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► **To cite this version:**

Jacques Alev, François Dumas. On enveloping skew fields of some Lie superalgebras. Journal of Algebra and Its Applications, 2016, 15 (4), 1650071, 16 pp. hal-01075789

HAL Id: hal-01075789

<https://hal.science/hal-01075789>

Submitted on 20 Oct 2014

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ON ENVELOPING SKEW FIELDS OF SOME LIE SUPERALGEBRAS

JACQUES ALEV AND FRANÇOIS DUMAS

ABSTRACT. We determine the skew fields of fractions of the enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1, 2)$ and of some significant sub-superalgebras of the Lie superalgebra $\mathfrak{osp}(1, 4)$. We compare the kinds of skew fields arising from this “super” context with the Weyl skew fields in the classical Gelfand-Kirillov property.

INTRODUCTION

This paper deals with the question of a possible analogue of the Gelfand-Kirillov property for the enveloping algebras of Lie superalgebras. Let us recall that a finite dimensional complex Lie algebra \mathfrak{g} satisfies the Gelfand-Kirillov property when its enveloping skew field, that is the skew field of fractions of the enveloping algebra $\mathcal{U}(\mathfrak{g})$, is isomorphic to a Weyl skew field over a purely transcendental extension of \mathbb{C} . A rich literature has developed on this topic from the seminal work [7] and we refer to the papers [3], [16] and their bibliographies for an overview on it.

A natural starting point for the same problem for a finite dimensional complex Lie superalgebra \mathfrak{g} is the classification of the classical simple Lie superalgebras (see [10], [13]) and more precisely the study of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2n)$ since this is the only case in the classification whose enveloping algebra is a domain (see [4], [9]). This topic is introduced and discussed by Musson in [12] who proves in particular that $\text{Frac}(\mathcal{U}(\mathfrak{osp}(1, 2n)))$ is not isomorphic to a Weyl skew field over a purely transcendental extension of \mathbb{C} when $n = 1$. We show here that the same is true for any n describing explicitly some classes of skew fields arising from this context. The even part \mathfrak{g}_0 of $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ is the Lie algebra $\mathfrak{sp}(2n)$ of the symplectic group, for which the Gelfand-Kirillov property remains an open question (see [16]). Therefore we concentrate in this exploratory paper on the case of $\mathfrak{osp}(1, 2)$ and on some significant sub-superalgebras of $\mathfrak{osp}(1, 4)$. We consider in $\mathfrak{osp}(1, 4)$ the Lie sub-superalgebras \mathfrak{n}^+ , \mathfrak{b}^+ and \mathfrak{p}^+

Date: October 20, 2014.

2010 Mathematics Subject Classification. Primary 17B35; Secondary 16S30, 16S85, 16K40.

Key words and phrases. Simple Lie superalgebra, enveloping algebra, Gelfand-Kirillov hypothesis, Weyl skew fields.

which have as even parts respectively the nilpotent positive part, the associated Borel subalgebra and the associated parabolic subalgebra in the triangular decomposition of the even part $\mathfrak{g}_{\overline{0}} = \mathfrak{sp}(4)$. Determining their enveloping skew fields is the content of sections 2 and 3 of the paper.

The skew fields appearing in this “super” context are skew fields of rational functions mixing classical Weyl relations $xy - yx = 1$ and “fermionic” relations $xy + yx = 1$ (or equivalently $xy = -yx$ up to rational equivalence) between the generators. A noteworthy fact is that these relations are braided and not necessarily pairwise separable up to isomorphism as in the case of the classical Weyl skew fields. The main properties of these skew fields, which already appeared in [2] and [18], are given in section 1.

We end this introduction by a short reminder on the para-Bose definition of the Lie superalgebra $\mathfrak{osp}(1, 2n)$ and its enveloping algebra (see [6], [14]). The basefield is \mathbb{C} . We fix an integer $n \geq 1$. We have $\mathfrak{osp}(1, 2n) = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ where the even part $\mathfrak{g}_{\overline{0}}$ is the Lie algebra $\mathfrak{sp}(2n)$ of the symplectic group and $\mathfrak{g}_{\overline{1}}$ is a vector space of dimension $2n$. As a Lie superalgebra, $\mathfrak{osp}(1, 2n)$ is generated by the $2n$ elements b_i^{\pm} ($1 \leq i \leq n$) of a basis of the odd part $\mathfrak{g}_{\overline{1}}$. The $2n^2 + n$ elements $\{b_j^{\pm}, b_k^{\pm}\}$ ($1 \leq j \leq k \leq n$) and $\{b_j^+, b_k^-\}$ ($1 \leq j, k \leq n$) form a basis of $\mathfrak{g}_{\overline{0}}$. The dimension of the vector space $\mathfrak{osp}(1, 2n)$ is $2n^2 + 3n$. The brackets are given by the so called “parabose” relations:

$$[\{b_j^{\xi}, b_k^{\eta}\}, b_{\ell}^{\epsilon}] = (\epsilon - \xi)\delta_{j\ell}b_k^{\eta} + (\epsilon - \eta)\delta_{k\ell}b_j^{\xi} \quad (1)$$

$$\begin{aligned} [\{b_i^{\xi}, b_j^{\eta}\}, \{b_k^{\epsilon}, b_{\ell}^{\varphi}\}] &= (\epsilon - \eta)\delta_{jk}\{b_i^{\xi}, b_{\ell}^{\varphi}\} + (\epsilon - \xi)\delta_{ik}\{b_j^{\eta}, b_{\ell}^{\varphi}\} \\ &+ (\varphi - \eta)\delta_{j\ell}\{b_i^{\xi}, b_k^{\epsilon}\} + (\varphi - \xi)\delta_{i\ell}\{b_j^{\eta}, b_k^{\epsilon}\}. \end{aligned} \quad (2)$$

By the PBW theorem (see [13]), the enveloping algebra $\mathcal{U}(\mathfrak{osp}(1, 2n))$ is generated by the $2n^2 + n$ elements:

$$b_i^{\pm}, k_i := \frac{1}{2}\{b_i^-, b_i^+\} \text{ for } 1 \leq i \leq n, \quad (3)$$

$$a_{ij}^{\pm} := \frac{1}{2}\{b_i^{\pm}, b_j^{\pm}\}, s_{ij} := \frac{1}{2}\{b_i^-, b_j^+\}, t_{ij} := \frac{1}{2}\{b_i^+, b_j^-\} \text{ for } 1 \leq i < j \leq n, \quad (4)$$

with commutation relations deduced from (1) and (2) taking $\{x, y\} = xy + yx$ if $x, y \in \mathfrak{g}_{\overline{1}}$, and $[x, y] = xy - yx$ otherwise. The enveloping algebra $\mathcal{U}(\mathfrak{sp}(2n))$ of the even part is the subalgebra of $\mathcal{U}(\mathfrak{osp}(1, 2n))$ generated by the $2n^2 + n$ elements $(b_i^{\pm})^2, k_i$ for $1 \leq i \leq n$, and $a_{ij}^{\pm}, s_{ij}, t_{ij}$ for $1 \leq i < j \leq n$.

