ , respectively. We also sampled parameters  $H_1 \sim \mathcal{U}(0, 1)$ ,  $H_2 \sim \mathcal{U}(H_1, 1)$ ,  $C_2 \sim |\mathcal{N}(0, 1)|$ , and  $C_1 \sim C_2 + |\mathcal{N}(0, 1)|$ . Fields of a same triplet were then defined as an AFBF with  $\beta \equiv H_1$ ,  $\tau \equiv 1$ , an AFBF with  $\tau \equiv 1$  and  $\beta = g_{H_1, H_2, \delta_1, \delta_2, \varphi}$ , and an AFBF with  $\tau = g_{C_1, C_2, \delta_1, \delta_2, \varphi}$  and  $\beta \equiv H_1$ . In this way, fields of a same triplet had the same order  $H_1$  of Hölder irregularity, and anisotropic fields had a same privileged direction  $\varphi$ . We simulated 2000 triplets (6000 fields), covering largely the range of the irregularity and orientation pair  $(H_1, \varphi)$ .

For testing the isotropy, we used quadratic variations obtained with different increments filters of Example 3.1. We took two filters of order 1 specified  $((L_1, L_2)$  in  $\{(0, 2), (1, 1)\}$ ), two of order 2  $((L_1, L_2)$  in  $\{(0, 3), (1, 2)\}$ ), three of order 3  $((L_1, L_2) \in \{(0, 4), (1, 3), (2, 2)\})$ . According to Theorem 3.4, filters of order 1, 2 and 3 are adapted to 0-IRF, 1-IRF, and 2-IRF, respectively. We did not consider filters  $(L_1, L_2)$  with  $L_1 > L_2$  since they are symmetric to filters  $(L_2, L_1)$  and lead to the same quadratic variations. For filters with  $L_1 \neq L_2$ , we computed

