A MINIMIZATION FORMULATION OF A BI-KINETIC SHEATH

Mehdi Badsi, Martin Campos Pinto, Bruno Després

Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

A MINIMIZATION FORMULATION OF A BI-KINETIC SHEATH

Mehdi Badsi, Martin Campos Pinto and Bruno Després

UPMC-Paris 06, CNRS UMR 7598
Laboratoire Jacques-Louis Lions
4, pl. Jussieu F75252 Paris cedex 05, France

Abstract. The mathematical description of the interaction between a plasma and a solid surface is a major issue that still remains challenging. In this paper, we model this interaction as a stationary and bi-kinetic Vlasov-Poisson-Ampère boundary value problem with boundary conditions that are consistent with the physics. In particular, we show that the wall potential can be determined from the ampibolarity of the particle flows as the unique solution of a non linear equation. Based on variational techniques, our analysis establishes the well-posedness of the model, provided that the incoming ion distribution satisfies a moment condition that generalizes the historical Bohm criterion of plasma physics. Quantitative estimates are also given, together with numerical illustrations that validate the robustness of our approach.

1. **Introduction.** The description of the plasma-wall interaction is a challenging issue with many practical applications, be it in the modeling of Tokamak walls or ionic engines for satellites. Thus, the mathematical study and the numerical simulation of physically consistent models is of interest. When a plasma is in contact with an isolated and partially absorbing wall, a thin net-charge layer develops spontaneously between the wall and the plasma. This layer of several Debye lengths is called a sheath [6, 8, 24] and it is usually understood as the way by which the plasma preserves its global neutrality. Indeed, because the electrons are a lighter species they are prone to exit the plasma at a higher rate than the heavier ions. As this phenomenon alone would result in an unstable situation, namely a positive charge built up in the core plasma, the negative charge accumulated at the isolated wall causes the electric potential to drop and repel a significant fraction of the electrons. The magnitude of the drop is then such that the flow is ambipolar, in the sense that positive and negative charges exit the core plasma at the same rate [8, 24].

Plasma-sheaths have been extensively studied in the last decades [6, 15, 8, 25, 22, 7], however several important questions do not have fully satisfactory answers on the mathematical level. For instance, we are not aware of a simple model that describes in a unified way the physical processes at play between the sheath and the core of the plasma. Nevertheless, a common observation that is supported by both theoretical and empirical evidence is that at the sheath entrance the average ion velocity must exceed its sound speed c_s,

$$u_i > c_s := \sqrt{\frac{kT_e}{m_i}} \tag{1}$$

Key words and phrases. Kinetic equation, Plasma sheath, Bohm criterion, Ambipolarity, Vlasov-Poisson system.

where k is the Boltzmann constant, T_e is the electronic temperature and m_i is the ion mass [8]. This definition of the ion sound velocity corresponds to the case where the ion temperature T_i is much smaller than the electron one. Another possible definition for the ion sound velocity is $c_s' := \sqrt{\dfrac{k(T_e+T_i)}{m_i}}$. This inequality is often referred to as the original *Bohm criterion* and several variants have been developed in the scope of more general models [21, 4]. For instance, in the case of a plasma consisting of a Poisson equation to define the electrostatic potential ϕ coupled to differential equations to define the ion and electron density n_i and n_e, it has been shown that these densities can both be expressed as functions of ϕ, and that at the sheath entrance (which is commonly defined as the limit between the non neutral region and the neutral region), the value ϕ_{se} of the potential must be such that

$$\frac{d}{d\phi}\left(n_i - n_e\right)(\phi_{se}) \leq 0. \tag{2}$$

The sheath-edge x_{se}, namely the entrance of the sheath, is then often defined as the position where $\phi(x_{se}) = \phi_{se}$, even though it is commonly admitted that the sheath-edge is a difficult place to define [21]. Overall, the inequality expresses the idea that at the entrance of the sheath the electron density decreases more rapidly than the ion density as the electric potential drops.

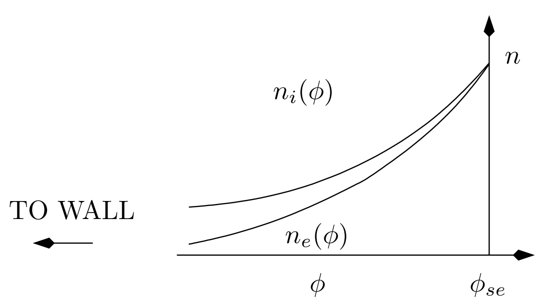


FIGURE 1. Schematic variations of the ion and electron densities with respect to some normalized potential $\phi \leq 0$ decreasing toward the wall. Here the plasma is supposed to be neutral at the sheath-edge x_{se}.

As for the boundary condition on the wall, most models describe the potential as having a "floating" value that adjusts itself according to the dynamics of the system. However no clear definition of a self consistent wall potential seems available. On the mathematical side some models have been proposed but they do not fully answer the above questions, see e.g. [16, 10].

In the present work we address this problem by considering a simple plasma-wall interaction model with a self-consistent potential and we show that it is well posed under the assumption that the incoming ion distribution satisfies a moment condition which generalizes the usual kinetic Bohm criterion. Moreover, our solutions share most of the properties of plasma sheaths, such as a decreasing potential and a positive charge density. In our model the ion and electron densities are solutions to one dimensional stationary Vlasov equations coupled with a self consistent Poisson equation. Boundary conditions are determined to reflect the physical properties of this simplified model: in particular, the wall potential is determined so that the Ampère equation holds for the stationary solutions. A surprising result is that the resulting potential is only well-defined for incoming ions satisfying an *upper bound* on their average velocity. This constraint is shown to be compatible with the Bohm criterion thanks to the large mass ratio between ions and electrons.

To allow some generality, we consider that electrons are re-emitted with probability $\alpha \leq 1$ while ions are totally absorbed. Ions and electrons are assumed to enter the plasma with given velocity distributions. Since the core of the plasma is well described by a full Maxwellian, we have chosen to consider (semi-) Maxwellian distributions for the incoming electrons. At the numerical level we then observe that the resulting velocity distribution is very close to a full Maxwellian when far from the wall, in good qualitative agreement with the results from [24, p. 75].

The plan of the paper is as follows. In section 2 we begin with a presentation of the model and write down its mathematical structure. Under a decreasing assumption on ϕ, we prove the formal equivalence between the Vlasov-Poisson system and a non linear Poisson equation. An important result of this section is a necessary and sufficient condition for the wall potential to be uniquely determined by the physical parameters of the problem. We also give an a priori lower bound for the wall potential. The result holds for re-emission coefficient $\alpha < 1$. The case of total electron re-emission ($\alpha = 1$) corresponds to a Boltzmannian electron density and turns out to be degenerate.

In section 3 we set up the mathematical framework that is used in the rest of the paper and state the main result (well posedness under a condition that generalizes the usual kinetic Bohm criterion, and quasi-neutrality estimates). The proof relies on reformulating the non linear Poisson equation as a minimization problem, and our generalized Bohm inequality appears naturally as a local convexity condition for the energy functional.

In section 4, we briefly describe the numerical method employed to solve the problem. Then we illustrate the main result with a physically based sheath problem.

Final comments about the range of applicability of this work are provided as a conclusion in section 5.

2. Description of the model.

2.1. Physical setting.

We consider a plasma at equilibrum made of one species of ions and electrons. This plasma is assumed to be contained in a one dimensional chamber. This model only describes a portion of the chamber of length L. Physical quantities will often be denoted with upper case while normalized ones will be denoted with lower case. Our system is subject to the following physical considerations:

1. The plasma is assumed to be non-collisional.
2. The effect of the self-consistent magnetic field is neglected.
3. The physical quantities that describe the plasma state such as, the ionic distribution, the electronic distribution and the electric potential (that we will denote F_i,F_e and Φ) depend (in space) exclusively on the longitudinal variable denoted X.
4. At $X = 0$ we consider:
   (a) that the potential Φ is arbritrarily set to zero;

  (b) ions and electrons entering the domain with non negative velocities which are described through velocity distributions denoted respectively F_i^{in} and F_e^{in};
  (c) an arbitrary charge imbalance denoted P_0 (normalized value ρ_0);
5. At the wall, that is at $X = L$ (or $x = 1$ in normalized variables), we consider:
  (a) purely absorbing conditions for ions, i.e, ions are not re-emitted from the wall;
  (b) electrons re-emitted with probability $\alpha \in [0,1]$.

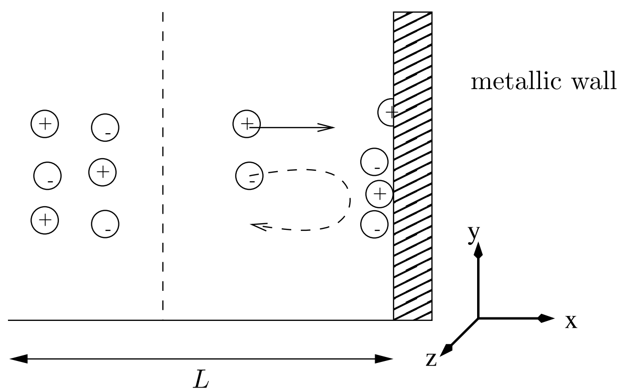


FIGURE 2. Schematic illustration. Ions and electrons are going toward the wall, some electrons reaching the wall are re-emitted with a probability $\alpha \in [0,1]$ while ions are totally absorbed.

2.2. **Kinetic modeling of the stationary plasma wall interaction.** We first write the equations in physical variables and then derive a dimensionless model. The unknowns are respectively the electric potential $\Phi : [0,L] \to \mathbb{R}$, the ion distribution function $F_i : [0,L] \times \mathbb{R} \to \mathbb{R}^+$ and the electron distribution function $F_e : [0,L] \times \mathbb{R} \to \mathbb{R}^+$. We denote by $N_i(X) := \int_{\mathbb{R}} F_i(X,V)dV$ (respectively $N_e(X) := \int_{\mathbb{R}} F_e(X,V)dV$) the ionic (respectively the electronic) density at $X \in [0,L]$ and $\Gamma_i(X) := \int_{\mathbb{R}} F_i(X,V)VdV$ (respectively $\Gamma_e(X) := \int_{\mathbb{R}} F_e(X,V)VdV$) the ionic (respectively the electronic) flux at $X \in [0,L]$. The equations governing the ion and electron transport in the plasma, with an electric field $E = -\frac{d}{dX}\Phi$ are assumed to be stationary Vlasov equations and write

$$V\partial_X F_e(X,V) + \frac{q}{m_e}\frac{d}{dX}\Phi(X)\partial_V F_e(X,V) = 0 \quad \forall (X,V) \in (0,L) \times \mathbb{R}, \tag{3}$$

$$V\partial_X F_i(X,V) - \frac{q}{m_i}\frac{d}{dX}\Phi(X)\partial_V F_i(X,V) = 0 \quad \forall (X,V) \in (0,L) \times \mathbb{R}, \tag{4}$$

with the boundary conditions

$$F_e(0,V) = F_e^{in}(V) \quad \text{ for } V > 0, \tag{5}$$

$$F_e(L,V) = \alpha F_e(L,-V) \quad \text{for } V < 0, \tag{6}$$

$$F_i(0,V) = F_i^{in}(V) \quad \text{for } V > 0, \tag{7}$$

$$F_i(L,V) = 0 \quad \text{for } V < 0. \tag{8}$$

Here q is the electric charge and m_i (respectively m_e) denotes the ionic (respectively the electronic) mass. Furthermore, a formal integration of equations (3)-(4) with respect to the velocity variable shows that the current density $J(X) := q\left(\Gamma_i(X) - \Gamma_e(X)\right)$ must be constant in space, and so $J(X) = J(L) = J(0)$ for all $X \in [0,L]$. It is therefore natural to require

$$J(X) = 0 \quad \forall X \in [0,L], \tag{9}$$

since (by ambipolarity) the current has to be zero at the wall. We further stress that equation (9) is necessary if one wants to construct stationary solutions compatible with the Maxwell-Ampère equation. The electric potential is determined from the densities through the Gauss law

$$-\frac{d^2}{dX^2}\Phi(X) = \frac{q}{\varepsilon_0}\left(N_i(X) - N_e(X)\right) \quad \forall X \in (0,L) \tag{10}$$

with boundary conditions

$$\Phi(0) = 0, \qquad \Phi(L) = \Phi_W. \tag{11}$$

Here the vacuum permittivity is ε_0 and Φ_W denotes the wall potential. Its value will be determined so that equation (9) holds. As far as our model is concerned, we will show in Section 2.5 that Φ_W can be determined from the previous physical parameters.

It is convenient to rescale the equations and to this end we introduce the dimensionless variables x, v and the dimensionless functions ϕ, f_i and f_e defined as:

$$x := \frac{X}{L}, \quad v := \frac{V}{c_s},$$

$$f_i(x,v) := Lc_sF_i(Lx, c_sv), \quad f_e(x,v) := Lc_sF_e(Lx, c_sv), \quad \phi(x) := \frac{q}{kT_e}\Phi(Lx),$$

where k is the Boltzmann constant, T_e is a reference electron temperature and $c_s := \sqrt{\frac{kT_e}{m_i}}$ the ion sound speed. We also define the dimensionless quantities

$$n_i(x) := \int_{\mathbb{R}} f_i(x,v)dv, \quad n_e(x) := \int_{\mathbb{R}} f_e(x,v)dv,$$

$$\gamma_i(x) := \int_{\mathbb{R}} f_i(x,v)vdv, \quad \gamma_e(x) := \int_{\mathbb{R}} f_e(x,v)vdv,$$

$$f_e^{in}(v) := Lc_sF_e^{in}(c_sv), \quad f_i^{in}(v) := Lc_sF_i^{in}(c_sv).$$

The coupled boundary value problem (3)-(11) is then equivalent to the following boundary value problem:

$$\begin{cases} v\partial_x f_e(x,v) + \dfrac{m_i}{m_e}\dfrac{d}{dx}\phi(x)\partial_v f_e(x,v) = 0 \quad \forall (x,v) \in (0,1)\times\mathbb{R}, & (12) \\ v\partial_x f_i(x,v) - \dfrac{d}{dx}\phi(x)\partial_v f_i(x,v) = 0 \quad \forall (x,v) \in (0,1)\times\mathbb{R}, & (13) \\ -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = n_i(x) - n_e(x) \quad \forall x \in (0,1), & (14) \end{cases}$$

complemented with the boundary conditions

$$\begin{cases} f_e(0,v) = f_e^{in}(v) \quad \forall v > 0, \quad f_e(1,v) = \alpha f_e(1,-v) \quad \forall v < 0, & (15) \\ f_i(0,v) = f_i^{in}(v) \quad \forall v > 0, \quad f_i(1,v) = 0 \quad \forall v < 0, & (16) \\ \phi(0) = 0, \quad \phi(1) = \phi_w & (17) \end{cases}$$

and the additional constraint (derived from the Ampère equation)

$$\gamma_i(x) - \gamma_e(x) =: j(x) = 0 \quad \forall x \in [0,1]. \tag{18}$$

We remind that the value of $\phi_w = \frac{q\Phi_w}{kT_e}$ will be determined later, see Section 2.5. Here, we have set $\varepsilon := \lambda_D\sqrt{\dfrac{N_0}{L}}$ where $N_0 := \displaystyle\int_{\mathbb{R}^+} F_e^{in}(V)dV$ denotes an electron reference density and $\lambda_D := \sqrt{\dfrac{\varepsilon_0 kT_e}{q^2 N_0}}$ is the Debye length.