1. SOME SKEW FIELDS

1.1. Definitions and notations. We fix the basefield to be \mathbb{C} . As usual \mathbf{A}_1 is the Weyl algebra, that is the algebra generated over \mathbb{C} by two generators x, y satisfying the commutation law $xy - yx = 1$. We also define \mathbf{A}^1 as the algebra generated over \mathbb{C} by two generators u, v satisfying the commutation law $uv + vu = 1$. For any nonnegative integers r, s , we denote by \mathbf{A}_r^s the

\mathbb{C} -algebra:

$$\mathbf{A}_r^s = \underbrace{\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_1}_r \otimes \underbrace{\mathbf{A}^1 \otimes \mathbf{A}^1 \otimes \cdots \otimes \mathbf{A}^1}_s$$

This is clearly a noetherian domain, so we can define $\mathbf{D}_r^s = \text{Frac } \mathbf{A}_r^s$. For any integer $t \geq 0$, we consider the polynomial algebra $\mathbb{C}[z_1, \dots, z_t]$, the non-commutative noetherian domain $\mathbf{A}_{r,t}^s = \mathbf{A}_r^s \otimes \mathbb{C}[z_1, \dots, z_t]$ and its skew field of fractions $\mathbf{D}_{r,t}^s = \text{Frac } \mathbf{A}_{r,t}^s$. In particular $\mathbf{D}_r^s = \mathbf{D}_{r,0}^s$. If $s = 0$ and $r \geq 1$, $\mathbf{D}_{r,t}^0$ is the classical Weyl skew field usually denoted by $D_r(\mathbb{C}(z_1, \dots, z_t))$ or $\mathcal{D}_{r,t}(\mathbb{C})$. The following lemma asserts that the $\mathbf{D}_{r,t}^s$ are a particular case of the mixed Weyl skew fields considered in [2].

1.2. Lemma. *The skew field $\mathbf{D}_{r,t}^s$ is isomorphic to the skew field of fractions of the algebra $\widehat{\mathbf{A}}_{r,t}^s$ generated over \mathbb{C} by $2r + 2s + t$ generators $x_1, \dots, x_r, y_1, \dots, y_r, u_1, \dots, u_s, w_1, \dots, w_s, z_1, \dots, z_t$ satisfying the commutation relations:*

$$\begin{aligned} [x_i, y_i] &= 1, [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 & (1 \leq i \neq j \leq r), \\ u_i w_i &= -w_i u_i, [u_i, w_j] = [u_i, u_j] = [w_i, w_j] = 0 & (1 \leq i \neq j \leq s), \\ [x_i, u_j] &= [x_i, w_j] = [y_i, u_j] = [y_i, w_j] = 0 & (1 \leq i \leq r, 1 \leq j \leq s) \\ [x_i, z_k] &= [y_i, z_k] = [u_j, z_k] = [w_j, z_k] = [z_k, z_\ell] = 0 & (1 \leq i \leq r, 1 \leq j \leq s, \\ & & 1 \leq k, \ell \leq t). \end{aligned}$$

Proof. For any $1 \leq i \leq s$, let us consider the copy of \mathbf{A}^1 generated by u_i, v_i with relation $u_i v_i + v_i u_i = 1$. The element $w_i := u_i v_i - v_i u_i = 2u_i v_i - 1$ of \mathbf{A}^1 satisfies $w_i u_i = -u_i w_i$ and $w_i v_i = -v_i w_i$. In the skew field of fractions, the subfield generated by u_i, v_i is isomorphic to the subfield generated by u_i, w_i since $v_i = \frac{1}{2} u_i^{-1} (w_i + 1)$. Hence the proof is complete. \square

We sum up in the following proposition some basic facts about the skew fields $\mathbf{D}_{r,t}^s$. It shows in particular that for $s \neq 0$ the skew fields $\mathbf{D}_{r,t}^s$ are not isomorphic to classical Weyl skew fields.

1.3. Proposition. *Let r, s, t be any nonnegative integers. Then:*

- (i) *the Gelfand-Kirillov transcendence degree of $\mathbf{D}_{r,t}^s$ equals to $2r + 2s + t$;*
- (ii) *the center of $\mathbf{D}_{r,t}^s$ is $\mathbb{C}(u_1^2, \dots, u_s^2, w_1^2, \dots, w_s^2, z_1, \dots, z_t)$, with the notations of lemma 1.2;*
- (iii) *$\mathbf{D}_{r,t}^s$ is isomorphic to a classical Weyl skew field $\mathbf{D}_{r',t'}^0$ if and only if $s = 0, r = r'$ and $t = t'$.*

Proof. The algebra $\widehat{\mathbf{A}}_{r,t}^s$ of lemma 1.2 is a particular case of the algebras $S_{n,r}^\Lambda$ studied in [18]. Explicitly $\widehat{\mathbf{A}}_r^s = S_{n,r}^\Lambda$ for $n = r + 2s + t$ and $\Lambda = (\lambda_{ij})$ the $n \times n$ matrix with entries in \mathbb{C} defined by $\lambda_{r+2k-1, r+2k} = \lambda_{r+2k, r+2k-1} = -1$ for any $1 \leq k \leq s$, and $\lambda_{i,j} = 1$ in any other case. Then points (i) and (ii) follow respectively from proposition 1.1.4 and proposition 3.3.1 of [18]. Suppose now that $\mathbf{D}_{r,t}^s$ is isomorphic to $\mathbf{D}_{r',t'}^0$ for some $r' \geq 1, t' \geq 0$. Denote

$G(L) = (L^\times)' \cap \mathbb{C}^\times$ the trace on \mathbb{C}^\times of the commutator subgroup of the group of nonzero elements of L for any skew field L over \mathbb{C} . It follows from theorem 3.10 of [1] that $G(\mathbf{D}_{r',t'}^0) = \{1\}$ while it is clear by lemma 1.2 that $-1 \in G(\mathbf{D}_{r,t}^s)$ if $s \geq 1$. Hence $s = 0$. Then comparing the centers we deduce $t = t'$ and comparing the Gelfand-Kirillov transcendence degrees we conclude $r = r'$. \square

1.4. Remark. Each copy in $\mathbf{D}_{r,t}^s$ of the algebra $\widehat{\mathbf{A}}^1$ generated over \mathbb{C} by two generators u, w satisfying $uw = -wu$ can be viewed as the enveloping algebra of the nilpotent Lie superalgebra $\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1$ where $\mathfrak{f}_0 = \mathbb{C}z \oplus \mathbb{C}t$ and $\mathfrak{f}_1 = \mathbb{C}u \oplus \mathbb{C}w$ with brackets $\{u, u\} = z$, $\{w, w\} = t$, $\{u, w\} = 0$.

The argument used in the proof of point (iii) of the previous proposition allows to show the following proposition, as predicted in [12].

1.5. Proposition. *For any integer $n \geq 1$, $\text{Frac}(\mathcal{U}(\mathfrak{osp}(1, 2n)))$ is not isomorphic to a classical Weyl skew field $\mathbf{D}_{r,t}^0$ for any $r \geq 1, t \geq 0$.*

More generally, for any subsuperalgebra \mathfrak{g} of $\mathfrak{osp}(1, 2n)$ containing the generators b_i^+ and b_i^- for some $1 \leq i \leq n$, $\text{Frac}(\mathcal{U}(\mathfrak{g}))$ is not isomorphic to a classical Weyl skew field $\mathbf{D}_{r,t}^0$ for any $r \geq 1, t \geq 0$.

Proof. If \mathfrak{g} contains b_i^+ and b_i^- , it contains the element $k_i = \frac{1}{2}\{b_i^+, b_i^-\}$. Then $\mathcal{U}(\mathfrak{g})$ contains the element $z_i = b_i^+ b_i^- - b_i^- b_i^+ + 1 = 2b_i^+ b_i^- - 2k_i + 1$. Using relation (1), we have $[k_i, b_i^+] = b_i^+$. An obvious calculation gives $z_i b_i^+ = -b_i^+ z_i$. It follows that $-1 \in G(\text{Frac}(\mathcal{U}(\mathfrak{g})))$; as at the end of the proof of proposition 1.3 we conclude that $\text{Frac}(\mathcal{U}(\mathfrak{g}))$ cannot be isomorphic to a classical Weyl skew field. \square

The skew fields $\mathbf{D}_{r,t}^s$ are the most simple and natural way to mix classical Weyl skew fields $\mathcal{D}_{r,t}(\mathbb{C})$ with “fermionic” relations $uw = -wu$. However we will see in the following that they are not sufficient to describe the rational equivalence of enveloping algebras of Lie superalgebras. Some “braided” versions of mixed skew fields are necessary. The low dimensional examples useful for the following results are introduced in [2]. Their generalization in any dimension are the subject of a systematic study in the article [18]. We recall here their definitions and main properties.

