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# ON THE CANCELLATION PROBLEM FOR ALGEBRAIC TORI

ADRIEN DUBOULOZ

*Dedicated to Wlodek Danielewski.*

ABSTRACT. We address a variant of Zariski Cancellation Problem, asking whether two varieties which become isomorphic after taking their product with an algebraic torus are isomorphic themselves. Such cancellation property is easily checked for curves, is known to hold for smooth varieties of log-general type by virtue of a result of Iitaka-Fujita and more generally for non  $\mathbb{A}_*^1$ -uniruled varieties. We show in contrast that for smooth affine factorial  $\mathbb{A}_*^1$ -ruled varieties, cancellation fails in any dimension bigger or equal to two.

Since the late seventies, the Cancellation Problem is usually understood in its geometric form as the question whether two algebraic varieties  $X$  and  $Y$  with isomorphic cylinders  $X \times \mathbb{A}^1$  and  $Y \times \mathbb{A}^1$  are isomorphic themselves. This problem is intimately related to the geometry of rational curves on  $X$  and  $Y$ : in particular, if  $X$  or  $Y$  are smooth quasiprojective and not  $\mathbb{A}^1$ -uniruled, in the sense that they do not admit any dominant generically finite morphism from a variety of the form  $Z \times \mathbb{A}^1$ , then every isomorphism  $\Phi : X \times \mathbb{A}^1 \xrightarrow{\sim} Y \times \mathbb{A}^1$  descends to an isomorphism between  $X$  and  $Y$ , a property which is sometimes called strong cancellation. Over an algebraically closed field of characteristic zero, the non  $\mathbb{A}^1$ -uniruledness of a smooth quasi-projective variety  $X$  is guaranteed in particular by the existence of pluri-forms with logarithmic poles at infinity on suitable projective completions of  $X$ , a property which can be read off from the non-negativity of a numerical invariant of  $X$ , called its (logarithmic) Kodaira dimension  $\kappa(X)$ , introduced by S. Iitaka [11] as the analogue of the usual notion of Kodaira dimension for complete varieties. In this setting, it was established by S. Iitaka et T. Fujita [12] that strong cancellation does hold for a large class of smooth varieties, namely whenever  $X$  or  $Y$  has non-negative Kodaira dimension. This general result implies in particular that cancellation holds for smooth affine curves, due to the fact that the affine line  $\mathbb{A}^1$  is the only such curve with negative Kodaira dimension.

All these assumptions turned out to be essential, as shown by a famous unpublished counter-example due to W. Danielewski [3] of a pair of non-isomorphic smooth complex  $\mathbb{A}^1$ -ruled affine surfaces with isomorphic cylinders. The techniques introduced by W. Danielewski have been the source of many progress on the Cancellation Problem during the last decade but, except for the case of the affine plane  $\mathbb{A}^2$  which was solved earlier affirmatively by M. Miyanishi and T. Sugie [16], the question whether cancellation holds for the complex affine space  $\mathbb{A}^n$  remains one of the most challenging and widely open problem in affine algebraic geometry. In contrast, the same question in positive characteristic was recently settled by the negative by N. Gupta [8], who checked using algebraic methods developed by A. Crachiola and L. Makar-Limanov that a three-dimensional candidate constructed by T. Asanuma [1] was indeed a counter-example.

In this article, we consider another natural cancellation problem in which  $\mathbb{A}^1$  is replaced by the punctured affine line  $\mathbb{A}_*^1 \simeq \text{Spec}(\mathbb{C}[x^{\pm 1}])$  or, more generally, by an algebraic torus  $\mathbb{T}^n = \text{Spec}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ ,  $n \geq 1$ . The question is thus whether two, say smooth quasi-projective, varieties  $X$  and  $Y$  such that  $X \times \mathbb{T}^n$  is isomorphic to  $Y \times \mathbb{T}^n$  are isomorphic themselves. In contrast with the usual Cancellation Problem, this version seems to have received much less attention, one possible reason being that the analogue in this context of the Cancellation Problem for  $\mathbb{A}^n$ , namely the question whether an affine variety  $X$  such that  $X \times \mathbb{T}^n$  is isomorphic to  $\mathbb{T}^{n+m}$  is itself isomorphic to the torus  $\mathbb{T}^n$ , admits an elementary positive answer derived from the knowledge of the structure of the automorphism groups of algebraic tori: indeed, the action of the torus  $\mathbb{T}^n = \text{Spec}(\mathbb{C}[M'])$  by translations on the second factor of  $X \times \mathbb{T}^n$  corresponds to a grading of the algebra of the torus  $\mathbb{T}^{n+m} = \text{Spec}(\mathbb{C}[M])$  by the lattice  $M'$  of characters of  $\mathbb{T}^n$ , induced by a surjective homomorphism  $\sigma : M \rightarrow M'$  from the lattice of characters  $M$  of  $\mathbb{T}^{n+m}$ . Letting  $\tau : M' \rightarrow M$  be a section of  $\sigma$ ,  $M'' = M/\tau(M')$  is a lattice of rank  $m$  for which we have isomorphisms of algebraic quotients  $X \simeq X \times \mathbb{T}^n // \mathbb{T}^n \simeq \mathbb{T}^{n+m} // \mathbb{T}^n \simeq \mathbb{T}^m = \text{Spec}(\mathbb{C}[M''])$ .

Without such precise information on automorphism groups, the question for general varieties  $X$  and  $Y$  is more complicated. Of course, appropriate conditions on the structure of invertible functions on  $X$  and  $Y$  can be imposed to

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guarantee that cancellation holds (see [7] for a detailed discussion of this point of view): this is the case for instance when either  $X$  or  $Y$  does not have non constant such functions. Indeed, given an isomorphism  $\Phi : X \times \mathbb{A}_*^1 \xrightarrow{\sim} Y \times \mathbb{A}_*^1$ , the restriction to every closed fiber of the first projection  $\text{pr}_1 : X \times \mathbb{A}_*^1 \rightarrow X$  of the composition of  $\Phi$  with the second projection  $\text{pr}_2 : Y \times \mathbb{A}_*^1 \rightarrow \mathbb{A}_*^1$  induces an invertible function on  $X$ , implying that  $\Phi$  descends to an isomorphism between  $X$  and  $Y$  as soon as every such function on  $X$  is constant.

But from a geometric point of view, it seems that the cancellation property for  $\mathbb{A}_*^1$  is again related to the nature of affine rational curves contained in the varieties  $X$  and  $Y$ , more specifically to the geometry of images of the punctured affine line  $\mathbb{A}_*^1$  on them. It is natural to expect that strong cancellation holds for varieties which are not dominantly covered by images of  $\mathbb{A}_*^1$ , but this property is harder to characterize in terms of numerical invariants. In particular, in every dimension  $\geq 2$ , there exists smooth  $\mathbb{A}_*^1$ -uniruled affine varieties  $X$  of any Kodaira dimension  $\kappa(X) \in \{-\infty, 0, 1, \dots, \dim X - 1\}$ . In contrast, a smooth complex affine variety  $X$  of log-general type, i.e. of maximal Kodaira dimension  $\kappa(X) = \dim X$ , is not  $\mathbb{A}_*^1$ -uniruled, and another general result of I. Itaka and T. Fujita [12] does indeed confirm that strong cancellation holds for products of algebraic tori with smooth affine varieties of log-general type. Combined with the fact that  $\mathbb{A}_*^1$  is the unique smooth affine curve of Kodaira dimension 0, this is enough for instance to conclude that cancellation holds for smooth affine curves.