The set of equations (12)-(18) is the model problem and we will refer to it as the Vlasov-Poisson-Ampère problem. It contains the main physical parameters ε, α, ρ_0, f_e^{in} and f_i^{in}. The Vlasov-Poisson problem is made of equations (12)-(17) which can be considered as the main equations while the Ampère equation (18) can be considered as an additionnal constraint. To our knowledge, this stationary and bi-kinetic boundary-value problem has never been studied in full details. For example in [20], a model of plane diode is studied. It is consists of a one single stationary Vlasov equation for electrons coupled with the Poisson equation for the electrostatic potential, the well-posedness is studied for a large class of electron boundary conditions. In [12], the non-stationary version of the plane diode is studied.

2.3. **Reformulation as a non-linear Poisson equation.** Thanks to the one-dimensional structure of the Vlasov-Poisson problem (12)-(17), it is possible to reformulate as a non linear Poisson equation. When the potential ϕ is given both Vlasov equations for ions and electrons are linear advection equations, and their solutions are determined by transport along the characteristics of the (incoming) boundary conditions. In this section we assume $\phi \in W^{2,\infty}(0,1)$ to be given which is a sufficient condition for the characteristics curves to be well-defined [1]. Moreover, we assume $\phi(0) = 0$, $\phi(1) = \phi_w$ and $\phi' < 0$. In such a configuration, we have the following formal result.

Proposition 1 (Formal equivalence)**.** *Let $\phi \in W^{2,\infty}(0,1)$ be such that $\phi' < 0$ with $\phi(0) = 0$ and $\phi(1) = \phi_w$. Then there exists f_i,f_e such that the Vlasov-Poisson* *system holds if and only if ϕ is a solution to*

$$\text{(NLP)} \ : \begin{cases} -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = (n_i - n_e)(x) \quad \forall x \in (0,1) \\ \phi(0) = 0, \quad \phi(1) = \phi_w \\ \ \textit{with} \\ n_i(x) = \displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2 - 2\phi(x)}} dv \\ n_e(x) = 2\displaystyle\int_{\sqrt{-\frac{2m_i}{m_e}\phi(x)}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2m_i}{m_e}\phi(x)}} dv \\ \qquad\qquad -(1-\alpha)\displaystyle\int_{\sqrt{-\frac{2m_i}{m_e}\phi_w}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2m_i}{m_e}\phi(x)}} dv. \end{cases} \tag{19}$$

We will give a formal proof of the above proposition. Especially, we do not want to discuss regularity and integrability issues. The proof is decomposed in two parts. The necessary condition is shown in Section 2.3.1 and the sufficient one is established in Section 2.3.2.

2.3.1. *Necessary condition.* We assume that there is ϕ with $\phi' < 0$, f_i and f_e solutions of (12)-(17) . Let us show that ϕ is solution of (NLP). To this effect, we determine an explicit representation of f_i and f_e by means of the characteristics curves.

Electrons trajectories. The characteristics trajectories of electrons (12) are the curves which satisfy the ordinary differential system of equations

$$(C_e) : \begin{cases} \dot{\mathcal{X}}(t) = \mathcal{V}(t) \\ \dot{\mathcal{V}}(t) = \frac{m_i}{m_e}\frac{d}{dx}\phi(\mathcal{X}(t)) \\ \mathcal{X}(s) = x \\ \mathcal{V}(s) = v \end{cases}$$

for $t \geq s$ and for an arbitrary initial data $(s,x,v) \in \mathbb{R} \times [0,1] \times \mathbb{R}$. A geometry of the characteristics is illustrated in Figure 3.

Under the decreasing assumption on ϕ, one can identify the solutions to (C_e) with the curves $\left\{(x,v) \in [0,1] \times \mathbb{R} \mid \frac{v^2}{2} - \frac{m_i}{m_e}\phi(x) = k\right\}$ for $k \geq 0$. The phase-space $[0,1] \times \mathbb{R}$ is then splitted into two subdomains which are separated by the characteristic curve of equation

$$\frac{1}{2}v^2 - \frac{m_i}{m_e}\phi(x) = -\frac{m_i}{m_e}\phi_w \quad \Leftrightarrow \quad v^2 = \frac{2m_i}{m_e}\left(\phi(x) - \phi_w\right).$$

One can write $[0,1] \times \mathbb{R} = D_1 \cup D_2$ with $D_1 := \{(x,v) \in [0,1] \times \mathbb{R} \mid v \geq -\sqrt{\frac{2m_i}{m_e}\left(\phi(x) - \phi_w\right)}\}$ and $D_2 := \{(x,v) \in [0,1] \times \mathbb{R} \mid v < -\sqrt{\frac{2m_i}{m_e}\left(\phi(x) - \phi_w\right)}\}$. For $(x,v) \in D_1$ there exists $w > 0$ and a characteristic curve passing through (x,v) which originates from $(0,w)$. Conversely, for $(x,v) \in D_2$ there exists $w < 0$ and a characteristic curve passing through (x,v) which originates from $(1,w)$. Since the electron distribution function f_e is constant along the characteristics we can determine $f_e(x,v)$ for every $(x,v) \in [0,1] \times \mathbb{R}$. To this end, consider the two following cases. If $(x,v) \in D_1$ one has $\frac{1}{2}v^2 - \frac{m_i}{m_e}\phi(x) = \frac{1}{2}w^2$ for some $w > 0$. Then $w = \sqrt{v^2 - \frac{2m_i}{m_e}\phi(x)}$ and $f_e(x,v) = f_e(0,w) = f_e^{in}\left(\sqrt{v^2 - \frac{2m_i}{m_e}\phi(x)}\right)$. Conversely

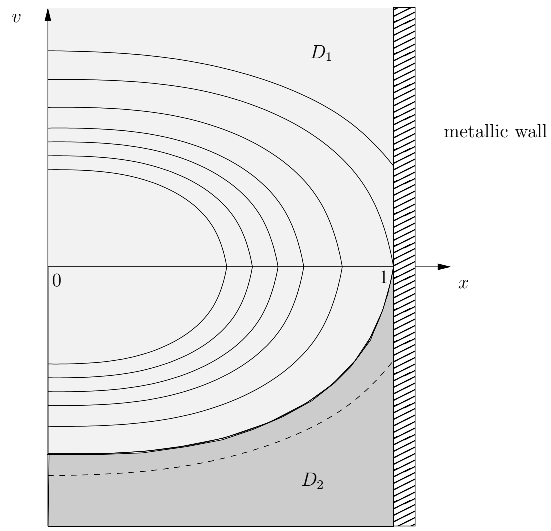


FIGURE 3. Schematic characteristic electron trajectories associated with a decreasing potential ϕ. The dashed line corresponds to a characteristic curve which originates at the wall with a negative velocity. Because of the boundary condition at the wall, particles following this curve were originally at $x = 0$ with a positive velocity.

if $(x, v) \in D_2$ one has $\frac{1}{2}v^2 - \frac{m_i}{m_e}\phi(x) = \frac{1}{2}w^2 - \frac{m_i}{m_e}\phi_w$ for some $w < 0$. Then $w = -\sqrt{v^2 - \frac{2m_i}{m_e}(\phi(x) - \phi_w)}$ and $f_e(x, v) = f_e(1, w) = \alpha f_e\left(1, \sqrt{v^2 - \frac{2m_i}{m_e}(\phi(x) - \phi_w)}\right) = \alpha f_e\left(0, \sqrt{v^2 - \frac{2m_i}{m_e}\phi(x)}\right) = \alpha f_e^{in}\left(\sqrt{v^2 - \frac{2m_i}{m_e}\phi(x)}\right)$. We thus obtain that the solution of (12) is given by

$$f_e(x, v) = \begin{cases} f_e^{in}\left(\sqrt{v^2 - \frac{2m_i}{m_e}\phi(x)}\right) & \text{if } (x, v) \in D_1 \\ \alpha f_e^{in}\left(\sqrt{v^2 - \frac{2m_i}{m_e}\phi(x)}\right) & \text{if } (x, v) \in D_2. \end{cases} \tag{20}$$

For $x \in [0, 1]$ we split $\mathbb{R} = (-\infty, -\sqrt{\frac{2m_i}{m_e}(\phi(x) - \phi_w)}) \cup [-\sqrt{\frac{2m_i}{m_e}(\phi(x) - \phi_w)}, +\infty)$. Making the change of variable $w = \sqrt{v^2 - \frac{2m_i}{m_e}\phi(x)}$ and integrating in velocity (20) leads to

$$n_e(x) = 2\int_{\sqrt{-\frac{2m_i}{m_e}\phi(x)}}^{+\infty} \frac{f_e^{in}(w)w}{\sqrt{w^2 + \frac{2m_i}{m_e}\phi(x)}}dw - (1 - \alpha)\int_{\sqrt{-\frac{2m_i}{m_e}\phi_w}}^{+\infty} \frac{f_e^{in}(w)w}{\sqrt{w^2 + \frac{2m_i}{m_e}\phi(x)}}dw. \tag{21}$$

Ion trajectories. The characteristics trajectories of ions (13) are the curves which satisfy the ordinary differential system of equations

$$(C_i):\begin{cases}\dot{\mathcal{X}}(t)=\mathcal{V}(t)\\ \dot{\mathcal{V}}(t)=-\frac{d}{dx}\phi(\mathcal{X}(t))\\ \mathcal{X}(s)=x\\ \mathcal{V}(s)=v\end{cases}$$

for $t\geq s$ and for an arbitrary initial data $(s,x,v)\in\mathbb{R}\times[0,1]\times\mathbb{R}$. A geometry of the characteristics is illustrated in Figure 4. Again, under the decreasing assumption

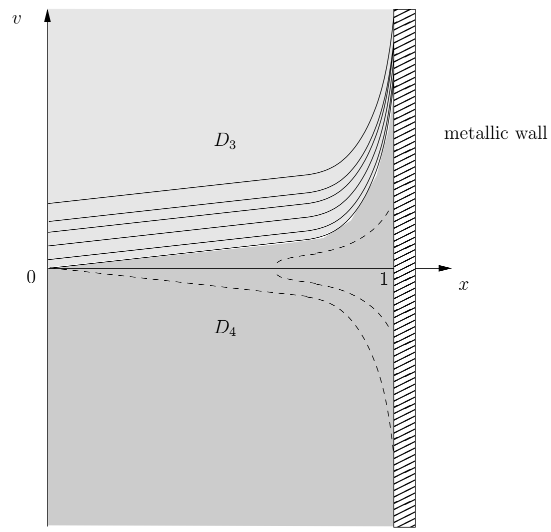


FIGURE 4. Schematic characteristic ions trajectories associated with a decreasing potential ϕ. Here the dashed lines correspond to characteristic curves originating from the wall with negative velocities, and they span the darker gray region D_4. Because of the boundary condition at the wall, no particles travel on these curves and hence f_i vanishes outside D_3 (lighter gray).

on ϕ the solutions to (C_i) are the curves $\left\{(x,v)\in[0,1]\times\mathbb{R}\mid\frac{v^2}{2}+\phi(x)=k\right\}$ for $k\geq\phi_w$. The phase space $[0,1]\times\mathbb{R}$ is then splitted into two subdomains which are separated by the characteristic curve of cartesian equation

$$\frac{1}{2}v^2+\phi(x)=0.$$

One can write $[0,1]\times\mathbb{R}=D_3\cup D_4$ with $D_3:=\{(x,v)\in[0,1]\times\mathbb{R}\mid v>\sqrt{-2\phi(x)}\}$ and $D_4:=\{(x,v)\in[0,1]\times\mathbb{R}\mid v\leq\sqrt{-2\phi(x)}\}$. For $(x,v)\in D_3$ there exists $w>0$ and a characteristic curve which originates from $(0,w)$. Conversely for $(x,v)\in D_4$ there exists $w<0$ and a characteristic curve passing through (x,v) which originates from $(1,w)$. Since the ion distribution function f_i is constant along the characteristic curves we can determine $f_i(x,v)$ for every $(x,v)\in[0,1]\times\mathbb{R}$. Again consider the two following cases. If $(x,v)\in D_3$ one has $\frac{1}{2}v^2+\phi(x)=\frac{1}{2}w^2$ for some $w>0$. Then $w=\sqrt{v^2+2\phi(x)}$ and $f_i(x,v)=f_i(0,w)=f_i^{in}(\sqrt{v^2+2\phi(x)})$. Conversely if $(x,v)\in D_4$

$\frac{1}{2}v^2 + \phi(x) = \frac{1}{2}w^2 + \phi_w$ for some $w < 0$. Then $w = -\sqrt{v^2 + 2(\phi(x) - \phi_w)}$ and $f_i(x,v) = f_i(1,w) = 0$. We thus obtain that the solution of (13) is given by

$$f_i(x,v) = \begin{cases} f_i^{in}(\sqrt{v^2 + 2\phi(x)}) \text{ if } (x,v) \in D_3 \\ 0 \text{ if } (x,v) \in D_4. \end{cases} \tag{22}$$

For each $x \in [0,1]$ we split $\mathbb{R} = (-\infty, \sqrt{-2\phi(x)}) \cup [\sqrt{-2\phi(x)}, +\infty)$. Making the change of variable $w = \sqrt{v^2 + \phi(x)}$ and integrating in velocity (22) leads to

$$n_i(x) = \int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2 - 2\phi(x)}} dv. \tag{23}$$

Formulas (21)-(23) show that ϕ is indeed a solution to (NLP), which ends the proof of the necessary condition.

2.3.2. *Sufficient condition.* Assume there is ϕ decreasing and solution of (NLP). Then $\phi_w \le \phi(x) \le 0$ for all $x \in [0,1]$, f_e and f_i defined in (20), (22) are well defined. We can check that (ϕ, f_i, f_e) is a weak solution of the Vlasov-Poisson system in the sense of A.1 provided f_i^{in} and f_e^{in} are regular enough.

Remark 1. It is easy to see there is a little change in the geometry of the characteristics when ϕ' is permitted to vanish on some interval. However, when ϕ is not non increasing, it is possible that the characteristics curves are closed and never intersect the boundaries. This can lead to the presence of trapping sets of non zero-measure (see [3] for a definition of trapping sets) which results in (12)-(17) being ill-posed.

2.4. **Semi-Maxwellian electron boundary condition and charge imbalance.** As mentioned in the introduction, electrons in the core of the plasma are well described by a full Maxwellian distribution. As a matter of fact, in this work we shall consider Maxwellian boundary conditions for electrons which takes the form

$$F_e^{in}(V) := N_0 \sqrt{\frac{2m_e}{\pi k T_e}} e^{-\dfrac{m_e V^2}{2kT_e}} \qquad \text{for } V > 0. \tag{24}$$

It gives in terms of dimensionless variables

$$f_e^{in}(v) := n_0 \sqrt{\frac{2m_e}{\pi m_i}} e^{-\dfrac{m_e v^2}{2m_i}} \qquad \text{for } v > 0 \tag{25}$$

where $n_0 := \int_{\mathbb{R}^+} f_e^{in}(v) dv = N_0 L$ is an electron reference density. The electron density (21) is then given for all $x \in [0,1]$ by

$$n_e(x) = \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{\phi(x)} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} v}{\sqrt{v^2 + 2\phi(x)}} dv \right). \tag{26}$$

The electron flux is constant in space and given for all $x \in [0,1]$ by

$$\gamma_e(x) = \gamma_e(0) = \gamma_e = (1-\alpha) \int_{\sqrt{-\frac{2m_i}{m_e}\phi_w}}^{+\infty} f_e^{in}(v) v dv = (1-\alpha) \sqrt{\frac{2m_i}{\pi m_e}} n_0 e^{\phi_w}. \tag{27}$$

Notice that the electron density is close to a Boltzmannian density but not equal. It contains a perturbation that represents the truncation of the Maxwellian distribution due to the electron loss at the wall. The Boltzmannian density corresponds to the case $\alpha = 1$. In this case, the Ampère equation will be shown to be degenerate see Proposition 2 and Remark 3, below.

