

# Rhythmic generation of trees and languages

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## Abstract

This work builds on the notion of *breadth-first signature* of infinite trees and (prefix-closed) languages introduced by the authors in a previous work. We focus here on periodic signatures, a case coming from the study of rational base numeration systems; the language of integer representations in base  $\frac{p}{q}$  has a purely periodic signature whose period is derived from the Christoffel word of slope  $\frac{p}{q}$ . Conversely, we characterise languages whose signature are purely periodic as representations of integers in such number systems with non-canonical alphabets of digits.

## 1 Introduction

In this work, we study a family of (infinite) trees and languages that are defined by means of a new technique: the *breadth-first search* that we have introduced in a recent paper [11]. In that paper, we have explained that an ordered tree of finite degree  $\mathcal{T}$  can be characterised by the infinite sequence of the degrees of its nodes visited in the order given by the breadth-first search, called the *signature*  $\mathbf{s}$  of  $\mathcal{T}$ . This signature  $\mathbf{s}$ , together with an infinite sequence  $\boldsymbol{\lambda}$  of letters taken in an ordered alphabet characterises then a *labelled tree*  $\mathcal{T}$ .

If the sequence  $\boldsymbol{\lambda}$  is consistent with the signature  $\mathbf{s}$  — we call the pair  $(\mathbf{s}, \boldsymbol{\lambda})$  a *labelled signature* — the breadth-first search of  $\mathcal{T}$  corresponds to the enumeration in the *radix order* of the prefix-closed language  $L_{\mathcal{T}}$  of branches of  $\mathcal{T}$ . And we have shown ([11, Th. 1]) that regular trees or (prefix-closed) regular languages are characterised by those labelled signatures that are substitutive sequences.

Here, we consider and study the simplest possible signatures, and labelled signatures, when seen as infinite words, namely the purely periodic ones. The labelled tree — call it  $\mathcal{I}_{\frac{3}{2}}$  — shown at Fig. 1 gives an example of a labelled tree having such a periodic labelled signature. The nodes of  $\mathcal{I}_{\frac{3}{2}}$  are numbered by integers in the order of a breadth-first search and, with exception of the root 0 for a reason that will be explained later, even nodes have two children and odd nodes one, which results in the sequence  $2, 1, 2, 1, \dots = 21^\omega$  for the signature.

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Moreover, the sequence of labels of the arcs in the same breadth-first search is  $0, 2, 1, 0, 2, 1, \dots = 021^\omega$ .

The branch language of  $\mathcal{I}_{\frac{3}{2}}$ , that is, the set of words that label the paths from the root to every node, is the language  $L_{\frac{3}{2}}$  of the representations of the integers in the so-called *numeration system in base  $\frac{3}{2}$*  that has been introduced and studied in [1].

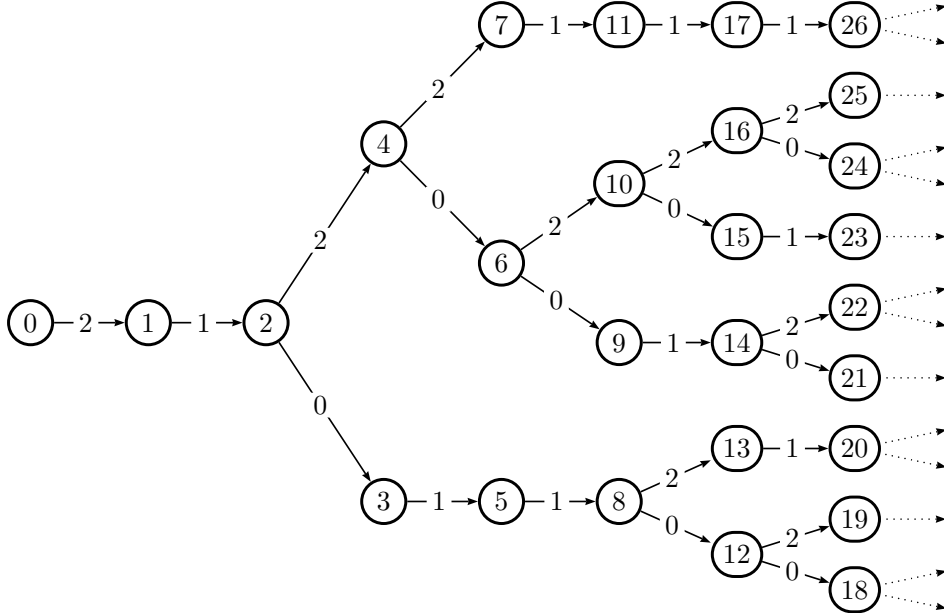


Figure 1: The tree  $\mathcal{I}_{\frac{3}{2}}$ , representation of the language  $L_{\frac{3}{2}}$

This language  $L_{\frac{3}{2}}$  and more generally the languages  $L_{\frac{p}{q}}$  of the representations of the integers in base  $\frac{p}{q}$  are indeed the starting point of this work. It is a challenge to better understand their structure, both from a number theoretic point of view — as we did in [1] — and from a formal language theory point of view as they seem to be at the same time ‘very regular’ and completely orthogonal to all classification and usual tools of classical formal language theory. In particular, none of them are regular languages and they even defeat any kind of iteration lemma. In a former work we have shown that these languages enjoy a kind of ‘autosimilarity property’ [2]. Here, we show that they are somehow characterised by their periodic labelled signature.

First, we show that the labelled signature of these languages are periodic (Theorem 23). Stating the converse requires some more definitions.

We call *rhythm of directing parameter*  $(q, p)$  a  $q$ -tuple of integers whose sum is  $p$ :  $\mathbf{r} = (r_0, r_1, \dots, r_{q-1})$ . We then describe how such rhythm  $\mathbf{r}$  allows to generate a tree such that the  $n$ -th node (in the breadth-first order) has  $r_k$  successors, where  $k$  is congruent to  $n$  modulo  $q$  (plus a special rule for the root).

The main result of this paper reads then (below we give the definition of FLIP languages, which, roughly speaking, are languages that meet no kind of iteration lemma):

**Theorem 12..** *Let  $K_{\mathbf{r}}$  be the branch language of the tree generated by a rhythm  $\mathbf{r}$  of directing parameter  $(q, p)$ .*

- a. If  $\frac{p}{q}$  is an integer, then  $K_{\mathbf{r}}$  is a regular language;
- b. If  $\frac{p}{q}$  is not an integer, then  $K_{\mathbf{r}}$  is a FLIP language.

In the general case, the rhythm of  $L_{\frac{p}{q}}$  corresponds to *the most equitable way of parting  $p$  objects into  $q$  cases* (with a bias to the left when necessary). We call it the *Christoffel rhythm* associated with  $\frac{p}{q}$ , as it can be derived from the more classical notion of Christoffel word of slope  $\frac{p}{q}$  (cf. [3]), that is, the canonical way to approximate the line of slope  $\frac{p}{q}$  on a  $\mathbb{Z} \times \mathbb{Z}$  lattice.

The proof of Theorem 12 is the purpose of Section 5 and consists of the reduction of any structure generated by a rhythm to the number system whose base is the growth ratio of this rhythm. In fact, the language generated by a rhythm is simply a non-canonical representation of the integers in this base, in the sense that the integers are represented on a non-canonical alphabet. Using the existing work on alphabet conversion in rational base number systems (cf. [1] or [6]) it allows to conclude that both languages are basically as complicated (or as simple, in the degenerate case where the growth ratio happens to be an integer).

This article is organised as follows. In the preliminaries, we present the three notions used in the sequel: the numeration system in base  $\frac{p}{q}$ , the Finite Left Iteration Property, and the trees and their signature. In Section 3 we give a precise definition of the breadth-first generation of infinite trees and language by a rhythm. Then in Section 4, we describe how this process can be used to generate the language of the representation of integers in a rational base numeration system. Finally, in Section 5, we prove that any language build by a rhythm is in some sense a non-canonical representation of the integers in some underlying rational base.

## 2 Preliminaries and Notation

Given two *positive integers*  $n$  and  $m$ , we denote by  $\frac{n}{m}$  their division in  $\mathbb{Q}$ ; by  $n \div m$  and  $n \% m$  respectively the quotient and the remainder of the Euclidean division of  $n$  by  $m$ , that is,  $n = (n \div m)m + (n \% m)$  and  $0 \leq (n \% m) < m$ . Additionally, we denote by  $\llbracket n, m \rrbracket$  the integer interval  $\{n, (n+1), \dots, m\}$ .

### 2.1 Rational Base Numeration Systems

Let  $p$  be an integer,  $p \geq 2$ , and  $A_p = \llbracket 0, p-1 \rrbracket$  the alphabet of the  $p$  first digits. Every word  $w = a_n a_{n-1} \dots a_0$  of  $A_p^*$  is given a value in  $\mathbb{N}$  by the *evaluation function*  $\pi_p$ :

$$\pi_p(a_n a_{n-1} \dots a_0) = \sum_{i=0}^n a_i p^i ,$$

and  $w$  is a  $p$ -development of  $n$ . Every  $n$  in  $\mathbb{N}$  has a unique  $p$ -development without leading 0's in  $A_p^*$ : it is called the  $p$ -representation of  $n$  and is denoted by  $\langle n \rangle_p$ . The  $p$ -representation of  $n$  can be computed from left-to-right by a greedy algorithm, and also from *right-to-left* by iterating the Euclidean division of  $n$  by  $p$ , the digits  $a_i$

being the successive remainders. The language of the  $p$ -representations of the integers is the rational language  $L_p = \{\langle n \rangle_p \mid n \in \mathbb{N}\} = (A_p \setminus 0) A_p^*$ .