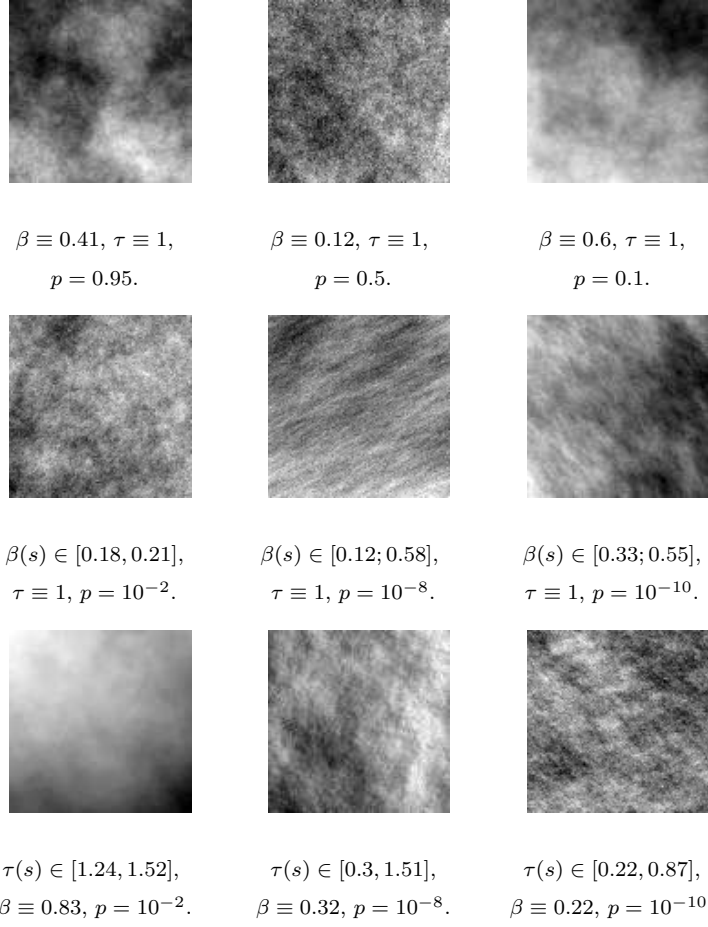


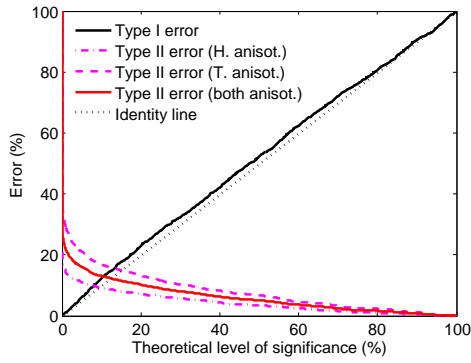
Figure 2: Examples of simulated fields. On the first, second, and third rows, fields are (isotropic) fractional Brownian fields, AFBF with a constant topothesy function, and AFBF with a constant Hurst function, respectively. P-values of the isotropy test (filter (0,2)) applied to each realizations are indicated by  $p$  below each image.

quadratic variations on images transformed by composition of rotations of angles 0 (*i.e.* for  $u \propto (1, 0)$ ),  $\pi/4$  ( $u \propto (1, 1)$ ),  $\pi/2$  ( $u \propto (0, 1)$ ), or  $3\pi/4$  ( $u \propto (1, -1)$ ), and rescalings of factors  $|u|^2$  with  $1 \leq |u| \leq 6$ . Hence, we had 6 quadratic variations in vertical or horizontal directions ( $u \propto (1, 0)$  or  $u \propto (0, 1)$ ), and 4 in each diagonal direction ( $u \propto (1, 1)$  or  $u \propto (1, -1)$ ). For the filter (1, 1), we only used rotations of angles 0 and  $\pi/4$ . Indeed, due to the symmetry of filters for which  $L_1 = L_2$ , quadratic variations are the same up to a rotation of angle  $\pi/2$ . For each filter, we computed the Fisher statistic  $F^N$  defined in (38). Given a threshold  $s_\alpha$ , a realization could then be considered as isotropic whenever  $F^N \leq s_\alpha$ , and anisotropic in the opposite case. Following Theorem 3.4, under the null assumption  $\mathcal{H}_0$ , Fisher statistics are theoretically distributed according to a Fisher law of degrees (3, 15) if  $(L_1 \neq L_2)$  and (1, 7) if  $L_1 = L_2$ . Hence, the threshold  $s_\alpha$  could be set to ensure a theoretical level of significance  $P_{\mathcal{H}_0}(F^N > s_\alpha) \leq \alpha$ , for  $\alpha \in (0, 1)$ .

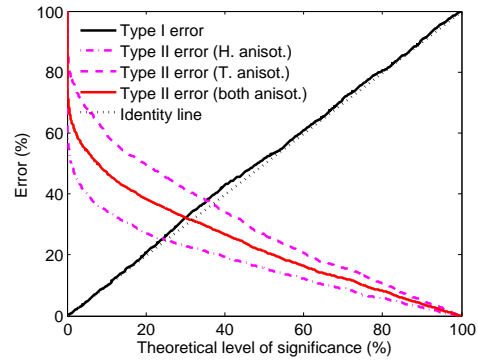
To assess the performance of a test, we estimated the probability distribution  $P_{\mathcal{H}_1}(F^N \leq s_\alpha)$  of errors of type II. For that, we computed the ratio between the number of misclassified anisotropic cases and the total number of anisotropic case, for different significance levels  $\alpha$ . This was done for the two anisotropy types separately and jointly. We also computed the empirical distribution of errors of type I ( $P_{\mathcal{H}_0}(F^N > s_\alpha)$ ) to check its adequacy to the theoretical one. Evaluation results are shown in Figure 3.

We observe that empirical distributions of type I errors (black lines) are slightly above the significance level. This means that values of the Fisher statistics tend to be higher than expected on isotropic realizations. For each filter, we tested the adequacy of the empirical distribution of  $F^N$  (under  $\mathcal{H}_0$ ) to its theoretical one using a one-way Kolmogorov-Smirnov test. Adequacy assumptions were validated with p-values 0.03, 0.08, 0.3, 0.15, 0.22, 0.28, 0.48 for filters (0, 2), (1, 1), (0, 3), (1, 2), (0, 4), (1, 3) and (2, 2), respectively.