Our main result, which can be summarized as follows, shows that similarly as in the case of the usual Cancellation Problem for  $\mathbb{A}^1$ , these assumptions are essential:

**Theorem.** *In every dimension  $d \geq 2$ , there exists non isomorphic smooth factorial affine  $\mathbb{A}_*^1$ -uniruled varieties  $X$  and  $Y$  of dimension  $d$  and Kodaira dimension  $d - 1$  with isomorphic  $\mathbb{A}_*^1$ -cylinders  $X \times \mathbb{A}_*^1$  and  $Y \times \mathbb{A}_*^1$ .*

In dimension  $d \geq 3$ , these families are obtained in the form of total spaces of suitable Zariski locally trivial  $\mathbb{A}_*^1$ -bundles over smooth affine varieties of log-general type. The construction guarantees the isomorphy between the corresponding  $\mathbb{A}_*^1$ -cylinders thanks to a fiber product argument reminiscent to the famous Danielewski fiber product trick in the case of the usual Cancellation Problem. The two-dimensional counter-examples are produced along the same lines, at the cost of replacing the base varieties of the  $\mathbb{A}_*^1$ -bundles involved in the construction by appropriate orbifold curves. The article is organized as follows: in the first section, we establish a variant of Itaka-Fujita strong cancellation Theorem for Zariski locally trivial  $\mathbb{T}^n$ -bundles over smooth affine varieties of log-general type. This criterion is applied in the second section to deduce the existence of families of Zariski locally trivial  $\mathbb{A}_*^1$ -bundles over smooth affine varieties of log-general type with non isomorphic total spaces but isomorphic  $\mathbb{A}_*^1$ -cylinders. The two-dimensional case is treated in a separate sub-section. The last section contains a generalization of some of these constructions to the cancellation problem for higher dimensional tori  $\mathbb{T}^n$  over varieties of dimension at least three, and a complete discussion of the cancellation problem for  $\mathbb{A}_*^1$  in the special case of smooth factorial affine surfaces.

## 1. A CRITERION FOR CANCELLATION

**1.1. Recollection on locally trivial  $\mathbb{T}^n$ -bundles.** In what follows, we denote by  $\mathbb{T}^n$  the spectrum of the Laurent polynomial algebra  $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$  in  $n$  variables. We use the notation  $\mathbb{T}^n$  to indicate that we consider  $\mathbb{T}^n$  as the product  $\mathbb{G}_m^n$ , i.e.  $\mathbb{T}^n$  equipped with its natural algebraic group product structure. The automorphism group  $\text{Aut}(\mathbb{T}^n)$  of  $\mathbb{T}^n$  is isomorphic to the semi-direct product  $\mathbb{T}^n \rtimes \text{GL}_n(\mathbb{Z})$ , where  $\mathbb{T}^n$  acts on  $\mathbb{T}^n$  by translations and where  $\text{GL}_n(\mathbb{Z})$  acts by  $(a_{ij})_{i,j=1,\dots,n} \cdot (t_1, \dots, t_n) = (\prod_{i=1}^n t_i^{a_{1i}}, \dots, \prod_{i=1}^n t_i^{a_{ni}})$ .

**Definition 1.** A *Zariski locally trivial  $\mathbb{T}^n$ -bundle* over a scheme  $X$ , is an  $X$ -scheme  $p : P \rightarrow X$  for which every point of  $X$  has a Zariski open neighbourhood  $U \subset X$  such that  $p^{-1}(U) \simeq U \times \mathbb{T}^n$  as schemes over  $U$ .

**1.1.1.** Isomorphy classes of Zariski locally trivial  $\mathbb{T}^n$ -bundle over  $X$  are in one-to-one correspondence with elements of the Čech cohomology group  $\check{H}^1(X, \text{Aut}(\mathbb{T}^n))$ . Furthermore, letting  $\text{GL}_n(\mathbb{Z})_X$  denote the locally constant sheaf  $\text{GL}_n(\mathbb{Z})$  on  $X$ , we derive from the short exact sequence  $0 \rightarrow \mathbb{T}^n \rightarrow \text{Aut}(\mathbb{T}^n) \rightarrow \text{GL}_n(\mathbb{Z})_X \rightarrow 0$  of sheaves over  $X$  the following long exact sequence in Čech cohomology

$$0 \rightarrow \check{H}^0(X, \mathbb{T}^n) \rightarrow \check{H}^0(X, \text{Aut}(\mathbb{T}^n)) \rightarrow \check{H}^0(X, \text{GL}_n(\mathbb{Z})_X) \rightarrow \check{H}^1(X, \mathbb{T}^n) \rightarrow \check{H}^1(X, \text{Aut}(\mathbb{T}^n)) \rightarrow \check{H}^1(X, \text{GL}_n(\mathbb{Z})_X).$$

If  $X$  is irreducible, then  $\check{H}^1(X, \text{GL}_n(\mathbb{Z})_X) = 0$  and so, every Zariski locally trivial  $\mathbb{T}^n$ -bundle can be equipped with the additional structure of a principal homogeneous  $\mathbb{T}^n$ -bundle. Moreover, two principal homogeneous  $\mathbb{T}^n$ -bundles have isomorphic underlying  $\mathbb{T}^n$ -bundles if and only if their isomorphy classes in  $\check{H}^1(X, \mathbb{T}^n) \simeq H^1(X, \mathbb{T}^n)$  belong to the same orbit of the natural action of  $\check{H}^0(X, \text{GL}_n(\mathbb{Z})_X) \simeq \text{GL}_n(\mathbb{Z})$  which, for every  $(a_{ij})_{i,j=1,\dots,n} \in \text{GL}_n(\mathbb{Z})$ , sends the isomorphy class of the  $\mathbb{T}^n$ -bundle  $p : P \rightarrow X$  with action  $\mathbb{T}^n \times P \rightarrow P$ ,  $((t_1, \dots, t_n), p) \mapsto (t_1, \dots, t_n) \cdot p$  to the isomorphy class of  $p : P \rightarrow X$  equipped with the action  $((t_1, \dots, t_n), p) \mapsto ((a_{ij})_{i,j=1,\dots,n} \cdot (t_1, \dots, t_n)) \cdot p$ . In other words, for an irreducible  $X$ , isomorphy classes of Zariski locally trivial  $\mathbb{T}^n$ -bundles over  $X$  are in one-to-one correspondence with elements of  $H^1(X, \mathbb{T}^n)/\text{GL}_n(\mathbb{Z})$ .

**1.2. Cancellation for  $\mathbf{T}^n$ -bundles over varieties of log-general type.** Recall that the (*logarithmic*) Kodaira dimension  $\kappa(X)$  of a smooth complex algebraic variety  $X$  is the Iitaka dimension of the invertible sheaf  $\omega_{\overline{X}/\mathbb{C}}(\log B) = (\det \Omega_{\overline{X}/\mathbb{C}}^1) \otimes \mathcal{O}_{\overline{X}}(B)$  on a smooth complete model  $\overline{X}$  of  $X$  with reduced SNC boundary divisor  $B = \overline{X} \setminus X$ . So  $\kappa(X)$  is equal to  $\text{tr.deg}_{\mathbb{C}}(\bigoplus_{m \geq 0} H^0(\overline{X}, \omega_{\overline{X}/\mathbb{C}}(\log B)^{\otimes m})) - 1$  if  $H^0(\overline{X}, \omega_{\overline{X}/\mathbb{C}}(\log B)^{\otimes m}) \neq 0$  for sufficiently large  $m$ , and, by convention to  $-\infty$  otherwise. The so-defined element of  $\{-\infty\} \cup \{0, \dots, \dim_{\mathbb{C}} X\}$  is independent of the choice of a smooth complete model  $(\overline{X}, B)$  [11] and coincides with the usual Kodaira dimension in the case where  $X$  is complete. A smooth variety  $X$  such that  $\kappa(X) = \dim_{\mathbb{C}} X$  is said to be of log-general type.

The following Proposition is a variant for Zariski locally trivial bundles of Iitaka-Fujita's strong Cancellation Theorem [12, Theorem 3] for products of varieties of log-general type with affine varieties of Kodaira dimension equal to 0, such as algebraic tori  $\mathbf{T}^n$ .