It will be convenient for the mathematical discussion to denote by ρ_0 the charge imbalance at $x = 0$. By definition it writes

$$\rho_0 := n_i(0) - n_e(0) = \int_{\mathbb{R}^+} f_i^{in}(v)dv - \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv \right). \quad (28)$$

We observe that n_0 can be expressed as

$$n_0 = \sqrt{\frac{\pi}{2}} \frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0}{\left(\sqrt{2\pi} - (1-\alpha) \displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv \right)}. \quad (29)$$

Remark 2. In the mathematical analysis Section 3, we will study the well-posedness of the above problems and consider ρ_0 as a given parameter. The value of n_0 will then be *defined* by (29). In order that n_0 be positive we observe that ρ_0 and f_i^{in} must be chosen such that $\rho_0 < \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv$. Also in the next section, we will show that ϕ_w only depends on ρ_0, α and f_i^{in} and hence, so does n_0. Also remark that $\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv = \sqrt{\frac{\pi}{2}}(1 - \mathrm{erf}(\sqrt{-\phi_w}))$ where erf denotes the error function.

2.5. **Equation of the wall potential.** In general the potential at the wall cannot be a priori specified as a physical parameter. Therefore it is important to understand how it is determined in this model from other physical parameters. As mentioned in the introduction, the wall potential adjusts itself so that equal numbers of ions and electrons reach the wall per second. Following the idea in [24] Section 2.6 page 79, its value is determined from the ambipolarity of the flow which can be also seen as a consequence of the stationary Maxwell-Ampère equation (18). Using (22) the ion flux is constant in space and given for all $x \in [0,1]$ by

$$\gamma_i(x) = \gamma_i(0) = \gamma_i = \int_{\mathbb{R}^+} f_i^{in}(v) v dv, \quad (30)$$

and the electron flux is given in (27). Then for all $\rho_0 \in \mathbb{R}$ and $\alpha \in [0,1)$ the ambipolarity $\gamma_i = \gamma_e$ (18) yields

$$(1-\alpha)\sqrt{\frac{2m_i}{\pi m_e}} n_0 e^{\phi_w} = \int_{\mathbb{R}^+} f_i^{in}(v) v dv. \quad (31)$$

Substituting the expression (29) of n_0 in (31) leads to the equivalent non linear relation

$$\mathcal{W}(\phi_w) = b \quad (32)$$

where $\mathcal{W} : \mathbb{R}^- \to \mathbb{R}$ is defined by

$$\mathcal{W}(\psi) := \sqrt{\frac{m_i}{m_e}} e^{\psi} \left(\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0 \right) + \int_{\mathbb{R}^+} f_i^{in}(v) v dv \int_{\sqrt{-2\psi}}^{+\infty} e^{-\frac{v^2}{2}} dv \quad (33)$$

and $b = \dfrac{\sqrt{2\pi}}{1-\alpha} \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv$. Note that here we have considered $\alpha < 1$, see Remark 3 below. Then for all $\rho_0 \in \mathbb{R}$ and $\alpha \in [0,1)$, ϕ_w must be solution of the non linear equation (32). We remember that due to the definition of n_0 one has of course $\rho_0 < \int_{\mathbb{R}^+} f_i^{in}(v)dv$. Therefore using standard arguments one has the following

Proposition 2. *Let $\rho_0 \in \mathbb{R}$ and $\alpha \in [0,1)$. The equation (32) has a unique non positive solution $\phi_w = \phi_w(\rho_0, \alpha)$ if and only if*

$$0 < \frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0} \leq \frac{(1-\alpha)}{(1+\alpha)} \sqrt{\frac{m_i}{m_e}} \sqrt{\frac{2}{\pi}}. \tag{34}$$

Moreover the solution is in the interval,

$$\ln\left(\frac{\sqrt{2\pi}}{(1-\alpha)\left(\sqrt{2} + \sqrt{\frac{m_i}{m_e}}\left(\dfrac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv}\right)\right)} \right) \leq \phi_w \leq 0. \tag{35}$$

Proof. Since $\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0 > 0$, the function $\mathcal{W}$ is continuous and increasing with $\lim\limits_{-\infty} \mathcal{W} = 0$. Consequently, $\mathcal{W}$ is a bijection from $(-\infty, 0]$ to $(0, \mathcal{W}(0)]$ and the equation (32) admits a unique solution if and only if $b = \dfrac{\sqrt{2\pi}}{1-\alpha} \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv \in (0, \mathcal{W}(0)]$ which leads to the inequality (34). Now we prove the bounds (35). The upper bound is straightforward from the definition of the domain of $\mathcal{W}$. For the lower bound, after a change of variable in $\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv$ we obtain

$$\mathcal{W}(\phi_w) = \sqrt{\frac{m_i}{m_e}} e^{\phi_w} \left(\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0 \right) + \sqrt{2} \int_{\mathbb{R}^+} f_i^{in}(v) v dv \int_{\sqrt{-\phi_w}}^{+\infty} e^{-v^2} dv.$$

Then using the inequality $\displaystyle\int_{\sqrt{-\phi_w}}^{+\infty} e^{-u^2} du \leq \frac{e^{\phi_w}}{\sqrt{-\phi_w} + 1}$ (see [19] page 163) we obtain

$$\mathcal{W}(\phi_w) \leq e^{\phi_w} \left(\sqrt{\frac{m_i}{m_e}} \left(\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0 \right) + \frac{\sqrt{2} \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv}{\sqrt{-\phi_w} + 1} \right).$$

A simpler and easily computable bound is then given by

$$\mathcal{W}(\phi_w) \leq e^{\phi_w} \left(\sqrt{\frac{m_i}{m_e}} \left(\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0 \right) + \sqrt{2} \int_{\mathbb{R}^+} f_i^{in}(v) v dv \right)$$

and we conclude using the equality $\mathcal{W}(\phi_w) = \dfrac{\sqrt{2\pi}}{1-\alpha} \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv$. □

Let us make a list of remarks about this result.

Remark 3. When $\alpha = 1$ (which corresponds to a total re-emission of electrons at the wall) the only integrable boundary condition that satisfies (31) is $f_i^{in} \equiv 0$ and the model is of no interest.

Remark 4. The inequality (34) also re-writes $\gamma_i \le \dfrac{(1-\alpha)}{(1+\alpha)}\sqrt{\dfrac{m_i}{m_e}}\sqrt{\dfrac{2}{\pi}}n_e(0)$ and since $\gamma_i = \gamma_e$ this means that ion and electrons flow must be bounded and cannot exceed the upper bound $\gamma^{limit} := \dfrac{(1-\alpha)}{(1+\alpha)}\sqrt{\dfrac{m_i}{m_e}}\sqrt{\dfrac{2}{\pi}}n_e(0)$.

Remark 5. The case of equality $\dfrac{\int_{\mathbb{R}^+} f_i^{in}(v)v dv}{\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0} = \dfrac{(1-\alpha)}{(1+\alpha)}\sqrt{\dfrac{m_i}{m_e}}\sqrt{\dfrac{2}{\pi}}$ is equivalent to $\phi_w = 0$ is the solution to (32).

For the sake of simplicity and from now, we omit to precise the dependence of $\phi_w(\rho_0,\alpha)$ on ρ_0 and α and simply denote $\phi_w = \phi_w(\rho_0,\alpha)$.

2.6. **A variational approach to the non linear Poisson problem.** We remember that (NLP) is formally equivalent to the Vlasov-Poisson system when the electrostatic potential is decreasing, see Proposition 1. In the following section we will study the well posedness of (NLP) in the case of an incoming Maxwellian electron distribution. To this end, we will consider that $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and f_i^{in} satisfying (34) are given parameters as well as the normalized Debye length ε. The wall potential $\phi_w \le 0$ will then be the solution of (32) and the electron boundary condition will be of the form (25) where the reference density n_0 is defined by (29). In particular, the (NLP) problem reformulates as follows : Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, f_i^{in} satisfying (34) and $\varepsilon > 0$. Find $\phi : [0,1] \to \mathbb{R}$ solution of

$$\text{(NLP-M)} \ : \begin{cases} -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = (n_i - n_e)(x) \text{ for all } x \in (0,1) \\ \ \text{with Dirichlet boundary conditions} \\ \phi(0) = 0 \text{ and } \phi(1) = \phi_w \\ \ \text{where} \\ n_i(x) = \displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2 - 2\phi(x)}}dv, \\ n_e(x) = \dfrac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}e^{\phi(x)} \\ \qquad\qquad - \dfrac{2n_0}{\sqrt{2\pi}}(1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}v}{\sqrt{v^2+2\phi(x)}}dv. \end{cases} \tag{36}$$

From a mathematical point of view, one would notice the analogy between the non linear Poisson equation and classical motion equations of a single particle in a potential force field. Indeed, the opposite of the right hand side of (36) derives from an abstract potential function $\mathcal{U} : [\phi_w, 0] \to \mathbb{R}$ given by

$$\mathcal{U}(\psi) := \int_{\mathbb{R}^+} f_i^{in}(v)v\sqrt{v^2 - 2\psi}dv + \frac{2n_0}{\sqrt{2\pi}}\left(\sqrt{2\pi}e^{\psi} - (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}}v\sqrt{v^2+2\psi}dv\right). \tag{37}$$

The non linear Poisson equation (36) rewrites in the form

$$-\varepsilon^2 \frac{d^2}{dx^2}\phi(x) = -\mathcal{U}'(\phi(x)) \quad \forall x \in (0,1), \tag{38}$$

indeed for all $x \in [0,1]$ $-\mathcal{U}'(\phi(x)) = (n_i - n_e)(x)$. Moreover, it can eventually be re-written into a variational form. Indeed, solutions to (36) are critical point of the energy functional

$$J_\varepsilon(\phi) = \int_0^1 \left(\frac{\varepsilon^2}{2} |\phi'(x)|^2 + \mathcal{U}(\phi(x)) \right) dx \tag{39}$$

defined on the adequate functional space. Namely critical points of J_ε are solutions of $dJ_\varepsilon(\phi) \equiv 0$ where dJ_ε denotes the Fréchet derivative of J_ε. Then variational techniques constitute a convenient mathematical tool to solve the non linear Poisson equation. Historically speaking, variational methods to treat stationary transport problems were used in [18] to deal with neutron diffusion problems such as the Milne problem.

3. **Mathematical analysis.** In this section, we study the well-posedness of the non linear Poisson problem (NLP-M) which corresponds to the Vlasov-Poisson-Ampère problem in the case where the incoming electron distribution is Maxwellian. We will use variational principles and the theory of Nemytskii's operator to study the functional J_ε. The results we need are reminded in the appendix.

3.1. **Mathematical setting and main result.** Let us define for all $\alpha \in [0,1)$

$$s_1(\alpha) := \frac{(1-\alpha)}{(1+\alpha)} \sqrt{\frac{m_i}{m_e}} \sqrt{\frac{2}{\pi}}, \tag{40}$$

which is the upper bound in (34). In the case $\alpha = 1$ the problem has no interest (see Remark 3). The physically interesting case corresponds to $\phi_w < 0$, so we shall only consider distributions f_i^{in} satisfying (34) with a strict inequality. Let us therefore define the functional framework that will be used in the following. We shall consider ion boundary conditions that are bounded, integrable and of finite kinetic energy so that $\mathcal{U}$ is well defined. We denote the set of such functions

$$\mathcal{I} := \left\{ h \in \left(L^1 \cap L^\infty\right)(\mathbb{R}^+; \mathbb{R}^+) \text{ such that } \int_{\mathbb{R}^+} h(v) v^2 dv < +\infty \right\}. \tag{41}$$

For $\rho_0 \in \mathbb{R}$ and $\alpha \in [0,1)$ given, we define the set of admissible ion boundary conditions

$$\mathcal{I}_{ad}(\rho_0, \alpha) := \left\{ h \in \mathcal{I} \text{ such that } 0 < \frac{\displaystyle\int_{\mathbb{R}^+} h(v) v dv}{\displaystyle\int_{\mathbb{R}^+} h(v) dv - \rho_0} < s_1(\alpha) \right\}, \tag{42}$$

as well as the set of admissible potential

$$V := V(\rho_0, \alpha) = \{\phi \in V_0 \mid \phi_w \le \phi \le 0 \text{ with } \phi(1) = \phi_w\}, \tag{43}$$

where $V_0 := \left\{\phi \in H^1(0,1) \mid \phi(0) = 0\right\}$ is a Hilbert space equipped with the inner product $(\phi, \varphi)_{V_0} := \int_0^1 \phi'(x) \varphi'(x) dx$ for any $(\phi, \varphi) \in V_0 \times V_0$ and with the induced norm defined by $\|\phi\|_{V_0} = \sqrt{(\phi, \phi)_{V_0}} = \|\phi\|_{H_0^1}$ for all $\phi \in V_0$. We also denote H^{-1} the dual space of $H_0^1(0,1)$, and we remind that the norm on H^{-1} is defined by $\|\mathcal{L}\|_{H^{-1}} := \sup_{\varphi \in H_0^1, \varphi \neq 0} \frac{|\langle \mathcal{L}, \varphi \rangle|}{\|\varphi\|_{H_0^1}}$ for all $\mathcal{L} \in H^{-1}$.

Remark 6. It is important to notice that in the definition of V the decreasing assumption on ϕ does not appear. It is not necessary for $\mathcal{U}$ to be well-defined, hence we decide to relax it. It will be shown in Theorem 3.2 that the solution ϕ of (NLP-M) is non increasing.

Finally, since the mass ratio always satisfies $\dfrac{m_e}{m_i} < \dfrac{2}{\pi}$ we define a critical re-emission coefficient

$$\alpha_c := \frac{1 - \sqrt{\frac{\pi}{2}}\sqrt{\frac{m_e}{m_i}}}{1 + \sqrt{\frac{\pi}{2}}\sqrt{\frac{m_e}{m_i}}} \tag{44}$$

which is such that $s_1(\alpha_c) = 1$ and $0 < \alpha_c < 1$.

Definition 3.1. (Sheath solutions) Let (f_i, f_e, ϕ) be a solution to the Vlasov-Poisson-Ampère system (12)-(18). We say that it is a sheath-type solution on $(x^*, 1]$ with $0 \leq x^* < 1$ if on that interval ϕ is decreasing and $n_i > n_e$, and if $n_i(x^*) = n_e(x^*)$.

We are now in position to state our main result.