Let  $p$  and  $q$  be two co-prime integers,  $p > q > 1$ . In [1], we have generalised these classical, and obvious, statements to the more exotic case of *numeration system with rational base*  $\frac{p}{q}$ . Given a positive integer  $n$ , let us define  $N_0 = N$  and, for all  $i > 0$ ,

$$qN_i = pN_{(i+1)} + a_i \quad , \quad (1)$$

where  $a_i$  is the remainder of the Euclidean division of  $qN_i$  by  $p$ , hence in  $A_p = \llbracket 0, p-1 \rrbracket$ . Since  $p > q$ , the sequence  $(N_i)_{i \in \mathbb{N}}$  is strictly decreasing and eventually stops at  $N_{k+1} = 0$ . Moreover, it holds that

$$N = \sum_{i=0}^k \frac{a_i}{q} \left(\frac{p}{q}\right)^i \quad .$$

The *evaluation function*  $\pi_{\frac{p}{q}}$  is derived from this formula. Given a word  $a_n a_{n-1} \dots a_0$  over  $A_p$ , and indeed over any alphabet of digits, its *value* is defined by

$$\pi_{\frac{p}{q}}(a_n a_{n-1} \dots a_0) = \sum_{i=0}^n \frac{a_i}{q} \left(\frac{p}{q}\right)^i \quad . \quad (2)$$

Conversely, a word  $u$  in  $A_p^*$  is called a  $\frac{p}{q}$ -*representation* of an integer  $x$  if  $\pi(u) = x$ . Since the representation is unique up to leading 0's (see [1, Theorem 1]) the  $\frac{p}{q}$ -representation of  $x$  which does not start with a 0 is denoted by  $\langle x \rangle_{\frac{p}{q}}$  and can be computed with the modified Euclidean division algorithm above. By convention, the representation of 0 is the empty word  $\varepsilon$ . The set of  $\frac{p}{q}$ -representations of integers is denoted by  $L_{\frac{p}{q}}$ :

$$L_{\frac{p}{q}} = \left\{ \langle n \rangle_{\frac{p}{q}} \mid n \in \mathbb{N} \right\} \quad .$$

It is immediate that  $L_{\frac{p}{q}}$  is prefix-closed (since, in the modified Euclidean division algorithm  $\langle N \rangle = \langle N_1 \rangle . a_0$ ) and right-extendable (for every representation  $\langle n \rangle$ , there exists (at least) an  $a$  in  $A_p$  such that  $q$  divides  $(np + a)$  and then  $\langle \frac{np+a}{q} \rangle = \langle n \rangle . a$ ). As a consequence,  $L_{\frac{p}{q}}$  can be represented as an infinite tree; it is shown at Figure 1, in the introduction. By abuse of language, in the following we will write that  $n \xrightarrow[L_{\frac{p}{q}}]{u} m$  (or  $n \xrightarrow{u} m$ , for short) if  $\langle m \rangle = \langle n \rangle . u$ ; it should be noted that, with this notation, the following equation hold.

$$\forall n \in \mathbb{N}, \forall m \in \mathbb{N}, \forall a \in A_p \quad n \xrightarrow{a} m \quad \iff \quad a = qm - pn \quad . \quad (3)$$

It is known that  $L_{\frac{p}{q}}$  is not rational and not even context-free (*cf.* [1]). In fact  $L_{\frac{p}{q}}$  defeats any reasonable kind of pumping lemma; it possesses the *Finite Left Iteration Property*, discussed in the next Section 2.2.

**REMARK 1.** *Even though we separated the notions of rational and integer base number systems in order to give specific statements, it should be noted that the*

former extends naturally the latter. Indeed, in the case where  $q = 1$ , the definitions of  $\pi_{\frac{p}{q}}$ ,  $\langle \cdot \rangle_{\frac{p}{q}}$  and  $L_{\frac{p}{q}}$  respectively coincide with those of  $\pi_p$ ,  $\langle \cdot \rangle_p$  and  $L_p$ . In the sequel, we will consider the base  $\frac{p}{q}$  such that  $p > q \geq 1$ , that is indifferently one number system or the other.

**REMARK 2.** It should be noted that a rational base number systems is not a  $\beta$ -numeration — where the representation of a number is computed by the (greedy) Rényi algorithm (cf. [8, Chapter 7]) — in the special case where  $\beta$  is a rational number. In such a system, the digit set is  $\{0, 1, \dots, \lfloor \frac{p}{q} \rfloor\}$  and the weight of the  $i$ -th leftmost digit is  $(\frac{p}{q})^i$ ; whereas in the rational base number system, they are  $\{0, 1 \dots (p-1)\}$  and  $\frac{1}{q}(\frac{p}{q})^i$  respectively.

## 2.2 The Finite Left Iteration Property (FLIP)

We define here a strong ‘non-iteration’ property of languages that will be closely related to the breadth-first generation process (by rhythm) later on.<sup>1</sup>

**DEFINITION 3.** A language  $L$  of  $A^*$  has the Finite Left Iteration Property, or is a FLIP language for short, if for all  $u, v$  in  $\mathbb{N}$ ,  $|v| \geq 1$ ,

$uv^i$  is prefix of a word of  $L$  for only **finitely** many  $i$  in  $\mathbb{N}$ .

Clearly, a FLIP language is neither regular, nor context-free. In [1], it has been shown that the languages  $L_{\frac{p}{q}}$ , for all coprime  $p$  and  $q$ , are FLIP languages.

Although, or because, the Finite Left Iteration Property is the strongest way of contradicting any kind of iteration lemma, it is difficult to find natural examples of FLIP languages in the classical formal language theory.

The set of the prefixes of any infinite aperiodic word is a FLIP language, but this is rather trivial an example (as a language) since the number of words of every length is 1.

**EXAMPLE 4.** Let  $L_{\text{fib}} = \{\varphi^i(0) \mid i \in \mathbb{N}\}$  be the language of Fibonacci words, defined by the Fibonacci morphism  $\varphi: \varphi(0) = 01$  and  $\varphi(1) = 0$ . As the (infinite) Fibonacci word is power 4-free,  $L_{\text{fib}}$  has no prefix of the form  $uv^4$  and is a FLIP language. Since the power, that is 4, is independent of  $u$  and  $v$ ,  $L_{\text{fib}}$  a fortiori possess the Bounded Left Iteration Property.

The Finite Left Iteration Property is related by the following statement to another property of language theory called ‘IRS’ (for *Infinite Regular Subset*, cf. [7]): a language is IRS if it does not contains any rational sublanguage.

**PROPOSITION 5.** For a language  $L$ , the following statements<sup>2</sup> are equivalent:

- (i)  $L$  is a FLIP language.
- (ii)  $\text{Pre}(L)$  is IRS.

<sup>1</sup>We have introduced this property in [10] under the name of *Bounded Left Iteration Property*, or *BLIP* for short. The term ‘bounded’ is indeed improper according to usual terminology and was a mistake.

<sup>2</sup> $\text{Pre}(L)$  denotes the closure of  $L$  by prefix:  $\text{Pre}(L) = \{u \in A^* \mid \exists v \in A^* \quad uv \in L\}$ .

(iii) *The topological closure of  $L$  contains aperiodic (infinite) words only.*

*Proof.* (i)  $\Rightarrow$  (ii) If  $\text{Pre}(L)$  contains an infinite rational sublanguage, it contains a subset  $uv^*w$  for some words  $u, v$  and  $w$ , hence, for infinitely many integers  $i$ ,  $uv^i$  is a prefix of some words of  $L$ , a contradiction.

(ii)  $\Rightarrow$  (iii) Let us assume that  $w = uv^\omega$  belongs to the topological closure of  $L$ . It implies that, for every integer  $i$ ,  $uv^i$  is the prefix of a word of  $L$ , hence  $uv^*$  is a sublanguage of  $\text{Pre}(L)$ , a contradiction.

The proof of the implication (iii)  $\Rightarrow$  (i) is analogous.  $\square$

**LEMMA 6.** *The class of FLIP languages is stable by finite union, arbitrary intersection, sublanguage, concatenation and inverse morphism image.*

*Proof for concatenation.* Let  $L$  and  $M$  be two FLIP languages. Let  $u$  and  $v$  be two words, and  $J$  the set of integers  $j$  such that the prefix  $w_j$  of length  $j$  of  $uv^\omega$  is in  $L$ . Since  $L$  is FLIP language,  $J$  is finite. For every  $j$  in  $J$ , the ultimately periodic word  $(w_j)^{-1}uv^\omega$  is not in the topological closure of  $M$ , hence there are only finitely many  $k$  such that  $(w_j)^{-1}uv^k$  is a prefix of a word of  $M$ . Summing up for all  $j$  in  $J$ , there are only finitely many  $k$  such that  $uv^k$  is a prefix of a word of  $LM$ .  $\square$

It is easy to give examples showing that the class of FLIP languages is not closed under complementation, star, transposition and direct morphism image.

## 2.3 On Trees and Signatures

Classically, trees are undirected graphs in which any two vertices are connected by exactly one path (*cf.* [4], for instance). Our view differs in two respects.