**Proposition 2.** *Let  $X$  and  $Y$  be smooth algebraic varieties and let  $p : P \rightarrow X$  and  $q : Q \rightarrow Y$  be Zariski locally trivial  $\mathbf{T}^n$ -bundles. If either  $X$  or  $Y$  is of log-general type then for every isomorphism of abstract algebraic varieties  $\Phi : P \xrightarrow{\sim} Q$  between the total spaces of  $P$  and  $Q$ , there exists an isomorphism  $\varphi : X \xrightarrow{\sim} Y$  such that the diagram*

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & Q \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{\varphi} & Y \end{array}$$

*commutes.*

*Proof.* The proof is very similar to that of [12, Theorem 1]. We may assume without loss of generality that  $Y$  is of log-general type. Since  $p$  has local sections in the Zariski topology, it is enough to show that  $q \circ \Phi$  is constant on the fibers of  $p$  to guarantee that the induced set-theoretic map  $\varphi : X \rightarrow Y$  is a morphism. Furthermore, since  $\Phi$  is an isomorphism,  $\varphi$  will be bijective whence an isomorphism by virtue of Zariski Main Theorem [5, 8.12.6]. Since  $p : P \rightarrow X$  is Zariski locally trivial and  $\kappa(\mathbf{T}^n) = 0$ , it follows from [11] that for every prime Weil divisor  $D$  on  $X$ , the Kodaira dimension  $\kappa(p^{-1}(D_{\text{reg}}))$  of the inverse image of the regular part of  $D$  is at most equal to  $\dim D$ . This implies in turn that the restriction of  $q \circ \Phi$  to  $p^{-1}(D)$  cannot be dominant since otherwise we would have  $\kappa(Y) \leq \kappa(p^{-1}(D_{\text{reg}})) < \dim X = \dim Y$ , in contradiction with the assumption that  $\kappa(Y) = \dim Y$ . So there exists a prime Weil divisor  $D'$  on  $Y$  such that the image of  $p^{-1}(D)$  by  $\Phi$  is contained in  $q^{-1}(D')$ , whence is equal to it since they are both irreducible of the same dimension. Now given any closed point  $x \in X$ , we can find a finite collection of prime Weil divisors  $D_1, \dots, D_n$  such that  $D_1 \cap \dots \cap D_n = \{x\}$ . Letting  $D'_i$  be a collection of prime Weil divisors on  $Y$  such that  $\Phi(p^{-1}(D_i)) = q^{-1}(D'_i)$  for every  $i = 1, \dots, n$ , we have

$$q^{-1}\left(\bigcap_{i=1, \dots, n} D'_i\right) = \bigcap_{i=1, \dots, n} q^{-1}(D'_i) = \bigcap_{i=1, \dots, n} \Phi(p^{-1}(D_i)) \simeq \Phi\left(\bigcap_{i=1, \dots, n} p^{-1}(D_i)\right) \simeq \Phi(\{x\} \times \mathbf{T}^n) \simeq \mathbf{T}^n.$$

So the intersection of the  $D'_i$ ,  $i = 1, \dots, n$ , consists of a unique closed point  $y \in Y$  for which we have by construction  $\Phi(p^{-1}(x)) = q^{-1}(y)$ , as desired.  $\square$

*Remark 3.* The proof above shows in fact that the conclusion of the Theorem holds under the more geometric hypothesis that either  $X$  or  $Y$  is not  $\mathbb{A}_*^1$ -uniruled, i.e., does not admit any dominant generically finite morphism from a variety of the form  $Z \times \mathbb{A}_*^1$ . In particular strong cancellation holds for products of algebraic tori  $\mathbf{T}^n$  with non  $\mathbb{A}_*^1$ -uniruled varieties.

**Corollary 4.** *Two smooth curves  $C$  and  $C'$  admit Zariski locally trivial  $\mathbf{T}^n$ -bundles  $p : P \rightarrow C$  and  $p' : P' \rightarrow C'$  with isomorphic total spaces  $P$  and  $P'$  if and only if they are isomorphic.*

*Proof.* If either  $C$  or  $C'$  is of log-general type, then the assertion follows from Proposition 2. Note further that  $C$  is affine if and only if so is  $P$ . Indeed,  $p : P \rightarrow C$  is an affine morphism and conversely, if  $P$  is affine, then viewing  $P$  as a principal homogeneous  $\mathbb{T}^n$ -bundle with geometric quotient  $P//\mathbb{T}^n \simeq C$ , the affineness of  $C$  follows from the fact that the algebraic quotient morphism  $P \rightarrow P//\mathbb{T}^n = \text{Spec}(\Gamma(P, \mathcal{O}_P)^{\mathbb{T}^n})$  is a categorical quotient in the category of algebraic varieties, so that  $C \simeq \text{Spec}(\Gamma(P, \mathcal{O}_P)^{\mathbb{T}^n})$ . Thus  $C$  and  $C'$  are simultaneously affine or projective. In the first case,  $C$  and  $C'$  are isomorphic to either the affine line  $\mathbb{A}^1$  or the punctured affine line  $\mathbb{A}_*^1$  which both have a trivial Picard group. So  $P$  and  $P'$  are trivial  $\mathbf{T}^n$ -bundles and the isomorphy of  $C$  and  $C'$  follows by comparing invertible function on  $P$  and  $P'$ . In the second case, if either  $C$  or  $C'$  has non negative genus, say  $g(C') \geq 0$ , then, being rational, the image of a fiber of  $p : P \rightarrow C$  by an isomorphism  $\Phi : P \xrightarrow{\sim} P'$  must be contained in a fiber of  $p' : P' \rightarrow C'$ . We conclude similarly as in the proof of the previous Proposition that  $\Phi$  descends to an isomorphism between  $C$  and  $C'$ .  $\square$

**1.2.1.** The automorphism group  $\text{Aut}(X)$  of a scheme  $X$  acts on the set of isomorphism classes of principal homogeneous  $\mathbb{T}^n$ -bundles over  $X$  via the linear representation

$$\eta : \text{Aut}(X) \rightarrow \text{GL}(H^1(X, \mathbb{T}^n)), \quad \psi \mapsto \eta(\psi) = \psi^* : H^1(X, \mathbb{T}^n) \xrightarrow{\sim} H^1(X, \mathbb{T}^n),$$

where  $\psi^*$  maps the isomorphism class of principal homogeneous  $\mathbb{T}^n$ -bundle  $p : P \rightarrow X$  to the one of the  $\mathbb{T}^n$ -bundle  $\text{pr}_2 : P \times_{p, X, \psi} X \rightarrow X$ . This action commutes with natural action of  $\text{GL}_n(\mathbb{Z})$  introduced in § 1.1.1 above, and Proposition 2 implies the following characterization:

**Corollary 5.** *Over a smooth variety  $X$  of log-general type, the set  $H^1(X, \mathbb{T}^n)/(\text{Aut}(X) \times \text{GL}_n(\mathbb{Z}))$  parametrizes isomorphism classes as abstract varieties of total spaces of Zariski locally trivial  $\mathbf{T}^n$ -bundles  $p : P \rightarrow X$ .*

## 2. NON-CANCELLATION FOR THE 1-DIMENSIONAL TORUS

Candidates for non-cancellation of the 1-dimensional torus  $\mathbf{T} = \mathbb{A}_*^1 = \text{Spec}(\mathbb{C}[t^{\pm 1}])$  can be constructed along the following lines: given say a smooth quasi-projective variety  $X$  and a pair of non-isomorphic principal homogeneous  $\mathbb{G}_m$ -bundles  $p : P \rightarrow X$  and  $q : Q \rightarrow X$  whose classes generate the same subgroup of  $H^1(X, \mathbb{G}_m)$ , the fiber product  $W = P \times_X Q$  is a principal homogeneous  $\mathbb{T}^2$ -bundle over  $X$ , which inherits the structure of a principal  $\mathbb{G}_m$ -bundle over  $P$  and  $Q$  simultaneously, via the first and the second projection respectively. Since the classes of  $P$  and  $Q$  generate the same subgroup of  $H^1(X, \mathbb{G}_m)$ , it follows that the classes of  $\text{pr}_1 : W \simeq p^*Q \rightarrow P$  and  $\text{pr}_2 : W \simeq q^*P \rightarrow Q$  in  $H^1(P, \mathbb{G}_m)$  and  $H^1(Q, \mathbb{G}_m)$  respectively are both trivial and so, we obtain isomorphisms  $P \times \mathbb{G}_m \simeq W \simeq Q \times \mathbb{G}_m$  of locally trivial  $\mathbf{T}^2$ -bundles over  $X$ .

