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HOLDERIAN WEAK INVARIANCE PRINCIPLE FOR STATIONARY MIXING SEQUENCES

DAVIDE GIRAUDO

ABSTRACT. We provide some sufficient mixing conditions on a strictly stationary sequence in order to guarantee the weak invariance principle in Hölder spaces. Strong mixing, ρ -mixing and interlaced ρ -mixing conditions are investigated as well as τ -dependent sequences. The main tools are moment and Fuk-Nagaev type inequalities for mixing sequences and a truncation argument.

1. INTRODUCTION

1.1. Context and notations. Let $(X_j)_{j \geq 0}$ be a strictly stationary sequence of real valued random variables with zero mean and finite variance, and for an integer $n \geq 1$, $S_n := \sum_{j=1}^n X_j$ denotes the n -th partial sum. Its variance is denoted by σ_n^2 . Let us consider the partial sum process

$$(1) \quad S_n^{\text{pl}}(t) := \sum_{j=1}^{\lfloor nt \rfloor} X_j + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1}, \quad n \geq 1, t \in [0, 1].$$

We are interested in the asymptotic behavior of $\sigma_n^{-1} S_n^{\text{pl}}(\cdot)$ viewed as a random function in some function spaces.

Notation 1.1. If $T: \Omega \rightarrow \Omega$ is a bi-measurable measure preserving map, we define for $f: \Omega \rightarrow \mathbb{R}$ and a positive integer n the n th partial sum $S_n(f) := \sum_{j=1}^n f \circ T^j$ and $\sigma_n^2(f) := \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$ denotes its variance. We shall also consider $S_n^{\text{pl}}(f)$ defined in a similar way as in (1), that is,

$$(2) \quad S_n^{\text{pl}}(f, t) := S_{\lfloor nt \rfloor}(f) + (nt - \lfloor nt \rfloor) f \circ T^{\lfloor nt \rfloor + 1}.$$

In all the paper, the involved sequences of random variable are assumed to be strictly stationary.

When $(X_j)_{j \geq 0}$ is an independent identically distributed sequence, Donsker showed (cf. [Don51]) that $(n^{-1/2}(\mathbb{E}[X_1^2])^{-1/2} S_n^{\text{pl}})_{n \geq 1}$ converges in distribution in the space of continuous functions on the unit interval to a standard Brownian motion W . An intensive research has then been performed to extend this result to stationary weakly dependent sequences. We refer the reader to [MPU06] for the main theorems in this area.

In this paper, we rather focus on the convergence in distribution of the partial sum in other function spaces.

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1.2. Hölder spaces. It is well known that standard Brownian motion's paths are almost surely Hölder regular of exponent α for each $\alpha \in (0, 1/2)$, hence it is natural to consider the random function defined in (2) as an element of $\mathcal{H}_\alpha[0, 1]$ and try to establish its weak convergence to a standard Brownian motion in this function space.

Before stating the results in this direction, let us define for $\alpha \in (0, 1)$ the Hölder space $\mathcal{H}_\alpha[0, 1]$ of functions $x: [0, 1] \rightarrow \mathbb{R}$ such that $\sup_{s \neq t} |x(s) - x(t)| / |s - t|^\alpha$ is finite. The analogue of the continuity modulus in $C[0, 1]$ is w_α , defined by

$$w_\alpha(x, \delta) = \sup_{0 < |t-s| < \delta} \frac{|x(t) - x(s)|}{|t - s|^\alpha}.$$

We then define $\mathcal{H}_\alpha^0[0, 1]$ by $\mathcal{H}_\alpha^0[0, 1] := \{x \in \mathcal{H}_\alpha[0, 1], \lim_{\delta \rightarrow 0} w_\alpha(x, \delta) = 0\}$. We shall essentially work with the space $\mathcal{H}_\alpha^0[0, 1]$ which, endowed with $\|\cdot\|_\alpha: x \mapsto w_\alpha(x, 1) + |x(0)|$, is a separable Banach space (while $\mathcal{H}_\alpha[0, 1]$ is not). Since the canonical embedding $\iota: \mathcal{H}_\alpha^0[0, 1] \rightarrow \mathcal{H}_\alpha[0, 1]$ is continuous, each convergence in distribution in $\mathcal{H}_\alpha^0[0, 1]$ also takes place in $\mathcal{H}_\alpha[0, 1]$.

Let us denote by D_j the set of dyadic numbers in $[0, 1]$ of level j , that is,

$$D_0 := \{0, 1\}, \quad D_j := \{(2l - 1)2^{-j}; 1 \leq l \leq 2^{j-1}\}, j \geq 1.$$

If $r \in D_j$ for some $j \geq 0$, we define $r^+ := r + 2^{-j}$ and $r^- := r - 2^{-j}$. For $r \in D_j$, $j \geq 1$, let Λ_r be the function whose graph is the polygonal path joining the points $(0, 0)$, $(r^-, 0)$, $(r, 1)$, $(r^+, 0)$ and $(1, 0)$. We can decompose each $x \in C[0, 1]$ as

$$x = \sum_{r \in D} \lambda_r(x) \Lambda_r = \sum_{j=0}^{+\infty} \sum_{r \in D_j} \lambda_r(x) \Lambda_r,$$

and the convergence is uniform on $[0, 1]$. The coefficients $\lambda_r(x)$ are given by

$$\lambda_r(x) = x(r) - \frac{x(r^+) - x(r^-)}{2}, \quad r \in D_j, j \geq 1,$$

and $\lambda_0(x) = x(0)$, $\lambda_1(x) = x(1)$.

Ciesielski proved (cf. [Cie60]) that $\{\Lambda_r; r \in D\}$ is a Schauder basis of $\mathcal{H}_\alpha^0[0, 1]$ and the norms $\|\cdot\|_\alpha$ and the sequential norm defined by

$$\|x\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(x)|,$$

are equivalent.

Considering the sequential norm, we can show that a sequence $(\xi_n)_{n \geq 1}$ of random elements of \mathcal{H}_α^0 vanishing at 0 is tight if and only if for each positive ε ,

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(\xi_n)| > \varepsilon \right\} = 0.$$

Notation 1.2. In the sequel, we will denote $r_{k,j} := k2^{-j}$ and $u_{k,j} := \lfloor nr_{k,j} \rfloor$ (or r_k and u_k for short). Notice that $u_{k+1,j} - u_{k,j} = \lfloor nr_{k,j} + n2^{-j} \rfloor - u_{k,j} \leq 2n2^{-j}$ if $j \leq \log n$, where $\log n$ denotes the binary logarithm of n and for a real number x , $\lfloor x \rfloor$ is the unique integer for which $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

The following is Theorem 14 in [Suq99]. It essentially shows that in terms of tightness, the role of w_α in $\mathcal{H}_\alpha^0[0, 1]$ is the same as those of the classical modulus of continuity in $C[0, 1]$.

Theorem 1.3. Let $\xi := (\xi_n)_{n \geq 1}$ be a sequence of random elements of $\mathcal{H}_\alpha^0[0, 1]$ where $0 < \alpha < 1$. The sequence ξ is tight in $\mathcal{H}_\alpha^0[0, 1]$ if and only if both conditions are fulfilled:

- a) $\lim_{A \rightarrow +\infty} \sup_{n \geq 1} \mu \{ \|\xi_n\|_\infty > A \} = 0$;
- b) for each positive ε and η , there exist $\delta \in (0, 1)$ and a positive integer n_0 such that for each $s \in [0, 1)$,

$$\frac{1}{\delta} \mu \left\{ \sup_{s < t \leq s + \delta} \frac{|\xi_n(t) - \xi_n(s)|}{(t - s)^\alpha} \geq \varepsilon \right\} \leq \eta, \quad n \geq n_0.$$

In the particular case $\xi_n = \sigma_n^{-1} S_n^{\text{pl}}(f)$ where $S_n^{\text{pl}}(f)$ is defined by (2) and $(f \circ T^j)_{j \geq 0}$ is strictly stationary, we can translate the conditions in terms of partial sums.

Corollary 1.4. Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence. Assume that for each positive ε and η , there exists $\delta \in (0, 1)$ and an integer n_0 such that

$$(3) \quad \mu \left\{ \max_{1 \leq i < j \leq [n\delta]} \frac{|S_j(f) - S_i(f)|}{(j - i)^\gamma} > \sigma_n n^{-\gamma} \varepsilon \right\} \leq \delta \eta, \quad n \geq n_0.$$

Then the sequence $(\sigma_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$ is tight in $\mathcal{H}_\gamma^0[0, 1]$.

We now state a tightness criterion for $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$ using the sequential norm. This relies on the estimate

$$\mu \left\{ \sup_{\log n \leq j} 2^{(1/2-1/p)j} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f \circ T^i \right| > n^{1/2} \varepsilon \right\} \leq C(p, \varepsilon) n \cdot \mu \left\{ |f| > n^{1/p} \varepsilon \right\},$$

valid for each positive integer n and each positive ε , and used by Račkauskas and Suquet in [RS03].