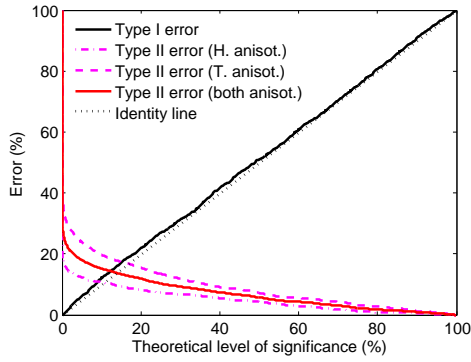
Filters had different performances in terms of type II errors (red lines). The symmetric filter (1, 1) had the worst results. This filter mixes increments in two orthogonal directions, which leads to a lost of directional information. The filter (2, 2) had comparable poor performances, which are not presented in the figure. The filter (0, 3) (resp. (0, 4)) outperformed the filter (1, 2) (resp. (1, 3))



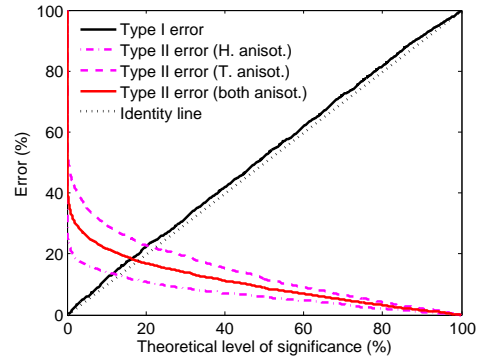
Filter (0,2)



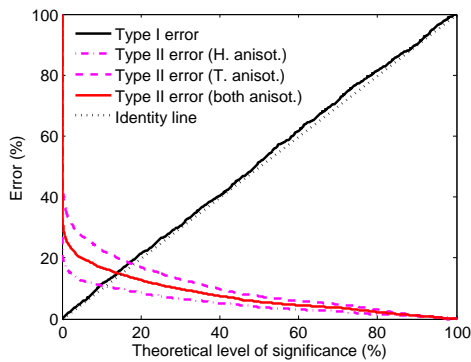
Filter (1,1)



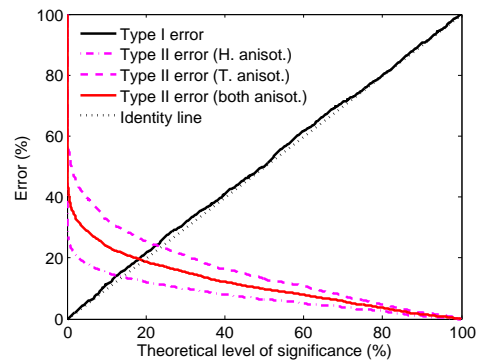
Filter (0,3)



Filter (1,2)



Filter (0,4)



Filter (1,3)

Figure 3: Evaluation of the isotropy tests on 6000 simulated fields.



suggesting that the most directional filters of the form  $(0, L_2)$  are preferable to the other ones. Choosing those filters led to fairly good results. For a significance level of 0.01 (*i.e.* less than 1% of misclassified isotropic cases), we detected 78.3%, 76.6%, 73.3% percents of anisotropic cases for  $(0, 2)$ ,  $(0, 3)$  and  $(0, 4)$ , respectively. The performance of these filters decreased as the filter order increased. This was probability due to the fact that the number of points ( $n_e$ ) used for computing quadratic variations decreases as the order increases. Nevertheless, the detection score obtained with the filter  $(0, 4)$  of higher order was still close to the filter  $(0, 2)$ . Hence, we conclude that anisotropic cases can be well-detected using filters of the form  $(0, L_2)$  even when a polynomial trend is present in images.

Besides, let us notice that percents of detected cases were significantly lower for cases with a toposy anisotropy (red dot lines) than with the toposy one (red dash lines). This performance difference will be explained below.

We further investigated effects of simulation and analysis parameters on the isotropic test. For that, we classified simulations of anisotropic fields into isotropic and anisotropic by applying the test of filter  $(0, 2)$  with a level of significance  $\alpha = 0.05$ . We then computed numbers and percentages of detected anisotropic cases by range of parameter values.