1.6. Definitions and notations. Let \mathbf{S}_3 be the algebra generated over \mathbb{C} by three generators x, y, z satisfying:

$$xy - yx = 1, \quad xz = -zx, \quad yz = -zy.$$

Crossing two copies of \mathbf{S}_3 , we define \mathbf{S}_4 as the algebra generated over \mathbb{C} by four generators x_1, x_2, y_1, y_2 satisfying:

$$\begin{aligned} x_1 y_1 - y_1 x_1 &= 1, & x_1 y_2 &= -y_2 x_1, & x_1 x_2 &= -x_2 x_1 \\ x_2 y_2 - y_2 x_2 &= 1, & x_2 y_1 &= -y_1 x_2, & y_1 y_2 &= -y_2 y_1. \end{aligned}$$

The algebras \mathbf{S}_3 and \mathbf{S}_4 are obviously noetherian domains. We denote $\mathbf{F}_3 = \text{Frac } \mathbf{S}_3$ and $\mathbf{F}_4 = \text{Frac } \mathbf{S}_4$.

The algebra \mathbf{S}_4 is the case $n = 2$ of the family of quantum Weyl algebras $A_n^{\bar{q}, \Lambda}$ introduced in [1] when all nontrivial entries λ_{ij} of Λ are equal to -1 and all entries q_i of \bar{q} are equal to 1. They have been intensively studied (we refer to [8] and to section 1.3.3 of [17] for a survey and references), are simple of center \mathbb{C} and have the same Hochschild homology and cohomology as the classical Weyl algebra $A_n(\mathbb{C})$. A similar study for \mathbf{S}_3 lies in sections 5 and 7 of [17].

1.7. Proposition. *The following holds for the skew fields \mathbf{F}_3 and \mathbf{F}_4 :*

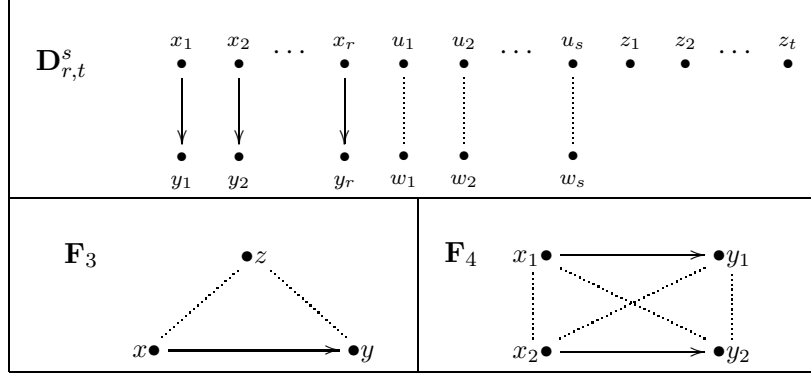
- (i) *the Gelfand-Kirillov transcendence degrees of \mathbf{F}_3 and \mathbf{F}_4 are 3 and 4 respectively ;*
- (ii) *the center of \mathbf{F}_3 is $\mathbb{C}(z^2)$, and the center of \mathbf{F}_4 is \mathbb{C} ;*
- (iii) *\mathbf{F}_3 and \mathbf{F}_4 are not isomorphic to $\mathbf{D}_{r,t}^s$, for any $r, s, t \geq 0$.*

Proof. These properties are proved under slightly different assumptions in section 3 of [2]. With the notation of [18], we have $\mathbf{S}_3 = S_{2,1}^\Lambda$ and $\mathbf{S}_4 = S_{2,2}^\Lambda$ for $\Lambda = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then points (i) and (ii) follow respectively from proposition 1.1.4 and proposition 3.3.1 of [18]. Suppose that \mathbf{F}_3 is isomorphic to some $\mathbf{D}_{r,t}^s$. Comparing the Gelfand-Kirillov transcendence degree, we necessarily have $(r, s, t) = (0, 0, 3)$, $(1, 0, 1)$ or $(0, 1, 1)$. The first case is obviously excluded since \mathbf{F}_3 is not commutative. The second case is impossible because, with the notation $G(L)$ recalled in the proof of proposition 1.3, we know that $G(\mathbf{D}_{1,1}^0) = \{1\}$ by theorem 3.10 of [1], while it is clear that $-1 \in G(\mathbf{F}_3)$. The third case is also impossible because, denoting $E(L) = [L, L] \cap \mathbb{C}$ the trace on \mathbb{C} of the subspace generated by the commutation brackets for any skew field L over \mathbb{C} , we have $E(\mathbf{D}_{0,1}^1) = \{0\}$ by proposition 3.9 of [1], and $E(\mathbf{F}_3) = \mathbb{C}$ since $D_1(\mathbb{C}) \subset \mathbf{F}_3$. Suppose now that \mathbf{F}_4 is isomorphic to $\mathbf{D}_{r,t}^s$. Since the transcendence degree of the center of $\mathbf{D}_{r,t}^s$ is at least t , it follows from point (ii) that $s = t = 0$. Therefore \mathbf{F}_4 would be isomorphic to the usual Weyl skew field $\mathbf{D}_{2,0}^0$. One more time this is impossible because $G(\mathbf{D}_{2,0}^0) = \{1\}$ and $-1 \in G(\mathbf{F}_4)$. \square

1.8. Remarks. Let us consider the algebra $\widehat{\mathbf{A}}_{r,0}^s$ with the notations of lemma 1.2. If $r \geq 1$ and $s \geq 1$, the subfield of \mathbf{D}_r^s generated by $x_1 w_1, y_1 w_1^{-1}$ and u_1 is isomorphic to \mathbf{F}_3 . If $r \geq 2$ and $s \geq 1$, the subfield of \mathbf{D}_r^s generated by $x_1 w_1, y_1 w_1^{-1}, x_2 u_1$ and $y_2 u_1^{-1}$ is isomorphic to \mathbf{F}_4 . In other words, \mathbf{F}_3 can be embedded in any \mathbf{D}_r^s such that $r \geq 1, s \geq 1$ and \mathbf{F}_4 can be embedded in any \mathbf{D}_r^s such that $r \geq 2, s \geq 1$. More deeply it follows from proposition 5.3.3 of [18] that \mathbf{F}_4 cannot be embedded in some $\mathbf{D}_{r,t}^s$ for $r \leq 1$.

1.9. Illustration. We illustrate the definitions of the skew fields under consideration by the following graphs, stressing the particular nature of the relevant relations. The vertices are parametrized by some system of generators.

A directed edge $a \bullet \longrightarrow \bullet b$ between two generators a and b means that $ab - ba = 1$, an undirected edge $a \bullet \cdots \cdots \bullet b$ means that $ab = -ba$, and no edge between two generators means that they commute.



2. THE ENVELOPING SKEW FIELD OF THE LIE SUPERALGEBRA $\mathfrak{osp}(1,2)$

2.1. Notations. Applying (1) and (3) for $n = 1$, the algebra $\mathcal{U}(\mathfrak{osp}(1,2))$ is generated by b^+, b^-, k with relations:

$$b^+ - b^+k = b^+, \quad kb^- - b^-k = -b^-, \quad b^-b^+ = -b^+b^- + 2k. \quad (5)$$

It is clearly an iterated Ore extension $\mathcal{U}(\mathfrak{osp}(1,2)) = \mathbb{C}[b^+][k; \delta][b^-; \tau, d]$, where δ is the derivation $b^+ \partial_{b^+}$ in $\mathbb{C}[b^+]$, τ is the automorphism of $\mathbb{C}[b^+][k; \delta]$ defined by $\tau(b^+) = -b^+$ and $\tau(k) = k + 1$, and d is the τ -derivation of $\mathbb{C}[b^+][k; \delta]$ defined by $d(b^+) = 2k$ and $d(k) = 0$.