Proposition 1.5. Let $p > 2$ and let $(f \circ T^j)_{j \geq 0}$ be a sequence of centered random variables such that $\lim_{t \rightarrow +\infty} t^p \mu \{ |f| > t \} = 0$. Assume that for each positive ε ,

$$(4) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{J \leq j \leq \log n} 2^{(1/2-1/p)j} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f \circ T^i \right| > n^{1/2} \varepsilon \right\} = 0.$$

Then the sequence $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$ is tight in $\mathcal{H}_{1/2-1/p}^0[0, 1]$.

As mentioned before, the random function defined in (1) can be viewed as an element of $\mathcal{H}_\alpha[0, 1]$, $\alpha \in (0, 1/2)$ and we can try to establish the weak convergence of the sequence $(\sigma_n^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$ to a standard Brownian motion in this function space. To the best of our knowledge, it seems that the study of this kind of convergence was not as intensive as in the space of continuous functions or the Skorohod space. The first result in this direction was established by Lamperti in [Lam62]: if $(X_j)_{j \geq 0}$ is an i.i.d. sequence with $\mathbb{E}[X_0] = 0$, $\mathbb{E}[X_0^2] = 1$ and $\mathbb{E}|X_0|^p$ is finite, then the sequence $(n^{-1/2} S_n^{\text{pl}})_{n \geq 1}$ converges to a standard Brownian motion in $\mathcal{H}_\gamma^0[0, 1]$ for each $\gamma < 1/2 - 1/p$. Later, Račkauskas and Suquet improved this result (cf. [RS03]), showing that for an i.i.d. zero mean sequence, a necessary and sufficient condition to obtain the invariance principle in $\mathcal{H}_{1/2-1/p}^0[0, 1]$ is $\lim_{t \rightarrow \infty} t^p \mu \{ |X_0| > t \} = 0$ (in [RS04] they considered the case of more general Hölder spaces, where the role of $t \mapsto t^\alpha$ is played by $t \mapsto t^\alpha L(t)$ with some conditions on L).

Thus, establishing the weak convergence of the partial sum process in Hölder spaces requires, even in the independent case, finite moment of order greater than 2 and the moment condition depends on the exponent of the considered Hölder space. It is a natural question to ask about generalizations of the result by Račkauskas and Suquet for dependent sequences. In this paper, we focus on strictly stationary sequences satisfying some mixing conditions (see next section).

1.3. Mixing conditions. We present the mixing conditions involved in the paper.

Let \mathcal{A} and \mathcal{B} be two sub- σ -algebras of \mathcal{F} , where $(\Omega, \mathcal{F}, \mu)$ is a probability space. We define the α -mixing coefficients as introduced by Rosenblatt in [Ros56]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \}.$$

The ρ -mixing coefficients were introduced by Hirschfeld [Hir35] and are defined by

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \{ |\text{Corr}(f, g)|, f \in \mathbb{L}^2(\mathcal{A}), g \in \mathbb{L}^2(\mathcal{B}), f \neq 0, g \neq 0 \},$$

where $\text{Corr}(f, g) := [\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)] [\|f - \mathbb{E}(f)\|_{\mathbb{L}^2} \|g - \mathbb{E}(g)\|_{\mathbb{L}^2}]^{-1}$.

The coefficients are related by the inequalities

$$(5) \quad 4\alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}).$$

For a strictly stationary sequence $(X_k)_{k \in \mathbb{Z}}$ and $n \geq 0$ we define $\alpha_X(n) = \alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$ where \mathcal{F}_u^v is the σ -algebra generated by X_k with $u \leq k \leq v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict). In the same way we define coefficients $\rho_X(n)$.

If I and J are two subsets of \mathbb{Z} , let $d(I, J)$ denote the infimum of the quantities $|i - j|$, $i \in I, j \in J$. Stein introduced in [Ste72] the interlaced ρ -mixing coefficients, denoted by $\rho_X^*(n)$ and defined by

$$\rho_X^*(n) := \sup \{ \rho(\sigma(X_i, i \in I), \sigma(X_j, j \in J)), d(I, J) \geq n \}.$$

When there will be no ambiguity, we shall simply write $\alpha(n)$, $\rho(n)$ and $\rho^*(n)$ for simplicity.

We say that the sequence $(X_k)_{k \in \mathbb{Z}}$ is α -mixing if $\lim_{n \rightarrow +\infty} \alpha(n) = 0$, and similarly we define ρ -mixing, ρ^* -mixing sequences.

α -mixing sequences were considered in the mentioned references, while ρ -mixing sequences first appeared in [KR60]. Inequality (5) translated in terms of mixing coefficients of a sequence states that for each positive integer n ,

$$4\alpha(n) \leq \rho(n) \leq \rho^*(n).$$

In particular, a ρ^* -mixing sequence is ρ -mixing, and a ρ -mixing sequence is α -mixing. In [Bra01], Bradley showed that given a sequence of real numbers $(\delta_n)_{n \geq 1}$ decreasing to 0, there exists a Markov chain $(X_k)_{k \geq 0}$ for which $\rho(n) \leq \delta_n$ and $\rho^*(n) = 1$ for each $n \geq 1$.

1.4. τ -dependence coefficient. In order to define the τ -dependence coefficients of a stationary sequence, we first need a result about conditional probability (see Theorem 33.3 of [Bil95]).

Lemma 1.6. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, \mathcal{M} a sub- σ -algebra of \mathcal{F} and X a real-valued random variable with distribution μ_X . There exists a function $\mu_{X|\mathcal{M}}$ from $\mathcal{B}(\mathbb{R}) \times \Omega$ to $[0, 1]$ such that*

- (1) *For any $\omega \in \Omega$, $\mu_{X|\mathcal{M}}(\cdot, \omega)$ is a probability measure on $\mathcal{B}(\mathbb{R})$.*
- (2) *For any $A \in \mathcal{B}(\mathbb{R})$, $\mu_{X|\mathcal{M}}(A, \cdot)$ is a version of $\mathbb{E}[\chi_{\{X \in A\}} | \mathcal{M}]$.*

We now introduce the τ -dependence coefficients as in [DP05]. We denote by $\Lambda_1(\mathbb{R})$ the collection of 1-Lipschitz functions from \mathbb{R} to \mathbb{R} and define the quantity

$$W(\mu_{X|\mathcal{M}}) := \sup \left\{ \left| \int f(x) \mu_{X|\mathcal{M}}(dx) - \int f(x) \mu_X(dx) \right|, f \in \Lambda_1(\mathbb{R}) \right\}.$$

For an integrable random variable X and a sub- σ -algebra \mathcal{M} , the coefficient τ is defined by

$$(6) \quad \tau(\mathcal{M}, X) = \|W(\mu_{X|\mathcal{M}})\|_1.$$

This definition can be extended to random variables with values in finite dimensional vector spaces. If d is a positive integer, we endow \mathbb{R}^d with the norm $\|x - y\| := \sum_{j=1}^d |x_j - y_j|$ and define $\Lambda_1(\mathbb{R}^d)$ as the set of 1-Lipschitz functions from \mathbb{R}^d to \mathbb{R} .

Definition 1.7. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, \mathcal{M} a sub- σ -algebra of \mathcal{F} and X an \mathbb{R}^d -valued random variable. We define

$$(7) \quad \tau(\mathcal{M}, X) := \sup \{ \tau(\mathcal{M}, f(X)), f \in \Lambda_1(\mathbb{R}^d) \}.$$

We can now introduce the τ -mixing coefficient for a sequence of real-valued random variables.

Definition 1.8. Let $(X_i)_{i \geq 1}$ be a sequence of random variables and $(\mathcal{M}_i)_{i \geq 1}$ a sequence of sub- σ -algebras of \mathcal{F} . For any positive integer k , define

$$(8) \quad \tau(i) := \max_{p, l \geq 1} \frac{1}{l} \sup \{ \tau(\mathcal{M}_p, (X_{j_1}, \dots, X_{j_l})), p + i \leq j_1 < \dots < j_l \}.$$

In the sequel, we shall focus on the case $\mathcal{M}_i := \sigma(X_k, k \leq i)$.

Notation 1.9. Let $X: \Omega \rightarrow \mathbb{R}$ be a random variable. We denote $Q_X(\cdot)$ the inverse function defined by $Q_X(u) := \inf \{ t, \mu \{ |X| > t \} \leq u \}$. If $(f \circ T^j)_{j \geq 0}$ is a strictly stationary sequence and $(\alpha(n))_{n \geq 1}$ is its sequence of α -mixing coefficients, we denote by $\alpha^{-1}(u)$ the number of indices n for which $\alpha(n) \geq u$. More generally, if $(\delta_i)_{i \geq 0}$ is a non-increasing sequence of non-negative numbers, we define $\delta^{-1}(u) := \inf \{ k \in \mathbb{N}, \delta_k \leq u \}$.

We can compare the τ -dependence coefficient with the α -mixing coefficients. The following is a simplified version of Lemma 7 of [DP04].

Lemma 1.10. Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence. Then for each integer i , the following inequality holds:

$$(9) \quad \tau(i) \leq 2 \int_0^{2\alpha(i)} Q_f(u) du.$$

In [DP04], "Application 1: causal linear processes" (p. 871), Dedecker and Prieur provide an example of a process whose τ -dependence coefficients converge to 0 as fast as 2^{-i} but $\alpha_i = 1/4$ for each positive integer i .