Range of $\varphi$	Total case number	Analysis in four directions		Analysis in two directions	
		Detected cases		Detected cases	
		H. anisot.	T. anisot.	H. anisot.	T. anisot.
[-1.57,-1.37]	137	125 (91%)	114 (83%)	131 (96%)	122 (89%)
[-1.37,-0.98]	261	231 (89%)	204 (78%)	227 (87%)	196 (75%)
[-0.98,-0.59]	233	211 (91%)	184 (79%)	140 ( <b>60%</b> )	88 ( <b>38%</b> )
[-0.59,-0.20]	240	214 (89%)	177 (74%)	214 (89%)	167 (70%)
[-0.20, 0.20]	246	220 (89%)	194 (79%)	225 (91%)	202 (82%)
[0.20, 0.59]	222	205 (92%)	175 (79%)	202 (91%)	164 (74%)
[0.59, 0.98]	289	255 (88%)	228 (79%)	165 ( <b>57%</b> )	113 ( <b>39%</b> )
[0.98, 1.37]	238	213 (89%)	195 (82%)	213 (89%)	190 (80%)
[1.37, 1.57]	134	120 (90%)	108 (81%)	121 (90%)	116 (87%)

Table 1: Effect of the anisotropy orientation on the detection.

We first focus on the effect of the anisotropy direction  $\varphi$ . In Table 1, we report results obtained with two isotropy tests. The first one corresponds to the

test described initially ("analysis in four directions"), whereas the second one ("analysis in two directions") is based on variations computed in the horizontal and vertical directions only. Concerning the first test, we observe that, for both anisotropy types, the number of detected cases were quite uniform with respect to the direction. This suggests that the detection is independent of this factor. This independence was confirmed by a  $\chi^2$ -test with a large p-value of 0.56. This clearly differed from the second test. Without the diagonal directions, the test performances were much lower when the anisotropy orientation  $\varphi$  was away from the vertical and horizontal directions (especially when  $\varphi \in (-0.98, -0.59)$  or  $\varphi \in (0.59, 0.98)$ ).

Range of $\varphi$	Total case number	Test 1 of [59]		Test 2 of [59]	
		Detected cases		Detected cases	
		H. anisot.	T. anisot.	H. anisot.	T. anisot.
[-1.57,-1.37]	137	21 (15%)	8 (6%)	30 (22%)	9 (7%)
[-1.37,-0.98]	261	37 (14%)	8 (3%)	45 (17%)	12 (5%)
[-0.98,-0.59]	233	41 (18%)	12 (5%)	20 (9 %)	4 (2%)
[-0.59,-0.20]	240	41 (17%)	8 (3%)	29 (12%)	21 (9%)
[-0.20, 0.20]	246	47 (19%)	13 (5%)	37 (15%)	22 (9%)
[0.20 , 0.59]	222	52 (23%)	14 (6%)	34 (15%)	21 (9%)
[0.59 , 0.98]	289	50 (17%)	16 (6%)	17 (6 %)	18 (6%)
[0.98 , 1.37]	238	33 (14%)	11 (5%)	40 (17%)	15 (6%)
[1.37 , 1.57]	134	19 (14%)	5 (4%)	31 (23%)	4 (3%)

Table 2: Evaluation of isotropy tests of Richard and Biermé [59].

On the same data, we also evaluated the two isotropy tests proposed in [59] (see Table 2). Both of these tests had much lower detection scores than the new ones. Their performances were particularly low when the anisotropy was induced by the topothesis or oriented away from the vertical and horizontal directions.

In Table 3, we deal with the order of Hölder irregularity  $H_1$ . We notice that the detection was worser for fields with a very low regularity ( $0 < H_1 \leq 0.2$ ) than for the more regular one. But, for a regularity above 0.2, detection performances seem quite uniform with respect to  $H_1$ . We applied a  $\chi^2$ -test on all anisotropic cases for which  $H_1 > 0.2$ . This test confirmed the independence of the irregularity and the detection with a p-value of 0.36. In other words, the irregularity effect