First, a tree is a *directed* graph  $\mathcal{T} = (V, \Gamma)$  such that there exist a *unique* vertex, called *root*, which has no incoming arc, and there is a *unique (oriented) path* from the root to every other vertex. Elements of the tree  $\mathcal{T}$  gets particular names: vertices are called *nodes*; if  $(x, y)$  is an arc,  $y$  is called a *child* of  $x$  and  $x$  *the father* of  $y$ ; a node without children is a *leaf*. We draw trees with the root on the left, and arcs rightwards.

Second, our trees are *ordered*, that is, that there is a total order on the set of children of every node. The order will be implicit in the figures, with the convention that lower children are smaller (according to this order).

It will prove to be extremely convenient to have a slightly different look at trees and to consider that the root of a tree is also a *child of itself* that is, bears a loop onto itself. This convention is sometimes taken when implementing tree-like structures (for instance the unix/linux file system). We call such a structure an *i-trees*. It is so close to a tree that we pass from tree to i-tree (or conversely) with no further ado. Fig.1 shows a tree and Fig.4 shows the associated i-tree.

The degree of a node is the number of its children. In the sequel, we consider infinite ordered (i-)trees of finite degree, that is, all nodes of which have finite degree. The breadth-first search of such a tree defines a total ordering of its

nodes. We then consider that the set of nodes of an (i-)tree is always the set of integers  $\mathbb{N}$ . The root is 0 and  $n$  is the  $(n + 1)$ -th node visited by the search.

Let  $\mathcal{T}$  be an ordered (i-)tree of finite degree. The sequence  $\mathbf{s}$  of the degrees of the nodes of  $\mathcal{T}$  visited in the breadth-first search of  $\mathcal{T}$  is called the *signature* of  $\mathcal{T}$  and is *characteristic* of  $\mathcal{T}$ , that is, one can compute, or build,  $\mathcal{T}$  from  $\mathbf{s}$ . By convention, and whether  $\mathcal{T}$  be a tree or an i-tree, *the signature is always that of the i-tree*.

### 3 Rhythmic trees and languages

#### 3.1 Rhythms and their geometric representation

**DEFINITION 7.** *Let  $p$  and  $q$  be two integers with  $p > q \geq 1$ .*

1. *We call rhythm of directing parameter  $(q, p)$ , a  $q$ -tuple  $\mathbf{r}$  of non-negative integers whose sum is  $p$ :*

$$\mathbf{r} = (r_0, r_1, \dots, r_{q-1}) \quad \text{and} \quad \sum_{i=0}^{q-1} r_i = p .$$

2. *We say that a rhythm  $\mathbf{r}$  is valid if it satisfies the following equation:*

$$\forall j \in \llbracket 0, q-1 \rrbracket \quad \sum_{i=0}^j r_i > j+1 . \quad (4)$$

3. *We call growth ratio of  $\mathbf{r}$  the rational number  $z = \frac{p}{q}$ , also written  $z = \frac{p'}{q'}$  where  $p'$  and  $q'$  are the quotients of  $p$  and  $q$  by their greatest common divisor (gcd), hence coprime.*

The growth ratio of a rhythm is always greater than 1. Examples of rhythms of growth ratio  $\frac{5}{3}$  are  $(2, 2, 1)$ ,  $(3, 0, 2)$ ,  $(1, 2, 2)$ ,  $(2, 2, 1, 2, 2, 1)$ ,  $(2, 1, 3, 0, 0, 4)$ ; all but the third one are valid; the directing parameter is  $(3, 5)$  for the first three, and  $(6, 10)$  for the last two.

Rhythms are given a very useful geometric representation as *paths* in the  $\mathbb{Z} \times \mathbb{Z}$ -lattice and such paths are coded by *words* of  $\{x, y\}^*$  where  $x$  denotes a unit horizontal segment and  $y$  a unit vertical segment. Hence the name *path* given to a *word* associated with a rhythm.

**DEFINITION 8.** *With a rhythm  $\mathbf{r} = (r_0, r_1, \dots, r_{q-1})$  of directing parameter  $(q, p)$ , we associate the word  $\text{path}(\mathbf{r})$  of  $\{x, y\}^*$ :*

$$\text{path}(\mathbf{r}) = y^{r_0} x y^{r_1} x y^{r_2} \dots x y^{r_{q-1}} x$$

*which corresponds to a path from  $(0, 0)$  to  $(q, p)$  in the  $\mathbb{Z} \times \mathbb{Z}$ -lattice.*

Fig.2 shows the paths associated with three of the above rhythms. It then appears clearly that Definition 7.2 can be restated as ‘a rhythm is valid if the associated path is strictly above the line of slope 1’.

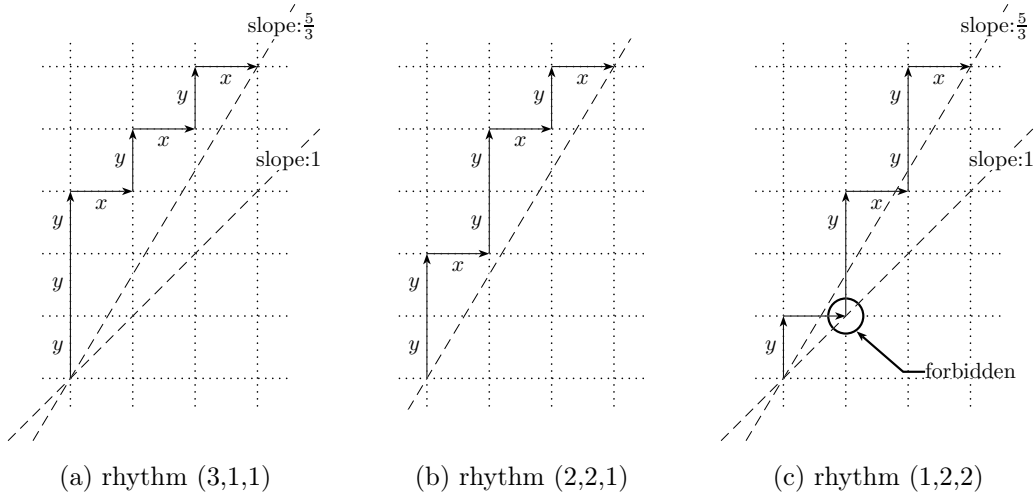


Figure 2: Words and paths associated with rhythms of directing parameter (5,3)

### 3.2 Generating trees by rhythm

As said above, we have describe in [11] the procedure that reconstructs a tree  $\mathcal{T}$  from its signature  $\mathbf{s}$ . We present here this construction in the case where  $\mathbf{s}$  is a purely periodic word  $\mathbf{s} = \mathbf{r}^\omega$ . First, a static description of the result.

**DEFINITION 9.** Let  $\mathbf{r} = (r_0, r_1, \dots, r_{q-1})$  be a (valid) rhythm. The tree  $\mathcal{I}_{\mathbf{r}}$  generated by  $\mathbf{r}$  is defined by:

- the root 0 has  $(r_0 - 1)$  children: the nodes 1, 2, ..., and  $(r_0 - 1)$ ;
- for every  $n > 0$ , the node  $n$  has  $r_{n \% q}$  children: the nodes  $(m + 1)$ ,  $(m + 2)$ , ..., and  $(m + r_{n \% q})$ , where  $m$  is the greatest child of the node  $(n - 1)$ .

Fig. 3a shows  $\mathcal{I}_{(3,1,1)}$ , Fig. 1 shows  $\mathcal{I}_{(2,1)}$  (if one forgets the labels on the arcs).

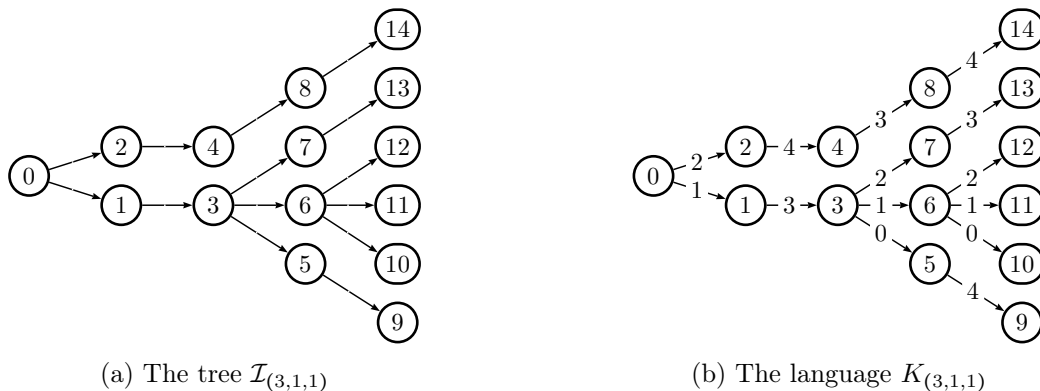


Figure 3: Tree and language generated by the rhythm (3, 1, 1)

If we had treated the root 0 in the same way as the other nodes, we would have said that 0 has  $r_0$  children: the nodes 0, 1, 2, ..., and  $(r_0 - 1)$  and we would