1.5. Some existing results on stationary mixing sequences. In this section, we review some results about sufficient mixing conditions for strictly stationary sequences which guarantee the weak invariance principle in the space $C[0, 1]$. They contain also results on the asymptotic behavior of the variance of the n th partial sum. We then state some concentration and moment inequalities for partial sums of stationary mixing sequences of random variables. In the most of original papers, the results are stated for non-stationary sequence. Since we only use them in the strictly stationary context, we state them accordingly.

Let us begin by the case of α -mixing random variables.

Doukhan, Massart and Rio have shown in [DMR95] a functional central limit theorem in $C[0, 1]$ for strictly stationary sequences satisfying the condition

$$(10) \quad \int_0^1 \alpha^{-1}(u) Q^2(u) du < \infty.$$

Notation 1.11. If $(f \circ T^j)_{j \geq 0}$ is a (strictly stationary) sequence of random variables, we define

$$(11) \quad s_N^2(f) := \sum_{i=1}^N \sum_{j=1}^N |\text{Cov}(f \circ T^i, f \circ T^j)|.$$

We state an extension of the Fuk-Nagaev inequality for sequences of α -mixing random variables (cf. Theorem 6.2 of [Rio00]).

Theorem 1.12. *Let $(f \circ T^j)_{j \geq 0}$ be a sequence of real-valued and centered random variables with finite variance. Set $R(u) = \alpha^{-1}(u) Q_f(u)$ and let $H(u) = R^{-1}(u)$ denote the generalized inverse function of R . Then, for any positive λ , any integer $N \geq 1$ and any $r \geq 1$,*

$$(12) \quad \mu \left\{ \max_{1 \leq k \leq N} |S_k(f)| \geq 4\lambda \right\} \leq 4 \left(1 + \frac{\lambda^2}{r s_N^2(f)} \right)^{-r/2} + 4N\lambda^{-1} \int_0^{H(\lambda/r)} Q_f(u) du.$$

The following is Theorem 6.3 in [Rio00].

Proposition 1.13. *Let $p \geq 2$ and $(f \circ T^j)_{j \geq 0}$ be a sequence of centered random variables. For each integer $N \geq 1$, the following inequality holds:*

$$(13) \quad \mathbb{E} \left[\max_{1 \leq k \leq N} |S_k(f)|^p \right] \leq a_p s_N(f)^p + N b_p \int_0^1 [\alpha_X^{-1}(u)]^{p-1} Q_f^p(u) du,$$

where a_p and b_p depend only on p .

The following result gives an upper bound for $s_N(f)$.

Proposition 1.14. *Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence of centered random variables which satisfies (10). Then:*

- the sequence $(\sigma_n^2(f)/n)_{n \geq 1}$ has a limit as n goes to infinity;
- for each positive integer N ,

$$(14) \quad s_N^2(f) \leq 8N \int_0^1 \alpha^{-1}(u) Q_f(u)^2 du.$$

The first assertion is contained in Corollary 1.2. of [Rio00] while the second one follows from Theorem 1.2 of [Rio93].

Notation 1.15. Let Y be an integrable random variable. We denote by G_Y the generalized inverse of $x \mapsto \int_0^x Q_Y(u) du$.

The following result is a version of Theorem 1.12 for τ -dependence sequences. It is a version of Theorem 2 of [DP04] for stationary sequences.

Theorem 1.16. *Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence of centered and square integrable random variables. Let $R := ((\tau/2)^{-1} \circ G_f^{-1}) Q_f$ and $S = R^{-1}$. For any $\lambda > 0$, any integer $N \geq 1$ and any $r \geq 1$,*

$$(15) \quad \mu \left\{ \max_{1 \leq j \leq N} |S_N(f)| \geq 5\lambda \right\} \leq 4 \left(1 + \frac{\lambda^2}{r s_N^2(f)} \right)^{-r/2} + \frac{4N}{\lambda} \int_0^{S(\lambda/r)} Q_f(u) du.$$

Integrating as done in [Rio00] p.88, we derive the following moment inequality.

Corollary 1.17. *Under the same assumptions as in Theorem 1.16, the following inequalities hold:*

$$(16) \quad \mathbb{E} \left[\max_{1 \leq j \leq N} |S_j|^p \right] \leq a_p s_N(f)^p + N b_p \int_0^{\|f\|_1} ((\tau/2)^{-1}(u))^{p-1} \circ G_f(u) du,$$

where a_p and b_p depend only on p and

$$(17) \quad s_N^2(f) \leq 4N \int_0^{\|f\|_1} (\tau/2)^{-1}(u)^{p-1} Q_f^{p-1} \circ G_f(u) du.$$

For ρ -mixing sequences, the following inequality is available.

Proposition 1.18 (Shao, [Sha95]). *Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence of centered random variables and $q \geq 2$.*

Then there exists a constant K depending only on q and the sequence $(\rho(n))_{n \geq 1}$ such that for each integer n ,

$$(18) \quad \mathbb{E} |S_n(f)|^q \leq K \cdot n^{q/2} \exp \left(K \sum_{i=0}^{\lfloor \log n \rfloor} \rho(2^i) \right) \|f\|_2^q \\ + K \cdot n \cdot \exp \left(K \sum_{i=0}^{\lfloor \log n \rfloor} \rho^{2/q}(2^i) \right) \|f\|_q^q.$$

Remark 1.19. If we use this inequality with the sequence $(h \circ f \circ T^j)_{j \geq 0}$ instead of $(f \circ T^j)_{j \geq 0}$ but the same q , the constant K' which will be involved in (18) will be smaller or equal to K .

Remark 1.20. In order to get a Rosenthal type inequality, we have to assume the convergence of the series $\sum_{i=0}^{\infty} \rho^{2/q}(2^i)$. It turns out that in our application, we will only need the convergence of $\sum_{i=0}^{\infty} \rho(2^i)$.

Let us recall Theorem 4.1. in [Pel82].

Theorem 1.21. *Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence of square integrable random variables. Suppose that the series $\sum_{i \geq 1} \rho(2^i)$ is convergent and $\lim_{n \rightarrow \infty} \sigma_n(f) = \infty$. Then the sequence $(\sigma_n^2(f)/n)_{n \geq 1}$ converges to a positive constant.*

Definition 1.22. *A sequence $(c_n)_{n \geq 1}$ is slowly varying if there exists a continuous function $h: \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ such that $c_n = h(n)$ for each positive integer n and for each positive x , $\lim_{t \rightarrow \infty} h(tx)/h(t) = 1$.*

Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence of centered random variables with $\mathbb{E}[f^2] < +\infty$ and $\rho(n) \rightarrow 0$. Convergence of finite dimensional distributions of $(\sigma_n(f)^{-1} S_n^{\text{pl}}(f))_{n \geq 1}$ and the equality $\sigma_n^2(f) = n \cdot h(n)$ with h slowly varying are established in Theorem 2.1 of [Ibr75].

To conclude this section, we consider ρ^* -mixing sequences.

Theorem 1.23. *Suppose that $(f \circ T^j)_{j \geq 0}$ is a strictly stationary ρ^* -mixing sequence of random variables such that $\mathbb{E}[f] = 0$, $0 < \mathbb{E}[f^2] < \infty$ and $\sigma_n(f) \rightarrow \infty$ as $n \rightarrow \infty$. Then $\sigma^2(f) = \lim_{n \rightarrow \infty} \sigma_n^2(f)/n$ exists in $(0, \infty)$ and $S_n(f)/(n^{1/2}\sigma)$ converges in distribution to a standard normal distribution as n goes to infinity.*

This can be seen from Theorems 1, 3 and 4 of [Bra92].

A Rosenthal-type inequality was established by Peligrad and Gut (cf. [PG99]).

Proposition 1.24. *Let $(f \circ T^j)_{j \geq 0}$ be a sequence of centered random variables satisfying $\mathbb{E}|f|^q < \infty$ for some $q \geq 2$. Suppose that $\rho^*(N) < 1$ for some $N \geq 1$. Then there exists a positive constant C depending only on N , $\rho^*(N)$ and q such that for each integer $n \geq 1$,*

$$(19) \quad \mathbb{E}|S_n(f)|^q \leq C \left(n\mathbb{E}|f|^q + n^{q/2} (\mathbb{E}[f]^2)^{q/2} \right).$$

2. MAIN RESULTS

In this section, we present the main results of the paper. The first one gives a sufficient condition for tightness of the partial sum process associated to a strictly stationary sequence. This condition is expressed in terms of moments of the partial sums.

A similar result was used by Lamperti in [Lam62] for a process with linear variance. We provide an extension to the case where the variance of the n th partial sum behaves like $n \cdot h(n)$ where h is slowly varying.

The condition is satisfied for example by ρ -mixing strictly stationary sequences with a moment condition for which the variance of the n th partial sum diverge to infinity.

This result allows to derive tightness in $\mathcal{H}_\gamma^0[0, 1]$ for each $\gamma < 1/2 - 1/p$, where p is such that $(n^{-1/2}S_n(f))_{n \geq 1}$ is bounded in \mathbb{L}^p . However, even for processes with linear variance, we cannot conclude tightness of $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$ in $\mathcal{H}_{1/2-1/p}^0[0, 1]$.