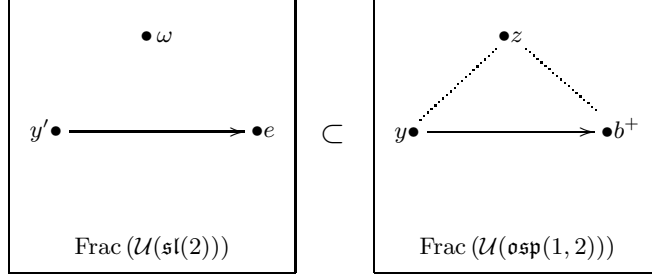
2.2. Proposition. $\text{Frac} \mathcal{U}(\mathfrak{osp}(1,2))$ is isomorphic to \mathbf{F}_3 .

Proof. By obvious calculations using (5), the element $z := b^+b^- - b^-b^+ + 1 = 2b^+b^- - 2k + 1$ satisfies $zb^+ = -b^+z$ and $zk = kz$. Since $b^- = \frac{1}{2}(b^+)^{-1}(z + 2k - 1)$ in the algebra $\mathcal{U}' := \mathbb{C}(b^+)[k; \delta][b^-; \tau, d]$, we have $\mathcal{U}' = \mathbb{C}(b^+)[k; \delta][z; \tau']$ with $kb^+ - b^+k = b^+$, $zk = kz$ and $zb^+ = -b^+z$. Setting $y := (b^+)^{-1}k$, we obtain $\mathcal{U}' = \mathbb{C}(b^+)[y; \partial_{b^+}][z; \tau']$ with $yb^+ - b^+y = 1$, $zb^+ = -b^+z$ and $zy = -yz$. Hence $\text{Frac} \mathcal{U}' = \text{Frac} \mathcal{U}(\mathfrak{osp}(1,2))$ is isomorphic to \mathbf{F}_3 . \square

2.3. Remarks. We know that $\mathcal{U}(\mathfrak{sl}(2))$ is the subalgebra of $\mathcal{U}(\mathfrak{osp}(1,2))$ generated by $(b^+)^2, (b^-)^2$ and k . Actually up to a change of notations $e := \frac{1}{2}(b^+)^2$ and $f := -\frac{1}{2}(b^-)^2$ it follows from (5) that $[k, e] = 2e$, $[k, f] = -2f$ et $[e, f] = k$. We introduce $\omega := 4ef + k^2 - 2k$ the usual Casimir in $\mathcal{U}(\mathfrak{sl}(2))$.

(i) With the notations used in the proof of the previous proposition, we have in \mathcal{U}' the identities $f = \frac{1}{4}e^{-1}(\omega - k^2 + 2k)$ and $[\frac{1}{2}e^{-1}k, e] = 1$. Therefore $\text{Frac} \mathcal{U}(\mathfrak{sl}(2))$ is the subfield of $\text{Frac} \mathcal{U}(\mathfrak{osp}(1,2))$ generated by $e = \frac{1}{2}(b^+)^2$, $y' := (b^+)^{-2}k = (b^+)^{-1}y$ and ω with relations $y'e - ey' = 1$, $\omega e = e\omega$ and $\omega y' = y'\omega$. We recover the well known Gelfand-Kirillov property that

$\text{Frac}\mathcal{U}(\mathfrak{sl}(2))$ is a classical Weyl skew field D_1 over a center $\mathbb{C}(\omega)$ of transcendence degree one. With the conventions of 1.9, we can illustrate this skew fields embedding by:



(ii) By previous proposition 2.2 and point (ii) of proposition 1.7, the center of $\text{Frac}(\mathcal{U}(\mathfrak{osp}(1, 2)))$ is $\mathbb{C}(z^2)$. The element z lying in $\mathcal{U}(\mathfrak{osp}(1, 2))$, it follows that the center of $\mathcal{U}(\mathfrak{osp}(1, 2))$ is $\mathbb{C}[z^2]$. A straightforward calculation shows that $z^2 = 4\omega - 2z + 3 = 4\omega - 2(z - 1) + 1$, or equivalently $(z + 1)^2 = 4(\omega + 1)$. Since $z - 1 = b^+b^- - b^-b^+$ by definition of z , we recover the well known property, see [15], that the center of $\mathcal{U}(\mathfrak{osp}(1, 2))$ is $\mathbb{C}[\theta]$ for θ the super Casimir operator

$$\theta := \omega - \frac{1}{2}(b^+b^- - b^-b^+), \quad (6)$$

with ω the usual Casimir operator of the even part $\mathcal{U}(\mathfrak{sl}(2))$. On one hand the above expression of z^2 becomes $z^2 = 4\theta + 1$. On the other hand, (6) implies $z - 1 = 2\omega - 2\theta$. We deduce that $(2\omega - 2\theta + 1)^2 = 4\theta + 1$, or equivalently:

$$\omega^2 - (2\theta - 1)\omega + \theta(\theta - 2) = 0. \quad (7)$$

This relation of algebraic dependance between θ and ω is exactly the one given in proposition 1.2 of [15] up to a normalization of the coefficients.

3. ENVELOPING SKEW FIELDS OF SOME LIE SUBSUPERALGEBRAS OF $\mathfrak{osp}(1, 4)$

3.1. Definitions and notations. We apply for $n = 2$ the description of $\mathfrak{osp}(1, 2n)$ recalled at the end of the introduction. We have $\mathfrak{osp}(1, 4) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where \mathfrak{g}_1 is a vector space of dimension 4 with basis $b_1^+, b_2^+, b_1^-, b_2^-$ and \mathfrak{g}_0 is the Lie algebra $\mathfrak{sp}(4)$ of dimension 10 with basis:

$$\begin{aligned} c_1^+ &= \frac{1}{2}\{b_1^+, b_1^+\}, & c_2^+ &= \frac{1}{2}\{b_2^+, b_2^+\}, & c_1^- &= \frac{1}{2}\{b_1^-, b_1^-\}, & c_2^- &= \frac{1}{2}\{b_2^-, b_2^-\}, \\ a^+ &= \frac{1}{2}\{b_1^+, b_2^+\}, & a^- &= \frac{1}{2}\{b_1^-, b_2^-\}, & s &= \frac{1}{2}\{b_1^-, b_2^+\}, & t &= \frac{1}{2}\{b_1^+, b_2^-\}, \\ k_1 &= \frac{1}{2}\{b_1^-, b_1^+\}, & k_2 &= \frac{1}{2}\{b_2^-, b_2^+\}. \end{aligned} \quad (8)$$

The brackets between these 14 generators of $\mathfrak{osp}(1, 4)$ are computed by the relations (1) et (2). By (3) and (4) the algebra $\mathcal{U}(\mathfrak{osp}(1, 4))$ is generated

by the 10 elements $b_1^+, b_2^+, b_1^-, b_2^-, k_1, k_2, a^+, a^-, s, t$. The enveloping algebra $\mathcal{U}(\mathfrak{sp}(4))$ of the even part of $\mathfrak{osp}(1,4)$ is the subalgebra generated by $(b_1^+)^2, (b_2^+)^2, (b_1^-)^2, (b_2^-)^2, k_1, k_2, a^+, a^-, s, t$.

We describe now some subsuperalgebras of $\mathfrak{osp}(1,4)$ whose enveloping skew field we study in the following. The even part of each of them satisfies the usual Gelfand-Kirillov property.