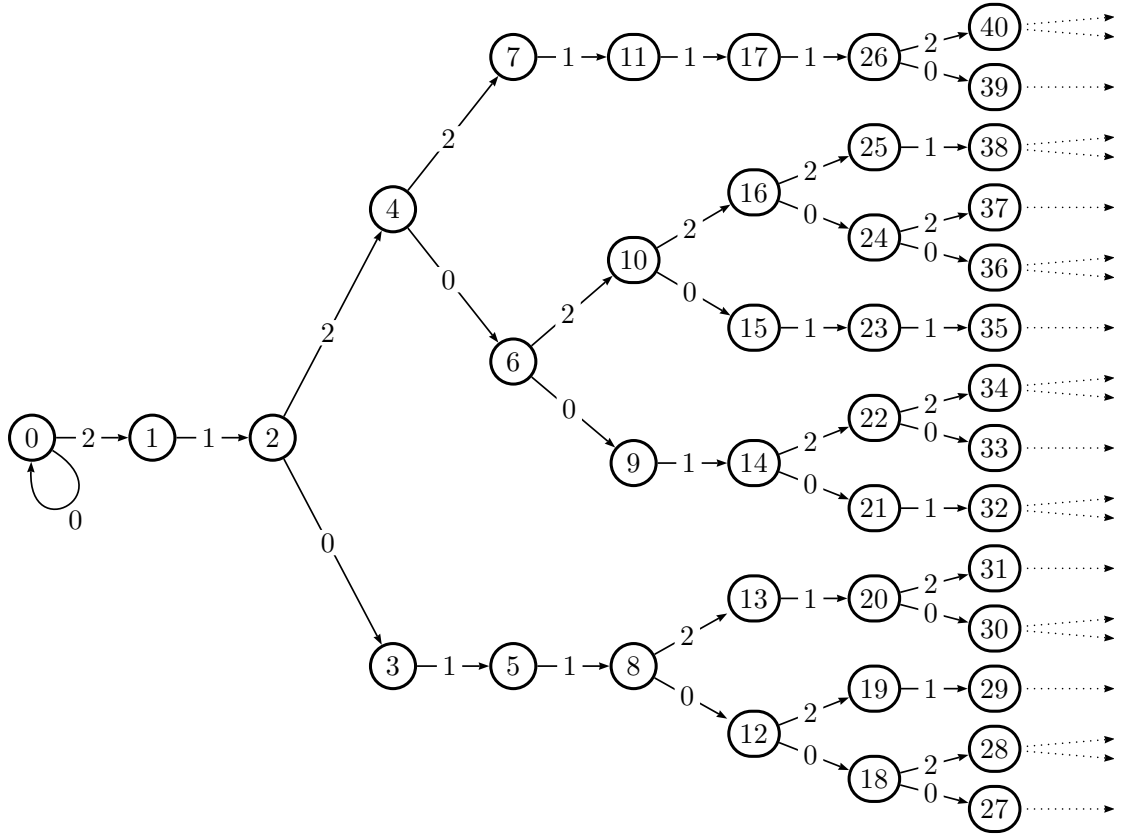


Figure 4: The  $i$ -tree associated with  $\mathcal{I}_{\frac{3}{2}}$

have defined the  $i$ -tree (associated with)  $\mathcal{I}_{\mathbf{r}}$ . For instance, the  $i$ -tree  $\mathcal{I}_{(2,1)}$  is shown at Fig. 4.

The description of a procedure that builds  $\mathcal{I}_{\mathbf{r}}$  from  $\mathbf{r}$  gives a more dynamical view on the process. The procedure maintains two integers,  $n$  and  $m$ , both initialised to 0:  $n$  is *the node to be processed* and  $m$  is *the next node to be created*. At every step of the procedure,  $r_{(n \% q)}$  nodes are created: the nodes  $m, (m + 1), \dots$ , and  $(m + r_{(n \% q)} - 1)$  and  $r_{(n \% q)}$  arcs are created, from  $n$  to every new node. Then  $n$  is incremented by 1, and  $m$  by  $r_{(n \% q)}$ .

This procedure indeed builds the  $i$ -tree  $\mathcal{I}_{\mathbf{r}}$  since its first step creates a loop  $0 \rightarrow 0$ . The *tree*  $\mathcal{I}_{\mathbf{r}}$  is obtained by removing this loop, and the root 0 has then  $(r_0 - 1)$  children only. Fig. 5 shows the first five and the tenth steps of the procedure for the rhythm  $(3, 1, 1)$ . It is an easy verification that the tree built by that procedure meets the tree defined at Definition 9.

The *validity* of the rhythm is the necessary and sufficient condition for  $m$  always be greater than  $n$  in the course of the execution of the procedure, that is, a node is always ‘created’ before being ‘processed’, or, equivalently, for the tree described at Definition 9 be infinite.

A direct consequence of the building of  $\mathcal{I}_{\mathbf{r}}$  by the procedure is that  $q$  consecutive nodes of  $\mathcal{I}_{\mathbf{r}}$  (in the breadth-first search) have  $p$  (consecutive) children, hence the name *growth* given to the ratio  $\frac{p}{q}$ . More precisely, the following holds.

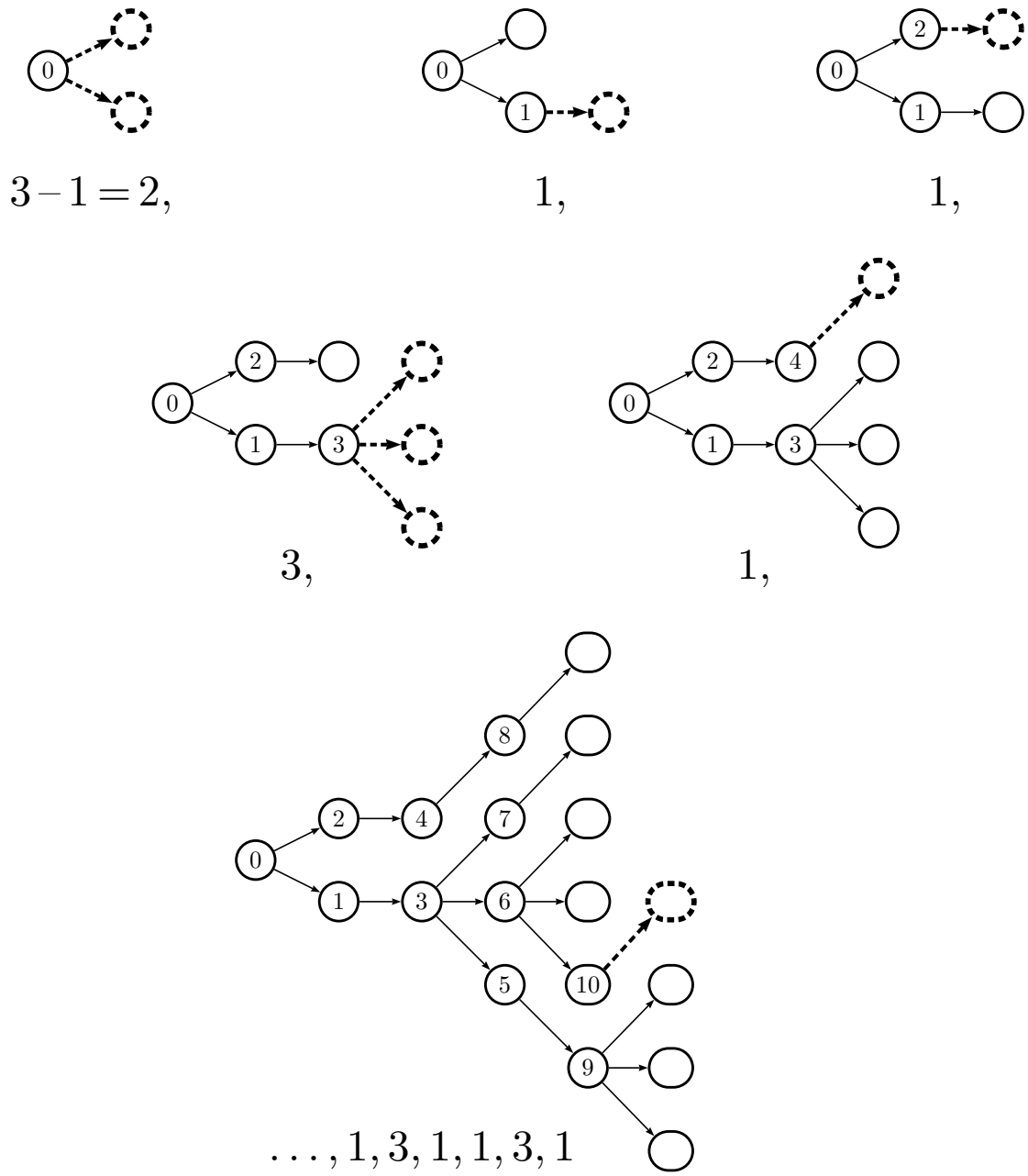


Figure 5: Building  $\mathcal{I}_{(3,1,1)}$  from the rhythm  $(3, 1, 1)$

**LEMMA 10.** *Let  $\mathcal{I}_{\mathbf{r}}$  be the tree generated by the rhythm  $\mathbf{r}$  of directing parameter  $(q, p)$ . Then, for all  $n, m$  in  $\mathbb{N}$ :*

$$n \xrightarrow{\mathcal{I}_{\mathbf{r}}} m \iff (n + q) \xrightarrow{\mathcal{I}_{\mathbf{r}}} (m + p) .$$

### 3.3 Labelling of Rhythmic Trees

An ordered tree  $\mathcal{T}$  defines its signature  $\mathbf{s}$ , and we have seen how to reconstruct  $\mathcal{T}$  from  $\mathbf{s}$  (at least in the case where  $\mathbf{s} = \mathbf{r}^\omega$ ). In the same way, a labelled ordered tree  $\mathcal{T}$  defines the sequence  $\boldsymbol{\lambda}$  of the labels of the arcs as they are visited in the

breadth-first search, and  $\mathcal{T}$ , and hence its branch language  $L_{\mathcal{T}}$ , will be determined by the pair  $(\mathbf{s}, \boldsymbol{\lambda})$ .