Theorem 3.2. *Let $\alpha \in [0, \alpha_c]$, $\rho_0 = 0$, $f_i^{in} \in \mathcal{I}_{ad}(0, \alpha)$ and $\varepsilon > 0$. Let ϕ_w be the unique solution of* (32)*. Assume the kinetic Bohm criterion*

$$\frac{\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv} < \frac{\left(\sqrt{2\pi} + (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv\right)}{\left(\sqrt{2\pi} - (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv\right)}. \tag{45}$$

Then the Vlasov-Poisson-Ampère system (12)-(18) *is well-posed, with a Maxwellian incoming electron distribution f_e^{in} defined by* (25), (29)*. More precisely, there is a unique $\phi_\varepsilon \in V$ solution of (NLP-M). In addition,*

1. *The densities f_e and f_i defined in* (20) *and* (22) *belong to $(L^1 \cap L^\infty)([0,1] \times \mathbb{R}; \mathbb{R}^+)$ and are weak solutions of the Vlasov equations in the sense of Definition A.1.*
2. *There exists $x^* \in [0, 1)$ such that $(f_e, f_i, \phi_\varepsilon)$ is a sheath-type solution on $(x^*, 1]$ in the sense of Definition 3.1.*
3. *At the wall the values of n_i, n_e, ϕ_w and the velocity distributions f_i, f_e do not depend on the normalized Debye length ε.*
4. *ϕ_ε is $C^2[0, 1]$, concave and we have the quantitative estimates*

$$\|\phi_\varepsilon\|_{V_0} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) \quad \text{and} \quad \|n_i - n_e\|_{H^{-1}} = \mathcal{O}(\varepsilon). \tag{46}$$

The proof of this theorem is proved in Section 3.2.2. Let us make a list of general but somewhat useful remarks about this theorem.

Remark 7. A sufficient condition for (45) is

$$\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv < \int_{\mathbb{R}^+} f_i^{in}(v) dv.$$

This inequality still re-writes in physical variables

$$\int_{\mathbb{R}^+} \frac{F_i^{in}(V)}{V^2} dV < \frac{1}{c_s^2} \int_{\mathbb{R}^+} F_i^{in}(V) dV.$$

It coincides with the standard kinetic Bohm criterion, see [24, Section 2.4].

Remark 8. The re-emission coefficient α_c is said to be critical because when $\alpha > \alpha_c$ we are not able to prove the existence of admissible boundary condition satisfying the kinetic Bohm criterion (45). On the contrary when $\alpha \leq \alpha_c$ we are able to do so (see theorem 3.7 and corollary 1).

Remark 9. In practice α_c is close to 1 and it allows to consider a wide range of material, even those with a high re-emission coefficient. As an example, for a Deuterium plasma $\dfrac{m_i}{m_e} = 3672$ and the critical re-emission coefficient is $\alpha_c \approx 0.95$.

Remark 10. In the theorem we have considered $\rho_0 = 0$ which corresponds to the neutrality $n_i(0) = n_e(0)$. It is an usual assumption in the physics literature. In the case $\rho_0 \neq 0$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ for some $\alpha \in [0,1)$ and $\varepsilon > 0$, we are able to establish the existence of a non increasing minimizer for J_ε see theorem 3.8 and proposition 8. However, since V is a strict and closed convex subset of V_0, minimizers are not necessarily critical points.

Remark 11. The kinetic Bohm criterion (45) expresses the strict convexity of $\mathcal{U}$ in the vicinity of $\psi = 0$. Moreover, we have seen that $(n_i - n_e)(x) = -\mathcal{U}'(\phi(x))$. It follows that $n_i - n_e$ is a function of the electrostatic potential and we can verify

$$\frac{d}{d\phi}(n_i - n_e)(x = 0) = -\mathcal{U}''(\phi(0)).$$

In particular (45) is equivalent to $\frac{d}{d\phi}(n_i - n_e)(0) < 0$ which is an usual sheath criterion [8, 21].

Remark 12. The kinetic Bohm criterion implies that $v \mapsto \dfrac{f_i^{in}(v)}{v^2} \in L^1(\mathbb{R}^+;\mathbb{R}^+)$. This means there is essentially no ions with null velocity at $x = 0$. In such a configuration, minimizers of J_ε are concave and non increasing solutions of the non linear Poisson equation. Thus f_e and f_i defined in (20) and (22) are weak solutions of the Vlasov equations and the Vlasov-Poisson-Ampère system is well posed. The uniqueness for the Poisson equation is proven by a reduction to a first order differential equation.

Remark 13. In the limit $\varepsilon \to 0$, the estimates (46) are mathematical expression of the quasi-neutrality.

3.2. **Well posedness of the Vlasov-Poisson-Ampère problem.** Here we use the variational formulation of (NLP-M) and treat the following minimization problem.

$$\begin{cases} \text{Let } \rho_0 \in \mathbb{R},\ \alpha \in [0,1),\ f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha) \text{ and } \varepsilon > 0. \\ \text{Find } \phi_\varepsilon^* \in V \text{ such that} \\ \phi_\varepsilon^* = \arg\min\limits_{\phi \in V} J_\varepsilon(\phi). \end{cases} \tag{47}$$

This minimization problem is a constrained problem and we can see it is a non linear variant of the obstacle problem [23]. Let us remember that

$$J_\varepsilon(\phi) = \int_0^1 \left(\frac{\varepsilon^2}{2}|\phi'(x)|^2 + \mathcal{U}(\phi(x))\right) dx$$

where the real valued function $\mathcal{U}$ is defined for all $\psi \in [\phi_w, 0]$ by

$$\mathcal{U}(\psi) := \int_{\mathbb{R}^+} f_i^{in}(v) v \sqrt{v^2 - 2\psi} dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{\psi} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v \sqrt{v^2 + 2\psi} dv \right).$$

This function belongs to $C^1[\phi_w, 0]$ and the Nemytskii operator associated with $\mathcal{U}$ is denoted $T_{\mathcal{U}}$ and defined for all $\phi \in V$ by $T_{\mathcal{U}}(\phi)(x) := \mathcal{U}(\phi(x))$ for all $x \in [0,1]$ see A.4. To prove the Theorem (3.2), we shall need some preliminary results.

3.2.1. *Preliminary part.* Notice that J_ε is made of strictly convex part

$$\phi \in V \mapsto E_\varepsilon(\phi) = \frac{\varepsilon^2}{2} \|\phi\|_{V_0}^2 \tag{48}$$

plus a perturbation that is not necessarily convex

$$\phi \in V \mapsto F(\phi) = \int_0^1 T_{\mathcal{U}}(\phi)(x) dx. \tag{49}$$

All the analysis is based on the properties of the perturbation (49). Using the theory of Nemytskii operators and more precisely Theorem A.6 we have the following.

Proposition 3. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. Then $T_{\mathcal{U}} : V \to C^0[0,1]$ is of class C^1. Its Fréchet derivative is given by*

$$dT_{\mathcal{U}}(\phi)(h) = \mathcal{U}'(\phi) h \quad \forall (\phi, h) \in V \times V_0.$$

Moreover the perturbation (49) is compact as we prove in the following.

Proposition 4. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. Then $T_{\mathcal{U}}$ is compact.*

Proof. We endow $C^0[0,1]$ with the norm $\phi \mapsto \|\phi\|_\infty := \max_{x \in [0,1]} |\phi(x)|$. One has $T_{\mathcal{U}} = \tilde{T_{\mathcal{U}}} \circ i$ where $i : V \to C^0([0,1]; [\phi_w, 0])$ is the Rellich compact embedding and $\tilde{T_{\mathcal{U}}}$ is the restriction to $C^0([0,1]; [\phi_w, 0])$ of $T_{\mathcal{U}}$. Since i is compact and $\tilde{T_{\mathcal{U}}}$ is continuous, we conclude that $T_{\mathcal{U}}$ is compact. □

A direct consequence of the above proposition which results from Lemma A.3 is the following.

Proposition 5. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. Then $T_{\mathcal{U}}$ is (sequentially) weakly lower semicontinuous.*

Lemma 3.3. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. Then $J_\varepsilon : V \to \mathbb{R}$ is well-defined, of class C^1 and (sequentially) wealky lower semicontinuous. Its Fréchet derivative is given by*

$$dJ_\varepsilon(\phi)(h) = \int_0^1 \varepsilon^2 \phi'(x) h'(x) + \mathcal{U}'(\phi(x)) h(x) dx \quad \textit{for all} \quad (\phi, h) \in V \times V_0. \tag{50}$$

Proof. First notice that for all $\phi \in V$

$$J_\varepsilon(\phi) = E_\varepsilon(\phi) + F(\phi)$$

where E_ε and F were given in (48)-(49). From Proposition 3 we deduce that F is C^1 over V and since E_ε is also C^1 over V thus J_ε is also. For the weak lower semicontinuity, we notice that E_ε is convex and continuous for the strong topology, consequently applying the Mazur lemma [9, p. 562] we deduce that E_ε is sequentially weakly lower semicontinuous. Applying Proposition 5 we also deduce that F is weakly lower semicontinuous and thus J_ε is. $\square$

We shall also need techniqual inequalities that will be useful to study the motonicity of $\mathcal{U}$. In a general manner, the two following inequalities are obtained by a convexity argument.

Lemma 3.4. *For all $\eta > 0$ and $t^* \in (-\frac{1}{2\eta}, +\infty)$ we have*

$$\frac{e^t}{\sqrt{1+2\eta t}} \geq \frac{e^{t^*}}{\sqrt{1+2\eta t^*}} + \frac{e^{t^*}(t-t^*)}{\sqrt{1+2\eta t^*}} - \frac{\eta e^{t^*}(t-t^*)}{(1+2\eta t^*)^{\frac{3}{2}}}, \quad \forall t \in (-\frac{1}{2\eta}, +\infty). \tag{51}$$

Proof. For all $\eta > 0$, the function $h : t \in (-\frac{1}{2\eta}, +\infty) \mapsto \frac{e^t}{\sqrt{1+2\eta t}}$ is convex over $(-\frac{1}{2\eta}, +\infty)$. Indeed one has $h''(t) = \frac{e^t}{(1+2\eta t)^{\frac{5}{2}}}\left(4\eta^2t^2 + 4\eta(1-\eta)t + 3\eta^2 - 2\eta + 1\right)$.
The polynomial $t \mapsto 4\eta^2t^2 + 4\eta(1-\eta)t + 3\eta^2 - 2\eta + 1$ has for discriminant $\Delta = -32\eta^4$, hence if $\eta > 0$ then $\Delta < 0$ and $h''(t) > 0$. The conclusion follows from

$$h(t) \geq h(t^*) + (t-t^*)h'(t^*) \text{ for all } t, t^* \in (-\frac{1}{2\eta}, +\infty).$$

$\square$

Lemma 3.5. *For all $\eta > 0$ and $t^* \in (-\infty, \frac{1}{2\eta})$, , we have*

$$\frac{e^t}{\sqrt{1-2\eta t}} \geq \frac{e^{t^*}}{\sqrt{1-2\eta t^*}} + \frac{e^{t^*}(t-t^*)}{\sqrt{1-2\eta t^*}} + \frac{\eta e^{t^*}(t-t^*)}{(1-2\eta t^*)^{\frac{3}{2}}} \quad \forall t \in (-\infty, \frac{1}{2\eta}) \tag{52}$$

Proof. The proof is similar to the previous one. For all $\eta > 0$, the function $h : t \in (-\infty, \frac{1}{2\eta}) \mapsto \frac{e^t}{\sqrt{1-2\eta t}}$ is convex over $(-\infty, \frac{1}{2\eta})$. Indeed one has $h''(t) = \frac{e^t}{(1-2\eta t)^{\frac{5}{2}}}\left(4\eta^2t^2 - 4\eta(\eta+1)t + 3\eta^2 + 2\eta + 1\right)$.
The polynomial $t \mapsto 4\eta^2t^2 - 4\eta(\eta-1)t + 3\eta^2 + 2\eta + 1$ has for discriminant $\Delta = -32\eta^4$, hence if $\eta > 0$ then $\Delta < 0$ and $h''(t) > 0$. The conclusion follows from

$$h(t) \geq h(t^*) + (t-t^*)h'(t^*) \text{ for all } t, t^* \in (-\infty, \frac{1}{2\eta}).$$

$\square$

Lemma 3.6. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. Then $\mathcal{U}$ is positive.*

Proof. Let $\psi \in [\phi_w, 0]$. Making the change of variable $w := \sqrt{v^2 + 2\psi}$ leads to

$$\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v\sqrt{v^2+2\psi}\,dv = e^{\psi}\int_{\sqrt{-2(\phi_w-\psi)}}^{+\infty} e^{-\frac{w^2}{2}} w^2 dw$$

and $\int_{\sqrt{-2(\phi_w-\psi)}}^{+\infty} e^{-\frac{w^2}{2}} w^2 dw \leq \int_0^{+\infty} e^{-\frac{w^2}{2}} w^2 dw = \frac{\sqrt{2\pi}}{2}$. We obtain finally

$$\mathcal{U}(\psi) \geq \int_{\mathbb{R}^+} f_i^{in}(v)v^2 dv + n_0(1+\alpha)e^{\phi_w} > 0.$$

$\square$

Proposition 6. *Let $\rho_0 \in \mathbb{R}^+$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$. If*

$$\rho_0 > \int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv - \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv \right) \tag{53}$$

then $\mathcal{U}$ is decreasing. If the inequality is large then $\mathcal{U}$ is non increasing.

Remark 14. When $\rho_0 = 0$ the inequality (53) is nothing but the kinetic Bohm criterion (45).

Proof. It is convenient to make the change of variable $u := -\psi$ and to define the function $u \in [0, -\phi_w] \mapsto \tilde{\mathcal{U}}(u) := \mathcal{U}(-u)$. We have $\tilde{\mathcal{U}} \in C^1[0,-\phi_w]$ and for all $u \in [0, -\phi_w]$

$$\frac{d}{du}\tilde{\mathcal{U}}(u) = e^{-u} \left(\underbrace{\int_{\mathbb{R}^+} \frac{f_i^{in}(v) e^u}{\sqrt{1+\frac{2u}{v^2}}} dv}_{:=A} \right) + e^{-u} \left(-\frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi} + \frac{2n_0}{\sqrt{2\pi}} \left((1-\alpha) \underbrace{\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} e^u}{\sqrt{1-\frac{2u}{v^2}}} dv}_{:=B} \right) \right).$$

We shall give a lower bound for A and B. Applying respectively inequalities (51) and (52) to the integrands of A and B with $u^* = 0$ and $\eta = \frac{1}{v^2}$, we obtain

$$\begin{aligned} \frac{d}{du}\tilde{\mathcal{U}}(u) \geq & e^{-u} \left(\int_{\mathbb{R}^+} f_i^{in}(v) \left(1 + u(1-\frac{1}{v^2})\right) dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi} \right) + \\ & e^{-u} \left(\frac{2n_0}{\sqrt{2\pi}} (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} \left(1 + u(1+\frac{1}{v^2})\right) dv \right) \end{aligned}$$

we therefore obtain

$$\begin{aligned} \frac{d}{du}\tilde{\mathcal{U}}(u) \geq & e^{-u}\rho_0 \\ & + e^{-u} u \left[\rho_0 + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv \right) - \int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv \right]. \end{aligned}$$

By hypothesis $\rho_0 \geq 0$ hence for all $u \in (0, -\phi_w]$ we have that $\frac{d}{du}\tilde{\mathcal{U}}(u)$ is positive if the bracket is positive and non negative if the bracket vanishes. □

For the self-consistency of the analysis we investigate the existence of admissible ion boundary conditions f_i^{in} that satisfy the kinetic Bohm criterion (45). Let us define for all $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(0,\alpha)$

$$s_2(\alpha) := \frac{\left(\sqrt{2\pi} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv \right)}{\left(\sqrt{2\pi} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv \right)}. \tag{54}$$

Notice that $s_2(\alpha) > 1$ for all $\alpha \in [0,1)$. We have the following characterization of existence result.