We then provide conditions in the spirit of Doukhan et al. for the invariance in $\mathcal{H}_{1/2-1/p}^0[0, 1]$. The cases of ρ -mixing with logarithmic mixing rate and interlaced mixing are investigated.

Theorem 2.1. *Let $p > 2$ and $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence such that $\mathbb{E}[f] = 0$. Suppose that*

- (1) *the finite dimensional distributions of $(\sigma_n^{-1}S_n^{\text{pl}}(f))_{n \geq 1}$ converge to those of a standard Brownian motion, and*
- (2) *the sequence $(\sigma_n^2(f)/n)_{n \geq 1}$ is slowly varying, and*
- (3) *the sequence $(\mathbb{E}|S_n(f)|^p / \sigma_n^p)_{n \geq 1}$ is bounded.*

Then for each $\alpha < 1/2 - 1/p$ the sequence $(\sigma_n^{-1}S_n^{\text{pl}}(f))_{n \geq 1}$ converges in distribution in $\mathcal{H}_\alpha^0[0, 1]$ to a standard Brownian motion.

Corollary 2.2. *Let $p > 2$, $(f \circ T^j)_{j \geq 0}$ be a zero-mean strictly stationary sequence with $\lim_{t \rightarrow +\infty} t^p \mu\{|f| > t\} = 0$. Suppose that $\sigma_n(f) \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \rho(n) = 0$.*

Then for each $\gamma < \frac{1}{2} - \frac{1}{p}$, the following convergence takes place:

$$\frac{1}{\sigma_n} S_n^{\text{pl}}(f) \rightarrow W \text{ in distribution in } \mathcal{H}_\gamma^0[0, 1].$$

However, boundedness of the sequence of p th moments of the normalized partial sums is not enough to guarantee tightness in $\mathcal{H}_{1/2-1/p}[0, 1]$.

Theorem 2.3. *Let $p > 2$. There exists a strictly stationary sequence $(f \circ T^j)_{j \geq 0}$ such that*

- *the finite dimensional distributions of $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$ converge to those of a standard Brownian motion,*

- the sequence $(\mathbb{E}|S_n(f)|^p/n^{p/2})_{n \geq 1}$ is bounded and
- the process $(S_n^{\text{pl}}(f)/\sqrt{n})_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2-1/p}[0, 1]$.

Theorem 2.4. Let $p > 2$ and let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary sequence of centered random variables such that $\lim_{t \rightarrow \infty} t^p \mu\{|f| > t\} = 0$. Assume that one of the following conditions is satisfied:

C.1 $\lim_{t \rightarrow +\infty} t^p \text{Leb}\{u \mid \alpha^{-1}(u)Q_f(u) > t\} = 0;$

C.2 $\int_0^1 \alpha^{-1}(u)^{p-1} Q_f(u)^p du < \infty$ and $\sum_k k^{p+\eta-1} \alpha(k)$ converges for some positive $\eta;$

C.3 $\lim_{t \rightarrow +\infty} t^p \text{Leb}\left\{u \mid ((\tau/2)^{-1} \circ G_f^{-1})(u)Q_f(u) > t\right\} = 0;$

C.4 for some positive $\eta,$

$$(20) \quad \int_0^1 [(\tau/2)^{-1} \circ G_f^{-1}(u)]^{p+\eta-1} Q_f(u)^p du < +\infty;$$

C.5 the series $\sum_{i=1}^{+\infty} \rho(2^i)$ is convergent;

C.6 $\lim_{n \rightarrow \infty} \rho^*(n) = 0.$

Then

$$(21) \quad \frac{1}{\sqrt{n}} S_n^{\text{pl}}(f) \rightarrow \sigma W \text{ in distribution in } \mathcal{H}_{1/2-1/p}^0[0, 1],$$

where $\sigma^2(f) = \lim_{n \rightarrow \infty} \sigma_n^2(f)/n.$

We conclude this section by some remarks and comments about the previous result.

Remark 2.5. The conditions about ρ and ρ^* -mixing do not involve the exponent $p.$

Example 2.6. Let $\varepsilon: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be such that $\lim_{t \rightarrow +\infty} \varepsilon(t) = 0.$ Let $(f \circ T^j)_{j \geq 0}$ be a strictly stationary zero-mean sequence such that $t^p \mu\{|f| > t\} \leq \varepsilon(t).$ Defining $\delta(u) := \inf\{s > 0 \mid \varepsilon(su^{-1/p}) \leq s^p\},$ we obtain $Q(u) \leq u^{-1/p} \delta(u),$ hence condition **C.2** is fulfilled if

$$\int_0^1 \frac{\alpha^{-1}(u) \delta(u)^p}{u} du < \infty.$$

Remark 2.7. • Assume that $(X_j)_{j \geq 0}$ is an m -dependent sequence strictly stationary sequence. If U is uniformly distributed on the unit interval, then $Q(U)$ has the same distribution as $X_0.$ Consequently, condition **C.1** reads $t^p \mu\{|X_0| > t\} \rightarrow 0,$ which is the necessary and sufficient condition derived in [RS03]. On the other hand, condition **C.2** is equivalent to $X_0 \in \mathbb{L}^p,$ which is more restrictive.

- If $\alpha^{-1}(u) \stackrel{u \rightarrow 0}{\sim} \log u$ and $Q(u)\alpha^{-1}(u) \geq u^{-1/p},$ then condition **C.2** holds (with $\eta = 1/2$) while condition **C.1** is not satisfied.
- For a given $p > 2,$ Theorem 17 in [Ham00] provides a sufficient mixing and moment condition in order to guarantee the weak invariance principle in $\mathcal{H}_\gamma[0, 1]$ for each $\gamma < 1/2 - 1/p.$ This requires moments of order $p + \varepsilon$ for some positive $\varepsilon,$ while it is not necessarily the case in Theorem 2.4. Indeed, if $\mathbb{E}[|X_0|^p (\log(1 + |X_0|))^{p-1}]$ is finite and $\alpha_X(n) \leq C \cdot a^n$ for some positive C and $a \in (0, 1),$ then **C.2** holds.
- We do not know whether the assumption $\sum_{k=1}^{+\infty} k^{p+\eta-1} \alpha(k) < \infty$ for some positive η can be removed in **C.2.** (we only know that the invariance principle takes place in $\mathcal{H}_\gamma[0, 1]$ for each $\gamma < 1/2 - 1/p$).

3. PROOFS

Proof of Theorem 2.1. Let $h(n) := \sigma_n^2/n$. If $h(n) = 1$ for each n , then after a use of the sequential norm and some computations, the tightness in $\mathcal{H}_\alpha^0[0, 1]$ for $\alpha < 1/2 - 1/p$ follows from the convergence of the series with general term $2^{j(1+q(\alpha-1/2))}$ with $(1/2 - \alpha)^{-1} < q < p$. We shall see that the role of the slowly varying function does not influence the argument too much. This time we do not use the sequential explicitly norm but we use a dyadic argument (the proof can be carried out in both ways).

Let α be a positive real number strictly smaller than $1/2 - 1/p$ and ε a positive fixed number. For an integer $n \geq 1$ and $\delta \in (0, 1)$, we define

$$P(n, \delta) := \mu \left\{ \max_{1 \leq i < j \leq [n\delta]} \frac{|S_j(f) - S_i(f)|}{(j-i)^\alpha} > \frac{\sigma_n}{n^\alpha} \varepsilon \right\}.$$

Our goal is to show that given a positive η there is some $\delta \in (0, 1)$ and a positive integer n_0 such that for each $n \geq n_0$, $P(n, \delta) \leq \delta\eta$. In this way, we will be able to use Corollary 1.4. Let $q \in (1/2 - \alpha, p)$. Using Lemma 3.3 in [MR10] (which holds for strictly stationary and not necessarily independent sequences), we derive the inequalities

$$\begin{aligned} P(n, \delta) &\leq 2 \sum_{j=1}^{\log[n\delta]} 2^j \mu \left\{ \max_{1 \leq k \leq 2^{[n\delta]2^{-j}}} |S_k(f)| > \frac{\sigma_n}{n^\alpha} \varepsilon ([n\delta]2^{-j})^\alpha \right\} \\ &\leq 2 \sum_{j=1}^{\log[n\delta]} 2^j \mathbb{E} \left[\max_{1 \leq k \leq 2^{[n\delta]2^{-j}}} |S_k(f)|^q \right] \left(\frac{\sigma_n}{n^\alpha} \varepsilon ([n\delta]2^{-j})^\alpha \right)^{-q}. \end{aligned}$$