Range of $H_1$	Detected cases	
	H. anisotropy	T. anisotropy
[0, 0.1]	175/206 (85%)	95/206 (46%)
[0.1, 0.2]	197/203 (97%)	152/203 (75%)
[0.2, 0.3]	194/202 (96%)	160/202 (79%)
[0.3, 0.4]	186/194 (96%)	152/194 (78%)
[0.4, 0.5]	207/212 (98%)	176/212 (83%)
[0.5, 0.6]	186/203 (92%)	179/203 (88%)
[0.6, 0.7]	178/188 (95%)	161/188 (86%)
[0.7, 0.8]	164/181 (91%)	157/181 (87%)
[0.8, 0.9]	176/197 (89%)	163/197 (83%)
[0.9, 1]	131/214 (61%)	184/214 (86%)

Table 3: Effect of the irregularity on the anisotropy detection.

is limited to the lowest values of  $H_1$ .

Notice however that the detection was low (61%) for fields with a Hurst anisotropy when  $H_1 > 0.9$ . This was not directly due to the field irregularity, but rather to the low amplitude  $\delta\beta = H_2 - H_1$  of the Hurst function, which is necessarily below 0.1 when  $H > 0.9$ .

H. anisotropy		T. anisotropy	
$\delta\beta$	Detected cases	$\delta\tau$	Detected cases
[0, 0.01]	19/128 (15%)	[1, 1.1]	9/140 (6%)
[0.01, 0.02]	17/91 (19%)	[1.1, 1.2]	39/129 (30%)
[0.02, 0.03]	41/66 (62%)	[1.2, 1.3]	67/119 (56%)
[0.03, 0.04]	70/81 (86%)	[1.3, 1.4]	87/120 (72%)
[0.04, 0.07]	152/168 (90%)	[1.4, 1.8]	323/385 (84%)
[0.07, 1]	1427/1466 (97%)	[1.8, +∞]	1054/1107 (95%)

Table 4: Effect of the anisotropy amplitude on the detection.

The effect of this amplitude is analyzed into more details in Table 4. We clearly see that the detection depended on this factor up to a bound, which was about 0.07. When the amplitude is above this bound, the detection was almost perfect (97% of detected cases). The same phenomenon occurred for the detection of cases with a topothesy anisotropy. For these cases, we defined the

amplitude as  $\delta\tau = C_2/C_1$ . The anisotropic cases were almost perfectly detected when the amplitude was above 1.8. Hence, results obtained for these cases could have been better if we had chosen to simulate fields with higher amplitudes.

## 6 Discussion

We constructed an original statistical methodology to test whether the texture of an irregular image is isotropic or not. Statistics of our isotropy tests were built upon the well-known quadratic variations. However, specific to our approach, these variations were defined with increments computed both in different orientations and at different scales. Hence, variations could bring useful directional information. We developed the asymptotic theory of tests in a general context of intrinsic random fields which allows us to take into account the presence of arbitrary polynomial trends in images. We established an asymptotic Gaussian linear relationship between variation and scale logarithms which involves direction-dependent intercepts. We then designed isotropy tests using Fisher statistics which check the equality of these intercepts. The methodology was validated using 6000 realizations of anisotropic fractional Brownian fields simulated on a grid of size  $100 \times 100$ . Results showed that our tests could detect more than 70% of anisotropic cases with less than 1% of misclassified isotropic cases in the presence of a polynomial trend of order 2. They also revealed that test performances were not dependent on the anisotropy direction, and outperformed those of tests proposed in [59].

The asymptotical result was obtained by applying a multivariate version of the Breuer-Major (see Theorem 3.4). Proved in [14], this Breuer-Major theorem was already applied for a variation-based estimation of the Hölder irregularity in the case when the field is Gaussian with stationary increments and has a spectral density satisfying Condition (3) with a constant Hurst function. We thus extended its application to both a more general framework of IRF satisfying Condition (3) with non-constant Hurst functions, and the analysis of multi-oriented variations.

Our test methodology is semi-parametric. It is theoretically established for a large class of Gaussian random fields whose spectral densities satisfy two weak

conditions. The first one is a generic non-parametric integrability condition at low frequency. The second is a condition at high frequency constraining densities to be close to the one of an AFBF. This condition involves some functional parameters (topothesy and Hurst functions). However, the test procedure does not involve any estimation of these parameters.