Theorem 3.7. *Let $\alpha \in [0,1)$. There exists $f_i^{in} \in \mathcal{I}_{ad}(0,\alpha)$ satisfying the kinetic Bohm criterion (45) if and only if $s_1(\alpha)^2 s_2(\alpha) > 1$.*

Proof. Let $\alpha \in [0,1)$. We begin with showing the necessary condition. Assume there exists $f_i^{in} \in \mathcal{I}$ satisfying

$$\frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv} < s_1(\alpha) \quad \text{and} \quad \frac{\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2}dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv} < s_2(\alpha). \tag{55}$$

Applying twice the Cauchy Schwarz inequality yields

$$\int_{\mathbb{R}^+} f_i^{in}(v)dv \leq \left(\int_{\mathbb{R}^+} f_i^{in}(v)vdv\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2}dv\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}^+} f_i^{in}(v)dv\right)^{\frac{1}{4}}.$$

Using the previous inequalities (55) we obtain $1 < s_1(\alpha)^2 s_2(\alpha)$. Let us now prove the sufficient condition. Assume $s_1^2(\alpha)s_2(\alpha) > 1$. Then we claim that the function f_i^{in} defined for all $v \in \mathbb{R}^+$ by $f_i^{in}(v) = \mathbf{1}_{(v_{min},v_{max})}(v)$ with $v_{min} = \frac{1}{\sqrt{s_2(\alpha)}}$ and $v_{max} = s_1(\alpha)$ is a solution. □

A direct consequence of the previous result is the following.

Corollary 1. *Let $\alpha \in [0,\alpha_c]$. Then there exists $f_i^{in} \in \mathcal{I}_{ad}(0,\alpha)$ satisfying the kinetic Bohm criterion (45).*

Proof. For all $\alpha \in [0,\alpha_c]$, $s_1(\alpha) \geq 1$. Since $s_2(\alpha) > 1$ we deduce $s_1^2(\alpha)s_2(\alpha) > 1$ and Theorem 3.7 applies. □

We are now equipped to prove the main result, Theorem 3.2.

3.2.2. *Proof of the main result.*

Theorem 3.8 (Existence of minimizers)**.** *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$ and $\varepsilon > 0$. There is $\phi_\varepsilon^* \in V$ such that $J_\varepsilon(\phi_\varepsilon^*) \leq J_\varepsilon(\phi)$ for all $\phi \in V$. Moreover, the following estimate holds*

$$\|\phi_\varepsilon^*\|_{V_0} = \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{56}$$

Proof. We apply Theorem A.2. By definition V_0 is a reflexive Banach space and V is a closed convex subset. By Lemma 3.3, J_ε is sequentially weakly lower semicontinuous and since $\mathcal{U}$ is positive (Lemma 3.6) for all $\phi \in V$ we have

$$\frac{\varepsilon^2}{2}\|\phi\|_{V_0}^2 \leq J_\varepsilon(\phi).$$

By comparison $J_\varepsilon(\phi) \to +\infty$ as $\|\phi\|_{V_0} \to +\infty$ and thus J_ε is coercive. Therefore, there is $\phi_\varepsilon^* \in V$ such that $J_\varepsilon(\phi_\varepsilon^*) \leq J(\phi)$ for all $\phi \in V$. Finally, taking $x \in [0,1] \mapsto \phi(x) := x\phi_w$ which belongs to V we obtain

$$\|\phi_\varepsilon^*\|_{V_0} \leq \sqrt{\phi_w^2 + \frac{2}{\varepsilon^2}\int_0^1 \mathcal{U}(x\phi_w)dx} = \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{57}$$

□

This theorem states the existence of global minimizers but does not ensure they are criticial point of J_ε. Let us give more properties of minimizers that will be useful in the sequel.

Proposition 7 (First order condition)**.** *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. Let $\phi^* := \phi^*_\varepsilon \in V$ be a minimizer of J_ε. Then the following variational inequality holds*

$$dJ_\varepsilon(\phi^*)(h) \geq 0 \qquad \text{for all } h \in V_0 \text{ such that } \phi^* + h \in V. \tag{58}$$

Moreover, we have $\phi^* \in W^{2,\infty}(0,1) \cap C^1[0,1]$ and

$$-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) = -\mathcal{U}'(\phi^*(x)) \qquad \text{a.e in } \mathcal{O} := \{x \in (0,1) \mid \phi_w < \phi^*(x) < 0\}, \tag{59}$$

$$-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) \leq -\mathcal{U}'(\phi^*(x)) \qquad \text{a.e in } \mathcal{F}_1 := \{x \in (0,1) \mid \phi^*(x) = 0\}, \tag{60}$$

$$-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) \geq -\mathcal{U}'(\phi^*(x)) \qquad \text{a.e in } \mathcal{F}_2 := \{x \in (0,1) \mid \phi^*(x) = \phi_w\}. \tag{61}$$

Proof. Since ϕ^* is a minimizer it is straightforward from the C^1-regularity of J_ε that we have the variational inequality (58). Then the regularity property $\phi^* \in W^{2,\infty}(0,1) \cap C^1[0,1]$ is obtained following exactly the same ideas as in [23, p. 113]. The equality (59) is obtained as follows. Choose $h \in H^1_0(0,1)$ with $\mathrm{supp}(h) \subset \mathcal{O}$ then there is $|\tau|$ sufficiently small such that $\phi^* + \tau h \in V$ and $\tau dJ_\varepsilon(\phi^*)(h) \geq 0$ for both positive and negative τ. Then $dJ_\varepsilon(\phi^*) \equiv 0$ and the result then follows from (50) and the regularity property $\phi^* \in W^{2,\infty}(0,1)$. Inequalities (60) and (61) can also be obtained from the first order condition (58) using adequate test functions $h \in H^1_0(0,1)$ with a support included respectively in $\mathcal{F}_1$ and $\mathcal{F}_2$. □

Proposition 8 (Non increasing property)**.** *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. The minimizers of J_ε are non increasing functions. More precisely, if $\phi \in V$ is a minimizer and $0 \leq x \leq y \leq 1$ then $\phi(y) \leq \phi(x)$.*

Proof. Based on Theorem 1.1 of [5], we observe that the monotone decreasing rearrangement $\hat{\phi} : [0,1] \to \mathbb{R}$ of ϕ (that is the unique non increasing function such that for all $t \in \mathbb{R}$ $\mathrm{meas}\left(\{x \in (0,1) \mid \phi(x) \geq t\}\right) = \mathrm{meas}\left(\{x \in (0,1) \mid \hat{\phi}(x) \geq t\}\right)$) belongs to V and satisfies $F(\hat{\phi}) = F(\phi)$ and $E_\varepsilon(\hat{\phi}) \leq E_\varepsilon(\phi)$. In particular, if ϕ is not non increasing then the previous inequality is strict and $J_\varepsilon(\hat{\phi}) < J_\varepsilon(\phi)$ which contradicts the minimality of ϕ. □

Proposition 9. *Let $\rho_0 \leq 0$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$.*
Let $\phi^* := \phi^*_\varepsilon \in V$ a minimizer of J_ε. Assume $\mathcal{U}'(\phi_w) \leq 0$. Then $\phi^* \in C^2[0,1]$ and is solution of (NLP-M).

Proof. Let $\phi^* \in V$ a minimizer of J_ε, we will show that (59) holds on the all interval $(0,1)$. To do so we note that since ϕ^* is continuous and non increasing (see Proposition 8) there is $0 \leq \delta < \delta' \leq 1$ such that $\mathcal{F}_1 = (0,\delta]$ and $\mathcal{F}_2 = [\delta', 1)$ (where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the sets of Proposition 7). If $\mathcal{F}_1$ is non-empty then $\frac{d}{dx^2}\phi^* \equiv 0$ and (60) implies $0 \leq -\mathcal{U}'(0)$. Since $\mathcal{U}'(0) = -\rho_0 \geq 0$ by hypothesis, it follows that necessarily $0 = -\mathcal{U}'(0)$. Hence, if $\rho_0 = 0$ then $-\varepsilon^2 \frac{d^2}{dx^2}\phi^* = -\mathcal{U}'(\phi^*)$ on $\mathcal{F}_1$, else if $\rho_0 < 0$ then $\mathcal{F}_1$ is empty. The same argument holds for $\mathcal{F}_2$ using (61). We deduce $-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) = -\mathcal{U}'(\phi^*(x))$ for almost every $x \in (0,1)$. Since $x \mapsto \mathcal{U}'(\phi^*(x)) \in C[0,1]$ we deduce from the Poisson equation that $\phi^* \in C^2[0,1]$. □

Unfortunately, when $\rho_0 > 0$ we are not able to conclude to the existence of a solution to (NLP-M), however we are able to describe the behavior of $\mathcal{U}$, see Section 3.2.3 and Theorem 3.10. Let us now prove the main result 3.2.

Proof of theorem 3.2. Let $\alpha \in [0, \alpha_c]$, $\rho_0 = 0$, $f_i^{in} \in \mathcal{I}_{ad}(0, \alpha)$ and $\varepsilon > 0$. Moreover, assume the kinetic Bohm criterion

$$\frac{\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv} < \frac{\left(\sqrt{2\pi} + (1-\alpha) \displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv\right)}{\left(\sqrt{2\pi} - (1-\alpha) \displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv\right)}.$$

The proof is splitted into two parts. Mainly, the first part deals with the existence of a solution the Vlasov-Poisson-Ampère system and the second part deals with the uniqueness.

Existence part. We apply the proposition 6, so that the function $\phi \in [\phi_w, 0] \mapsto \mathcal{U}(\phi)$ is decreasing and $\mathcal{U}'(\phi) < 0$ for all $\phi \in [\phi_w, 0)$. Combining theorem 3.8 and corollary 9 we obtain there is $\phi_\varepsilon \in V \cap C^2[0,1]$ non increasing solution of (NLP-M). Since $\varepsilon^2 \frac{d}{dx^2} \phi_\varepsilon(x) = \mathcal{U}'(\phi_\varepsilon(x)) \le 0$ for all $x \in (0,1)$ we deduce that ϕ_ε is concave on $[0,1]$. Now considering f_e and f_i defined in (20) and (22), it is easy to see that they belongs to $(L^1 \cap L^\infty)([0,1] \times \mathbb{R}; \mathbb{R}^+)$. We can also check they are weak solution of the Vlasov equations in the sense of definition A.1. Now it easy to observe that $(f_i, f_e, \phi_\varepsilon)$ is a sheath type solution on $(x^*, 1]$ where $x^* = \max\{x \in [0,1] \, / \, \phi_\varepsilon(x) = 0\}$. Besides, the Ampère equation (9) is satisfied by definition of ϕ_w and we therefore deduce that the Vlasov-Poisson-Ampère system (12)-(18) is well posed. In addition, it is straightforward from the equation (32) that ϕ_w does not depend on ε and so do $v \mapsto (f_i(1,v), f_e(1,v))$, $n_i(1)$ and $n_e(1)$. We shall now prove the estimates (46). The first one is obtained from (56). The second one is obtained as follows. Since $n_i - n_e$ is a continuous function and by the canonical injection $C^0[0,1] \hookrightarrow H^{-1}(0,1)$, it defines a linear and continuous form on the space $H_0^1(0,1)$ and we have for all $\psi \in H_0^1(0,1)$

$$\langle n_i - n_e, \psi \rangle_{H^{-1}, H_0^1} = \int_0^1 (n_i - n_e)(x) \psi(x) dx = \varepsilon^2 \int_0^1 \frac{d}{dx} \phi_\varepsilon(x) \frac{d}{dx} \psi(x) dx.$$

Using the Cauchy-Schwarz inequality and the estimate (57) we obtain

$$\left| \langle n_i - n_e, \psi \rangle_{H^{-1}, H_0^1} \right| \le \varepsilon^2 \|\phi_\varepsilon\|_{V_0} \|\psi\|_{H_0^1} \le \varepsilon^2 \sqrt{\phi_w^2 + \frac{2}{\varepsilon^2} F(x\phi_w)} \|\psi\|_{H_0^1}$$

which leads to $\|n_i - n_e\|_{H^{-1}} \le \varepsilon^2 \sqrt{\phi_w^2 + \frac{2}{\varepsilon^2} F(x\phi_w)} = \mathcal{O}(\varepsilon)$. □

Uniqueness part. The proof of the uniqueness result relies on a reduction of the non linear Poisson equation to a first order differential equation. We shall also need the following lemma whose proof is a consequence of the Cauchy-Lipschitz theorem.

Lemma 3.9. *Let $0 \le x_1 < x_2 \le 1$ and $\phi \in C^1([x_1, x_2]; [\phi_w, 0])$ solution of the initial Cauchy problem*

$$\begin{cases} \frac{d}{dx} \phi(x) = -\sqrt{g(\phi(x))} \text{ where } g : [\phi_w, 0] \mapsto (0, +\infty) \text{ is } C^1[\phi_w, 0] \\ \phi(x_2) = \phi_2 \in \mathbb{R}. \end{cases}$$

then it is unique.