By Theorem A.1104 in [Bra07] or using arguments of [Ser70], the q th moment of $\max_{1 \leq j \leq N} |S_j(f)|/\sigma_N$ is uniformly bounded with respect to N , say by $M/2$. We deduce the estimates

$$\begin{aligned} P(n, \delta) &\leq M \sum_{j=1}^{\log[n\delta]} 2^j \frac{\sigma_n^q}{\sigma_n^q} n^{q\alpha} [n\delta]^{-q\alpha} 2^{jq\alpha} \\ &= M \sum_{j=1}^{\log[n\delta]} 2^j \frac{(2[n\delta]2^{-j})^{q/2}}{n^{q/2}} \frac{h(2[n\delta]2^{-j})^{q/2}}{h(n)^{q/2}} \frac{n^{q\alpha}}{[n\delta]^{q\alpha}} 2^{jq\alpha} \\ &\leq M 2^{q/2} \delta^{q/2 - q\alpha} \sum_{j=1}^{\log[n\delta]} 2^{j(1+q(\alpha-1/2))} \left(\frac{h(2[n\delta]2^{-j})}{h(n)} \right)^{q/2}. \end{aligned}$$

Since

$$\frac{h(2[n\delta]2^{-j})}{h(n)} \leq \sup_{x \geq 1} \frac{h(2x)}{h(x2^j/\delta)} =: c_{j,\delta}$$

and for each $\delta \in (0, 1)$ and each integer $j \geq 1$,

$$c_{j+1,\delta} \leq \sup_{x \geq 1} \frac{h(2x)}{h(x2^j/\delta)} \cdot \sup_{t \geq 1} \frac{h(t2^j/\delta)}{h(t2^{j+1}/\delta)} \leq c_{j,\delta} \sup_{s \geq 2^j/\delta} \frac{h(s)}{h(2s)},$$

we obtain

$$c_{j,\delta} \leq c_{0,\delta} \prod_{k=1}^j \sup_{s \geq 2^k} \frac{h(s)}{h(2s)}.$$

Since h is slowly varying, we obtain for k large enough the inequality

$$\left(\sup_{s \geq 2^k} \frac{h(s)}{h(2s)} \right)^{q/2} \leq 2^{-(1+q(\alpha-1/2))/2},$$

hence by the ratio test the series $\sum_{j=1}^{+\infty} 2^{j(1+q(\alpha-1/2))} c_{j,\delta}^{q/2}$ is convergent. We conclude that for each integer n and each $\delta \in (0, 1)$,

$$P(n, \delta) \leq M \cdot C(h) \cdot 2^{q/2} \delta^{q/2 - q\alpha} c_{0,\delta} \sum_{j=1}^{+\infty} 2^{j(1+q(\alpha-1/2))/2},$$

where $C(h)$ depends only on the function h . To conclude, we need the following lemma.

Lemma 3.1. *Let $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ be a slowly varying function. Then for each positive β ,*

$$\lim_{\delta \rightarrow 0} \delta^\beta \sup_{x \geq 2} \frac{h(x)}{h(x/\delta)} = 0.$$

Proof. Let $\delta > 0$, and let k be the unique integer such that $2^{-k} \leq \delta < 2^{1-k}$. Then

$$\sup_{x \geq 2} \frac{h(x)}{h(x/\delta)} \leq \sup_{x \geq 2} \sup_{2^{k-1}x \leq t \leq 2^k x} \frac{h(x)}{h(t)}.$$

Since h is a slowly varying function, we have

$$\lim_{s \rightarrow +\infty} \frac{\sup_{s \leq u \leq 2s} h(u)}{\inf_{s \leq u \leq 2s} h(u)} = 1,$$

hence the inequality

$$\sup_{2^{k-1}x \leq t \leq 2^k x} \frac{h(x)}{h(t)} \leq 2 \frac{h(x)}{h(2^{k-1}x)}$$

holds if k is greater than some k_0 (which is independent on x). Therefore, it is enough to prove that

$$(22) \quad \lim_{k \rightarrow \infty} 2^{-k\beta} \sup_{x \geq 2} \frac{h(x)}{h(2^k x)} = 0.$$

To this aim, we notice that

$$\sup_{x \geq 2} \frac{h(x)}{h(2^k x)} \leq \prod_{i=1}^k \sup_{x \geq 2} \frac{h(2^{i-1}x)}{h(2^i x)} = \prod_{i=1}^k \sup_{x \geq 2^i} \frac{h(x)}{h(2x)}$$

Since $h(x)/h(2x)$ converges to 1 as x goes to infinity, we can find an integer i_0 such that $2^{-\beta} \sup_{x \geq 2^{i_0}} \frac{h(x)}{h(2x)} < 1$. It follows that

$$2^{-k\beta} \sup_{x \geq 2} \frac{h(x)}{h(2^k x)} \leq \left(2^{-\beta} \sup_{x \geq 2^{i_0}} \frac{h(x)}{h(2x)} \right)^k \left(\sup_{x \geq 2} \frac{h(x)}{h(2x)} \right)^{2i_0},$$

hence (22) holds. □

We obtain tightness of the sequence $\left(\sigma_n^{-1} S_n^{\text{pl}}(f) \right)_{n \geq 1}$ in $\mathcal{H}_\alpha^0[0, 1]$ applying Lemma 3.1 with $\beta := q/2 - q\alpha - 1$. This finishes the proof of Theorem 2.1. □

Proof of Theorem 2.3. We assume that $(\Omega, \mathcal{F}, \mu, T)$ is a non-atomic invertible measure preserving system. We shall first construct a function g such that:

- (1) the sequence $(\mathbb{E}|S_n(g - g \circ T)|^p / n^{p/2})_{n \geq 1}$ is bounded and
- (2) the process $(n^{-1/2} S_n^{\text{pl}}(g - g \circ T))_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2-1/p}^0[0, 1]$.

We then consider $f := m + g - g \circ T$, where m is such that $(m \circ T^j)_{j \geq 0}$ is a martingale difference sequence with $m \in \mathbb{L}^p$ and $m \neq 0$. This will guarantee the convergence of the finite dimensional distributions of $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$ to those of a scalar multiple of a standard Brownian motion, and Burkholder's inequality ensures boundedness of the sequence $(\mathbb{E}|S_n(f)|^p / n^{p/2})_{n \geq 1}$. We use a construction similar to that given in [VS00]. Let us consider two increasing sequences of integer $(K_l)_{l \geq 1}$ and $(N_l)_{l \geq 1}$ satisfying for each $l \geq 2$:

$$(23) \quad \lim_{l \rightarrow +\infty} N_l \sum_{l' > l} K_{l'} / N_{l'} = 0;$$

$$(24) \quad 4N_l^{-1/p} \cdot l \cdot N_{l-1} < 1;$$

$$(25) \quad \sum_{i=1}^l K_i^{1/2} \leq K_{l+1}^{1/2};$$

$$(26) \quad \sum_{l=1}^{+\infty} \frac{K_l}{K_{l+1}^{1/2}} < \infty.$$

We also assume that $4K_l \leq N_l$ for each l .

Let us fix an integer l . Using Rokhlin's lemma, we can find a set $A_l \in \mathcal{F}$ such that the set $T^i A_l$, $0 \leq i \leq N_l - 1$ are pairwise disjoint and $\mu\left(\bigcup_{i=0}^{N_l-1} T^i A_l\right) \geq 1/2$. We define

$$h_l := \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \cdot \chi\left(\bigcup_{j=1}^{K_l} T^{N_l-j} A_l\right);$$

$$g_l := \sum_{j=0}^{K_l-1} h_l \circ T^j = \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left(\sum_{j=1}^{K_l} j \chi(T^{N_l-j} A_l) + \sum_{j=K_l+1}^{2K_l-1} (2K_l - j) \chi(T^{N_l-j} A_l) \right);$$

$$g := \sum_{l=1}^{+\infty} g_l.$$

Assume that $\omega \in A_l$ and $N_l - K_l \leq i \leq N_l - 1$. Then $g_l \circ T^i(\omega) = N_l - i$. Consequently, for $i, i' \in \{N_k - K_l, \dots, N_l - 1\}$,

$$\left| g_l \circ T^i - g_l \circ T^{i'} \right| \geq \frac{N_l^{1/p}}{K_l^{1/2+1/p}} |i' - i| \chi(A_l).$$

Applying U^{-k} on both sides of the previous inequality for $0 \leq k \leq N_l - K_l$ and taking the maximum over these k , we obtain

$$\max_{0 \leq k \leq N_l - K_l} \left| g_l \circ T^{i-k} - g_l \circ T^{i'-k} \right| \geq |i' - i| \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \chi\left(\bigcup_{k=0}^{N_k - K_l} T^k(A_l)\right).$$

This implies

$$(27) \quad \frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|g_l \circ T^{i'} - g_l \circ T^i|}{(i' - i)^{1/2-1/p}} \geq \chi \left(\bigcup_{k=0}^{N_k - K_l} T^k(A_l) \right).$$

For $l' < l$, noticing that $|g_{l'}| \leq N_{l'}^{1/p} K_{l'}^{1/2-1/p}$, the following inequality takes place:

$$\max_{1 \leq i < j \leq N_l} |g_{l'} \circ T^j - g_{l'} \circ T^i| \leq 2N_{l'}^{1/p} K_{l'}^{1-1/p},$$

therefore,

$$\begin{aligned} \frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' < l} g_{l'} \circ T^{i'} - \sum_{l' < l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} &\leq \\ &\leq \frac{1}{N_l^{1/p}} \sum_{l' < l} 2N_{l'}^{1/p} K_{l'}^{1-1/p} \leq 2N_l^{-1/p} l N_{l-1}. \end{aligned}$$