Then we are left with finding appropriate choices of  $X$  and classes in  $H^1(X, \mathbb{G}_m)$  which guarantee that the total spaces of the corresponding principal homogeneous  $\mathbb{G}_m$ -bundles  $p : P \rightarrow X$  and  $q : Q \rightarrow X$  are not isomorphic as abstract algebraic varieties.

**2.1. Non-cancellation for smooth factorial affine varieties of dimension  $\geq 3$ .** A direct application of the above strategy leads to families of smooth factorial affine varieties of any dimension  $\geq 3$  for which cancellation fails:

**Proposition 6.** *Let  $X$  be the complement of a smooth hypersurface  $D$  of degree  $d$  in  $\mathbb{P}^r$ ,  $r \geq 2$ , such that  $d \geq r+2$  and  $|(\mathbb{Z}/d\mathbb{Z})^*| \geq 3$ , and let  $p : P \rightarrow X$  and  $q : Q \rightarrow X$  be the  $\mathbb{G}_m$ -bundles corresponding to the line bundles  $\mathcal{O}_{\mathbb{P}^n}(1)|_X$  and  $\mathcal{O}_{\mathbb{P}^n}(k)|_X$ ,  $k \in (\mathbb{Z}/d\mathbb{Z})^* \setminus \{\bar{1}, \bar{d}-\bar{1}\}$  under the isomorphism  $H^1(X, \mathbb{G}_m) \simeq \text{Pic}(X)$ . Then  $P$  and  $Q$  are not isomorphic as abstract algebraic varieties but  $P \times \mathbb{A}_*^1$  and  $Q \times \mathbb{A}_*^1$  are isomorphic as schemes over  $X$ .*

*Proof.* The Picard group of  $X$  is isomorphic to the group  $\mu_d \simeq \mathbb{Z}/d\mathbb{Z}$  of  $d$ -th roots of unity, generated by the restriction of  $\mathcal{O}_{\mathbb{P}^n}(1)$  to  $X$ . Since  $k$  is relatively prime with  $d$ ,  $\mathcal{O}_{\mathbb{P}^r}(k)|_X$  is also a generator of  $\text{Pic}(X)$ . This guarantees that  $P \times \mathbb{A}_*^1$  is isomorphic to  $Q \times \mathbb{A}_*^1$  by virtue of the previous discussion. Since  $d \geq r+2$ ,  $K_{\mathbb{P}^r} + D$  is linearly equivalent to a positive multiple of a hyperplane section, and so  $X$  is of log-general type. We can therefore apply Proposition 2 to deduce that for every isomorphism of abstract algebraic varieties  $\Phi : P \xrightarrow{\sim} Q$ , there exists an automorphism  $\varphi$  of  $X$  such that  $P$  is isomorphic to  $\varphi^*Q$  as a Zariski locally trivial  $\mathbb{A}_*^1$ -bundle over  $X$ . In view of Corollary 5, this means equivalently that as a  $\mathbb{G}_m$ -bundle over  $X$ ,  $\varphi^*Q$  is isomorphic to either  $P$  or its inverse  $P^{-1}$  in  $H^1(X, \mathbb{G}_m)$ . Since the choice of  $k$  guarantees that the  $\mathbb{G}_m$ -bundle  $Q$  is isomorphic neither to  $P$  nor to  $P^{-1}$ , the conclusion follows from the observation that the natural action of  $\text{Aut}(X)$  on  $H^1(X, \mathbb{G}_m)$  is the trivial one, due to the fact that every automorphism of  $X$  is the restriction of a linear automorphism of the ambient space  $\mathbb{P}^r$ . Indeed, through the open inclusion  $X \hookrightarrow \mathbb{P}^r$ , we may consider an automorphism  $\varphi$  of  $X$  as a birational self-map of  $\mathbb{P}^r$  restricting to an isomorphism outside  $D$ . If  $\varphi$  is not biregular on the whole  $\mathbb{P}^r$ , then  $D$  would be an exceptional divisor of  $\varphi^{-1}$ , in particular,  $D$  would be birationally ruled, in contradiction with the ampleness of its canonical divisor  $K_D$  guaranteed by the condition  $d \geq r+2$ .  $\square$

**Example 7.** In the setting of Proposition 6 above, an isomorphism  $P \times \mathbb{A}_*^1 \xrightarrow{\sim} Q \times \mathbb{A}_*^1$  can be constructed “explicitly” as follows. The complement  $X \subset \mathbb{P}^r = \text{Proj}(\mathbb{C}[x_0, \dots, x_r])$  of a smooth hypersurface  $D$  defined by an equation  $F(x_0, \dots, x_r) = 0$  for some homogeneous polynomial of degree  $d$  can be identified with the quotient of the smooth factorial affine variety  $\tilde{X} \subset \mathbb{A}^{r+1}$  with equation  $F(x_0, \dots, x_r) = 1$  by the free action of the group  $\mu_d = \text{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^d - 1))$  of  $d$ -th roots of unity defined by  $\varepsilon \cdot (x_0, \dots, x_r) = (\varepsilon x_0, \dots, \varepsilon x_r)$ . The  $\mathbb{G}_m$ -bundles over  $X$  corresponding to the line bundles  $\mathcal{O}_{\mathbb{P}^r}(k)|_X$ ,  $k \in \mathbb{Z}$ , then coincide with the quotients of the trivial  $\mathbb{A}_*^1$ -bundles  $\tilde{X} \times \mathbb{A}_*^1 = \tilde{X} \times \text{Spec}(\mathbb{C}[t^{\pm 1}])$  by the respective  $\mu_d$ -actions  $\varepsilon \cdot (x_0, \dots, x_r, t) = (\varepsilon x_0, \dots, \varepsilon x_r, \varepsilon^k t)$ ,  $k \in \mathbb{Z}$ . Now let  $q : Q \rightarrow X$  be the  $\mathbb{G}_m$ -bundle corresponding to  $\mathcal{O}_{\mathbb{P}^r}(k)|_X$  for some  $k \in \{2, \dots, d-2\}$  relatively prime with  $d$ , and let  $a, b \in \mathbb{Z}$  be such that  $ak - bd = 1$ . Then one checks that the following isomorphism

$$\tilde{\Phi} : \tilde{X} \times \mathbf{T}^2 = \tilde{X} \times \text{Spec}(\mathbb{C}[t_1^{\pm 1}, u_1^{\pm 1}]) \xrightarrow{\sim} \tilde{X} \times \mathbf{T}^2 = \tilde{X} \times \text{Spec}(\mathbb{C}[t_2^{\pm 1}, u_2^{\pm 1}]), \quad (t_1, u_1) \mapsto (t_2, u_2) = (t_1^k u_1, t_1^{bd} u_2^a)$$

of schemes over  $\tilde{X}$  is equivariant for the actions, say  $\mu_{d,1}$  and  $\mu_{d,k}$ , of  $\mu_d$  defined respectively by  $\varepsilon \cdot (x_0, \dots, x_r, t_1, u_1) = (\varepsilon x_0, \dots, \varepsilon x_r, \varepsilon t_1, u_1)$  and  $\varepsilon \cdot (x_0, \dots, x_r, t_2, u_2) = (\varepsilon x_0, \dots, \varepsilon x_r, \varepsilon^k t_2, u_2)$ . It follows that  $\tilde{\Phi}$  descends to an isomorphism

$$\Phi : (\tilde{X} \times \mathbf{T}^2) / \mu_{d,1} \simeq P \times \mathbb{A}_*^1 \xrightarrow{\sim} Q \times \mathbb{A}_*^1 \simeq (\tilde{X} \times \mathbf{T}^2) / \mu_{d,k}$$

of schemes over  $X \simeq \tilde{X} / \mu_d$ .