Proof. Since g is C^1 and $g > 0$, the function $\phi \in [\phi_w, 0] \mapsto -\sqrt{g(\phi)}$ is Lipschitz in ϕ and it suffices to apply the Cauchy-Lipschitz theorem to conclude. $\square$

We are now able to prove the uniqueness result. To this effect, let us multiply the non linear Poisson equation (38) by $\frac{d}{dx}\phi_\varepsilon$ and integrate over an arbitrary segment $[x, y] \subset [0, 1]$ with $0 \leq x < y \leq 1$. Then we obtain

$$\frac{\varepsilon^2}{2}\left(\left(\frac{d}{dx}\phi_\varepsilon(y)\right)^2 - \left(\frac{d}{dx}\phi_\varepsilon(x)\right)^2\right) = \mathcal{U}(\phi_\varepsilon(y)) - \mathcal{U}(\phi_\varepsilon(x)). \tag{62}$$

Further assume there is a solution $\psi_\varepsilon \in V \cap C^2[0, 1]$ concave, non increasing and different of ϕ_ε. Since ϕ_ε and ψ_ε have the same boundary conditions and are continuous, there exist $x_1, x_2 \in [0, 1]$ such that $x_1 < x_2$, $\phi_\varepsilon < \psi_\varepsilon$ on (x_1, x_2), and $\phi_\varepsilon(x_1) = \psi_\varepsilon(x_1)$, $\phi_\varepsilon(x_2) = \psi_\varepsilon(x_2) < 0$. Then $\frac{d}{dx}\phi_\varepsilon(x_1) \leq \frac{d}{dx}\psi_\varepsilon(x_1) \leq 0$ and $\frac{d}{dx}\psi_\varepsilon(x_2) \leq \frac{d}{dx}\phi_\varepsilon(x_2) < 0$. Since the previous relation (62) is valid for any $0 \leq x < y \leq 1$, chosing $x = x_1$ and $y = x_2$ leads to

$$\left(\frac{d}{dx}\psi_\varepsilon(x_2)\right)^2 - \left(\frac{d}{dx}\psi_\varepsilon(x_1)\right)^2 = \left(\frac{d}{dx}\phi_\varepsilon(x_2)\right)^2 - \left(\frac{d}{dx}\phi_\varepsilon(x_1)\right)^2,$$

and by a comparison argument we obtain $\frac{d}{dx}\psi_\varepsilon(x_1) = \frac{d}{dx}\phi_\varepsilon(x_1)$ and $\frac{d}{dx}\phi_\varepsilon(x_2) = \frac{d}{dx}\psi_\varepsilon(x_2)$. Eventually using the relation (62) for $y = x_2$ and $x_1 \leq x \leq x_2$ it is easy to notice that ϕ_ε and ψ_ε are both solutions of the Cauchy problem

$$\begin{cases} \dfrac{\varepsilon}{\sqrt{2}}\dfrac{d}{dx}w(x) = -\sqrt{\mathcal{U}(w(x)) - \mathcal{U}(\phi_\varepsilon(x_2)) + \dfrac{\varepsilon^2}{2}\dfrac{d}{dx}\phi_\varepsilon(x_2)^2} \text{ for } x \geq x_1 \\ w(x_1) = \phi_\varepsilon(x_1). \end{cases}$$

Notice that $\mathcal{U}(w(x)) - \mathcal{U}(\phi_\varepsilon(x_2)) + \frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2 = \mathcal{U}(w(x)) - \mathcal{U}(\phi_\varepsilon(x_1)) + \frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2$ and also that $\mathcal{U}(w(x)) \geq \mathcal{U}(\phi_\varepsilon(x_1))$ because $\mathcal{U}$ and w are non increasing. Finally remark that $\mathcal{U}(w(x)) - \mathcal{U}(\phi_\varepsilon(x_1)) + \frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2 \geq \frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2 > 0$ and conclude by invoking lemma 3.9.

3.2.3. *Complementary study when the Bohm criterion is violated.* In the theorem 3.2 we have considered $\rho_0 = 0$ and the kinetic Bohm criterion (45). In this section, we consider more general cases where either the kinetic Bohm criterion is violated or $\rho_0 \neq 0$. We remember that for $\rho_0 \in \mathbb{R}$, $\alpha \in [0, 1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ given, $\mathcal{U}$ is defined for all $\psi \in [\phi_w, 0]$ by

$$\mathcal{U}(\psi) := \int_{\mathbb{R}^+} f_i^{in}(v) v\sqrt{v^2 - 2\psi}dv + \frac{2n_0}{\sqrt{2\pi}}\left(\sqrt{2\pi}e^{\psi} - (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v\sqrt{v^2 + 2\psi}dv\right).$$

We intend to prove that the general situation is that $\mathcal{U}$ admits at most two local minima and two local maxima. Ultimately, it shows that whenever the Vlasov-Poisson-Ampère is well posed with ϕ non increasing, the charge density $x \in [0, 1] \mapsto (n_i - n_e)(x) = -\mathcal{U}'(\phi(x))$ can change sign at most in three distinct regions. This section is thus devoted to the study of the monotonicity of $\mathcal{U}$. It is convenient to make the change of variable $u := -\psi$ and to define the function

$u \in [0, -\phi_w] \mapsto \tilde{\mathcal{U}}(u) := \mathcal{U}(-u)$. We will assume in this section that $\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv < +\infty$ so that $\tilde{\mathcal{U}} \in C^1[0, -\phi_w] \cap C^2(0, -\phi_w)$ and for all $u \in (0, -\phi_w)$,

$$\tilde{\mathcal{U}}(u) = \int_{\mathbb{R}^+} f_i^{in}(v) v \sqrt{v^2 + 2u} dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{-u} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v \sqrt{v^2 - 2u} dv \right), \quad (63)$$

$$\frac{d}{du} \tilde{\mathcal{U}}(u) = \int_{\mathbb{R}^+} \frac{f_i^{in}(v) v}{\sqrt{v^2 + 2u}} dv + \frac{2n_0}{\sqrt{2\pi}} \left(-\sqrt{2\pi} e^{-u} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} v}{\sqrt{v^2 - 2u}} dv \right), \quad (64)$$

$$\frac{d^2}{du^2} \tilde{\mathcal{U}}(u) = -\int_{\mathbb{R}^+} \frac{f_i^{in}(v) v}{(v^2 + 2u)^{\frac{3}{2}}} dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{-u} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} v}{(v^2 - 2u)^{\frac{3}{2}}} dv \right). \quad (65)$$

Proposition 10. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. If $\tilde{\mathcal{U}}$ has a local minimum at $u^* \in (0, -\phi_w)$ then $\tilde{\mathcal{U}}$ is non decreasing over $[u^*, -\phi_w]$.*

Proof. Assume $\tilde{\mathcal{U}}$ attains a minimum at $u^* \in (0, -\phi_w)$ then one has the first and second order conditions $\frac{d}{du}\tilde{\mathcal{U}}(u^*) = 0$ and $\frac{d^2}{du^2}\tilde{\mathcal{U}}(u^*) \geq 0$ that is

$$\int_{\mathbb{R}^+} \frac{f_i^{in}(v) e^{u^*}}{\sqrt{1 + \frac{2u^*}{v^2}}} dv + \frac{2n_0}{\sqrt{2\pi}} \left((1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} e^{u^*}}{\sqrt{1 - \frac{2u^*}{v^2}}} dv - \sqrt{2\pi} \right) = 0 \quad (66)$$

and

$$\int_{\mathbb{R}^+} \frac{f_i^{in}(v) e^{u^*}}{v^2 \left(1 + \frac{2u^*}{v^2}\right)^{\frac{3}{2}}} dv \leq \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} e^{u^*}}{v^2 \left(1 - \frac{2u^*}{v^2}\right)^{\frac{3}{2}}} dv \right). \quad (67)$$

One has the decomposition for all $u \in [0, -\phi_w]$

$$\frac{d}{du} \tilde{\mathcal{U}}(u) = e^{-u} \left(\underbrace{\int_{\mathbb{R}^+} \frac{f_i^{in}(v) e^u}{\sqrt{1 + \frac{2u}{v^2}}} dv}_{:=A} \right) + e^{-u} \left(-\frac{2n_0}{\sqrt{2\pi}} \sqrt{2\pi} + \frac{2n_0}{\sqrt{2\pi}} \left((1-\alpha) \underbrace{\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} e^u}{\sqrt{1 - \frac{2u}{v^2}}} dv}_{:=B} \right) \right).$$

Let us now give a lower bound for A. Applying the inequality (51) we have

$$A \geq \int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv + (u-u^*)\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv$$
$$-(u-u^*)\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{v^2\left(1+\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv$$

then we use the second order condition (67) and obtain for all $u \geq u^*$

$$A \geq \int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv$$
$$+(u-u^*)\left(\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi} - \frac{2n_0}{\sqrt{2\pi}}(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{v^2\left(1-\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv\right).$$

We also have a lower bound for B. Indeed, using the inequality (52) we have

$$B \geq \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv$$
$$+(u-u^*)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv + (u-u^*)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{v^2\left(1-\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv.$$

Combining A and B one finally obtains

$$\frac{d}{du}\tilde{\mathcal{U}}(u) \geq e^{-u}\left(\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv + \frac{2n_0}{\sqrt{2\pi}}(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}\right)$$
$$+e^{-u}(u-u^*)\left(\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv + \frac{2n_0}{\sqrt{2\pi}}(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}\right)$$

and the right hand side is exactly zero so that $\dfrac{d}{du}\tilde{\mathcal{U}}(u) \geq 0$ for all $u \geq u^*$. □

From Proposition 10 one can establish the following result.

Proposition 11. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$ such that $\tilde{\mathcal{U}}$ is not locally constant. If $\tilde{\mathcal{U}}$ attains a local minimum over $(0,-\phi_w)$ then it is unique. Similarly, if $\tilde{\mathcal{U}}$ attains a local maxima in $(0,-\phi_w)$ then it is unique.*

Proof. We do the proof by contradiction. Assume $\tilde{\mathcal{U}}$ has at least two local minima at some points u_1 and u_2 belonging to $(0,-\phi_w)$. Without loss of generality we can assume $u_1 < u_2$. Since $\tilde{\mathcal{U}}$ is not locally constant there exists $u_1 < \delta < u_2$ such that $\tilde{\mathcal{U}}$ is decreasing over (δ, u_2) which is contradiction with Proposition 10. We can also prove that if $\tilde{\mathcal{U}}$ has a local maxima in $(0,-\phi_w)$ then it is unique. □

Theorem 3.10. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$ such that $\tilde{\mathcal{U}}$ is not locally constant. Then $\tilde{\mathcal{U}}$ admits at most two local minima over $[0,-\phi_w]$ and it also admits at most two local maxima over $[0,-\phi_w]$.*

Proof. We begin with showing that $\tilde{\mathcal{U}}$ has at most two local minima. First of all, it is clear that $\tilde{\mathcal{U}}$ admits a local minimum (which is in fact a global one) since it is continuous over the compact set $[0, -\phi_w]$. We shall now distinguish two cases. Suppose $\mathcal{U}$ attains a local minimum at $u_1 \in (0, -\phi_w)$ then from corollary 11 it is unique in the interval $(0, -\phi_w)$ and from proposition 10 $\tilde{\mathcal{U}}$ is non decreasing over $[u_1, -\phi_w]$ and necessarily any other local minimum is attained at $u_2 = 0$. On the contrary if $\tilde{\mathcal{U}}$ does not have any local minimum in $(0, -\phi_w)$ this implies that $\tilde{\mathcal{U}}$ has at most two local minima attained at $u_1 = 0$ and $u_2 = -\phi_w$. We can also show that $\tilde{\mathcal{U}}$ has at most two local maxima. □

It results from the above analysis that we can describe the variation of $\tilde{\mathcal{U}}$ depending on where local minima/maxima are located in $[0, -\phi_w]$. Moreover since $\mathcal{U}(\psi) = \tilde{\mathcal{U}}(-\psi)$ for all $\psi \in [\phi_w, 0]$, we are able to deduce the variation of $\mathcal{U}$. Finally, one has the illustrated possible behavior for $\tilde{\mathcal{U}}$ see Figure 5.

4. Numerical illustrations.

4.1. Description of a numerical method.

The velocity integrals are computed by means of quadrature formulas. More precisely, for a given integrand $g : \mathbb{R}^+ \to \mathbb{R}$ our numerical strategy consists in splitting the integral of g as follows,

$$\int_{\mathbb{R}^+} g(v)dv = \int_{(0,v_{min})} g(v)dv + \int_{(v_{min},v_{max})} g(v)dv + \int_{(v_{max},+\infty)} g(v)dv$$

where $0 \leq v_{min} < v_{max}$ are chosen such that $|g|$ is small out of the interval (v_{min}, v_{max}). Then, we treat each of these integrals with adapted quadrature formulas, typically Gauss-Legendre and Gauss-Laguerre quadratures. This splitting strategy is also convenient when the integrand is not smooth, typically if g is piecewise defined, it suffices to split the integral conveniently, so that a possible loss of precision due to a loss of regularity is avoided.

It is straight from the derivation of our problem, that before employing a numerical method to solve the non linear Poisson equation, one has to compute the wall potential solution of the non linear equation (32). The numerical method consists in two main steps : a preprocessing step and a solving step.

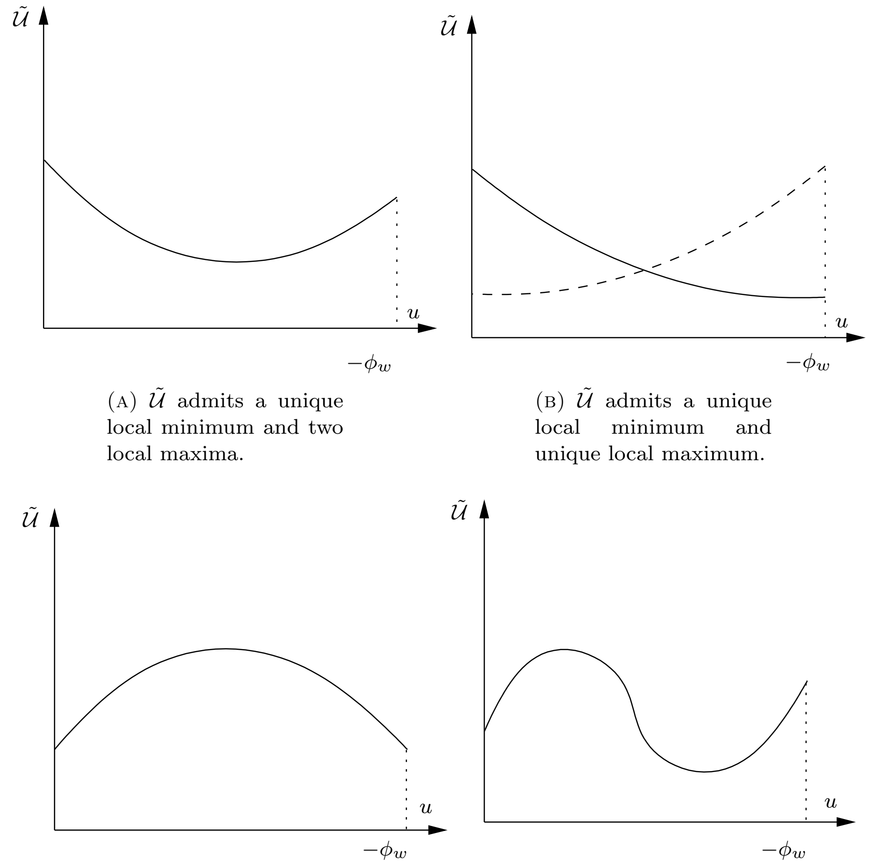
Pre-processing step. Consider $\rho_0 \in \mathbb{R}$, $\alpha \in [0, 1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. We use a standard Newton method to solve numerically the non linear equation (32). More precisely, we define $\tilde{\mathcal{W}}(\psi) = \mathcal{W}(\psi) - \dfrac{\sqrt{2\pi}}{1-\alpha} \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv$ for all $\psi \leq 0$. We then choose $\delta > 0$ and compute iteratively $(\phi_w^n)_{n\in\mathbb{N}}$ as follows

$$\begin{cases} \phi_w^0 \leq 0 \text{ well chosen} \\ \phi_w^{n+1} = \phi_w^n - \dfrac{\tilde{\mathcal{W}}(\phi_w^n)}{\tilde{\mathcal{W}}'(\phi_w^n)} \quad n \in \mathbb{N} \end{cases}$$

and stop as $\left|\tilde{\mathcal{W}}(\phi_w^n)\right| < \delta$.

Solving step. For an implementation reason, we lift the boundary condition and define $\bar{\phi} = \phi - x\phi_w$. For $\varepsilon > 0$, we solve the equivalent to (NLP-M) Poisson problem

$$\begin{cases} -\dfrac{\varepsilon^2}{2} \dfrac{d}{dx^2} \bar{\phi}(x) = -\dfrac{d}{d\psi} \bar{\mathcal{U}}(x, \bar{\phi}(x)) \quad \forall x \in (0,1) \\ \bar{\phi}(0) = 0 \text{ and } \bar{\phi}(1) = 0. \end{cases}$$



(A) $\tilde{\mathcal{U}}$ admits a unique local minimum and two local maxima.