By condition (24), we conclude that

$$(28) \quad \frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' < l} g_{l'} \circ T^{i'} - \sum_{l' < l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} \leq 1/2.$$

Moreover, notice that

$$\begin{aligned} \mu \left\{ N_l^{-1/p} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' > l} g_{l'} \circ T^{i'} - \sum_{l' > l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} \neq 0 \right\} \\ \leq \sum_{l' > l} \mu \left\{ \max_{1 \leq i < i' \leq N_l} |g_{l'} \circ T^{i'} - g_{l'} \circ T^i| \neq 0 \right\} \\ \leq N_l \sum_{l' > l} \mu \{g_{l'} \neq 0\}, \end{aligned}$$

hence

$$(29) \quad \mu \left\{ \frac{1}{N_l^{1/p}} \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' > l} g_{l'} \circ T^{i'} - \sum_{l' > l} g_{l'} \circ T^i|}{(i' - i)^{1/2-1/p}} \neq 0 \right\} \leq 2N_l \sum_{l' > l} K_{l'}/N_{l'}.$$

By (28) and (29), we get

$$\begin{aligned} & \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|g \circ T^{i'} - g \circ T^i|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p}/2 \right\} \geq \\ & \geq \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|\sum_{l' \geq l} (g_{l'} \circ T^{i'} - g_{l'} \circ T^i)|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p} \right\} \\ & \geq \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|(g_l \circ T^{i'} - g_l \circ T^i)|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p} \right\} - 2N_l \sum_{l' > l} K_{l'}/N_{l'}. \end{aligned}$$

Combining the previous inequality with (27), we obtain for each integer l ,

$$(30) \quad \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|g \circ T^{i'} - g \circ T^i|}{(i' - i)^{1/2-1/p}} \geq N_l^{1/p}/2 \right\} \geq \frac{1}{2} - \frac{K_l}{2N_l} - 2N_l \sum_{l' > l} K_{l'}/N_{l'},$$

and by (23), the inequality

$$(31) \quad \mu \left\{ \max_{1 \leq i < i' \leq N_l} \frac{|g \circ T^{i'} - g \circ T^i|}{(i' - i)^{1/2-1/p}} \geq \frac{N_l^{1/p}}{2} \right\} \geq \frac{1}{8}$$

holds for l large enough. We deduce that for such integers l ,

$$(32) \quad \mu \left\{ \frac{1}{\sqrt{N_l}} S_{N_l}^{\text{pl}}(g - g \circ T) \geq 1/2 \right\} \geq \frac{1}{8}.$$

Since the finite dimensional distributions of the process $(n^{-1/2} S_n^{\text{pl}}(g - g \circ T))_{n \geq 1}$ converge to 0, the below bound (3) proves that this process cannot be tight in $\mathcal{H}_{1/2-1/p}[0, 1]$.

It remains to show that the sequence $(n^{-1/2}(g - g \circ T^n))_{n \geq 1}$ is bounded in \mathbb{L}^p . Notice that for a fixed integer $l \geq 1$, the equalities

$$|g_l - g_l \circ T| = |h_l - h_l \circ T^{K_l}| = \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \cdot \chi \left(\bigcup_{j=1}^{2K_l} T^{N_l-j} A_l \right)$$

take place. This implies that

$$\begin{aligned} \|g_l - g_l \circ T\|_p &= \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \cdot \left[\mu \left(\bigcup_{j=1}^{2K_l} T^{N_l-j} A_l \right) \right]^{1/p} \\ &\leq \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left(\frac{2K_l}{N_l} \right)^{1/p} = 2^{1/p} K_l^{-1/2}, \end{aligned}$$

hence for each integer $n \geq 1$, $\|g_l - g_l \circ T^n\|_p \leq 2^{1/p} n K_l^{-1/2}$. Let us define

$$\tilde{g}_l := \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \sum_{j=1}^{K_l} j \chi(T^{N_l-j} A_l).$$

If $K_l \leq n$, then

$$\begin{aligned}\tilde{g}_l - \tilde{g}_l \circ T^n &= \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left(\sum_{j=1}^{K_l} j\chi(T^{N_l-j} A_l) - \sum_{j=1}^{K_l} j\chi(T^{N_l-j-n} A_l) \right) \\ &= \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left(\sum_{j=1}^{K_l} j\chi(T^{N_l-j} A_l) - \sum_{j=n+1}^{n+K_l} (j-n)\chi(T^{N_l-j} A_l) \right),\end{aligned}$$

hence

$$|\tilde{g}_l - \tilde{g}_l \circ T^n| \leq \frac{N_l^{1/p}}{K_l^{1/2+1/p}} \left(\sum_{j=1}^{K_l} j\chi(T^{N_l-j} A_l) + \sum_{j=n+1}^{n+K_l} (j-n)\chi(T^{N_l-j-n} A_l) \right),$$

and the following upper bound follows:

$$\mathbb{E} |\tilde{g}_l - \tilde{g}_l \circ T^n|^p \leq 2^{p-1} \sum_{j=1}^{K_l} j^p K_l^{-1-p/2} \leq 2^{p-1} K_l^{p/2}.$$

Treating in a similar manner the function $g_l - \tilde{g}_l$, we observe that the following inequality holds:

$$(33) \quad \|g_l - g_l \circ T^n\|_p \leq C_p \begin{cases} nK_l^{-1/2} & \text{if } K_l > n \\ K_l^{-1/2} & \text{otherwise,} \end{cases}$$

where C_p depends only on p (neither on n , nor on l). For a fixed integer n , we denote by $i(n)$ the unique integer satisfying the inequalities $K_{i(n)} \leq n < K_{i(n)+1}$.

By (33), we have

$$\begin{aligned}\|g - g \circ T^n\|_p &\leq \sum_{l=1}^{+\infty} \|g_l - g_l \circ T^n\|_p \\ &\leq C_p \left(\sum_{l=1}^{i(n)-1} K_l^{1/2} + K_{i(n)}^{1/2} + nK_{i(n)+1}^{-1/2} + \sum_{l=i(n)+2}^{+\infty} nK_l^{-1/2} \right) \\ &\leq 3C_p\sqrt{n} + C_p\sqrt{n} \sum_{l=i(n)+1}^{+\infty} \frac{K_l}{K_{l+1}^{1/2}} \\ &\leq C_p \left(3 + \sum_{l=1}^{+\infty} \frac{K_l}{K_{l+1}^{1/2}} \right) \sqrt{n},\end{aligned}$$

where we used (25) in the second inequality and condition (26) ensures finiteness of the right hand side in this inequality.

This concludes the proof of Theorem 2.3. \square

Remark 3.2. The constructed process has no reason to be α -mixing. However, this proves that in general, establishing tightness in $\mathcal{H}_{1/2-1/p}^0[0, 1]$ of $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$ cannot be done by proving boundedness in \mathbb{L}^p of the sequence $(n^{-1/2}S_n(f))_{n \geq 1}$. Thus other methods need to be used.

Before giving the proof of the conclusion of Theorem 2.4 under the six presented conditions, let us explain the outline of proof.

Each of these conditions implies the invariance principle in the space of continuous functions and a good asymptotic behavior of the variance of the n th partial sum, namely, an asymptotically linear variance. In this way, we obtain the convergence of the finite dimensional distribution to the limiting process $\sigma \cdot W$, where σ is like in the statement of the theorem. It remains to check the tightness of the partial sum process in the space $\mathcal{H}_{1/2-1/p}^0[0, 1]$. To this aim, we shall use the tightness criterion derived from the sequential norm. In order to check that it is satisfied in the different cases of the theorem, we shall use some truncations. The arguments of the proof under condition **C.1** are different from those of condition **C.2**, in particular no truncation is needed under **C.1**. In the proof under these two conditions, the notation $\alpha^{-1}(u)$ will refer to the inverse function of the original sequence $(f \circ T^j)_{j \geq 0}$ (we directly bound the corresponding inverse function of the truncated sequence). However, when we deal with the quantile function we shall always specify to which random variable we refer.