Some isotropy tests developed in [21, 37, 49] for spatial data could be applied to images. These tests enable to check if a stationary field is geometrically isotropic, *i.e.* if it has a rotation-invariant covariance. However, such an isotropy concerns all field frequencies whereas ours focuses on the asymptotic of the highest ones. Hence, applied to images, tests in [21, 37, 49] would probably detect anisotropic features which are not necessarily related to textures we deal with. Moreover, they are not suitable for IRF of arbitrary orders.

In Image Processing, the analysis of the texture anisotropy is often associated to a detection of a privileged anisotropy direction [2, 35, 36, 47, 53, 58, 45]. This topic is beyond the scope of the work presented here. But, we plan to extend our methodology to include confidence intervals and tests for anisotropy directions.

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## A A multivariate Breuer Major Theorem

The original Breuer-Major theorem was shown for stationary processes in [19], and extended to multivariate fields in [3]. Another formulation of the Breuer-Major Theorem is demonstrated in [14] (Theorem 3.2) using the Malliavin calculus. We state a specific version of this theorem which is sufficient for the proof of Theorem 3.4.

**Theorem A.1** (Breuer-Major theorem). *Let  $d, l \in \mathbb{N}^*$ , and  $X_N = (X_N[k])_{k \in \mathbb{Z}^d}$  be centered Gaussian stationary fields with values in  $\mathbb{R}^l$ . Assume that there exist functions  $g_{a,b}^N$  in  $L^2([0, 2\pi]^d)$  (spectral densities) such that, for all  $a, b \in \llbracket 1, l \rrbracket$  and  $k \in \mathbb{Z}^d$ ,*

$$\text{Cov}(X_a^N[k], X_b^N[0]) = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} e^{i\langle w, k \rangle} g_{a,b}^N(w) dw.$$

*Further assume that, for all  $a, b \in \llbracket 1, l \rrbracket$ ,  $g_{a,b}^N$  converges in  $L^2([0, 2\pi]^d)$  to a function  $g_{a,b}$  as  $N$  tends to  $+\infty$ . Define*

$$\forall k \in \mathbb{Z}^d, r_{a,b}[k] = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} e^{i\langle w, k \rangle} g_{a,a}(w) dw,$$

*and assume that  $r_{a,b}[0] = 1$ . Then,*

$$\frac{1}{N^{d/2}} \sum_{k \in \llbracket 1, N \rrbracket^d} ((X^N[k])^2 - \mathbf{1}) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \Sigma),$$

*where  $\mathbf{1}$  is the unit vector of size  $l$  and  $\Sigma$  is a  $l \times l$ -matrix having terms*

$$\Gamma_{a,b} = 2 \sum_{k \in \mathbb{Z}^d} (r_{a,b}[k])^2 = \frac{2}{(2\pi)^d} \int_{[0, 2\pi]^d} |g_{a,b}(w)|^2 dw.$$

## B Covariance estimation

In this section, we construct an estimate of the covariance matrix  $\Sigma^N$  of log-variations  $Y^N$  involved in the linear model (32). According to the proof in Section 4, the random vector  $Y^N$  has the same asymptotical covariance  $\Sigma$  as the

random vector  $U^N$  defined by Equation (39). Hence, we approximate  $\Sigma^N$  by an estimate of the covariance matrix of  $U^N$ . We have

$$\mathbb{E}(U_a^N U_b^N) = \frac{1}{N_e^2 \mathbb{E}(W_a^N) \mathbb{E}(W_b^N)} \sum_{p,q \in \mathcal{E}_N} \mathbb{E}((V_a^N[p])^2 (V_a^N[q])^2).$$

But  $(V_a^N[p], V_a^N[q])$  are centered Gaussian vectors. Thus,  $E((V_a^N[p])^2 (V_a^N[q])^2) = 2(E(V_a^N[p] V_a^N[q]))^2$ . Moreover,

$$E(V_a^N[p] V_a^N[q]) = \sum_{k,l} v[k] v[l] K_Z \left( \frac{p - T_a k - q + T_b l}{N} \right),$$

where  $K_Z$  is the generalized covariance of the IRF  $Z$ . Let  $\tilde{H}$  be an estimate of the Hölder irregularity of  $Z$  (*e.g.* an OLS estimate of  $H$  in the linear model (32)). Approximating the generalized covariance  $K_Z$  by the one of a fractional Brownian field of order  $\tilde{H}$ , it follows that

$$E(V_a^N[p] V_a^N[q]) \simeq C_{\tilde{H}} \sum_{k,l} v[k] v[l] |p - T_a k - q + T_b l|^{2\tilde{H}}.$$