The (finite) alphabet of labels is ordered as well — we consider the case of digit alphabets only. Of course, we want the labelling of  $\mathcal{T}$  be consistent with the order of  $\mathcal{T}$ , that is, the breadth-first search of  $\mathcal{T}$  yield the *radix order* on  $L_{\mathcal{T}}$ , which is equivalent to the condition that the children of every node  $n$  are in the same order as the labels of the arcs that come from their father  $n$ .

We consider here periodic signatures  $\mathbf{s} = \mathbf{r}^\omega$  where  $\mathbf{r}$  is a rhythm of directing parameter  $(q, p)$ . We then will consider pairs  $(\mathbf{s}, \boldsymbol{\lambda})$  with  $\boldsymbol{\lambda} = \gamma^\omega$  where  $\gamma$  is a sequence of letters (digits) of length  $p$ . And we say that  $\mathcal{T}$ , and its branch language  $L_{\mathcal{T}}$ , are determined by the pair  $(\mathbf{r}, \gamma)$ .

It follows from Lemma 10 that the labelling is consistent on the whole tree if and only if it is consistent on the first  $q$  nodes, hence on the first  $p$  arcs, in which case we say that  $\gamma$  is *valid*. A first, and obvious, valid labelling is the sequence  $\gamma_p$  of the first  $p$  digits:  $\gamma_p = (0, 1, \dots, p-1)$ , which we call the *naive labelling*.

**DEFINITION 11.** *Let  $\mathbf{r}$  be a rhythm of directing parameter  $(q, p)$  and  $\gamma_p$  the naive labelling. We denote by  $K_{\mathbf{r}}$  the branch language of the labelled tree determined by the pair  $(\mathbf{r}, \gamma_p)$ , that is, the tree  $\mathcal{I}_{\mathbf{r}}$  labelled by:*

$$\forall n, m \in \mathbb{N} \quad n \xrightarrow[\mathcal{I}_{\mathbf{r}}]{a} m \quad \text{with } a = m \% p \text{ .}$$

Fig. 3b shows the language  $K_{(3,1,1)}$  while  $K_{(3,1)}$  is shown at Fig. 6a.

All elements of the main result of this paper are now defined: it states that the language built by a rhythm and the naive labelling is either regular or FLIP, according to whether the growth ratio of the rhythm is an integer or not.

**THEOREM 12.** *Let  $\mathbf{r}$  be a rhythm of directing parameter  $(q, p)$ .*

- a. *If  $\frac{p}{q}$  is an integer, then  $K_{\mathbf{r}}$  is a regular language.*
- b. *If  $\frac{p}{q}$  is not an integer, then  $K_{\mathbf{r}}$  is a FLIP language.*

The proof of the whole statement consists in a reduction to the case of the representation language of rational base numeration systems and occupies indeed the remainder of the paper (Theorem 23 and Theorem 27). However, Theorem 12.a can be established by a direct and simpler proof given below; it is also a corollary of the main result of [11].

*Proof of Theorem 12a.* The proof consists in considering the underlying tree  $\mathcal{I}_{\mathbf{r}}$  of  $K_{\mathbf{r}}$  as an *infinite* automaton and then proving that it has a finite number of classes in the Nerode equivalence. More precisely, we prove that two states  $n$  and  $m$ , with  $n$  and  $m$  strictly positive, are Nerode-equivalent if they are congruent modulo  $q$ .

For every integer  $i$ , we denote by  $\sim_i$  the following equivalence relation: given two states  $n$  and  $m$ , we write that  $n \sim_i m$  if, for all word  $u$  of  $A_p^*$  of length  $i$ ,  $n \cdot u$  exists  $\iff m \cdot u$  exists. Of course two states  $n$  and  $m$  are Nerode-equivalent if and only if  $n \equiv_i m$  for all integers  $i$ .

Let us consider two integer  $n$  and  $m$  such that  $n \equiv m [q]$ . By induction. The relation  $\sim_0$  is trivial and has only one equivalency class, hence  $n \sim_0 m$ .

Let now be  $i$  an integer strictly greater than 0. Since  $n \equiv m [q]$ , it follows from Lemma 10 that for every  $n'$  such that  $n \xrightarrow{a} n'$ , then there exists an integer  $m'$  such that  $m \xrightarrow{b} m'$  and  $n' \equiv m' [p]$ . Hence, from Definition 11,  $a = (n' \% p) = (m' \% p) = b$ .

Moreover, by hypothesis  $q \mid p$ , hence  $n' \equiv m' [q]$ . By induction hypothesis,  $n' \sim_{(i-1)} m'$ , hence  $n \sim_i m$ .

The automaton accepting  $K_r$  has then  $q + 1$  states: one for each congruency class modulo  $q$  for positive integers, plus one special state for 0 which is initial. Figure 6 shows the case of rhythm  $(3, 1)$ .  $\square$

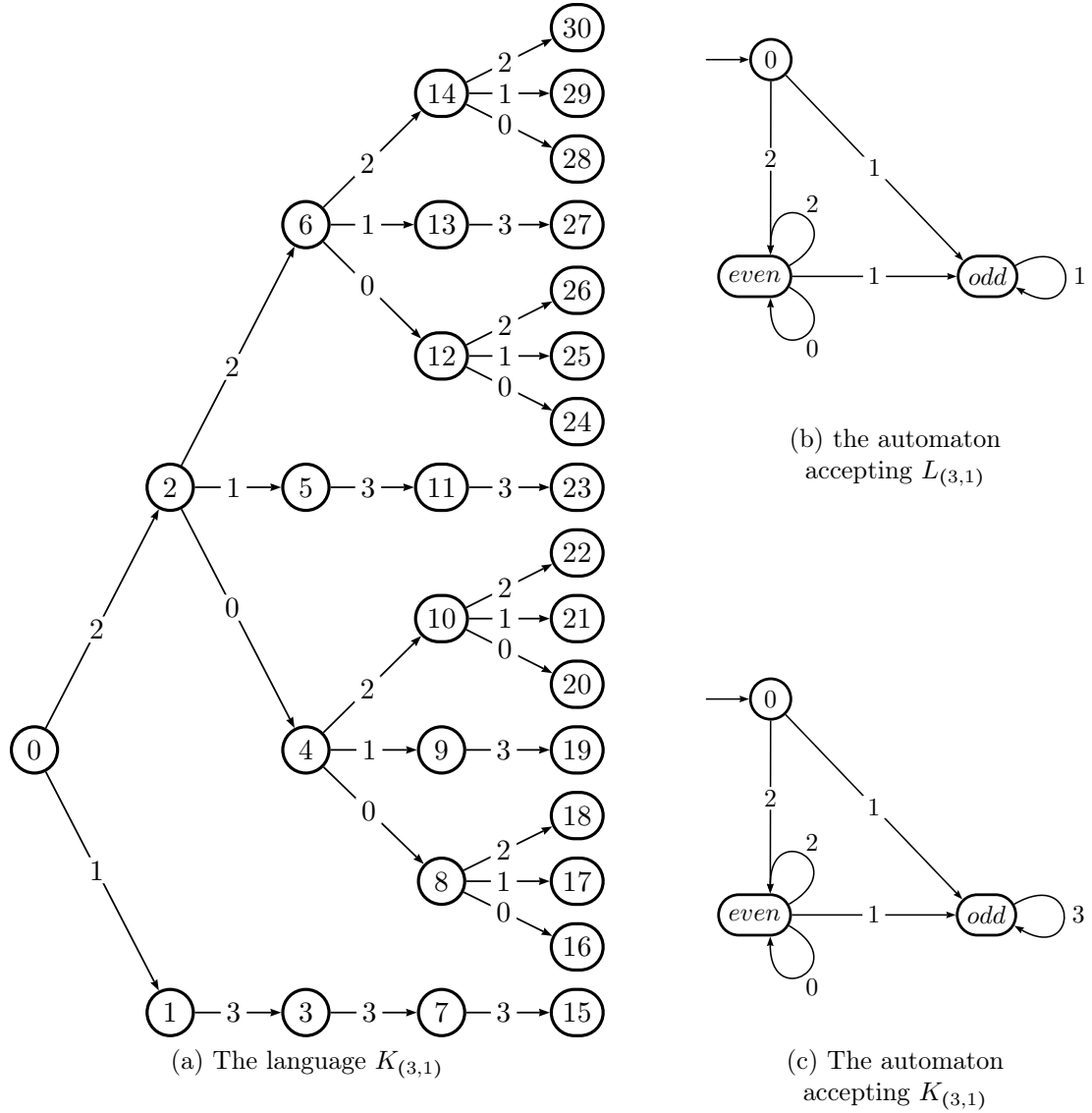


Figure 6: The case of the rhythm  $(3, 1)$  with integral growth ratio

**REMARK 13.** *If one considers the  $i$ -tree generated by rhythm  $\mathbf{r}$  (instead of the tree) then the special case for the state 0 of the previous proof is unnecessary, there are only  $q$  states and the congruency class  $(0 \% q)$  is initial. For instance, in Figure 6c, if there were a self-loop on the state 0, this state would be Nerode-equivalent to the state even.*

More generally, if  $\mathbf{r}$  is a rhythm of directing parameter  $(q, p)$  and  $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{p-1})$  a valid labelling, the language generated by  $(\mathbf{r}, \gamma)$  is the branch language of  $\mathcal{I}_{\mathbf{r}}$  labelled by

$$\forall n, m \in \mathbb{N} \quad n \xrightarrow[\mathcal{I}_{\mathbf{r}}]{a} m \quad \text{with } a = \gamma_m \%_p .$$

Of course, the naive labelling can be mapped onto any other (valid) labelling:

**PROPOSITION 14.** *Let  $L$  be the language of  $B^*$  generated by  $(\mathbf{r}, \gamma)$ . There exists a strictly alphabetic morphism  $\varphi: A_p^* \rightarrow B^*$  such that  $\varphi(K_{\mathbf{r}}) = L$ .*

We now define a labelling that will play a crucial role in the sequel.