Proof under condition C.1. For $q \in (2, p)$, the q th power of the map $u \mapsto \alpha^{-1}(u)Q_f(u)$ is integrable on the unit interval, hence (10) is satisfied. This guarantees, by Proposition 1.14, the convergence of the sequence $(\sigma_n^2(f)/n)_{n \geq 1}$ to a non-negative σ . Since (10) implies convergence of finite-dimensional distributions of $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$ to those of σW , the remaining task is to prove the tightness of this sequence in $\mathcal{H}_{1/2-1/p}[0, 1]$. To this aim, we shall use Proposition 1.5. This is possible since the facts that $\alpha^{-1}(u) \geq 1$ for each u and $Q_f(U)$ has the same distribution as $|f|$ if U is uniformly distributed on the unit interval imply that the random variable f satisfies $\lim_{t \rightarrow +\infty} t^p \mu \{|f| > t\} = 0$. Let us consider an arbitrary $\varepsilon > 0$ and define

$$P_{n,J} := \mu \left\{ \sup_{J \leq j \leq \log n} 2^{j(1/2-1/p)} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f \circ T^i \right| > 4n^{1/2}\varepsilon \right\}.$$

Notice that

$$(34) \quad P_{n,J} \leq \sum_{j=J}^{\log n} 2^j \max_{1 \leq k \leq 2^j} \mu \left\{ |S_{u_{k+1,j}} - S_{u_{k,j}}| > 4\varepsilon n^{1/2} 2^{-j(1/2-1/p)} \right\} =: \sum_{j=J}^{\log n} 2^j p_j.$$

Using Theorem 1.12 with $r := 2p$ and $u_{k+1,j} - u_{k,j}$ (which is smaller than $2n2^{-j}$, cf. Notation 1.2) instead of N together with the equality $\{u \in [0, 1] \mid u < H(t)\} = \{u \in [0, 1] \mid R(u) > t\}$, we infer that for some constant C depending only on p, ε and $\sup_t t^p \mu \{|f| > t\}$ changing from line to line,

$$\begin{aligned} p_j &\leq 4 \left(1 + \frac{(\varepsilon n^{1/2} 2^{-j(1/2-1/p)})^2}{r s_{2n2^{-j}}^2} \right)^{-r/2} \\ &\quad + 32n2^{-j} (\varepsilon n^{1/2} 2^{-(1/2-1/p)j})^{-1} \int_0^1 Q(u) \chi \left\{ u < H(\varepsilon n^{1/2} 2^{-(1/2-1/p)j} / (4r)) \right\} du \\ &\leq C 2^{rj(1/2-1/p)} 2^{-jr/2} + C 2^{j(1/2-1/p)} n^{1/2} 2^{-j} \cdot \text{Leb} \left\{ u \mid R(u) > \frac{\varepsilon n^{1/2} 2^{-j(1/2-1/p)}}{8p} \right\}^{1-1/p} \\ &\leq C 2^{-2j} + \\ &\quad + C n^{1/2} 2^{-j} 2^{j(1/2-1/p)} \left(n^{1/2} 2^{-j(1/2-1/p)} \right)^{-(p-1)} \left(\sup_{t \geq \varepsilon n^{1/p} / (8p)} t^p \text{Leb} \{u \mid R(u) > t\} \right)^{1-1/p}. \end{aligned}$$

Combining the later estimate with inequality (34), we derive that for some constant C depending only on p , ε and $\sup_t t^p \mu\{|f| > t\}$,

$$\begin{aligned} P_{n,J} &\leq C \sum_{j=J}^{+\infty} 2^{-j} + C n^{1-p/2} \sum_{j=1}^{\log n} 2^{j((1/2-1/p)+(p-1)(1/2-1/p))} \left(\sup_{t \geq \varepsilon n^{1/p}/(8p)} t^p \text{Leb}\{u \mid R(u) > t\} \right)^{1-1/p} \\ &= C 2^{-J} + C \left(\sup_{t \geq \varepsilon n^{1/p}/(8p)} t^p \text{Leb}\{u \mid R(u) > t\} \right)^{1-1/p} n^{1-p/2} \sum_{j=1}^{\log n} 2^{j(p/2-1)}. \end{aligned}$$

Since the quantity $n^{1-p/2} \sum_{j=1}^{\log n} 2^{j(p/2-1)}$ is bounded uniformly with respect to n , there exists a constant K depending only on p , ε and $\sup_t t^p \mu\{|f| > t\}$ such that for each n and each J ,

$$P_{n,J} \leq K 2^{-J} + K \left(\sup_{t \geq \varepsilon n^{1/p}/(8p)} t^p \text{Leb}\{u \mid R(u) > t\} \right)^{1-1/p},$$

hence by condition **C.1**

$$\limsup_{n \rightarrow \infty} P_{n,J} \leq K 2^{-J},$$

hence by Proposition 1.5, the sequence $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$ is tight in $\mathcal{H}_{1/2-1/p}^0[0, 1]$. This concludes the proof of Theorem 2.4 under condition **C.1**. \square

Proof under condition C.2. Notice that the convergence of the integral $\int_0^1 \alpha^{-1}(u)^{p-1} Q_f(u)^p du$ implies (10). The existence of the limit σ of the sequence $(\sigma_n^2(f)/n)_{n \geq 1}$ is given by Proposition 1.14.

Convergence of finite dimensional distributions of $(\sigma_n(f) S_n^{\text{pl}}(f))_{n \geq 1}$ to those of σW can be obtained by the main result of [DMR94].

The remaining part of the proof is to establish tightness of the sequence $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$. Let us fix two positive integers n and J with $J \leq \log n$ and $j \in \{J, \dots, \log n\}$. We define $q := p + \eta$, $r := 1/p - 1/q$ and

$$\begin{aligned} f' &:= f \chi\{|f| < 2^{jr} \delta\} - \mathbb{E}[f \chi\{|f| < 2^{jr} \delta\}] \\ f'' &:= f \chi\{|f| \geq 2^{jr} \delta\} - \mathbb{E}[f \chi\{|f| \geq 2^{jr} \delta\}]. \end{aligned}$$

Let C be a constant depending only on q and ε which may change from line to line. Using Proposition 1.14 and Theorem 1.13 with the exponent q instead of p ,

we obtain the following inequalities for $k \in \{1, \dots, 2^j\}$:

$$\begin{aligned}
\mu \left\{ \left| \sum_{i=u_{k,j}}^{u_{k+1,j}} f' \circ T^i \right| > n^{1/2} \varepsilon 2^{-(1/2-1/p)j} \right\} &\leq C(n^{1/2} 2^{-j(1/2-1/p)})^{-q} s_{u_{k+1,j}-u_{k,j}}^{q/2}(f') \\
&+ C(u_{k+1,j} - u_{k,j})(n^{1/2} 2^{-j(1/2-1/p)})^{-q} \int_0^1 \alpha^{-1}(u)^{q-1} \delta^q 2^{qrj} du \\
&\leq C(n^{1/2} 2^{-j(1/2-1/p)})^{-q} (n 2^{-j})^{q/2} \left(\int_0^1 \alpha^{-1}(u) Q_f^2(u) du \right)^{q/2} \\
&+ C n 2^{-j} n^{-q/2} 2^{jq(1/2-1/p)} 2^{jqr} \delta^q \int_0^1 \alpha^{-1}(u)^{q-1} du \\
&\leq C 2^{-jq/p} \left(\int_0^1 \alpha^{-1}(u) Q_f^2(u) du \right)^{q/2} \\
&+ C n^{1-q/2} 2^{j(q/2-2)} \delta^q \sum_{l=1}^{+\infty} (l+1)^{q-1} \alpha(l).
\end{aligned}$$

Taking the maximum over k , multiplying by 2^j and summing, we derive the estimate

$$\begin{aligned}
(35) \quad \mu \left\{ \sup_{J \leq j \leq \log n} 2^{(1/2-1/p)j} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f' \circ T^i \right| > n^{1/2} \varepsilon \right\} &\leq \\
&C \left(\int_0^1 \alpha^{-1}(u) Q_f^2(u) du \right)^{q/2} \sum_{j=J}^{+\infty} 2^{-jn/p} + C \delta^q \sum_{l=1}^{+\infty} (l+1)^{q-1} \alpha(l).
\end{aligned}$$

Using this time Theorem 1.13 with the exponent p , we get

$$\begin{aligned}
\mu \left\{ \left| \sum_{i=u_{k,j}}^{u_{k+1,j}} f'' \circ T^i \right| > n^{1/2} \varepsilon 2^{-(1/2-1/p)j} \right\} &\leq C n^{-p/2} 2^{jp(1/2-1/p)} (n 2^{-j})^{p/2} \int_0^1 \alpha^{-1}(u) Q_{f''}(u)^2 du + \\
&+ C n 2^{-j} (n^{1/2} 2^{-j(1/2-1/p)})^{-p} \int_0^1 \alpha^{-1}(u)^{p-1} Q_{f''}(u)^p du.
\end{aligned}$$

An application of Hölder's inequality with the exponent $p-1$ to the functions $\alpha^{-1}(u) Q_{f''}(u)^{p/(p-1)}$ and $Q_{f''}(u)^{2-p/(p-1)}$ yields

$$\begin{aligned}
\int_0^1 \alpha^{-1}(u) Q_{f''}(u)^2 du &\leq \left(\int_0^1 \alpha^{-1}(u)^{p-1} Q_{f''}(u)^p du \right)^{1/(p-1)}. \\
&(\mathbb{E} |f \chi \{|f| \geq 2^{jr} \delta\} - \mathbb{E} [f \chi \{|f| \geq 2^{jr} \delta\}]|)^{\frac{p-2}{p-1}} \\
&\leq C_p \left(\int_0^1 \alpha^{-1}(u)^{p-1} Q_f(u)^p du \right)^{1/(p-1)} (\mathbb{E} |f|^p)^{\frac{p-2}{p-1}} (2^{-jr} \delta)^{p-2},
\end{aligned}$$

hence

$$(36) \quad \mu \left\{ \left| \sum_{i=u_{k,j}}^{u_{k+1,j}} f'' \circ T^i \right| > n^{1/2} \varepsilon 2^{-(1/2-1/p)j} \right\} \\ \leq C \left(\int_0^1 \alpha^{-1}(u)^{p-1} Q_f(u)^p du \right)^{1/(p-1)} (\mathbb{E} |f|^p)^{\frac{p-2}{p-1}} (2^{-jr} \delta)^{p-2} 2^{-j} + \\ + n^{1-p/2} 2^{j(p/2-2)} \int_0^1 \alpha^{-1}(u)^{p-1} Q_{f''}(u)^p du.$$