Using the same approximation, we also have

$$\mathbb{E}(W_a^N) = \mathbb{E}((V_a^N[0])^2) \simeq \sum_{k,l} v[k] v[l] |T_a(l - k)|^{2\tilde{H}}.$$

Hence, we get

$$\Sigma_{a,b}^N \simeq \sum_{\delta \in \Delta \mathcal{E}_N} \frac{N_\delta \left( \sum_{k,l} v[k] v[l] |\delta - T_a k + T_b l|^{2\tilde{H}} \right)^2}{N_e^2 \left( \sum_{k,l} v[k] v[l] |T_a(l - k)|^{2\tilde{H}} \right) \left( \sum_{k,l} v[k] v[l] |T_b(l - k)|^{2\tilde{H}} \right)},$$

where  $\Delta \mathcal{E}_N = \{\delta = p - q, p, q \in \mathcal{E}_N\}$  and  $N_\delta$  is the number of couples  $(p, q) \in \mathcal{E}_N^2$  for which  $\delta = p - q$ .

## C Proofs of Propositions

*Proof of Proposition 3.1.* By definition, the kernel  $v$  leads to  $K$ -order increments of  $Z$  if and only if  $\sum_{k \in [0, L]^d} v[k] (m - k)^l = 0, \forall m \in \mathbb{Z}^d, \forall l \in \llbracket 0, K \rrbracket^d, |l| \leq K$ , which is also equivalent to

$$\sum_{k \in L} v[k] k^l = 0, \forall l \in \llbracket 0, K \rrbracket^d, |l| \leq K.$$

Besides, we have

$$\frac{\partial^{|l|} \mathcal{Q}_v}{\partial z^l}(z) = \sum_{k_1=l_1}^{L_1} \cdots \sum_{k_d=l_d}^{L_d} v[k] \prod_{j_1=0}^{l_1-1} (k_1 - j_1) \cdots \prod_{j_d=0}^{l_d-1} (k_d - j_d) z^{k-l},$$

using the convention that  $\prod_{j_i=0}^{l_i-1} (k_i - j_i) = 1$  if  $l_i = 0$ . From that, we deduce the recurrence equations

$$\frac{\partial^{|l|} \mathcal{Q}_v}{\partial z^l}(z) = \sum_{k \in \llbracket 0, L \rrbracket^d} v[k] k^l z^{k-l} - \sum_{j \in \llbracket 1, d \rrbracket, l_j \geq 1} \frac{l_j - 1}{z_j} \frac{\partial^{|l|-1} \mathcal{Q}_v}{\partial z^{l-e_j}}(z),$$

where  $e_j$  is the  $j$ th vector of the canonical basis of  $\mathbb{R}^d$ . In particular,

$$\forall j, l, \frac{\partial^{|l|} \mathcal{Q}_v}{\partial z^l}(1, \dots, 1) + \sum_{j \in \llbracket 1, d \rrbracket, l_j \geq 1} (l_j - 1) \frac{\partial^{|l|-1} \mathcal{Q}_v}{\partial z^{l-e_j}}(1, \dots, 1) = \sum_{k \in \llbracket 0, L \rrbracket^d} v[k] k^l.$$

We conclude the proof by recurrence on the order of the partial derivatives of  $\mathcal{Q}$ .  $\square$

*Proof of Proposition 3.2.* Let  $P$  be a polynomial of degree  $l \leq K$ . We notice

$$\sum_k v[k] P\left(\frac{m - T_u k}{N}\right) = \sum_k v[k] P \circ T_u\left(\frac{m' - k}{N}\right), \text{ with } m' = T_u^{-1} m.$$