**2.2. Non-cancellation for smooth factorial affine surfaces.** Since the Picard group of a smooth affine curve  $C$  of log-general type is either trivial if  $C$  is rational or of positive dimension otherwise, there no direct way to adapt the previous construction using principal  $\mathbb{G}_m$ -bundles over algebraic curves to produce 2-dimensional candidate counterexamples for cancellation by  $\mathbb{A}_*^1$ . Instead, we will use locally trivial  $\mathbb{A}_*^1$ -bundles over certain orbifold curves  $\tilde{C}$  which arise from suitably chosen  $\mathbb{A}_*^1$ -fibrations  $\pi : S \rightarrow C$  on smooth affine surfaces  $S$ .

**2.2.1.** Let  $S$  be a smooth affine surface  $S$  equipped with a flat fibration  $\pi : S \rightarrow C$  over a smooth affine rational curve  $C$  whose fibers, closed or not, are all isomorphic to  $\mathbb{A}_*^1$  over the corresponding residue fields when equipped with their reduced structure<sup>1</sup>. It follows from the description of degenerate fibers of  $\mathbb{A}_*^1$ -fibrations given in [15, Theorem 1.7.3] that  $S$  admits a relative completion into a  $\mathbb{P}^1$ -fibered surface  $\bar{\pi} : \bar{S} \rightarrow C$  obtained from a trivial  $\mathbb{P}^1$ -bundle  $\text{pr}_1 : C \times \mathbb{P}^1 \rightarrow C$  with a fixed pair of disjoint sections  $H_0$  and  $H_\infty$  by performing finitely many sequences of blow-ups of the following type: the first step consists of the blow-up of a closed point  $c_i \in H_0$ ,  $i = 1, \dots, s$ , with exceptional divisor  $E_{1,i}$  followed by the blow-up of the intersection point of  $E_{1,i}$  with the proper transform of the fiber  $F_i = \text{pr}_1^{-1}(\text{pr}_1(c_i))$ , the next steps consist of the blow-up of an intersection point of the last exceptional divisor produced with the proper transform of the union of  $F_i$  and the previous ones, in such a way that the total transform of  $F_i$  is a chain of proper rational curves with the last exceptional divisors produced, say  $E_{i,n_i}$ , as the unique irreducible component with self-intersection  $-1$ . The projection  $\text{pr}_1 : C \times \mathbb{P}^1 \rightarrow C$  lifts on the so-constructed surface  $\bar{S}$  to a  $\mathbb{P}^1$ -fibration  $\bar{\pi} : \bar{S} \rightarrow C$  and  $S$  isomorphic to the complement of the union of the proper transforms of  $H_0$  and  $H_\infty$  and of the divisors  $F_i \cup E_{i,1} \cup \dots \cup E_{i,n_i-1}$ ,  $i = 1, \dots, s$ . The restriction of  $\bar{\pi}$  to  $S$  is indeed an  $\mathbb{A}_*^1$ -fibration  $\pi : S \rightarrow C$  with  $s$  degenerate fibers  $\pi^{-1}(\text{pr}_1(c_i))$  isomorphic to  $E_{i,n_i} \cap S \simeq \mathbb{A}_*^1$  when equipped with their reduced structure and whose respective multiplicities depend on the sequences of blow-ups performed.

**2.2.2.** It follows in particular from this construction that  $S$  admits a proper action of the multiplicative group  $\mathbb{G}_m$  which lifts the one on  $(C \times \mathbb{P}^1) \setminus (H_0 \cup H_\infty) \simeq C \times \mathbb{A}_*^1$  by translations on the second factor. The local descriptions given in [6] can then be re-interpreted for our purpose as the fact that the  $\mathbb{A}_*^1$ -fibration  $\pi : S \rightarrow C$  factors through an étale locally trivial  $\mathbb{A}_*^1$ -bundle  $\tilde{\pi} : \tilde{S} \rightarrow \tilde{C}$  over an orbifold curve  $\delta : \tilde{C} \rightarrow C$  obtained from  $C$  by replacing the finitely many points  $c_1, \dots, c_s$  over the which the fiber  $\pi^{-1}(c_i)$  is multiple, say of multiplicity  $m_i > 1$ , by suitable orbifold points  $\tilde{c}_i$  depending only on the multiplicity  $m_i$ . More precisely,  $\tilde{C}$  is a smooth separated Deligne-Mumford stack of dimension 1, of finite type over  $\mathbb{C}$  and with trivial generic stabilizer, which, Zariski locally around  $\delta^{-1}(c_i)$  looks like the quotient stack  $[\tilde{U}_{c_i} / \mathbb{Z}_{m_i}]$ , where  $\tilde{U}_{c_i} \rightarrow U_{c_i}$  is a Galois cover of order  $m_i$  of a Zariski open neighborhood  $U_{c_i}$  of  $c_i$ , totally ramified over  $c_i$  and étale elsewhere [2].

**Example 8.** Let  $\mathbb{G}_m$  act on  $\mathbb{A}_*^2 = \text{Spec}(\mathbb{C}[x, y]) \setminus \{(0, 0)\}$  by  $t \cdot (x, y) = (t^2 x, t^5 y)$ . The quotient  $\mathbb{P}(2, 5) = \mathbb{A}_*^2 / \mathbb{G}_m$  is isomorphic to  $\mathbb{P}^1$  and the quotient morphism  $q : \mathbb{A}_*^2 \rightarrow \mathbb{P}^1 = \mathbb{A}_*^2 / \mathbb{G}_m$  is an  $\mathbb{A}_*^1$ -fibration with two degenerate fibers  $q^{-1}([0 : 1])$  and  $q^{-1}([1 : 0])$  of multiplicities 5 and 2 respectively, corresponding to the orbits of the points  $(0, 1)$  and  $(1, 0)$ . In contrast, the quotient stack  $[\mathbb{A}_*^2 / \mathbb{G}_m]$  is the Deligne-Mumford curve  $\mathbb{P}[2, 5]$  obtained from  $\mathbb{P}^1$  by replacing the points  $[0 : 1]$  and  $[1 : 0]$  by “stacky points” with respective Zariski open neighborhoods isomorphic to the quotients  $[\mathbb{A}^1 / \mathbb{Z}_5]$  and  $[\mathbb{A}^1 / \mathbb{Z}_2]$  for the actions of  $\mathbb{Z}_5$  and  $\mathbb{Z}_2$  on  $\mathbb{A}^1 = \text{Spec}(\mathbb{C}[z])$  given by  $z \mapsto \exp(2i\pi/5)z$  and  $z \mapsto -z$ . The quotient morphism  $q : \mathbb{A}_*^2 \rightarrow \mathbb{P}^1$  factors through the canonical morphism  $\tilde{q} : \mathbb{A}_*^2 \rightarrow \mathbb{P}[2, 5]$  which is an étale local trivial  $\mathbb{A}_*^1$ -bundle, and the induced morphism  $\delta : \mathbb{P}[2, 5] \rightarrow \mathbb{P}^1 = \mathbb{A}_*^2 / \mathbb{G}_m$  is an isomorphism over the complement of the points  $[0 : 1]$  and  $[1 : 0]$ .

In the next paragraphs, we construct two smooth affine surfaces  $S_1$  and  $S_2$  with an  $\mathbb{A}_*^1$ -fibration  $\pi_i : S_i \rightarrow \mathbb{A}^1$ ,  $i = 1, 2$ , factoring through a locally trivial  $\mathbb{A}_*^1$ -bundle  $\tilde{\pi} : S_i \rightarrow \mathbb{A}^1[2, 5]$  over the affine Deligne-Mumford curve  $\mathbb{A}^1[2, 5]$  obtained from the one  $\mathbb{P}[2, 5]$  of Example 8 above by removing a general scheme-like point.