Since

$$Q_{f''}(u) \leq \mathbb{E}[|f| \chi \{|f| \geq 2^{Jr} \delta\}] + Q_f(u) \chi \{u \leq \mu \{|f| > 2^{Jr} \delta\}\},$$

the following inequality takes place:

$$(37) \quad \mu \left\{ \sup_{J \leq j \leq \log n} 2^{(1/2-1/p)j} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f'' \circ T^i \right| > n^{1/2} \varepsilon \right\} \\ \leq C \left(\int_0^1 \alpha_X^{-1}(u)^{p-1} Q_f(u)^p du \right)^{1/(p-1)} (\mathbb{E} |f|^p)^{\frac{p-2}{p-1}} \sum_{j=J}^{+\infty} (2^{-jr} \delta)^{p-2} + \\ + C \int_0^1 \alpha_X^{-1}(u)^{p-1} (\mathbb{E}[|f| \chi \{|f| \geq 2^{Jr} \delta\}] + Q_f(u) \chi \{u \leq \mu \{|f| > 2^{Jr} \delta\}\})^p du$$

Combining (35) and (37), we obtain the bound

$$(38) \quad \mu \left\{ \sup_{J \leq j \leq \log n} 2^{(1/2-1/p)j} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f \circ T^i \right| > n^{1/2} \varepsilon \right\} \\ \leq C \left(\int_0^1 \alpha^{-1}(u) Q_f^2(u) du \right)^{q/2} \sum_{j=J}^{+\infty} 2^{-j\eta/p} + C \delta^q \sum_{l=1}^{+\infty} (l+1)^{q-1} \alpha(l) + \\ + C \left(\int_0^1 \alpha^{-1}(u)^{p-1} Q_f(u)^p du \right)^{1/(p-1)} (\mathbb{E} |f|^p)^{\frac{p-2}{p-1}} \sum_{j=J}^{+\infty} (2^{-jr} \delta)^{p-2} + \\ + C \int_0^1 \alpha^{-1}(u)^{p-1} (\mathbb{E}[|f| \chi \{|f| \geq 2^{Jr} \delta\}] + Q_f(u) \chi \{u \leq \mu \{|f| > 2^{Jr} \delta\}\})^p du.$$

The right hand side of (38) does not depend on n . Taking the lim sup with respect to J , the first and third term converge to 0 as the remainder of a convergent series, and the fourth term converge to 0 by dominated convergence. Therefore, for each positive δ ,

$$(39) \quad \limsup_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{J \leq j \leq \log n} 2^{(1/2-1/p)j} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f \circ T^i \right| > n^{1/2} \varepsilon \right\} \\ \leq C \delta^q \sum_{l=1}^{+\infty} (l+1)^{q-1} \alpha_X(l),$$

hence by condition **C.2** the assumption of Propotion 1.5 is satisfied.

This concludes the proof of Theorem 2.4 under condition **C.2**. \square

Proof under condition C.3 or .C.4. This uses similar arguments as for conditions C.1 and C.2 respectively. \square

Proof under condition C.5. Convergence of the sequence $(\sigma_n^2(f)/n)_{n \geq 1}$ and that of finite dimensional distributions are obtained in Theorem 1.21 and [Sha88] respectively, hence we are reduced to show tightness of $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$ in $\mathcal{H}_{1/2-1/p}[0, 1]$. We use Proposition 1.5. Let us define $q := 2p$, $\Delta_{k,j} := u_{k+1,j} - u_{k,j}$ and

$$f_{n,\delta} := f\chi \left\{ |f| < n^{1/p}\delta \right\} - \mathbb{E}[f\chi \left\{ |f| < n^{1/p}\delta \right\}].$$

As the computations in [RS03] show, it is enough to check that the following variation of (4) holds:

$$(40) \quad \lim_{\delta \rightarrow 0} \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{J \leq j \leq \log n} 2^{(1/2-1/p)j} \max_{1 \leq k \leq 2^j} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} f_{n,\delta} \circ T^i \right| > n^{1/2}\varepsilon \right\} = 0.$$

An application of inequality (18) to the exponent q and the sequence $((f_{n,\delta} - \mathbb{E}[f_{n,\delta}]) \circ T^i)_{i \geq 0}$ yields

$$\begin{aligned} P(n, J, \delta, f) &:= \sum_{j=J}^{\log n} 2^{q(1/2-1/p)j} n^{-q/2} 2^j \max_{1 \leq k \leq 2^j} \mathbb{E} \left| \sum_{i=u_{k,j}+1}^{u_{k+1,j}} (f_{n,\delta} - \mathbb{E}(f_{n,\delta})) \circ T^i \right|^q \\ &\leq K \sum_{j=J}^{\log n} 2^{q(1/2-1/p)j} n^{-q/2} 2^j \max_{1 \leq k \leq 2^j} \Delta_{k,j}^{q/2} \exp \left(K \sum_{k=1}^{\log \Delta_{k,j}} \rho(2^k) \right) (\mathbb{E}(f_{n,\delta} - \mathbb{E}(f_{n,\delta}))^2)^{q/2} + \\ &+ K \sum_{j=J}^{\log n} 2^{q(1/2-1/p)j} n^{-q/2} 2^j \max_{1 \leq k \leq 2^j} \Delta_{k,j} \exp \left(K \sum_{k=1}^{\log \Delta_{k,j}} \rho^{2/q}(2^k) \right) \mathbb{E} |f_{n,\delta} - \mathbb{E}(f_{n,\delta})|^q. \end{aligned}$$

Exploiting the estimates

- $\Delta_{k,j} = u_{k+1,j} - u_{k,j} \leq 2n2^{-j}$;
- $\mathbb{E}(f_{n,\delta} - \mathbb{E}(f_{n,\delta}))^2 \leq \mathbb{E}[f^2]$;
- $\mathbb{E} |f_{n,\delta} - \mathbb{E}(f_{n,\delta})|^q \leq 2^{q-1} \mathbb{E} |f_{n,\delta}|^q \leq \frac{q}{q-p} n^{(q-p)/p} \delta^{q-p} \sup_{t \geq 0} t^p \mu \{ |f| \geq t \}$ (by convexity of $t \mapsto |t|^q$,

we derive

$$(41) \quad \begin{aligned} P(n, J, \delta, f) &\leq 4K \exp \left(K \sum_{k=1}^{+\infty} \rho(2^k) \right) \|f\|_2^2 \sum_{j=J}^{\log n} 2^{q(1/2-1/p)j} n^{-q/2} 2^j (2n2^{-j})^{q/2} + \\ &+ \frac{q2^{q-1}}{q-p} \sup_{t \geq 0} t^p \mu \{ |f| \geq t \} K \sum_{j=J}^{\log n} 2^{q(1/2-1/p)j} n^{-q/2} 2^j 2n2^{-j} \exp \left(K \sum_{k=1}^{\log[2n2^{-j}]} \rho^{2/q}(2^k) \right), \end{aligned}$$

and after having rearranged the exponents, we get that for a constant C depending only on the sequence $(\rho(k))_{k \geq 1}$, p and $\sup_{t \geq 0} t^p \mu\{|f| > t\}$, that

$$(42) \quad P(n, J, \delta, f) \leq C \sum_{j=J}^{\log n} 2^{j(1-q/p)} + \\ + C n^{q(1/2-1/p)} \sum_{l=1}^{\log n} 2^{-q(1/2-1/p)l} n^{-q/2} n^{q/p} \delta^{q-p} \exp \left(K \sum_{k=1}^l \rho^{2/q}(2^k) \right) \\ \leq C \sum_{j=J}^{\infty} 2^{-j} + C \delta^p \sum_{l=1}^{\log n} 2^{-(p-2)l} \exp \left(K \sum_{k=1}^l \rho^{1/p}(2^k) \right),$$

where we used the relationship $q = 2p$. Since $\lim_{k \rightarrow +\infty} \rho(2^k) = 0$ and $p > 2$, the convergence of the series $\sum_{l=1}^{+\infty} 2^{-(p-2)l} \exp \left(K \sum_{k=1}^l \rho^{2/q}(2^k) \right)$ follows from the ratio test. We derive the estimate

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow +\infty} P(n, J, \delta, f) \leq C \delta^p \sum_{l=1}^{+\infty} 2^{-(p-2)l} \exp \left(K \sum_{k=1}^l \rho^{1/p}(2^k) \right)$$

from which we deduce that (40) holds. This finishes the proof of Theorem 2.4 under condition **C.5**. \square

Proof under condition C.6. Using inequality (19) and Theorem 1.23, the proof under **C.6** is completely similar to those of the independent case in [RS03]. \square

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UNIVERSITÉ DE ROUEN, LMRS, AVENUE DE L'UNIVERSITÉ, BP 12 76801 SAINT-ÉTIENNE-DU-ROUVRAY CEDEX, FRANCE.

E-mail address: `davide.giraudoi@univ-rouen.fr`