**DEFINITION 15.** *Let  $\mathbf{r}$  be a rhythm of directing parameter  $(q, p)$  and  $p', q'$  the coprime integers such that  $\frac{p'}{q'} = \frac{p}{q}$ . Let  $I_{\mathbf{r}}$  be the index set of the partial sums of  $\mathbf{r}$ , that is,  $I_{\mathbf{r}} = \{r_0 + r_1 + \dots + r_k \mid \leq k < q - 1\}$ . We call special labelling associated with  $\mathbf{r}$ , and denote by  $\gamma_{\mathbf{r}}$ , the  $p$ -tuple  $\gamma_{\mathbf{r}} = (\gamma_0, \gamma_1, \dots, \gamma_{p-1})$  defined by*

$$\gamma_0 = 0, \quad \forall i \notin I_{\mathbf{r}} \quad \gamma_i = \gamma_{(i-1)} + q', \quad \forall i \in I_{\mathbf{r}} \quad \gamma_i = \gamma_{(i-1)} + q' - p' .$$

**EXAMPLE 16.** *For instance, if  $\mathbf{r} = (3, 1, 3, 3)$ , its directing parameter is  $(4, 10)$ ,  $p' = 5$ ,  $q' = 2$ ,  $I_{\mathbf{r}} = \{3, 4, 7\}$  and*

$$\gamma_{\mathbf{r}} = \left( \overbrace{0, 2, 4}^3, \overbrace{1}^1, \overbrace{-2, 0, 2}^3, \overbrace{-1, 1, 3}^3 \right) .$$

*As other examples:  $\gamma_{(3,1)} = (0, 1, 2, 1)$  and  $\gamma_{(3,1,1)} = (0, 3, 6, 4, 2)$ .*

It directly follows from the definition that *the special labelling is valid.*

## 4 Rational Base Numeration Systems are Rhythmic

In this section,  $p$  and  $q$  are two coprime integers,  $p > q \geq 1$ , which define the numeration system with base  $\frac{p}{q}$ . We introduce a special, and canonical, *rhythm* of directing parameter  $(q, p)$ ,  $\mathbf{r}_{\frac{p}{q}}$ , hence of growth ratio  $z = \frac{p}{q}$ , which relates to the classical notion of *Christoffel words*. We then characterise the *special labelling*  $\gamma_{\mathbf{r}_{\frac{p}{q}}}$  as the permutation  $\gamma_{\frac{p}{q}}$  resulting from the generation of  $\mathbb{Z}/p\mathbb{Z}$  by  $q$ . The remarkable fact is then that the representation language in the  $\frac{p}{q}$ -numeration system is generated by the labelled rhythm  $(\mathbf{r}_{\frac{p}{q}}, \gamma_{\frac{p}{q}})$  (Theorem 23).

## 4.1 Christoffel words, Christoffel rhythms

*Christoffel words* code some kind of ‘best approximation’ of segments the  $\mathbb{Z} \times \mathbb{Z}$ -lattice and have been studied in the field of combinatorics of words (cf. [3]). We translate them into rhythms. More precisely (Definition 17.a is taken from [3]):

**DEFINITION 17.** *Let  $p$  and  $q$  be two coprime positive integers.*

(a) *The (upper) Christoffel word associated with  $\frac{p}{q}$ , and denoted by  $\mathbf{w}_{\frac{p}{q}}$ , is the label of the path from  $(0,0)$  to  $(q,p)$  on the  $\mathbb{Z} \times \mathbb{Z}$  lattice, such that*

- *the path is above the line of slope  $\frac{p}{q}$  passing through the origin;*
- *the region enclosed by the path and the line contains no point of  $\mathbb{Z} \times \mathbb{Z}$ .*

(b) *The Christoffel rhythm associated with  $\frac{p}{q}$ , and denoted by  $\mathbf{r}_{\frac{p}{q}}$ , is the rhythm whose path is  $\mathbf{w}_{\frac{p}{q}}$ :  $\text{path}(\mathbf{r}_{\frac{p}{q}}) = \mathbf{w}_{\frac{p}{q}}$ , hence its directing parameter is  $(q,p)$ .*

Fig. 2b shows the path of  $\mathbf{w}_{\frac{5}{3}} = \overline{y}y x \overline{y}y x \overline{y}x$ , the Christoffel word associated with  $\frac{5}{3}$ ; then,  $\mathbf{r}_{\frac{5}{3}} = (2, 2, 1)$ . Other Christoffel words and their paths are shown at Fig. 7.

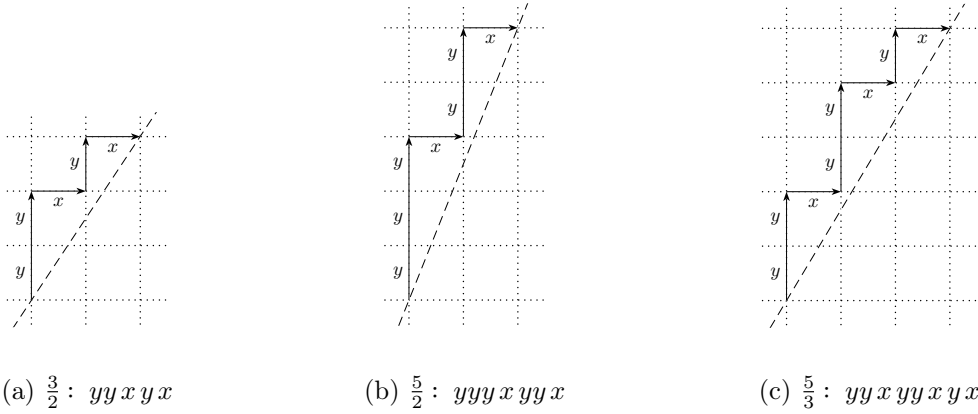


Figure 7: Christoffel words associated with three rational numbers

**PROPOSITION 18.** *Given a base  $\frac{p}{q}$  of rhythm  $\mathbf{r}_{\frac{p}{q}} = (r_0, r_1, \dots, r_{q-1})$ , and an integer  $0 < k \leq q$ , the partial sum  $r_0 + r_1 + \dots + r_{k-1}$ , of the first  $k$  components of  $\mathbf{r}$  is equal to the smallest integer greater than  $k\frac{p}{q}$ .*

Proposition 18 is a direct consequence of the following technical lemma which is basically a translation of the proposition into the Christoffel words and their geometric interpretation universe.

**LEMMA 19.** *Let us denote by  $\mathbf{w}_{\frac{p}{q}}$  the Christoffel word of slope  $\frac{p}{q}$ . If  $ux$  is prefix of  $\mathbf{w}_{\frac{p}{q}}$  then it corresponds to a path from  $(0,0)$  to  $(k, \lceil k\frac{p}{q} \rceil)$  in the  $\mathbb{Z} \times \mathbb{Z}$  lattice, where  $k$  is the number of  $x$ 's in  $ux$ .*

*Proof.* From Definition 17 of Christoffel word, there is no integer point between the path and the line of slope  $\frac{p}{q}$  and passing through the origin.

Since the point  $(k, k\frac{p}{q})$  is part of this line, the Christoffel path must pass through the point  $(k, \lceil k\frac{p}{q} \rceil)$ . Besides, the prefix of the Christoffel word reaching this point must end with an  $x$ ; were it ending with an  $y$  it would mean that the Christoffel path pass through the point  $(k, \lceil k\frac{p}{q} \rceil - 1)$  which is *below* the line of slope  $\frac{p}{q}$ , a contradiction.  $\square$

**LEMMA 20.** *Given a base  $\frac{p}{q}$ , we denote the associated Christoffel rhythm by  $\mathbf{r}$  and an integer  $k \in \llbracket 0, q-1 \rrbracket$ ,  $\sum_{i=0}^{k-1} r_{(q-1-k)} = \lfloor k\frac{p}{q} \rfloor$*

We define the sequence of integers  $e_0, e_1, \dots, e_{q-1}$  such that  $e_j$  is the difference between the approximation  $(r_0 + r_1 + \dots + r_{k-1})$  and the point of the associated line of the respective abscissa, that is  $(k\frac{p}{q})$ . This difference is a rational number smaller than 1 and whose denominator is  $q$ , in order to obtain an integer we multiply it by  $q$ :

$$\forall k \in \llbracket 0, q-1 \rrbracket \quad e_k = q \left( \sum_{i=0}^{k-1} r_i \right) - kp . \quad (5)$$

We describe on the example of the base  $\frac{5}{3}$  shown at Figure 8 a more diagrammatic way of characterising Christoffel rhythms. We associate  $p$  segments of length  $q$  (at the top in the figure) with  $q$  segments of length  $p$  such that each top segment is associated to the bottom segment in which it starts. The integer  $e_i$  is then the difference of length between the  $i$  left-most bottom segments (of length  $p \times i$ ) and the total number of top segments associated with them (of length  $q \times \sum_{j=0}^{i-1} r_j$ ).