3.1.1. *The nilpotent subsuperalgebra \mathfrak{n}^+ .* We define in \mathfrak{g}_1^+ the subspace $\mathfrak{g}_1^+ := \mathbb{C}b_1^+ \oplus \mathbb{C}b_2^+$ and in \mathfrak{g}_0^+ the subspace $\mathfrak{n}_0^+ := \mathbb{C}c_1^+ \oplus \mathbb{C}c_2^+ \oplus \mathbb{C}a^+ \oplus \mathbb{C}t$. We denote $\mathfrak{n}^+ := \mathfrak{n}_0^+ \oplus \mathfrak{g}_1^+$. We calculate in $\mathfrak{osp}(1,4)$ the 17 brackets between the 6 generators of \mathfrak{n}^+ :

$$\begin{aligned} [a^+, c_1^+] &= 0, & [a^+, c_2^+] &= 0, & [t, c_1^+] &= 0, & [t, c_2^+] &= 2a^+, \\ [c_1^+, c_2^+] &= 0, & [t, a^+] &= c_1^+, \\ \{b_1^+, b_1^+\} &= 2c_1^+, & \{b_2^+, b_2^+\} &= 2c_2^+, & \{b_1^+, b_2^+\} &= 2a^+, \\ [t, b_1^+] &= 0, & [t, b_2^+] &= b_1^+, & [a^+, b_1^+] &= 0, & [a^+, b_2^+] &= 0, \\ [b_1^+, c_1^+] &= 0, & [b_1^+, c_2^+] &= 0, & [b_2^+, c_1^+] &= 0, & [b_2^+, c_2^+] &= 0. \end{aligned} \tag{9}$$

It follows that \mathfrak{n}^+ is a Lie subsuperalgebra of $\mathfrak{osp}(1,4)$ and that \mathfrak{n}_0^+ is a Lie subalgebra of \mathfrak{g}_0^+ . Moreover setting

$$x_1 := t, \quad x_2 := c_2^+, \quad x_3 := 2a^+, \quad x_4 := 2c_1^+, \tag{10}$$

we rewrite the relations of the first two rows of (9) as:

$$\begin{aligned} [x_1, x_2] &= x_3, & [x_1, x_3] &= x_4, & [x_2, x_3] &= 0, \\ [x_1, x_4] &= 0, & [x_2, x_4] &= 0, & [x_3, x_4] &= 0, \end{aligned}$$

which are the relations between the Chevalley generators in the enveloping algebra of the nilpotent positive part corresponding to the root system of type B_2 . We conclude that in the Lie subsuperalgebra \mathfrak{n}^+ of $\mathfrak{g} = \mathfrak{osp}(1,4)$, the even part \mathfrak{n}_0^+ is isomorphic to the nilpotent positive part in the triangular decomposition of $\mathfrak{g}_0^+ = \mathfrak{sp}(4)$.

3.1.2. *The Borel subsuperalgebra \mathfrak{b}^+ .* We still denote $\mathfrak{g}_1^+ = \mathbb{C}b_1^+ \oplus \mathbb{C}b_2^+$ and we introduce in \mathfrak{g}_0^+ the subspaces $\mathfrak{h} := \mathbb{C}k_1 \oplus \mathbb{C}k_2$ and $\mathfrak{b}_0^+ := \mathfrak{n}_0^+ \oplus \mathfrak{h}$. We define $\mathfrak{b}^+ := \mathfrak{b}_0^+ \oplus \mathfrak{g}_1^+$. We calculate in $\mathfrak{osp}(1,4)$ the 30 brackets between the 8 generators of \mathfrak{b}^+ , adding to the 17 brackets of (9) the 13 brackets related

to k_1, k_2 , that is :

$$\begin{aligned}
 [k_1, b_1^+] &= b_1^+, & [k_1, b_2^+] &= 0, & [k_2, b_1^+] &= 0, & [k_2, b_2^+] &= b_2^+, \\
 [k_1, c_1^+] &= 2c_1^+, & [k_1, c_2^+] &= 0, & [k_2, c_1^+] &= 0, & [k_2, c_2^+] &= 2c_2^+, \\
 [k_1, a^+] &= a^+, & [k_1, t] &= t, & [k_2, a^+] &= a^+, & [k_2, t] &= -t, \\
 [k_1, k_2] &= 0.
 \end{aligned} \tag{11}$$

It follows that \mathfrak{b}^+ is a Lie subsuperalgebra of $\mathfrak{osp}(1, 4)$ and that \mathfrak{b}_0^+ is a Lie subalgebra of \mathfrak{g}_0 containing as direct summands the nilpotent Lie subalgebra \mathfrak{n}_0^+ and the abelian Lie subalgebra \mathfrak{h} . The change of basis (10) in \mathfrak{n}_0^+ and the change of basis

$$h_1 := k_2, \quad h_2 := k_1 - k_2, \tag{12}$$

in \mathfrak{h} allow to rewrite the action of \mathfrak{h} on \mathfrak{n}_0^+ as:

$$\begin{aligned}
 [h_1, x_1] &= -x_1, & [h_1, x_2] &= 2x_2, & [h_1, x_3] &= x_3, & [h_1, x_4] &= 0, \\
 [h_2, x_1] &= 2x_1, & [h_2, x_2] &= -2x_2, & [h_2, x_3] &= 0, & [h_2, x_4] &= 2x_4.
 \end{aligned} \tag{13}$$

We conclude that in the Lie subsuperalgebra \mathfrak{b}^+ of $\mathfrak{g} = \mathfrak{osp}(1, 4)$, the even part \mathfrak{b}_0^+ is isomorphic to the positive Borel subalgebra in the triangular decomposition of $\mathfrak{g}_0 = \mathfrak{sp}(4)$, and the abelian Lie subalgebra \mathfrak{h} is isomorphic to the corresponding Cartan subalgebra.

3.1.3. *The parabolic subsuperalgebra \mathfrak{p}^+ .* We introduce in the odd part \mathfrak{g}_1 of $\mathfrak{osp}(1, 4)$ the subspace $\mathfrak{p}_1^+ := \mathfrak{g}_1^+ \oplus \mathbb{C}b_2^- = \mathbb{C}b_1^+ \oplus \mathbb{C}b_2^+ \oplus \mathbb{C}b_2^-$ and in the even part \mathfrak{g}_0 the subspace $\mathfrak{p}_0^+ := \mathfrak{b}_0^+ \oplus \mathbb{C}c_2^- = \mathfrak{n}_0^+ \oplus \mathfrak{h} \oplus \mathbb{C}c_2^-$. We define $\mathfrak{p}^+ := \mathfrak{p}_0^+ \oplus \mathfrak{p}_1^+$. We calculate in $\mathfrak{osp}(1, 4)$ the 48 brackets between the 10 generators of \mathfrak{p}^+ , adding to the 30 brackets of (9) and (11) the 18 brackets related to b_2^-, c_2^- , that is :

$$\begin{aligned}
 \{b_2^-, b_2^+\} &= 2k_2, & [b_2^-, k_1] &= 0, & [b_2^-, a^+] &= b_1^+, \\
 \{b_2^-, b_1^+\} &= 2t, & [b_2^-, k_2] &= b_2^-, & [b_2^-, t] &= 0, \\
 [c_2^-, c_2^+] &= 4k_2, & [c_2^-, k_1] &= 0, & [c_2^-, a^+] &= 2t, \\
 [c_2^-, c_1^+] &= 0, & [c_2^-, k_2] &= 2c_2^-, & [c_2^-, t] &= 0, \\
 [c_2^-, b_1^+] &= 0, & [c_2^-, b_2^+] &= 2b_2^-, & [c_2^-, b_2^-] &= 0, \\
 \{b_2^-, b_2^-\} &= c_2^-, & [b_2^-, c_1^+] &= 0. & [b_2^-, c_2^+] &= 2b_2^+.
 \end{aligned} \tag{14}$$

It follows that \mathfrak{p}^+ is a Lie subsuperalgebra of $\mathfrak{osp}(1, 4)$ and that \mathfrak{p}_0^+ is a Lie subalgebra of \mathfrak{g}_0 containing as direct summands the Borel subalgebra \mathfrak{b}_0^+ and the line $\mathbb{C}c_2^-$. The changes of basis (10) and (12) allow to rewrite the

action of c_2^- on \mathfrak{b}_0^+ as:

$$\begin{aligned} [c_2^-, x_1] &= 4h_1, & [c_2^-, x_2] &= 0, & [c_2^-, x_3] &= -4x_2, & [c_2^-, x_4] &= 0, \\ [c_2^-, h_1] &= 2c_2^-, & [c_2^-, h_2] &= -2c_2^-. \end{aligned} \quad (15)$$

We conclude that in the Lie subsuperalgebra \mathfrak{p}^+ of $\mathfrak{g} = \mathfrak{osp}(1, 4)$, the even part \mathfrak{p}_0^+ is isomorphic to the positive parabolic subalgebra in the triangular decomposition of $\mathfrak{g}_0 = \mathfrak{sp}(4)$.