<sup>1</sup>In particular,  $\pi$  is an untwisted  $\mathbb{A}_*^1$ -fibration in the sense of [15].

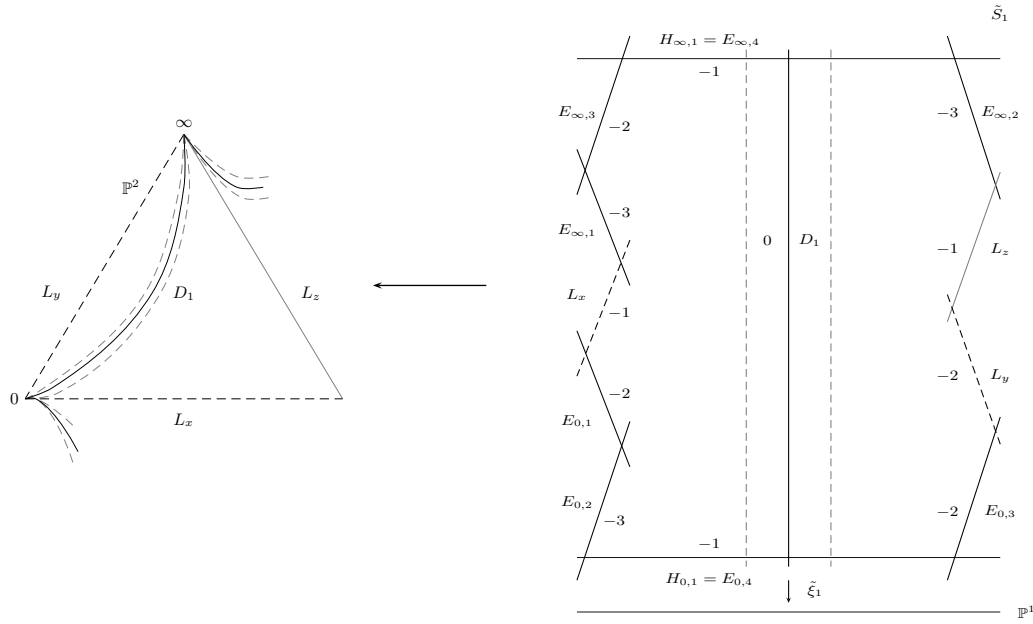


FIGURE 2.1. Minimal resolution of the pencil  $\xi_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ . The exceptional divisors  $E_{0,i}$  and  $E_{\infty,i}$ ,  $i = 1, \dots, 4$ , over the respective proper base points  $0 = [0 : 0 : 1]$  and  $\infty = [0 : 1 : 0]$  of  $\xi_1$  are numbered according to the order of their extraction.

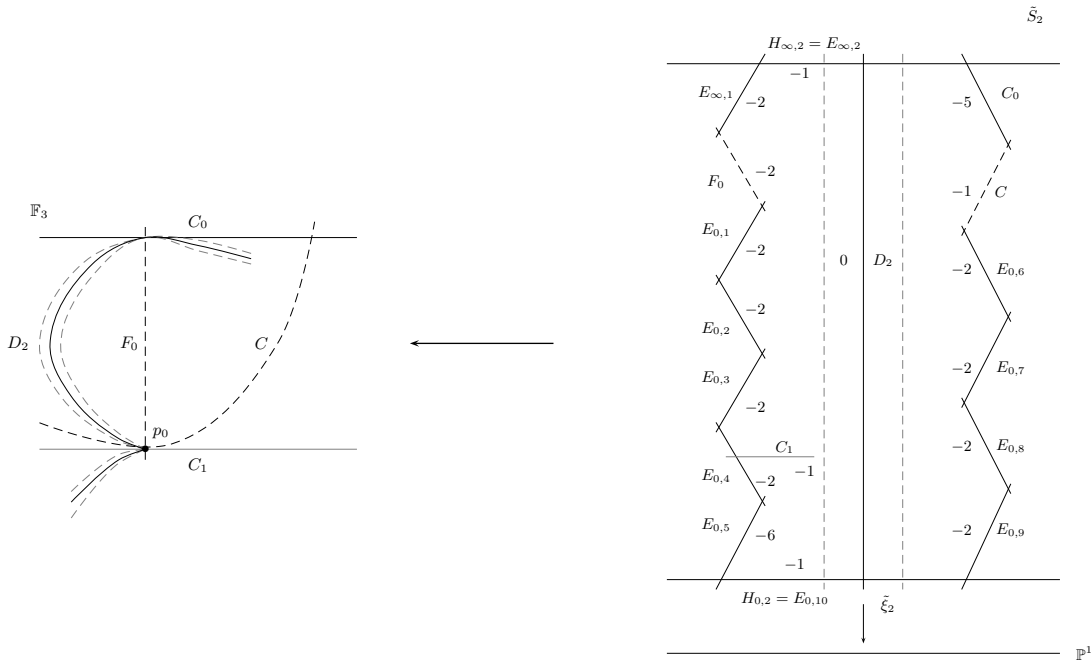


FIGURE 2.2. Minimal resolution of the pencil  $\xi_2 : \mathbb{F}_3 \dashrightarrow \mathbb{P}^1$ . The exceptional divisors  $E_{\infty,1}$ ,  $E_{\infty,2}$  and  $E_{0,i}$ ,  $i = 1, \dots, 10$ , over the respective proper base points  $\infty = F_0 \cap C_0$  and  $0 = p_0$  of  $\xi_2$  are numbered according to the order of their extraction.

**2.2.3.** The first one  $S_1$  is equal to the complement in  $\mathbb{P}^2 = \text{Proj}(\mathbb{C}[x, y, z])$  of the union of the cuspidal curve  $D_1 = \{x^5 - y^2z^3 = 0\}$  and the line  $L_z = \{z = 0\}$ . Equivalently,  $S_1$  is the complement in  $\mathbb{A}^2 = \text{Spec}(\mathbb{C}[x, y]) = \mathbb{P}^2 \setminus L_z$  of the curve  $D_1 \cap \mathbb{A}^2 = \{x^5 - y^2 = 0\}$ . This curve being an orbit with trivial isotropy of the  $\mathbb{G}_m$ -action  $t \cdot (x, y) = (t^2x, t^5y)$  on  $\mathbb{A}^2$ , the composition of the inclusion  $S_1 \hookrightarrow \mathbb{A}_*^2$  with the canonical morphism  $q : \mathbb{A}_*^2 \rightarrow \mathbb{P}[2, 5] = [\mathbb{A}_*^2/\mathbb{G}_m]$  defines a locally trivial  $\mathbb{A}_*^1$ -bundle  $\tilde{\pi}_1 : S_1 \rightarrow \mathbb{P}[2, 5] \setminus q(D_1 \cap \mathbb{A}_*^2) \simeq \mathbb{A}^1[2, 5]$ . The rational pencil  $\xi_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  induced by  $\pi_1 = \delta \circ \tilde{\pi}_1 : S_1 \rightarrow \mathbb{A}^1$  coincide with that generated by the pairwise linearly equivalent divisors  $D_1, 5L_x$  and  $3L_z + 2L_y$ , where  $L_x, L_y$  and  $L_z$  denote the lines  $\{x = 0\}, \{y = 0\}$  and  $\{z = 0\}$  in  $\mathbb{P}^2$  respectively. A minimal resolution  $\tilde{\xi}_1 : \tilde{S}_1 \rightarrow \mathbb{P}^1$  of  $\xi_1$  is depicted in Figure 2.1. A relatively minimal SNC completion  $(\overline{S}_1, B_1)$  of  $S_1$ , with boundary  $B_1 = D_1 \cup H_{\infty,1} \cup H_{0,1} \cup E_{\infty,3} \cup E_{\infty,1} \cup E_{0,1} \cup E_{0,2} \cup E_{\infty,2} \cup E_{0,3}$ , on which  $\pi_1$  extends to a  $\mathbb{P}^1$ -fibration  $\tilde{\pi}_1 : \overline{S}_1 \rightarrow \mathbb{P}^1$  is then obtained from  $\tilde{S}_1$  by contracting the proper transform of  $L_z$ .