(B) $\tilde{\mathcal{U}}$ admits a unique local minimum and unique local maximum.

(C) $\tilde{\mathcal{U}}$ has a unique local maximum and two local minima.

(D) $\tilde{\mathcal{U}}$ has one local maxima and one local minima in $(0, -\phi_w)$. It also has one local maximum at $u = -\phi_w$ and one local minimum at $u = 0$.

FIGURE 5. Plots of all possible behavior of the function $u \in [0, -\phi_w] \mapsto \tilde{\mathcal{U}}(u)$. The function studied in the scope of theorem 3.2 corresponds to the dashed line of Figure (B).

where $\bar{\mathcal{U}}(x, \psi) := \mathcal{U}(\psi + x\phi_w)$ for all $x \in [0,1]$ and $\psi \in [(1-x)\phi_w, -x\phi_w]$. We then look for $\bar{\phi}^* \in W = W(\rho_0, \alpha) := \left\{\bar{\phi} \in H_0^1(0,1) \mid (1-x)\phi_w \leq \bar{\phi} \leq -x\phi_w \text{ a.e in } (0,1)\right\}$ minimizing the functional

$$\bar{J}(\bar{\phi}) = \int_0^1 \left(\frac{\varepsilon^2}{2}|\bar{\phi}'(x)|^2 + \bar{\mathcal{U}}(x, \bar{\phi}(x))\right) dx. \tag{68}$$

Let $N \in \mathbb{N}^*$, the discretization consists of a mesh $(x_i := \frac{i}{(N+1)})_{i=0,\dots,N+1}$ with $h = \frac{1}{N+1}$ and the approximation of the Hilbert space $H_0^1(0,1)$ by a standard and conformous $\mathbb{P}_1$ finite element space V_0^h. More precisely

$$V_0^h := \left\{ \bar\phi_h \in C^0[0,1], \quad \bar\phi_h(0) = \bar\phi_h(1) = 0 \mid \forall i = 0, ..., N \quad \bar\phi_h \underset{|[x_i,x_{i+1}]}{\in} \mathbb{P}_1, \right\} \tag{69}$$

and the admissible potential set is approximated by

$$W^h := \left\{ \bar\phi_h \in V_0^h \mid (1-x)\phi_w \le \bar\phi_h \le -x\phi_w \forall x \in [0,1] \right\}. \tag{70}$$

We finally solve the minimization problem associated with $\bar J$ using a fixed step gradient algorithm. Namely, given $\eta > 0$ and $\delta > 0$, we compute iteratively

$$\begin{cases} \bar\phi_h^0 \in W^h \\ \bar\phi_h^{n+1} = \bar\phi_h^n - \eta \nabla \bar J(\bar\phi_h^n) \end{cases}$$

and stops when $\|\nabla \bar J(\bar\phi_h^n)\|_{H_0^1(0,1)} < \delta$. We have denoted $\nabla \bar J(\bar\phi_h^n) \in V_0^h$ the gradient of $\bar J$ at $\bar\phi_h^n$. It is the unique solution of the variational problem
$(\nabla \bar J(\bar\phi_h^n), \psi_h)_{H_0^1} = d\bar J(\bar\phi_h^n)(\psi_h)$ for all $\psi_h \in V_0^h$ where $(.,.)_{H_0^1}$ is the $H_0^1(0,1)$ inner product and

$$d\bar J(\bar\phi_h^n)(\psi_h) = \int_0^1 \left(\varepsilon^2 \frac{d}{dx}\bar\phi_h^n(x) \frac{d}{dx}\psi_h(x) + \frac{d}{d\psi}\bar{\mathcal{U}}(x, \bar\phi_h^n(x))\psi_h(x) \right) dx.$$

4.2. **Numerical results.** We carry out two numerical experiments. In the first one we perform numerical simulations that are in the scope of Theorem 3.2, that is in the case of a satisfied kinetic Bohm criterion. For these simulations we vary the parameters ε and α. In the second one, we perform numerical simulations with fixed values of ε and α but with an incoming ion boundary condition that violates the kinetic Bohm criterion (45). The data are :

- We set the mass ratio $\dfrac{m_e}{m_i} = \dfrac{1}{3672}$ for a Deuterium plasma. It results in $\alpha_c \approx 0.95$.
- We choose $\alpha \in [0, \alpha_c]$ and $f_i^{in}(v) = \min(1, \dfrac{v^2}{\eta}) \dfrac{e^{\frac{-(v-Z)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$ for all $v > 0$, where η is a small parameter, $\sigma^2 = \frac{T_i}{T_e}$ is the temperature ratio and Z is a macroscopic velocity adjusted with respect to the kinetic Bohm criterion (45). In the following simulations $\eta = 10^{-1}$ and $\sigma = \frac{1}{2}$. We also remember that f_e^{in} is given in (25).
- We set the neutrality $\rho_0 = 0$.
- We choose a mesh size $h = 2^{-11}$ and a tolerance parameter for our gradient algorithm $\delta = 10^{-6}$.

4.2.1. *The case of a satisfied Bohm criterion.* In this part we present the numerical solutions we obtained with a fixed value of Z chosen equal to $\frac{3}{2}$. The moments are computed numerically and we obtain :

$$\int_{\mathbb{R}^+} f_i^{in}(v)dv \approx 0.99, \quad \int_{\mathbb{R}^+} f_i^{in}(v)v dv \approx 1.4, \text{ and } \int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv \approx 0.74.$$

We can check numerically that both the admissibility condition (34) and the kinetic Bohm criterion (45) are satisfied. In figure 6 we have represented the ion incoming boundary condition f_i^{in}. We are now going to illustrate the behavior of the solution with respect to ε and α. We know from Theorem 3.2 that $n_i(x) \ge n_e(x)$ for all $0 \le x \le 1$. The general intuition is that when $\varepsilon > 0$ is small, one would expect n_i

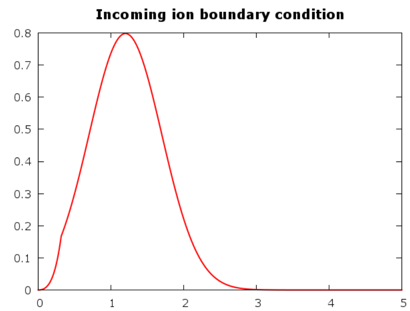


FIGURE 6. Plot of the incoming ion boundary condition for positive velocities with $Z = \frac{3}{2}$.

to be very close to n_e and ϕ_ε to be almost linear over some interval $[0, x^*(\varepsilon)]$ with $x^*(\varepsilon) > 0$. Then because of the potential drop the difference $n_i - n_e$ must become larger and larger as we approach the wall.
Case $\alpha = 0$ and varying ε. We fix the re-emission coefficient $\alpha = 0$, the results are presented in figures 7,8,9, 10 and 11. The a priori bound (35) on ϕ_w gives $\phi_w \geq -2.80$ and the numerically computed wall potential is $\phi_w \approx -2.78$. The electron reference density is $n_0 \approx 0.50$. If figure 7 we have represented the graph of $\mathcal{U}$ over its definition domain $[\phi_w, 0]$. In agreement with the theory it is a decreasing function. For the data we have chosen it also seems to be convex, however notice that from its expression (37) it is not straightforward.

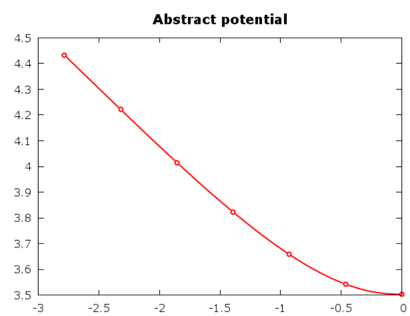
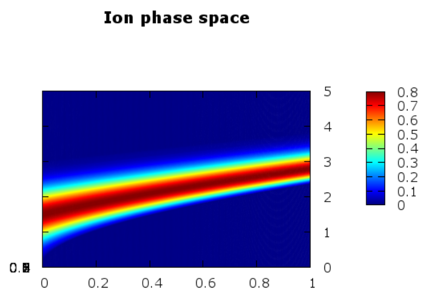
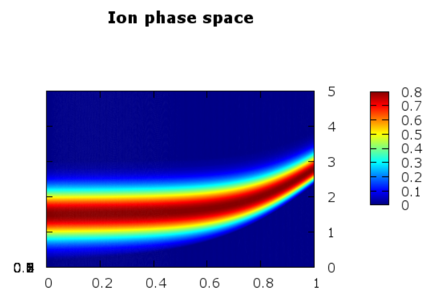


FIGURE 7. Plot of the potential function $\mathcal{U}$ over $[\phi_w, 0]$ for $\alpha = 0$.

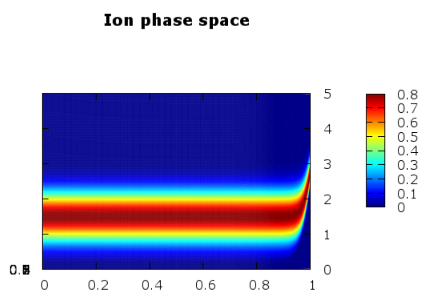
Figures 8,9, 10 and 11 represent the computed solution : f_i^ε, f_e^ε, n_i^ε, n_e^ε, ϕ_ε and $u_i^\varepsilon = \dfrac{\gamma_i^\varepsilon}{n_i^\varepsilon}$ for $\varepsilon \in \{1, 0.1, 0.01\}$.



(A) $\varepsilon = 1.0$



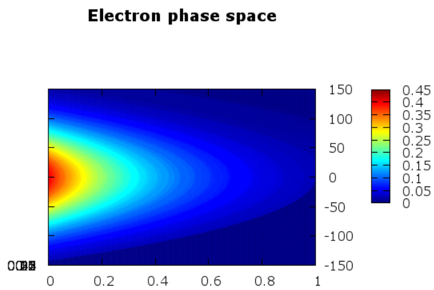
(B) $\varepsilon = 0.1$



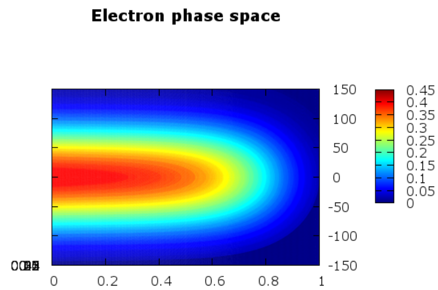
(C) $\varepsilon = 0.01$

FIGURE 8. Ion distribution functions in the phase space for various ε. Plot (a),(b) and (c) are represented in the phase space $[0,1] \times [0,5]$.

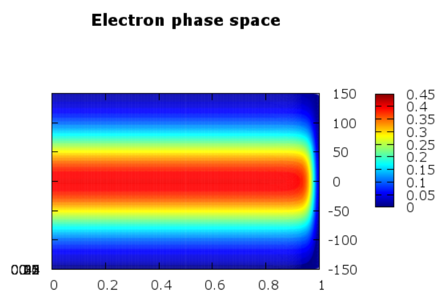
In addition, we observe that when ε is small a sheath of length of the order of ε develops near the wall and the sheath-edge denoted x^* varies with ε. In the simulations the sheath-edge corresponds to the point where approximately $|\phi_\varepsilon(x^*)| > 10^{-4}|\phi_w|$. We have $x^*(1) \approx 0$, $x^*(0.1) \approx 0.5$, $x^*(0.01) \approx 0.95$. For $x > x^*(\varepsilon)$, the plasma is significantly positively charged and there is a non negligible electric field that accelerates ions and decelerates electrons. Sufficiently fast electrons reach the wall and are absorbed. For $x \leq x_\varepsilon$, the plasma is almost neutral and there is no appreciable electric field, particles have constant velocities. These results are in good agreement with the physics, and confirms the commonly made assumption of semi-Maxwellian electron distribution function at the wall, see for example [17].



(A) $\varepsilon = 1.0$



(B) $\varepsilon = 0.1$



(C) $\varepsilon = 0.01$

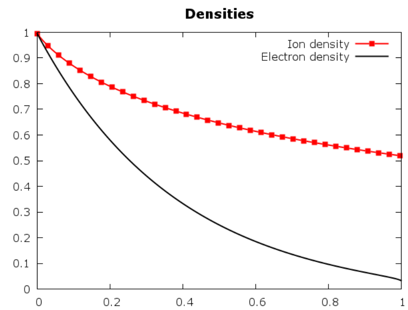
FIGURE 9. Electron distribution functions in the phase space for various ε. Plot (a),(b) and (c) are represented in the phase space $[0, 1] \times [-150, 150]$.

Case $\varepsilon = 0.1$ and varying α. We fix $\varepsilon = 0.1$ and vary $\alpha \in \{0, 0.5, 0.9\}$. The results are qualitatively the same with the difference that some electrons are re-emitted with negative velocities, therefore we decide only to plot the electron and ion densities n_e^α and n_i^α, see figure 12. We also gather the different values of the wall potential ϕ_w and the electron reference density n_0 with respect to α in the table 1. The ion

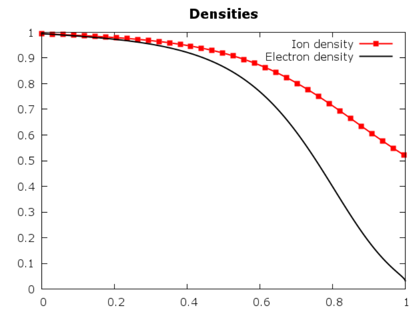
α	ϕ_w	n_0
0	-2.7	0.5
0.5	-2.1	0.5
0.9	-0.48	0.5

TABLE 1. Values of the wall potential and the reference density for various values of α.

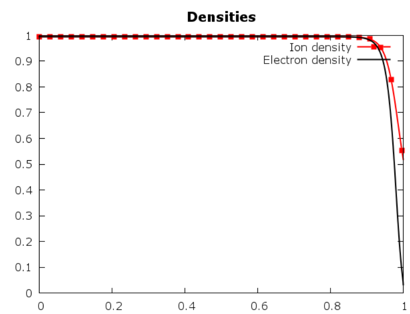
and electron densities seem to be respectively increasing functions of α.



(A) $\varepsilon = 1.0$

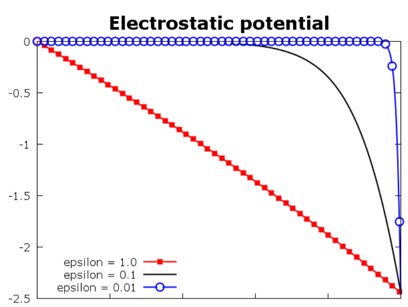


(B) $\varepsilon = 0.1$

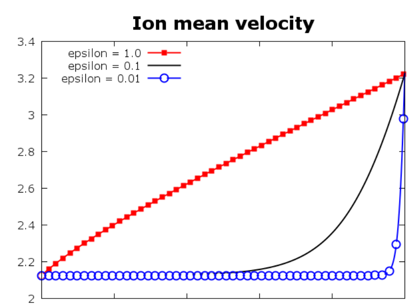


(C) $\varepsilon = 0.01$

FIGURE 10. Ion and electron densities in space for various ε. Plot (a),(b),(c) are represented in the space $[0, 1]$.