3.1.4. Remark: *the Levi subsuperalgebra \mathfrak{l} associated to \mathfrak{p}^+ .* It follows from relations (11) and (14) that the subspace $\mathfrak{l} := \mathfrak{l}_0 \oplus \mathfrak{l}_1$ with $\mathfrak{l}_1 := \mathbb{C}b_2^+ \oplus \mathbb{C}b_2^-$ in \mathfrak{p}_1^+ and $\mathfrak{l}_0 := \mathbb{C}c_2^+ \oplus \mathbb{C}k_2 \oplus \mathbb{C}c_2^-$ in \mathfrak{p}_0^+ is a Lie subsuperalgebra of $\mathfrak{osp}(1, 4)$. It is clear that \mathfrak{l} is isomorphic to $\mathfrak{osp}(1, 2)$. The Lie algebra \mathfrak{l}_0 is the Levi subalgebra associated to \mathfrak{p}_0^+ in \mathfrak{g}_0 and is isomorphic to $\mathfrak{sl}(2)$.

3.2. Proposition. *Frac $\mathcal{U}(\mathfrak{n}^+)$ is isomorphic to $\text{Frac}(\mathbf{A}_1 \otimes \mathbf{A}^1) = \mathbf{D}_1^1$.*

Proof. By 3.1.1, $\mathcal{U}(\mathfrak{n}^+)$ is an iterated Ore extension $\mathbb{C}[b_1^+, a^+][b_2^+; \tau, d][t; \delta]$ expressing the commutation relations

$$\begin{aligned} b_1^+ a^+ &= a^+ b_1^+, & b_2^+ a^+ &= a^+ b_2^+, & b_2^+ b_1^+ &= -b_1^+ b_2^+ + 2a^+, \\ t b_1^+ &= b_1^+ t, & t a^+ &= a^+ t + (b_1^+)^2, & t b_2^+ &= b_2^+ t + b_1^+. \end{aligned} \quad (16)$$

This is a particular case of the more general theorem 2.1 of [11]. In the algebra $\mathcal{U}' := \mathbb{C}(b_1^+)[a^+][b_2^+; \tau, d][t; \delta]$, the elements:

$$t' := (b_1^+)^{-2} t, \quad y := \frac{1}{2}(b_1^+ b_2^+ - b_2^+ b_1^+) = b_1^+ b_2^+ - a^+ \quad (17)$$

satisfy $\mathcal{U}' = \mathbb{C}(b_1^+)[a^+][y; \tau][t'; \delta']$ with relations:

$$\begin{aligned} b_1^+ a^+ &= a^+ b_1^+, & y a^+ &= a^+ y, & y b_1^+ &= -b_1^+ y, \\ t' b_1^+ &= b_1^+ t', & t' y &= y t', & t' a^+ - a^+ t' &= 1. \end{aligned} \quad (18)$$

Hence by lemma 1.2 we conclude that $\text{Frac}\mathcal{U}(\mathfrak{n}^+) = \text{Frac}\mathcal{U}'$ is isomorphic to \mathbf{D}_1^1 . \square

3.3. Remark. The enveloping algebra $\mathcal{U}(\mathfrak{n}_0^+)$ is the subalgebra of $\mathcal{U}(\mathfrak{n}^+)$ generated by $(b_1^+)^2, (b_2^+)^2, a^+, t$ with commutation relations coming from (9):

$$\begin{aligned} (b_1^+)^2 a^+ &= a^+ (b_1^+)^2, & (b_2^+)^2 a^+ &= a^+ (b_2^+)^2, & (b_2^+)^2 (b_1^+)^2 &= (b_1^+)^2 (b_2^+)^2, \\ t (b_1^+)^2 &= (b_1^+)^2 t, & t a^+ &= a^+ t + (b_1^+)^2, & t (b_2^+)^2 &= (b_2^+)^2 t + 2a^+. \end{aligned}$$

The element t' defined in (17) lies in $\text{Frac}\mathcal{U}(\mathfrak{n}_0^+)$ and the element y defined in (17) satisfies $y^2 = (a^+)^2 - (b_1^+)^2 (b_2^+)^2$ which also lies in $\text{Frac}\mathcal{U}(\mathfrak{n}_0^+)$. Hence

$\text{Frac}\mathcal{U}(\mathfrak{n}_0^+)$ is the subfield of $\text{Frac}\mathcal{U}(\mathfrak{n}^+)$ generated by $(b_1^+)^2, y, a^+, t'$ with more simple commutation relations:

$$\begin{aligned} (b_1^+)^2 a^+ &= a^+ (b_1^+)^2, & y^2 a^+ &= a^+ y^2, & y^2 (b_1^+)^2 &= (b_1^+)^2 y^2, \\ t' (b_1^+)^2 &= (b_1^+)^2 t', & t' a^+ - a^+ t' &= 1, & t' y^2 &= y^2 t'. \end{aligned}$$

We recover the well known Gelfand-Kirillov property that $\text{Frac}\mathcal{U}(\mathfrak{n}_0^+)$ is a classical Weyl skew field D_1 over a center $\mathbb{C}((b_1^+)^2, y^2)$ of transcendence degree two. We have $(b_1^+)^2 = \frac{1}{2}x_4$ et $y^2 = \frac{1}{4}(x_3^2 - 2x_2x_4)$ with notations (10). Up to a normalization we recover the well known expressions for the generators of the center of $\mathcal{U}(\mathfrak{n}_0^+)$ in terms of Chevalley generators.

The following theorem gives a decomposition of $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ into two commuting subfields respectively isomorphic to \mathbf{D}_1^0 and \mathbf{F}_4 .

3.4. Theorem. *$\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ is isomorphic to $\text{Frac}(\mathbf{A}_1 \otimes \mathbf{S}_4)$.*

Proof. By 3.1.2, $\mathcal{U}(\mathfrak{b}^+)$ is generated in $\mathcal{U}(\mathfrak{osp}(1, 4))$ by $\mathcal{U}(\mathfrak{n}^+)$ and $\mathcal{U}(\mathfrak{h})$ with the commutation relations (16) and the action of k_1 and k_2 on b_1^+, b_2^+, a^+, t coming from (11). Taking again the notations used in the proof of proposition 3.2, this action extends to $\mathcal{U}'' = \mathbb{C}(b_1^+, a^+)[y; \tau][t'; \delta']$ by:

$$\begin{aligned} [k_1, b_1^+] &= b_1^+, & [k_1, y] &= y, & [k_1, a^+] &= a^+, & [k_1, t'] &= -t', \\ [k_2, b_1^+] &= 0, & [k_2, y] &= y, & [k_2, a^+] &= a^+, & [k_2, t'] &= -t'. \end{aligned}$$

The change of variables:

$$k'_1 = (b_1^+)^{-1}(k_1 - k_2), \quad k'_2 = (a^+)^{-1}k_2 \quad (19)$$

gives:

$$k'_1 k'_2 = k'_2 k'_1, \quad [k'_1, b_1^+] = 1, \quad [k'_1, a^+] = 0, \quad [k'_2, b_1^+] = 0, \quad [k'_2, a^+] = 1. \quad (20)$$

That shows that the subalgebra \mathcal{W} generated by b_1^+, a^+, k'_1, k'_2 in $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ is isomorphic to the Weyl algebra $A_2 = \mathbf{A}_1 \otimes \mathbf{A}_1$. The commutation relations of these new generators k'_1, k'_2 with the generator y are $k'_1 y = -y k'_1$ and $k'_2 y = y k'_2 + y(a^+)^{-1}$. We replace y by:

$$y' := (a^+)^{-1}y = (a^+)^{-1}b_1^+ b_2^+ - 1, \quad (21)$$

which satisfies:

$$y' k'_2 = k'_2 y', \quad y' a^+ = a^+ y', \quad y' b_1^+ = -b_1^+ y', \quad y' k'_1 = -k'_1 y'. \quad (22)$$

We deduce with (20) that the subalgebra \mathcal{V} generated by $b_1^+, a^+, k'_1, k'_2, y'$ in $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ is isomorphic to the algebra $\mathbf{A}_1 \otimes \mathbf{S}_3$.