**2.2.4.** The second surface  $S_2$  is obtained as follows. In the Hirzebruch surface  $\rho : \mathbb{F}_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) \rightarrow \mathbb{P}^1$  with exceptional section  $C_0$  of self-intersection  $-3$ , we choose a section  $C$  of  $\rho$  in the linear system  $|C_0 + 4F|$ , where  $F$  denotes a general fiber of  $\rho$ , and a section  $C_1$  in the linear system  $|C_0 + 3F|$  intersecting  $C$  with multiplicity 4 in a unique point  $p_0$ . The fact that such pairs of sections exists follows for instance from [4, Lemma 3.2]. Let  $\xi_2 : \mathbb{F}_3 \dashrightarrow \mathbb{P}^1$  be the pencil generated by the linearly equivalent divisors  $C_0 + 5C$  and  $6C_1 + 2F_0$  where  $F_0 = \bar{\rho}^{-1}(\rho(p_0))$ . Let  $D_2$  be a general member of  $\xi$  and let  $S_2 \subset \mathbb{F}_3$  be the complement of  $C_0 \cup C_1 \cup D_2$ . A minimal resolution  $\tilde{\xi}_2 : \tilde{S}_2 \rightarrow \mathbb{P}^1$  of  $\xi_2 : \mathbb{F}_3 \dashrightarrow \mathbb{P}^1$  is depicted in Figure 2.2. A relatively minimal SNC completion  $(\overline{S}_2, B_2)$  of  $S_2$  with boundary  $B_2 = D_2 \cup H_{\infty,2} \cup H_{0,2} \cup E_{\infty,1} \cup E_{0,5} \cup C_0 \cup \bigcup_{i=6}^9 E_{0,i}$ , on which  $\pi_2$  extends to a  $\mathbb{P}^1$ -fibration  $\tilde{\pi}_2 : \overline{S}_2 \rightarrow \mathbb{P}^1$  is then obtained from  $\tilde{S}_2$  by contracting successively the proper transform of  $C_1, E_{0,4}, E_{0,3}, E_{0,2}$  and  $E_{0,1}$ . By construction, the restriction of  $\xi_2$  to  $S_2$  is an  $\mathbb{A}_*^1$ -fibration  $\pi_2 : S_2 \rightarrow \mathbb{A}^1 = \mathbb{P}^1 \setminus \xi(D_2)$  with two degenerate fibers: one of multiplicity 5 supported by  $C \cap S_2 \simeq \mathbb{A}_*^1$ , and one of multiplicity 2 supported by  $F_0 \cap S_2 \simeq \mathbb{A}_*^1$ . So by virtue of § 2.2.2,  $\pi_2$  factors through a locally trivial  $\mathbb{A}_*^1$ -bundle  $\tilde{\pi}_2 : S_2 \rightarrow \mathbb{A}^1[2, 5]$ .

**Proposition 9.** *The surfaces  $S_1$  and  $S_2$  are smooth affine rational and factorial. They are non isomorphic but  $S_1 \times \mathbb{A}_*^1$  is isomorphic to  $S_2 \times \mathbb{A}_*^1$ .*

*Proof.* Since  $S_1$  is a principal open subset of  $\mathbb{A}^2$ , it is smooth affine rational and factorial. The smoothness and the rationality of  $S_2$  are also clear. Since  $D_2$  belongs to the linear system  $6C_0 + 20F$ , it is ample by virtue of [10, Theorem 2.17]. This implies in turn that  $C_0 + C_1 + D_2$  is the support of an ample divisor, whence that  $S_2$  is affine. Since the divisor class group of  $\mathbb{F}_3$  is generated by  $C_0$  and  $F$ , the identity  $F \sim 7C_1 - C_0 - D_2$  in the divisor class group of  $\mathbb{F}_3$  implies that every Weil divisor on  $S_2$  is linearly equivalent to one supported on the boundary  $\mathbb{F}_3 \setminus S_2 = C_0 \cup C_1 \cup D_2$ . So  $S_2$  is factorial. By construction,  $S_1$  and  $S_2$  both have the structure of locally trivial  $\mathbb{A}_*^1$ -bundles  $\tilde{\pi}_i : S_i \rightarrow \mathbb{A}^1[2, 5]$ . The fiber product  $W = S_1 \times_{\mathbb{A}^1[2, 5]} S_2$  thus inherits via the first and the second projection respectively the structure of an étale locally trivial  $\mathbb{A}_*^1$ -bundle over  $S_1$  and  $S_2$ . Since  $H_{\text{ét}}^1(S_i, \mathbb{G}_m) \simeq H^1(S_i, \mathbb{G}_m)$  by virtue of Hilbert's Theorem 90 and  $S_i$  is factorial, it follows that  $W$  is simultaneously isomorphic to the trivial  $\mathbb{A}_*^1$ -bundles  $S_1 \times \mathbb{A}_*^1$  and  $S_2 \times \mathbb{A}_*^1$ . It remains to check that  $S_1$  and  $S_2$  are not isomorphic. Suppose on the contrary that there exists an isomorphism  $\varphi : S_1 \xrightarrow{\sim} S_2$  and consider its natural extension as a birational map  $\varphi : \overline{S}_1 \dashrightarrow \overline{S}_2$  between the smooth SNC completions  $(\overline{S}_1, B_1)$  and  $(\overline{S}_2, B_2)$  of  $S_1$  and  $S_2$  constructed above. Then  $\varphi$  must be a biregular isomorphism. Indeed, if either  $\varphi$  or  $\varphi^{-1}$ , say  $\varphi$ , is not regular then we can consider a minimal resolution  $\overline{S}_1 \xleftarrow{\sigma} X \xrightarrow{\sigma'} \overline{S}_2$  of it. By definition of the minimal resolution, there is no  $(-1)$ -curve in the union  $B$  of the total transforms of  $B_1$  and  $B_2$  by  $\sigma$  and  $\sigma'$  respectively which is exceptional for  $\sigma$  and  $\sigma'$  simultaneously, and  $\sigma'$  consists of the contraction of a sequence of successive  $(-1)$ -curves supported on  $B$ . The only possible  $(-1)$ -curves in  $B$  which are not exceptional for  $\sigma$  are the proper transforms of  $D_1$  and of the two sections  $H_{0,1}$  and  $H_{\infty,1}$  of the  $\mathbb{P}^1$ -fibration  $\tilde{\pi}_1 : \overline{S}_1 \rightarrow \mathbb{P}^1$ , but the contraction of any of these would lead to a boundary which would no longer be SNC, which is excluded by the fact that  $B_2$  is SNC. It follows that every isomorphism  $\varphi : S_1 \xrightarrow{\sim} S_2$  is the restriction of an isomorphism of pairs  $(\overline{S}_1, B_1) \xrightarrow{\sim} (\overline{S}_2, B_2)$ . But no such isomorphism can exist due the fact that the intersection forms of the boundaries  $B_1$  and  $B_2$  are different. Thus  $S_1$  and  $S_2$  are not isomorphic, which completes the proof.  $\square$