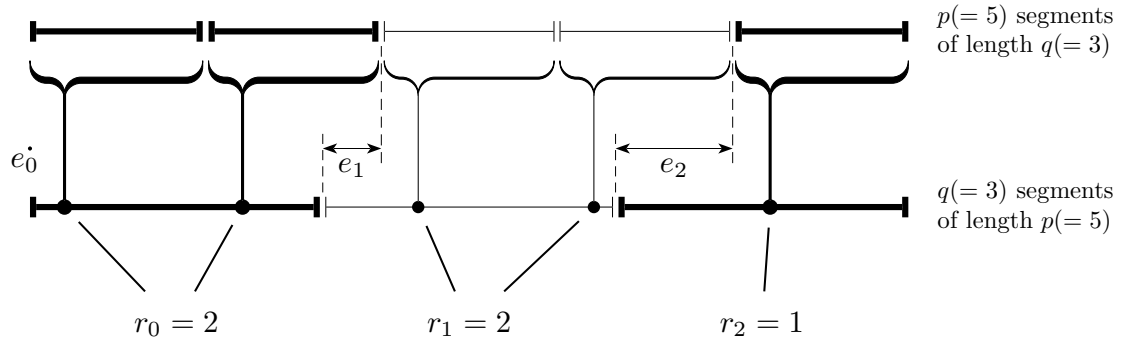


Figure 8: Diagrammatic interpretation of the rhythm  $(2, 2, 1)$  of base  $\frac{5}{3}$

The next lemma compiles basic properties of the  $r_j$ 's and  $e_j$ 's.

**LEMMA 21.** *Let  $\mathbf{r}_\frac{p}{q} = (r_0, r_1, \dots, r_{q-1})$  be the Christoffel rhythm of slope  $\frac{p}{q}$ . For every integer  $j$  in  $\llbracket 0, q-1 \rrbracket$ , it holds:*

- (a)  $e_j$  belongs to  $\llbracket 0, q-1 \rrbracket$ ;
- (b)  $r_j$  is the smallest integer such that  $qr_j + e_j \geq p$ ;
- (c)  $e_{j+1} = e_j + qr_j - p$ ;
- (d) all  $e_j$ 's are distinct;

$$(e) \quad q \sum_{k=i}^{j-1} r_k + e_i = (j-i)p + e_j ;$$

(f) for every  $i$  in  $\llbracket 1, q \rrbracket$ ,  $i > j$ , it holds:

$$\sum_{k=i}^{j-1} r_k = \left\lceil \frac{(j-i)p - e_i}{q} \right\rceil = \left\lfloor \frac{(j-i)p - e_j}{q} \right\rfloor .$$

*Proof.* Statements (a), (b), and (c) are simple consequences of the definition.

(d) Let  $i$  and  $j$  be in  $\llbracket 0, q-1 \rrbracket$  and suppose that  $e_i = e_j$ . It follows from (b) and (c) that  $r_i = r_j$  and then  $e_{i+1} = e_{j+1}$ . By iterating this process, it follows that the rhythm  $\mathbf{r}$  is periodic of period  $|j-i|$ , a contradiction.

(e) Follows from the iteration of (c).

(f) From (e),  $e_{i+j} = e_i + q \left( \sum_{k=i}^{j-1} r_k \right) - (j-i)p$ , hence

$$\sum_{k=i}^{j-1} r_k = \frac{(j-i)p}{q} - \frac{e_i}{q} + \frac{e_j}{q} .$$

Since the right-hand side of this equation is an integer and that  $e_i$  and  $e_j$  are smaller than  $q$ ,  $\frac{e_i}{q}$  and  $\frac{e_j}{q}$  are smaller than 1 and the whole statement follows.  $\square$

## 4.2 Generation of $L_{\frac{p}{q}}$ by rhythm and labelling

Since  $p$  and  $q$  are coprime integers,  $q$  is a generator of the (additive) group  $\mathbb{Z}/p\mathbb{Z}$ . We denote by  $\gamma_{\frac{p}{q}}$  the sequence induced by this generation process:

$$\gamma_{\frac{p}{q}} = (0, q \% p, (2q) \% p, \dots, ((p-1)q) \% p) .$$

**PROPOSITION 22.** *Let  $p$  and  $q$  be two coprime positive integers,  $p > q \geq 1$ . The sequence  $\gamma_{\frac{p}{q}}$  coincide with the special labelling  $\gamma_{\mathbf{r}_{\frac{p}{q}}}$ .*

*Proof.* We denote by  $(\gamma_0, \gamma_1, \dots, \gamma_{q-1})$  the special labelling  $\gamma_{\mathbf{r}_{\frac{p}{q}}}$ . By the very definition of special labelling, it is obvious that for all  $i \in \llbracket 0, q-1 \rrbracket$ ,  $\gamma_i \equiv iq \pmod{p}$ . It is then enough to prove that for every integer  $i \in \llbracket 0, q-1 \rrbracket$ ,  $0 \leq \gamma_i < p$ . Given an integer  $k$  we denote by  $m_k = r_0 + r_1 + \dots + r_{k-1}$  and by  $M_k = (m_{k+1} - 1) = r_0 + r_1 + \dots + r_{k-1} + r_k - 1$ .

Let us fix an integer  $i$  in the following; there exists an integer  $k$  such that  $m_k \leq i \leq M_k$ . From the definition of special labelling,  $\gamma_{m_k} = (q m_k) - (p k)$ ,  $\gamma_i = (q i) - (p k)$  and  $\gamma_{M_k} = (q M_k) - (p k)$ , hence  $\gamma_{m_k} \leq \gamma_i \leq \gamma_{M_k}$ .

From Proposition 18,  $m_k$  is the smallest integer greater than  $k \frac{p}{q}$ , hence  $m_k \geq k \frac{p}{q}$ , hence  $\gamma_{m_k} \geq 0$ . Using the same proposition for  $(k+1)$ ,  $m_{k+1}$  is the smallest integer greater than  $(k+1) \frac{p}{q}$ , hence  $M_k (= m_{k+1} - 1)$  is strictly smaller than  $(k+1) \frac{p}{q}$ ; that is,  $M_k < (k+1) \frac{p}{q}$ , hence  $\gamma_{M_k} < p$ .

Therefore  $0 \leq \gamma_{m_k} \leq \gamma_i \leq \gamma_{M_k} < p$ .

$\square$



As an immediate consequence of Proposition 22,  $\gamma_{\frac{p}{q}}$  is a *valid labelling*.

**THEOREM 23.** *Let  $p$  and  $q$  be two coprime integers,  $p > q \geq 1$ . The language  $L_{\frac{p}{q}}$  of  $\frac{p}{q}$ -representations of the integers is generated by the rhythm  $\mathbf{r}_{\frac{p}{q}}$  and labelling  $\gamma_{\frac{p}{q}}$ .*

For instance,  $L_{\frac{3}{2}}$ , shown at Fig. 1, is built with the rhythm  $(2, 1)$  and the labelling  $(0, 2, 1)$ . The proof of Theorem 23 relies mostly on the following statement itself being a consequence of the technical Lemma 21b.

**PROPOSITION 24.** *For every integer  $n > 0$  (resp.  $n = 0$ ), there is exactly  $r_{(n \% q)}$  (resp.  $(r_0 - 1)$ ) letters  $a$  of  $A_p$  such that  $\langle n \rangle.a$  is in  $L_{\frac{p}{q}}$ .*

*Proof.* We denote by  $j$  the congruency class of  $n$  modulo  $q$ . From Lemma 21b,  $r_j$  is the smallest integer such that  $qr_j + e_j > p$ . It follows that for all  $k$  in  $\llbracket 0, r_j - 1 \rrbracket$   $(e_j + qk) < p$  and  $e_j + qr_j > p$ .

From Lemma 21b,  $e_j$  is the smallest label of the state  $n$ , hence the state  $n$  has exactly  $r_j$  outgoing transitions, respectively labelled by  $e_j, e_j + q, \dots, (e_j + q(r_j - 1))$ .  $\square$

**LEMMA 25.** *Given a base  $\frac{p}{q}$  and an integer  $n$ , the smallest letter  $a$  of  $A_p$  such that  $\langle n \rangle.a$  is in  $L_{\frac{p}{q}}$ , is  $e_{(n \% q)}$ .*

*Proof.* Let us denote by  $n$  an integer and by  $j$  its congruency modulo  $q$ . Since  $e_j$  is in  $A_q$  (from Lemma 21a), it is enough to prove that  $e_j$  is an outgoing label of  $n$ , or (from Equation (3)) that  $np + e_j$  is a multiple of  $q$ ; or, equivalently that  $jp + e_j$  is a multiple of  $q$ . From Equation (5),  $jp + e_j = (q \sum_{i=0}^{j-1} r_i)$ , that is, a multiple of  $q$ .  $\square$

To complete the proof of Theorem 23, it remains to prove that for every integer  $n$ , the last digit of  $\langle n \rangle_{\frac{p}{q}}$  is  $(qn) \% p$ , which directly results from the definition of the modified Euclidean division algorithm (Equation (1)).

## 5 Reduction to Rational Base Numeration Systems

In this section,  $p$  and  $q$  are two integers,  $p > q \geq 1$ , not necessarily coprime, and  $\mathbf{r}$  is a rhythm of directing parameter  $(q, p)$ . As in Definition 7, we denote by  $p'$  and  $q'$  their respective quotient by their gcd, that is,  $\frac{p'}{q'}$  is the reduced fraction of  $\frac{p}{q}$ .