We have now to formulate the commutation relations of the last generator t' with the generators of \mathcal{V} . It is clear that $t' b_1^+ = b_1^+ t'$ and $t' k'_1 = k'_1 t'$; we compute $t' a^+ - a^+ t' = 1$ and $t' k'_2 = k'_2 t' - (a^+)^{-1} k'_2 + (a^+)^{-1} t'$. We try to replace t' by a generator of the form $a^+ t' + p$ commuting with a^+ and k'_2 , with $p \in \mathcal{W}$. A solution is given by $p = -a^+ k'_2$. In other words, the

element $u := a^+t' - a^+k'_2$ satisfies $[u, b_1^+] = [u, k'_1] = [u, a^+] = [u, k'_2] = 0$. We calculate:

$$\begin{aligned} uy' &= a^+t'y' - a^+k'_2y' = a^+t'(a^+)^{-1}y - y'a^+k'_2 \\ &= a^+((a^+)^{-1}t' - (a^+)^{-2})y - y'a^+k'_2 = t'y - (a^+)^{-1}y - y'a^+k'_2 \\ &= yt' - y' - y'a^+k'_2 = a^+y't' - y'a^+k'_2 - y' \\ &= y'(a^+t' - a^+k'_2) - y' = y'u - y'. \end{aligned}$$

This relation becomes $y't'' - t''y' = 1$ with notation:

$$t'' := (y')^{-1}u = (y')^{-1}a^+(b_1^+)^{-2}t - (y')^{-1}a^+k'_2. \quad (23)$$

Since u commutes with a^+, k'_2, b_1^+, k'_1 it follows from (22) that t'' commutes with a^+ and k'_2 , and anticommutes with b_1^+ and k'_1 .

To sum up, starting from the generators $b_1^+, a^+, k_2, k_1, b_2^+, t$ of $\mathcal{U}(\mathfrak{b}^+)$, we have proved that the elements $b_1^+, a^+, k'_2, k'_1, y', t''$ defined by (19), (21), (23) generate $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$. The subalgebra generated by k'_2 and a^+ is isomorphic to the Weyl algebra \mathbf{A}_1 , the subalgebra generated by k'_1, b_1^+, y' and t'' is isomorphic to the algebra \mathbf{S}_4 , each element of the first subalgebra commutes with each element of the second one, and $\text{Frac}\mathcal{U}(\mathfrak{b}^+) = \text{Frac}(\mathbf{A}_1 \otimes \mathbf{S}_4)$. Hence the proof is complete. \square

3.5. Remark. The enveloping algebra $\mathcal{U}(\mathfrak{b}_0^\pm)$ is the subalgebra of $\mathcal{U}(\mathfrak{b}^+)$ generated by $(b_1^+)^2, (b_2^+)^2, a^+, t, k_1, k_2$. With the notations used in the proof of theorem 3.4, the generators a^+, k'_2 lie in $\mathcal{U}(\mathfrak{b}_0^\pm)$ and we define in $\mathcal{U}(\mathfrak{b}_0^\pm)$ the elements $\ell_1 := (b_1^+)^{-2}(k_1 - k_2)$, $y'' := (y')^2 = -(a^+)^{-2}(b_1^+)^2(b_2^+)^2 + 1$ and $t''' := \frac{1}{2}(y'')^{-1}a^+(b_1^+)^{-2}t - \frac{1}{2}(y'')^{-1}a^+k'_2$. Then $\text{Frac}\mathcal{U}(\mathfrak{b}_0^\pm)$ is generated by $k'_2, a^+, (b_1^+)^2, \ell_1, y'', t'''$ and the brackets between these generators are $[k'_2, a^+] = [\ell_1, (b_1^+)^2] = [y'', t'''] = 1$ and 0 in all other cases. We recover the well known Gelfand-Kirillov property that $\text{Frac}\mathcal{U}(\mathfrak{b}_0^\pm)$ is a classical Weyl skew field D_3 over a trivial center \mathbb{C} .

The following theorem gives a decomposition of $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ into two commuting subfields respectively isomorphic to \mathbf{D}_2^0 and \mathbf{F}_3 .

3.6. Theorem. $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ is isomorphic to $\text{Frac}(\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{S}_3)$.

Proof. By 3.1.3, $\mathcal{U}(\mathfrak{p}^+)$ is generated in $\mathcal{U}(\mathfrak{osp}(1, 4))$ by $\mathcal{U}(\mathfrak{b}^+)$ and b_2^- with commutation relations coming from (11), (14) and (16). We start replacing in $\text{Frac}\mathcal{U}(\mathfrak{n}^+)$ the generators t and a^+ by:

$$u_1 := t, \quad v_1 := a^+(b_1^+)^{-2}, \quad (24)$$

which commute with b_1^+ and satisfy $u_1v_1 - v_1u_1 = 1$. Then we consider the enveloping algebra $\mathcal{U}(\mathfrak{l})$ of the Levi subalgebra generated by b_2^+, b_2^-, k_2 , see 3.1.4. In $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$, we replace b_2^+ and b_2^- by $m_2^+ := b_2^+ - a^+(b_1^+)^{-1}$ and

$m_2^- := b_2^- - t(b_1^+)^{-1}$ in order to simplify the commutation relations with the previous generators b_1^+, u_1, v_1 :

$$[m_2^\pm, u_1] = [m_2^\pm, v_1] = 0 \quad \text{et} \quad m_2^\pm b_1^+ = -b_1^+ m_2^\pm. \quad (25)$$

We define $\ell_2 := \frac{1}{2}(m_2^+ m_2^- + m_2^- m_2^+)$. A technical but straightforward calculation gives $\ell_2 = k_2 - ta^+(b_1^+)^{-2} + \frac{1}{2} = k_2 - u_1 v_1 + \frac{1}{2}$. Since $m_2^+ m_2^-$ and $m_2^- m_2^+$ commute with b_1^+, u_1, v_1 by (25), the same is true for ℓ_2 . Moreover we compute: $[\ell_2, m_2^+] = m_2^+$ and $[\ell_2, m_2^-] = -m_2^-$. The subalgebra generated by m_2^+, m_2^-, ℓ_2 is isomorphic to $\mathcal{U}(\mathfrak{osp}(1, 2))$ and we apply the method used in proposition 2.2 setting:

$$u_2 := -(m_2^-)^{-1} \ell_2, \quad v_2 := m_2^-, \quad z_2 := -2m_2^- m_2^+ + 2\ell_2 + 1. \quad (26)$$

To sum up, the subfield L of $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ generated by $b_1^+, t, a^+, b_2^-, k_2, b_2^+$ is also generated by $b_1^+, u_1, v_1, v_2, z_2, u_2$ with relations:

$$\begin{aligned} [u_1, v_1] &= 1, & [u_2, v_2] &= 1, & [u_1, v_2] &= [u_2, v_1] = [u_1, u_2] = [v_1, v_2] = 0, \\ b_1^+ u_2 &= -u_2 b_1^+, & b_1^+ v_2 &= -v_2 b_1^+, & [b_1^+, u_1] &= [b_1^+, v_1] = [b_1^+, z_2] = 0, \\ z_2 u_2 &= -u_2 z_2, & z_2 v_2 &= -v_2 z_2, & [z_2, u_1] &= [z_2, v_1] = 0. \end{aligned}$$

We can replace the generator b_1^+ by $w_1 := z_2^{-1} b_1^+$ which is central in L .