(A) Electrostatic potential in space for various ε.



(B) Ion mean velocity in space for various ε.

FIGURE 11. Electrostatic potential and ion mean velocity in the domain $[0, 1]$ for various values of ε.

4.2.2. *The case of a violated Bohm criterion.* We present numerical results when the kinetic Bohm criterion (45) is not satisfied. We mention that we are not in the scope of Theorem 3.2, thus we cannot ensure the existence and the uniqueness of a

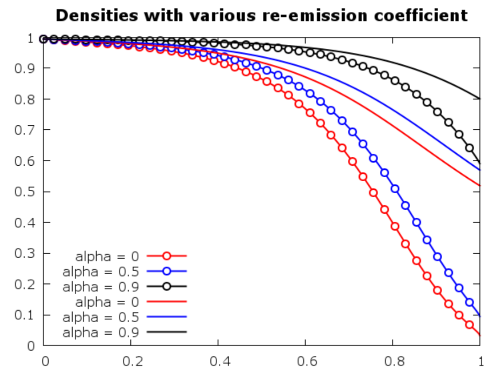


FIGURE 12. Ion and electron densities for $\varepsilon = 0.1$ and $\alpha \in \{0, 0.5, 0.9\}$. Pointed lines correspond to electron densities and thick lines to ion densities.

solution to (NLP-M). However, we can still minimize the functional J_ε and check a posteriori that the (numerically) computed minimizer is indeed a solution to (NLP-M). Consequently, for this numerical experiment we fix $\varepsilon = 0.01$, $\alpha = 0$. We choose $Z = 0.5$, for this value of Z the ion incoming boundary condition does not satisfy the Bohm criterion (45). Consequently, we know a priori that $\mathcal{U}''(0) < 0$ and thus the potential function $\mathcal{U}$ is locally concave near $\phi = 0$. In addition, because the slope at $\phi = 0$ is $\mathcal{U}'(0) = -\rho_0 = 0$ the potential function $\mathcal{U}$ is locally increasing near $\phi = 0$. Therefore the profile of $\mathcal{U}$ corresponds to one of the figure 5-(A). In figure 13 we have represented the function $\mathcal{U}$ over its domain of definition. The

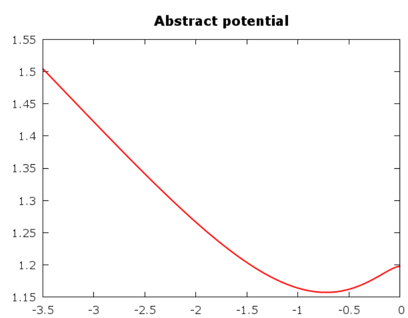
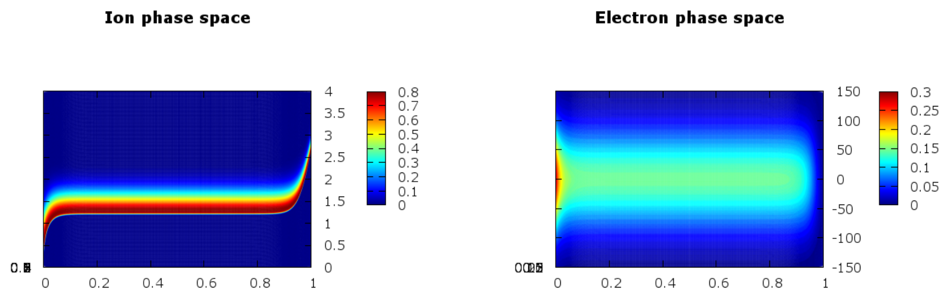


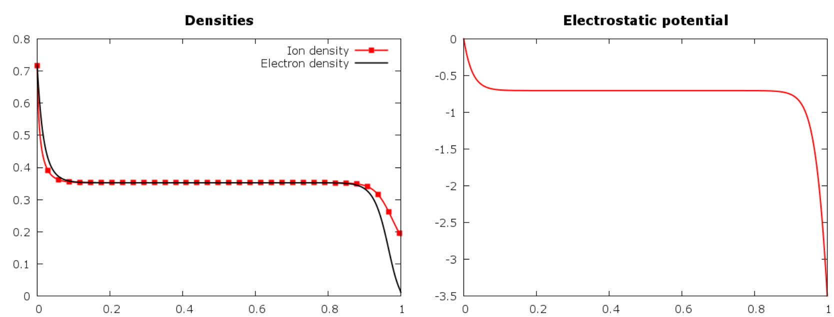
FIGURE 13. Plot of the potential function $\mathcal{U}$ over $[\phi_w, 0]$ for $\alpha = 0$.

figure 14 represents respectively the ion and electron distribution function in the phase space. The figure 15 represents respectively the macroscopic densities and the electrostatic potential. We see that when the Bohm criterion is violated, there is two boundary layer, one is at $x = 0$ and the other one is at $x = 1$. The charge density is negative near $x = 0$ while it is positive near $x = 1$. The physical interpretation of this numerical experiment is not obvious. However we mention that since the point $x = 0$ is assumed to represent a position somewhere in the (bulk) plasma, it seems to us that the boundary layer at $x = 0$ is unphysical. Consequently, this unphysical effect shows a limitation of our model. Lastly, we mention that for other values of



(A) Ion phase space (B) Electron phase space

FIGURE 14. Ion and electron distribution functions in the phase space. Plot (a) is represented in the phase space $[0,1] \times [0,5]$ while plot (b) is represented in the phase space $[0,1] \times [-150,150]$.



(A) Ion and electron densities in space. (B) Electrostatic potential in space.

FIGURE 15. Macroscopic densities and electrostatic potential in space. Plot (a),(b) are represented in the space $[0,1]$.

the macroscopic velocity Z which do not ensure the kinetic Bohm criterion (45) to be satisfied, the results are qualitatively the same.

5. **Conclusion.** We have proposed and studied a stationary and one dimensional plasma-wall interaction model, based on a bi-kinetic description of ions and electrons. Due to the presence of the wall, the electron phase space density is represented by a truncated Maxwellian distribution. As for the ions, our model supports a large class of incoming velocity distributions f_i^{in} and we have shown that it is well posed under a moment condition on f_i^{in} which generalizes the usual Bohm criterion. Furthermore, we have identified a second condition that must be satisfied by f_i^{in} for the wall potential to be well-defined. Surprisingly enough, this second condition takes the form of an upper bound on the average velocity of the incoming ions but thanks to the large mass ratio $\dfrac{m_i}{m_e}$ we have verified that it is not in

contradiction with the Bohm criterion. Our proof relies on a reformulation of the Vlasov-Poisson system into a non linear Poisson equation that we next study as a minimization problem. This approach also provides us with quantitative estimates for the boundary layer.

A physically based sheath problem was then illustrated with numerical simulations. Results show that when the neutrality is assumed at $x = 0$ and when the incoming ion distribution is admissible and satisfies the kinetic Bohm criterion, then for a vanishing normalized Debye length ε a sheath of length of the order of ε develops at the wall. Out of the sheath the plasma is almost neutral while in the sheath it is not, ions are accelerated and electrons decelerated. These results provide a strong numerical evidence for Theorem 3.2 and they are in good agreement with the simulations presented in [11]. We should add that this work takes full advantage of the one dimensional structure of our model. Although elementary, we hope this approach to be generic enough to be used in more general cases, including additional physics such as collision operators and magnetic fields [17, 14].

A. **Appendix.** Some fundamental results are reminded here for the self consistency of this work.

Definition A.1 (Weak solutions)**.**
1. Let $f_i^{in} \in \mathcal{I}$ and $f_i \in (L^1 \cap L^\infty)([0,1] \times \mathbb{R}; \mathbb{R}^+)$. We say that f_i is a weak solution of the Vlasov equation (13) iff for all $\varphi \in C_c^1([0,1] \times \mathbb{R})$ such that $\varphi(0,v) = 0$ for $v \leq 0$ and $\varphi(1,v) = 0$ for $v \geq 0$
$$\int_0^1 \int_{\mathbb{R}} f_i(x,v)\Phi(x,v)dvdx = \int_{\mathbb{R}^+} f_i^{in}(v)v\varphi(0,v)dv$$
where Φ is defined by $\Phi(x,v) = -v\partial_x\varphi(x,v) + \frac{d}{dx}\phi(x)\partial_v\varphi(x,v)$.
2. Let $\alpha \in [0,1]$, $f_e^{in} \in \mathcal{I}$ and $f_e \in (L^1 \cap L^\infty)([0,1] \times \mathbb{R}; \mathbb{R}^+)$. We say that f_e is a weak solution of the Vlasov equation (12) iff for all $\varphi \in C_c^1([0,1] \times \mathbb{R})$ such that $\varphi(0,v) = 0$ for $v \leq 0$ and $\varphi(1,v) = -\alpha\varphi(1,-v)$ for $v \geq 0$
$$\int_0^1 \int_{\mathbb{R}} f_e(x,v)\Phi(x,v)dvdx = -\int_{\mathbb{R}^+} f_e^{in}(v)v\varphi(0,v)dv$$
where Φ is defined by $\Phi(x,v) = v\partial_x\varphi(x,v) + \frac{m_i}{m_e}\frac{d}{dx}\phi(x)\partial_v\varphi(x,v)$.

Theorem A.2 (p. 135 [13])**.** *Let X be a reflexive Banach space, C a closed convex subset of X and $F : C \to \mathbb{R}$ a map. Moreover, assume*
1. *F is coercive, i.e $F(x) \to +\infty$ as $\|x\| \to +\infty$.*
2. *F is (sequentially) weakly lower semicontinuous, i.e for any sequences $(x_n)_{n\in\mathbb{N}} \subset C$ which converges to $x \in C$ for the weak topology, one has $x_n \rightharpoonup x \Rightarrow F(x) \leq \liminf F(x_n)$.*

then there exists $u \in C$ such that $F(u) := \inf_{v\in C} F(v)$.

Lemma A.3. *Let X and Y be two Banach spaces. Suppose X reflexive and $F : X \to Y$ is a compact mapping, then F is (sequentially) weakly-lower semi-continuous.*

The theory of Nemytskii operators provides continuity and differentiability results for some functional operators, see [2].

Definition A.4. Let be I a nonempty interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. The Nemytskii operator associated with f is the map which associates to any measurable function $u : (0,1) \to I$ the function $v := T_f(u)$ defined by $v(x) = f(u(x))$ for all $x \in (0,1)$.

Theorem A.5. *Let be I a nonempty interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a continuous function over I then the Nemytskii operator associated with f, $T_f : C^0([0,1]; I) \to C^0[0,1]$ is continuous from $C^0([0,1], I)$ to $C^0[0,1]$.*

Theorem A.6. *Let be I an nonempty interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a C^1-function over I then the Nemytskii operator associated with f, $T_f : C^0([0,1], I) \to C^0[0,1]$ is a C^1 mapping from $C^0([0,1], I)$ to $C^0[0,1]$ and its Fréchet derivative is given by*
$$dT_f(u)v = f'(u)v \quad \forall (u,v) \in C^0([0,1], I) \times C^0[0,1].$$

REFERENCES

[1] R.J. Diperna and P.L Lions, Ordinary differential equations, transport theory and Sobolev spaces *Inventiones mathematicae,1989.*

[2] J. Apell, The superposition operator in function spaces. A survey, *Expositiones Mathematicae, 1988.*

[3] A. Ambroso, X. Fleury, B. Lucquin-Desreux and P.A Raviart, Some remarks on a stationary Vlasov-Poisson system with source term arising in ion beam neutralization, *Centre de mathématiques appliquées R.I. Number 438, Janvier 2000*
[4] S. Baalrud and C. Hegna, Kinetic Theory of the Presheath and the Bohm Criterion, *Arxiv, 2013.*
[5] H. Beresticky and T. Lachant-Robert, Some properties of monotone rearrangement with applications to elliptic equations in cylinders, *Math. Nachr., 266: 3-19, 2004*
[6] D. Bohm, The characteristics of electrical discharges in magnetic fields, *New York: Mc Graw Hill,* ***Chap*** *3, 1949*
[7] R. Chalise and R. Khanal, A kinetic trajectory simulation for magnetized plasma sheath, *Institute of Physics Publishing, 2012.*
[8] F. Chen, Introduction to Plasma Physics, *Plenum press, 1974.*
[9] L.C Evans, Partial differential equations, *American Mathematical Society, Graduate Studies in Mathematics Volume 19, 2010.*
[10] M. Feldman, S.Y. HA and M. Slemrod, A Geometric level-set formulation of a plasma sheath interface, *Arch. Rat. Mech. Anal. 178(1), 81-123 2005.*
[11] Y. Güçlü, The plasma sheath for 1D-1V Vlasov-Poisson solvers, *Invited talk at the Numkin 2013 Conference, Garching, Germany,* `http://www.ipp.mpg.de/1525371/Gueclue.pdf`.
[12] Y. Guo, C-H. Shu, T. ZIE The dynamics of a plane diode SIAM J. MATH. ANAL. Vol. 35, No. 6
[13] O. Kavian, Introduction à la Théorie des Points critiques et applications aux problèmes Elliptiques, *Springer-Verlage, 1993.*
[14] H. Kohno, J. R. Myra and D. A. D'Ippolito, Radio-frequency sheath-plasma interactions with magnetic field tangency points along the sheath surface, *Physics of Plasmas, 20, 082514, 2013.*
[15] J.G. Laframboise, Theory of spherical and cylindrical Langmuir probes in a collision less, Maxwellian plasma at rest, *Institute for Aerospace Studies, University of Toronto, Report No. 100, 1966.*
[16] P.-H. Maire, Établissement et comparaison de modèles fluides pour un plasma faiblement ionisé quasi-neutre. Détermination des conditions aux limites à la paroi, *Thèse de Doctorat (in French) de l'Université Paris 6, 1996.*
[17] G. Manfredi and S. Devaux, Magnetized plasma-wall transition. Consequences for wall sputtering and erosion, *Institute of Physics Publishing, 2008.*
[18] R.E. Marshak, The Variational Method for Asymptotic Neutron Densities, *Physical review, May 1947.*
[19] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, NIST Handbook of Mathematical functions, *National Institute of Standards and Technology, Cambridge University Press, 2010.*
[20] P.A. Raviart and C. Greengard, A Boundary-Value problem for the stationary Vlasov-Poisson equations : The Plane Diode, *Communications on Pure and Applied Mathematics, 1990.*
[21] K.-U. Riemann, The Bohm criterion and sheath formation *Phys. Plasmas, 1991.*
[22] Terrence E. Sheridan, Solution of the plasma-sheath equation with a cool Maxwellian ion source, *AIP Publishing, 2001.*
[23] G. Stampachia and D. Kinderlehrer, An introduction to variational inequalities and their applications, *Academic Press, 1980.*
[24] P. Stangeby, The Plasma Boundary of Magnetic Fusion Devices, *Institute of Physics Publishing, 2000.*
[25] F. Valsaque and G. Manfredi, Numerical study of plasma wall transition in an oblique magnetic field, *Journal of Nuclear Materials, 2001.*

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: badsi@ljll.math.upmc.fr
E-mail address: campos@ljll.math.upmc.fr
E-mail address: despres@ljll.math.upmc.fr