But, any rescaling or rotation  $P \circ T_u$  of a polynomial  $P$  remains a polynomial of the same degree. Hence,  $\sum_k v[k] P\left(\frac{m - T_u k}{N}\right) = 0$  for all  $P$  of degree  $l \leq K$  if and only if  $\sum_k v[k] P\left(\frac{m' - k}{N}\right) = 0$  for all  $P$  of degree  $l \leq K$ . According to Proposition (3.1), this only holds if an and only if Condition (17) is satisfied.  $\square$

*Proof of Proposition 3.3.* Since, for  $a \in \tilde{\mathcal{F}}$  and  $m \in \mathbb{Z}^d$ ,  $V_a^N[m]$  is an increment of  $Z$  of order  $\geq M$ , it has zero mean. Moreover, for any  $a, b \in \tilde{\mathcal{F}}$  and  $m, n \in \mathbb{Z}^d$ , Theorem 2.3 yields

$$\mathbb{E}(V_a^N[m] V_b^N[n]) = \sum_{k, l \in \mathbb{Z}^d} v[k] v_b[l] K_Z \left( \frac{m - T_a k}{N} - \frac{n - T_b l}{N} \right) < +\infty,$$

where  $K_Z$  is a generalized covariance of the form (8). But, for any even polynomial  $P$  of degree  $2M$ , we can write  $P(x - y) = \sum_{|l|=0}^M q_l(y) x^l + \sum_{|l|=0}^M q_l(x) y^l$  where  $q_l$  are polynomials of degree up to  $2M$ . Hence, since  $V_a^N[m]$  and  $V_b^N[n]$  are increments of order  $K \geq M$ ,

$$\sum_{k, l \in \mathbb{Z}^d} v[k] v[l] P\left(\frac{m - T_a k}{N} - \frac{n - T_b l}{N}\right) = 0,$$

for any even polynomial  $P$  of degree  $2M$ , including  $P_M$  and  $Q$  of Equation (8).  
Therefore,

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sum_{k,l \in \mathbb{Z}^d} v[k]v[l] \cos \left( \left\langle \frac{m - T_a k}{N} - \frac{n - T_b l}{N}, w \right\rangle \right) f(w) dw,$$

and, since  $f$  is even,

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sum_{k,l \in \mathbb{Z}^d} v[k]v[l] e^{i \langle \frac{m-n}{N} - \frac{T_a k}{N} + \frac{T_b l}{N}, w \rangle} f(w) dw.$$

From this, we deduce

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{v} \left( \frac{T'_a w}{N} \right) \bar{\hat{v}}_b \left( \frac{T'_b w}{N} \right) e^{i \langle \frac{m-n}{N}, w \rangle} f(w) dw.$$

Since this expression exclusively depends on  $m - n$ , and not on  $m$  and  $n$ ,  $V^N$  is stationary. Using a variable change  $\zeta = w/N$ , and a decomposition of the integral domain, we further obtain

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \int_{[0, 2\pi]^d + 2k\pi} \hat{v}(T'_a \zeta) \bar{\hat{v}}_b(T'_b \zeta) e^{i 2\pi \langle m-n, \zeta \rangle} f(N\zeta) N^d d\zeta,$$

After a variable change  $\zeta = w + 2k\pi$  in each integral, we then get

$$\mathbb{E}(V_a^N[m]V_b^N[n]) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{[0, 2\pi]^d} g_{a,b}^N(w, z) dw d\Delta(z),$$

where  $d\Delta(u) = \sum_{k \in \mathbb{Z}^d} \delta_{2k\pi}(u)$  is a counting measure on  $\mathbb{R}^d$  and  $g_{a,b}^N(w, z) = N^d \hat{v}(T'_a w) \bar{\hat{v}}_b(T'_b w) e^{i \langle m-n, w \rangle} f(N(w+z))$ . Since  $\mathbb{E}(V_a^N[m]V_b^N[n]) < +\infty$ , the Lebesgue Fubini theorem implies that  $f_{a,b}^N(w) = \int_{\mathbb{R}^d} g_{a,b}^N(w, z) d\Delta(z)$  is almost everywhere defined, and that  $\int_{[0, 2\pi]^d} f_{a,b}^N(w) dw < +\infty$ .  $\square$

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