In the last step we look at the action of k_1 on L . Technical calculations using (24) and (26) show that, on one hand $[k_1, u_2] = [k_1, v_2] = [k_1, z_2] = 0$, and on the other hand $[k_1, u_1] = u_1, [k_1, v_1] = -v_1, [k_1, w_1] = w_1$. As in lemma 4 of [3], the last change of variable $k_1'' := (k_1 + u_1 v_1) w_1^{-1}$ doesn't change the first three relations and changes the last three into: $[k_1'', u_1] = [k_1'', v_1] = 0$ and $[k_1'', w_1] = 1$. We conclude that in $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ the subalgebra generated by k_1'', w_1 is isomorphic to \mathbf{A}_1 , the subalgebra generated by u_1, v_1 is also isomorphic to \mathbf{A}_1 , the subalgebra generated by u_2, v_2, z_2 is isomorphic to \mathbf{S}_3 , and $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ is isomorphic to $\text{Frac}(\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{S}_3)$. \square

3.7. Remark. The enveloping algebra $\mathcal{U}(\mathfrak{p}_0^+)$ is the subalgebra of $\mathcal{U}(\mathfrak{p}^+)$ generated by $(b_1^+)^2, (b_2^+)^2, a^+, t, k_1, k_2, (b_2^-)^2$. Computing the brackets between these generators we find exactly the table of the Lie algebra denoted by $L_{7,9}$ in [3] p. 565 up to the following change of variables:

$$\begin{aligned} e_0 &:= t, & e_1 &:= a^+, & e_2 &:= (b_1^+)^2, & e_3 &:= k_1, \\ x &:= \frac{1}{2}(b_2^-)^2, & y &:= -\frac{1}{2}(b_2^+)^2, & h &:= -k_2. \end{aligned}$$

It is proved in [3] that the Lie algebra $\mathfrak{p}_0^+ = L_{7,9}$ satisfies the Gelfand-Kirillov property with $\text{Frac}\mathcal{U}(\mathfrak{p}_0^+) = \text{Frac}(\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbb{C}[c])$. The central generator c and the pairs of elements p_i, q_i ($1 = 1, 2, 3$) described in [3] as generators of each copy of \mathbf{A}_1 correspond with our notations in the proof of

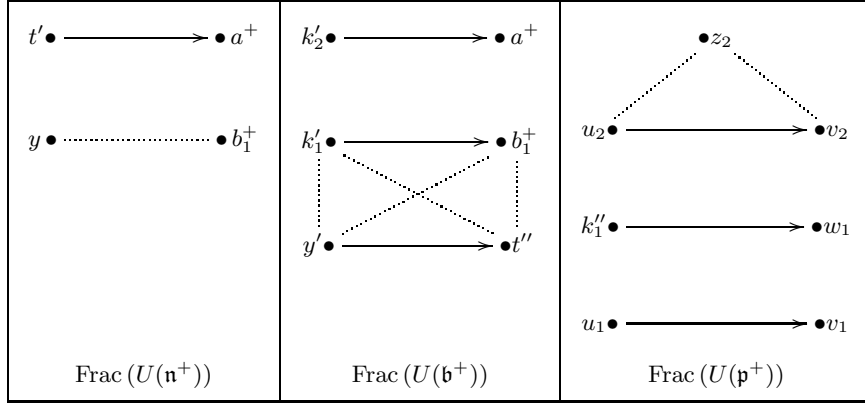
theorem 3.6 to:

$$\begin{aligned} p_1 &:= u_1, & p_2 &:= v_2^{-1}u_2, & p_3 &:= \frac{1}{2}k_1''w_1^{-1}z_2^{-2}, & c &:= \frac{1}{4}(z_2 + 1)^2 - 1, \\ q_1 &:= v_1, & q_2 &:= \frac{1}{2}(v_2)^2, & q_3 &:= w_1^2z_2^2, \end{aligned}$$

which gives an explicite description of the embedding:

$$\text{Frac}\mathcal{U}(\mathfrak{p}_0^\pm) = \text{Frac}(\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbb{C}[c]) \subset \text{Frac}\mathcal{U}(\mathfrak{p}^+) = \text{Frac}(\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{S}_3).$$

3.8. Illustration. With the conventions of remark 1.9, proposition 3.2, theorem 3.4 and theorem 3.6 can be represented by the following pictures:



3.9. Corollary. *The center of $\text{Frac}\mathcal{U}(\mathfrak{n}^+)$ is a purely transcendental extension of \mathbb{C} of degree two, the center of $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ is \mathbb{C} , and the center of $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ is a purely transcendental extension of \mathbb{C} of degree one.*

Proof. Follows directly from proposition 3.2, theorem 3.4 and theorem 3.6 applying the results on the centers 1.3.(ii) and 1.7.(ii). More explicitly with the notations used in the proofs, the center of $\text{Frac}\mathcal{U}(\mathfrak{n}^+)$ is $\mathbb{C}((b_1^+)^2, y^2)$ and the center of $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ is $\mathbb{C}(z_2^2)$. \square

3.10. Proposition. *The skew fields $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ and $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ are not isomorphic to $\mathbf{D}_{r,t}^s$ for any $r, s, t \geq 0$.*

Proof. Suppose that $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ is isomorphic to some skew field $\mathbf{D}_{r,t}^s$. Comparing the Gelfand-Kirillov transcendence degrees and the centers, we have $2r + 2s + t = 6$ and $2s + t = 0$, hence $\text{Frac}\mathcal{U}(\mathfrak{b}^+)$ would be isomorphic to the usual Weyl skew field $\mathbf{D}_{3,0}^0 = D_3(\mathbb{C})$ which is impossible because, as at the end of the proof of proposition 1.7, we have $G(D_3(\mathbb{C})) = \{1\}$ and $-1 \in G(\text{Frac}\mathcal{U}(\mathfrak{b}^+))$. Suppose now that $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ is isomorphic to some skew field $\mathbf{D}_{r,t}^s$. We obtain $2r + 2s + t = 7$ and $2s + t = 1$, hence $\text{Frac}\mathcal{U}(\mathfrak{p}^+)$ would be isomorphic to $\mathbf{D}_{3,1}^0$, which is impossible by the same argument. \square

3.11. Remark. The Lie algebra $\mathfrak{sp}(4)$ contains two non isomorphic parabolic subalgebras corresponding to the cases denoted by $L_{7,7}$ and $L_{7,9}$ in the classification of [3]. We have seen in 3.7 that the even part \mathfrak{p}_0^\pm of the parabolic subsuperalgebra \mathfrak{p}^\pm is isomorphic to $L_{7,9}$. But we can also define a subsuperalgebra \mathfrak{q}^\pm of $\mathfrak{osp}(1,4)$ whose even part is the alternative parabolic subalgebra $L_{7,7}$ of $\mathfrak{sp}(4)$. It is defined by $\mathfrak{q}^\pm = \mathfrak{q}_0^\pm \oplus \mathfrak{g}_1^\pm$ with $\mathfrak{g}_1^\pm = \mathbb{C}b_1^\pm \oplus \mathbb{C}b_2^\pm$ and $\mathfrak{q}_0^\pm = \mathfrak{b}_0^\pm \oplus \mathbb{C}s$, where s is defined in (8). A basis of \mathfrak{q}_0^\pm is $\{c_1^\pm, c_2^\pm, a^\pm, t, k_1, k_2, s\}$ and computing the brackets in $\mathfrak{osp}(1,4)$ we retrieve the table of $L_{7,7}$ in [3] up to the following change of notations:

$$\begin{aligned} e_0 &:= c_1^+, & e_1 &:= 2a^+, & e_2 &:= c_2^+, & e_3 &:= -\frac{1}{2}(k_1 + k_2), \\ x &:= t, & y &:= s, & h &:= k_1 - k_2. \end{aligned}$$

By a method similar to that of theorem 3.6, we can prove that $\text{Frac}\mathcal{U}(\mathfrak{q}^+)$ is also isomorphic to $\text{Frac}(\mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{S}_3)$.

ACKNOWLEDGEMENTS

We would like to thank Alfons Ooms for drawing our attention to the case of the second parabolic subalgebra considered in remark 3.11.

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