### 3. COMPLEMENTS AND OPEN QUESTIONS

**3.1. Non-cancellation for higher dimensional tori.** Continuing the same idea as in section 2 above, it is possible to construct more generally pairs of principal homogeneous  $\mathbb{T}^n$ -bundles over a given smooth variety  $X$  whose total spaces become isomorphic after taking their products with  $\mathbb{T}^n$  but not with any other lower dimensional tori. For instance, one can start with two collections  $\{[p_1], \dots, [p_n]\}$  and  $\{[q_1], \dots, [q_n]\}$  of classes in  $H^1(X, \mathbb{G}_m)$  which generate the same sub-group  $G$  of  $H^1(X, \mathbb{G}_m)$  and consider a pair of principal homogeneous  $\mathbb{T}^n$ -bundles  $p : P \rightarrow X$  and  $q : Q \rightarrow X$  representing the classes  $([p_1], \dots, [p_n])$  and  $([q_1], \dots, [q_n])$  in  $H^1(X, \mathbb{T}) \simeq H^1(X, \mathbb{G}_m)^{\oplus n}$ . Since  $G$  is contained in the kernels of the natural homomorphism  $p^* : H^1(X, \mathbb{G}_m) \rightarrow H^1(P, \mathbb{G}_m)$  and  $q^* : H^1(X, \mathbb{G}_m) \rightarrow H^1(Q, \mathbb{G}_m)$  (see





by  $\mathbb{A}^1$  and  $\mathbb{A}_*^1$ . Namely, let  $S = C \times \mathbb{A}^1$  and  $S' = C' \times \mathbb{A}^1$ , where  $C$  and  $C'$  are punctured affine lines. If either  $C$  or  $C'$ , say  $C$ , is not isomorphic to  $\mathbb{A}^1$ , then  $\kappa(C \times \mathbb{A}_*^1) = \kappa(C) \geq 0$  and so, by Iitaka-Fujita strong cancellation for  $\mathbb{A}^1$ , every isomorphism between  $S \times \mathbb{A}_*^1$  and  $S' \times \mathbb{A}_*^1$  descends to an isomorphism between  $C \times \mathbb{A}_*^1$  and  $C' \times \mathbb{A}_*^1$ . Since cancellation by  $\mathbb{A}_*^1$  holds for smooth affine curves, we deduce in turn that  $C$  and  $C'$  are isomorphic, whence that  $S$  and  $S'$  are isomorphic.

In the case where  $\kappa(S) = \kappa(S') = 0$ , we already observed that cancellation holds if every invertible function on  $S$  or  $S'$  is constant. Therefore, we may assume that  $S$  and  $S'$  both have non constant units whence, by virtue of [9, §5], belong up to isomorphism to the following list of surfaces:  $V_0 = \mathbb{A}_*^1 \times \mathbb{A}_*^1$ , and the complements  $V_k$  in the Hirzebruch surfaces  $\rho_k : \mathbb{F}_k \rightarrow \mathbb{P}^1$ ,  $k \geq 1$ , of a pair of sections  $H_{0,k}$  and  $H_{\infty,k}$  of  $\rho_k$  with self-intersection  $k$  intersecting each others in a unique point  $p_k$ , and a fiber  $F$  or  $\rho_k$  not passing through  $p_k$ . All the surfaces  $V_k$ ,  $k \geq 1$ , admit an  $\mathbb{A}_*^1$ -fibration  $\pi_k : V_k \rightarrow \mathbb{A}_*^1 = \text{Spec}(\mathbb{C}[x^{\pm 1}])$  induced by the restriction of the pencil on  $\mathbb{F}_k$  generated by  $H_{0,k}$  and  $H_{\infty,k}$ . The unique degenerate fiber of  $\pi_k$ , say  $\pi_k^{-1}(1)$  up to a linear change of coordinate on  $\mathbb{A}_*^1$ , is irreducible, consisting of the union of the intersection with  $V_k$  of the exceptional section  $C_{0,k}$  of  $\rho_k$ , and of the fiber  $F_k$  of  $\rho_k$  passing through  $p_k$ , counted with multiplicity  $k$ . Note that  $V_0$  does not contain any curve isomorphic to  $\mathbb{A}^1$  whereas each surface  $V_k$ ,  $k \geq 1$ , contains exactly two such curves: the intersections  $F_k \cap V_k$  and  $C_{0,k} \cap V_k$ . It follows in particular that  $V_0 \times \mathbb{A}_*^1$  cannot be isomorphic to any  $V_k \times \mathbb{A}_*^1$ ,  $k \geq 1$ . Now suppose that there exists an isomorphism  $\Phi : V_k \times \mathbb{A}_*^1 \xrightarrow{\sim} V_{k'} \times \mathbb{A}_*^1$  for some  $k, k' \geq 1$ . Since  $\kappa((F_k \cap V_k) \times \mathbb{A}_*^1) = \kappa((C_{0,k} \cap V_k) \times \mathbb{A}_*^1) = -\infty$ , their respective images by  $\Phi$  cannot be mapped dominantly on  $V_{k'}$  by the first projection and since  $F_{k'} \cap V_{k'}$  and  $C_{0,k'} \cap V_{k'}$  are the unique curves isomorphic to  $\mathbb{A}^1$  on  $V_{k'}$ , we conclude similarly as in the proof of Proposition 2 that  $\Phi$  map  $(\pi_k^{-1}(1))_{\text{red}} \times \mathbb{A}_*^1$  isomorphically onto  $(\pi_{k'}^{-1}(1))_{\text{red}} \times \mathbb{A}_*^1$ . This implies in turn that  $\Phi$  restricts to an isomorphism between the open subsets  $U_k = \pi_k^{-1}(\mathbb{A}_*^1 \setminus \{1\}) \times \mathbb{A}_*^1$  and  $U_{k'} = \pi_{k'}^{-1}(\mathbb{A}_*^1 \setminus \{1\}) \times \mathbb{A}_*^1$  of  $V_k \times \mathbb{A}_*^1$  and  $V_{k'} \times \mathbb{A}_*^1$  respectively. Now  $\mathbb{A}_*^1 \setminus \{1\}$  is of log-general type and since the restrictions of  $\pi_k$  and  $\pi_{k'}$  to  $U_k$  and  $U_{k'}$  are trivial  $\mathbb{A}_*^1$ -bundles, we deduce from Iitaka-Fujita strong cancellation Theorem [12] that the restriction of  $\Phi$  to  $U_k$  descends to an isomorphism  $\varphi : \mathbb{A}_*^1 \setminus \{1\} \xrightarrow{\sim} \mathbb{A}_*^1 \setminus \{1\}$  for which the following diagram commutes

$$\begin{array}{ccc} U_k \times \mathbb{A}_*^1 \simeq (\mathbb{A}_*^1 \setminus \{1\}) \times \mathbb{A}_*^1 \times \mathbb{A}_*^1 & \xrightarrow{\Phi} & (\mathbb{A}_*^1 \setminus \{1\}) \times \mathbb{A}_*^1 \times \mathbb{A}_*^1 \simeq U_{k'} \times \mathbb{A}_*^1 \\ \pi_k \circ \text{pr}_1 \downarrow & & \downarrow \pi_{k'} \circ \text{pr}_1 \\ \mathbb{A}_*^1 \setminus \{1\} & \xrightarrow{\varphi} & \mathbb{A}_*^1 \setminus \{1\}. \end{array}$$

Summing up, if it exists, an isomorphism  $\Phi : V_k \times \mathbb{A}_*^1 \xrightarrow{\sim} V_{k'} \times \mathbb{A}_*^1$  must be compatible with the  $\mathbf{T}^2$ -fibrations  $\pi_k \circ \text{pr}_1 : V_k \times \mathbb{A}_*^1 \rightarrow \mathbb{A}_*^1$  and  $\pi_{k'} \circ \text{pr}_1 : V_{k'} \times \mathbb{A}_*^1 \rightarrow \mathbb{A}_*^1$ . But this is impossible since the multiplicity of the irreducible component  $F_k \cap V_k$  of  $\pi_k^{-1}(1)$  is different for every  $k \geq 1$ . In conclusion, the surfaces  $V_k$ ,  $k \geq 0$ , are pairwise non isomorphic, with pairwise non isomorphic  $\mathbb{A}_*^1$ -cylinders, which completes the proof.  $\square$

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