**DEFINITION 26.** *We denote by  $L_{\mathbf{r}}$  the language generated by a rhythm  $\mathbf{r}$  and the associated special labelling  $\gamma_{\mathbf{r}}$ .*

If  $\mathbf{r}$  happens to be a Christoffel rhythm, then  $L_{\mathbf{r}}$  is by definition equal to  $L_{\frac{p}{q}}$  (which, in this case, is the same as  $L_{\frac{p'}{q'}}$ ). The key result of this work states that  $L_{\mathbf{r}}$  is indeed of the same kind as  $L_{\frac{p'}{q'}}$ .

**THEOREM 27.** *Let  $\mathbf{r}$  be a rhythm of directing parameter  $(q, p)$  and  $\frac{p'}{q'}$  the reduced fraction of  $\frac{p}{q}$ . Then, the language  $L_{\mathbf{r}}$  is a set of representations of the integers in the rational base  $\frac{p'}{q'}$ .*

Even though  $p$  and  $q$  are not coprime, the arcs of the tree  $L_{\mathbf{r}}$  satisfies essentially the same equation as  $L_{\frac{p'}{q'}}$  (cf. Equation (3)) as expressed by the following statement.

**LEMMA 28.** *Let  $\mathbf{r}$  be a rhythm of directing parameter  $(q, p)$  and  $\frac{p'}{q'}$  the reduced fraction of  $\frac{p}{q}$ . Then, for every integers  $n$  and  $m$  it holds:*

$$n \xrightarrow[L_{\mathbf{r}}]{a} m \implies a = q' m - p' n .$$

*Proof.* By induction on  $m$ . The implication obviously holds for the first arc of the tree  $L_{\mathbf{r}}$  as it is  $0 \xrightarrow[L_{\mathbf{r}}]{q'} 1$ .

Let us assume it holds for the  $m$ -th arc, that is,  $n \xrightarrow[L_{\mathbf{r}}]{a} m$  with  $a = q' m - p' n$ .

The  $(m+1)$ -th arc is either  $n \xrightarrow[L_{\mathbf{r}}]{b} (m+1)$  or  $(n+1) \xrightarrow[L_{\mathbf{r}}]{b} (m+1)$ .

- $n \xrightarrow[L_{\mathbf{r}}]{b} (m+1)$  corresponds to the case where  $(m+1) \% p < \sum_{i=0}^{n \% q} r_i$ , hence

$$b = \gamma_{((m+1) \% p)} = \gamma_{(m \% p)} + q' = q'(m+1) - p'n .$$

- $(n+1) \xrightarrow[L_{\mathbf{r}}]{b} (m+1)$  corresponds to the case where  $(m+1) \% p = \sum_{i=0}^{n \% q} r_i$ , hence

$$b = \gamma_{((m+1) \% p)} = \gamma_{(m \% p)} + q' - p' = q'(m+1) - p'(n+1) .$$

In both cases, the second equality follows from the definition of the special labelling  $\gamma_{\mathbf{r}}$ .  $\square$

If we call  $\mathbf{r}$ -representation of an integer  $n$ , and denote by  $\langle n \rangle_{\mathbf{r}}$ , the word that labels the path from the root 0 to the node  $n$  in the labelled tree defined by  $L_{\mathbf{r}}$ , Theorem 27 is equivalent to the following statement that is established by induction on the length of the  $\mathbf{r}$ -representation of  $n$ .

**PROPOSITION 29.** *Let  $\mathbf{r}$  be a rhythm of directing parameter  $(q, p)$ ,  $\frac{p'}{q'}$  the reduced fraction of  $\frac{p}{q}$  and  $\pi_{\frac{p'}{q'}}$  the evaluation function in the numeration system with rational base  $\frac{p'}{q'}$ . Then, for every integer  $n$  it holds:*

$$\pi_{\frac{p'}{q'}}(\langle n \rangle_{\mathbf{r}}) = n .$$

*Proof.* Let  $\langle n \rangle_{\mathbf{r}} = a_k a_{k-1} \dots a_0$  be the  $\mathbf{r}$ -representation of  $n$ . We then want to prove that

$$n = \sum_{i=0}^k \frac{a_i}{q'} \left( \frac{p'}{q'} \right)^i .$$

By induction on the length of  $\langle n \rangle_{\mathbf{r}}$ . The equality obviously holds true for  $\langle 0 \rangle_{\mathbf{r}} = \varepsilon$ .

Let  $m$  be an integer and  $\langle m \rangle_{\mathbf{r}} = a_{k+1} a_k a_{k-1} \cdots a_1 a_0$  its  $\mathbf{r}$ -representation, that is, a word of  $L_{\mathbf{r}}$ . The word  $a_{k+1} a_k a_{k-1} \cdots a_1$  is also in  $L_{\mathbf{r}}$ ; it is the  $\mathbf{r}$ -representation of an integer  $n$  strictly smaller than  $m$ , and such that:

$$n \xrightarrow[L_{\mathbf{r}}]{a_0} m .$$

By induction hypothesis,  $n = \sum_{i=1}^{k+1} \frac{a_i}{q} \left( \frac{p'}{q'} \right)^{i-1}$ . It follows from Lemma 28 that  $a_0 = q'm - p'n$ , or, equivalently, that  $m = \frac{np' + a_0}{q'}$ , hence

$$m = \frac{p'}{q'} \left( \sum_{i=1}^{k+1} \frac{a_i}{q'} \left( \frac{p'}{q'} \right)^{i-1} \right) + \frac{a_0}{q} = \left( \sum_{i=1}^{k+1} \frac{a_i}{q'} \left( \frac{p'}{q'} \right)^i \right) + \frac{a_0}{q} .$$

□

In other words,  $L_{\mathbf{r}}$  seen as an abstract numeration system is indeed a *positional* numeration system.

It has been shown in [1] that every numeration system in rational base  $\frac{p}{q}$  has the remarkable property that even though the representation language  $L_{\frac{p}{q}}$  is not a regular language, the *conversion* from any digit-alphabet  $B$  into the canonical alphabet  $A_p$  is realised by a *finite transducer* (indeed a letter-to-letter right sequential transducer), exactly as in the case of the numeration system in base  $p$  (*cf.* also [6]).

More precisely, let  $B_{\mathbf{r}}$  be the digit-alphabet of the special labelling  $\gamma_{\mathbf{r}}$ . Let  $\chi_{\mathbf{r}}$  be the function from  $B_{\mathbf{r}}^*$  into  $A_{p'}^*$  which maps every word of  $B_{\mathbf{r}}^*$  onto the word of  $A_{p'}^*$  which has the same value in the numeration system in the base  $\frac{p'}{q'}$ , that is,

$$\forall w \in B_{\mathbf{r}}^* \quad \pi_{\frac{p'}{q'}}(w) = \pi_{\frac{p'}{q'}}(\chi_{\mathbf{r}}(w)) .$$

Hence  $\chi_{\mathbf{r}}(L_{\mathbf{r}}) = L_{\frac{p'}{q'}}$ . And we can then state:

**THEOREM 30** ([1]). *The map  $\chi_{\mathbf{r}}$  is a rational function.*

We can now complete the proof of Theorem 12.b. Since  $K_{\mathbf{r}}$  is prefix-closed, it is a FLIP language if and only if it contains no infinite regular subset. Suppose that  $K_{\mathbf{r}}$  contains an infinite regular subset  $R$ . There exists a morphism  $\varphi_{\mathbf{r}}$  such that  $\varphi_{\mathbf{r}}(K_{\mathbf{r}}) = L_{\mathbf{r}}$  and a rational function  $\chi_{\mathbf{r}}$  such that  $\chi_{\mathbf{r}}(L_{\mathbf{r}}) = L_{\frac{p'}{q'}}$ ; hence the FLIP language  $L_{\frac{p'}{q'}}$  contains the infinite regular subset  $\chi_{\mathbf{r}}(\varphi_{\mathbf{r}}(R))$ , a contradiction.

## 6 Conclusion and future work

With this notion of labelled signature, we have somehow captured the ‘regularity’ of the representation languages in rational bases by showing that they have *periodic*

labelled signatures and that this periodicity is to some extent characteristic of these languages. A by-product of this characterisation is the remarkable fact that periodic labelled signatures yield either very simple languages (when the growth ratio is an integer) or very complex (when it is not).

It would be very tempting to get the same kind of results with periodic signatures only, that is, without bringing labelling into play. On one hand, it is easy to show that one gets a rational tree (that is, a tree with finite distinct subtrees) in the case of an integral growth ratio. But on the other hand, and as it is related to open problems in number theory, it would be certainly difficult to show, for instance, that all subtrees are distinct in the case of a non-integral growth ratio, although it is a reasonable conjecture. Hopefully, there are easier problems at hand.

For sake of simplicity, we have considered here purely periodic signatures and labelled signatures only. The generalisation to ultimately periodic ones raises no special difficulties but technical details to be settled. And the results established here readily extend.

The problem of the representation of negative integers in the  $\frac{p}{q}$ -numeration systems was considered (among others) in [5]. The characterisation of these  $\frac{p}{q}$ -numeration systems by the corresponding Christoffel words and the study of their combinatorial properties allow a new approach to this problem and yield new proof to some results of [5]; it is the purpose of forthcoming work of the first author [9].

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