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# The symmetric structure of the Green–Naghdi type equations

Dena Kazerani<sup>1 2 3</sup>

## Abstract

In this study, we generalize the notion of the symmetry classically defined for hyperbolic systems of conservation law, to evolution equations of conservative form. We first explain how such a symmetrization can work from an abstract point of view. We then apply it to the Green–Naghdi type equations which is a dispersive modification of the hyperbolic Saint-Venant system.

*Keywords* : Symmetric systems, Conservation law, Strict convexity, Variational derivative, Green–Naghdi equations.

## 1 Introduction

Incompressible Euler equations and water waves problem model free surface incompressible fluids under the influence of the gravity. The complexity of these systems leads to consider reduced complex geophysical models to describe coastal oceanic flows. In this work, we focus on a particular type of these reduced models called the Green–Naghdi type model [14]:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2) + \partial_x(gh^2/2 + \alpha h^2 \ddot{h}) = 0 \end{cases} \quad (1)$$

under the hypothesis that  $h(x, t) > 0$ . Here,  $\alpha$  is a positive real number and  $g$  is the gravity constant. The unknown  $h$  represents the fluid height while  $u$  is its average horizontal speed. Moreover, the material derivative  $\dot{(\cdot)}$  is defined by  $\dot{(\cdot)} = \partial_t(\cdot) + u\partial_x(\cdot)$ .

If  $\alpha = 0$ , the system is hyperbolic and equivalent to Saint–Venant or isentropic Euler equations. Therefore, the system with  $\alpha \neq 0$  is different from the Saint–Venant system by some dispersive terms contracted in  $\alpha h \ddot{h}$  term. System (1) has been derived

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for  $\alpha = \frac{1}{3}$  from water waves problem for irrotational flows by Li [20] and by Alvarez & Lannes [1]. In [16], Ionescu derived the same system by a variational method considering the Lagrangian formulation of the irrotational incompressible Euler equations. In [4], the authors obtain (1) for  $\alpha = \frac{1}{4}$  by a different but a formal method without any hypothesis on the irrotationality of the fluid.

It is worth remarking that (1) admits the following conservation law [10, 11, 4],

$$\partial_t E + \partial_x (u(E + p)) = 0, \quad (2)$$

where the energy  $E$  is defined by

$$E(h, u) = gh^2/2 + hu^2/2 + \alpha h^3(\partial_x u)^2/2, \quad (3)$$

and  $p$  by

$$p(h, u) = gh^2/2 + \alpha h^2 \ddot{h}. \quad (4)$$

Contrary to the case of hyperbolic systems, the energy  $E$  and the pressure  $p$  are not functions of unknowns but smooth operators acting on the space of functions the unknown belongs to. Let us note that during all this work, the partial derivative of a function  $f$  with respect to the variable  $x$  is denoted either by  $\partial_x f$  or  $f_x$ .

The main of this paper is to extend the notion of symmetry classically defined for hyperbolic systems, to more general type of equations. Thus, we first recall the definition of symmetrizability for hyperbolic systems as well as some classical results. Then, we generalize these results in Section 2. This allows us to symmetrize the Green–Naghdi equation in the small neighborhood of constant solutions. The symmetry of these equations is based on the strict convexity of the Hamiltonian around constant solutions. As a matter of fact, this Hamiltonian whose expression is recalled in Subsection 1.2, coincides with the spatial integral of the energy of the system.

## 1.1 Symmetric structure

This subsection is a brief review on results already known on hyperbolic systems of conservation laws. In fact, we consider the system

$$\partial_t U + \partial_x F(U) = 0 \quad (5)$$

where the flux  $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$  and the unknown  $U : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^N$  are smooth functions and  $N \geq 1$  an integer.

**Definition 1.1** *The hyperbolic system (5) is called symmetrizable if there exists a change of variable  $U \mapsto Q$  such that (5) is equivalent to*

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x Q = 0, \quad (6)$$

where  $A_0(Q)$  is a symmetric definite positive matrix and  $A_1(Q)$  is a symmetric one.

**Definition 1.2** *The pair  $(E, P)$  with  $\nabla_U^2 E$  definite positive, is an entropy pair for System (5) if the solution  $U$  satisfies*

$$\partial_t E(U) + \partial_x P(U) = 0, \quad (7)$$

or equivalently if

$$(\nabla_U F(U))^T \nabla_U E(U) = \nabla_U P(U). \quad (8)$$

The following proposition illustrates how the notions of entropy and symmetry are related. Let us mention that symbol  $M^T$  represents the transpose of the matrix  $M$ . Moreover,  $\nabla_V U$  denotes the Jacobian matrix of the change of variable  $V \mapsto U$ . Considering a scalar function  $E$  of  $U$ , the gradient and the Hessian of  $E$  are respectively symbolized by  $\nabla_U E$  and by  $\nabla_U^2 E$ .

**Proposition 1.3** [7, 13, 22, 3, 21] *Let us assume that the hyperbolic system (5) admits an entropy pair called  $(E, P)$ . Then, it is symmetrizable under any change of variable  $U \mapsto V$  under the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x V = 0,$$

where

$$A_0(V) = (\nabla_V U)^T \nabla_U^2 E \nabla_V U, \quad \text{and} \quad A_1(V) = (\nabla_V U)^T \nabla_U^2 E \nabla_U F(U) \nabla_V U. \quad (9)$$

**Proof** *Considering a change of variable  $U \mapsto V$ , System (5) becomes*

$$\nabla_V U \partial_t V + \nabla_U F(U) \nabla_V U \partial_x V = 0. \quad (10)$$

We now apply  $(\nabla_V U)^T \nabla_U^2 E$  from the left side and obtain

$$(\nabla_V U)^T \nabla_U^2 E \nabla_V U \partial_t V + (\nabla_V U)^T \nabla_U^2 E \nabla_U F(U) \nabla_V U \partial_x V = 0. \quad (11)$$

The strict convexity of the entropy implies the definite positivity of the matrix  $A_0(V) = (\nabla_V U)^T \nabla_U^2 E \nabla_V U$ . Moreover,  $A_0(V)$  is obviously symmetric. Therefore, we just need to prove the symmetry of  $\nabla_U^2 E \nabla_U F(U)$ . To do so, we consider the change of variable  $U \mapsto Q$  where

$$Q = \nabla_U E(U). \quad (12)$$

This change of variable is valid since  $E$  is strictly convex. This is also to say that the Legendre transform  $E^*$  of  $E$  [6] defined by

$$E^*(Q) = Q \cdot (\nabla_U E)^{-1}(Q) - E((\nabla_U E)^{-1}(Q)), \quad (13)$$

is such that

$$U = \nabla_Q E^*(Q). \quad (14)$$

Let us now define the scalar function  $\hat{P}$  by

$$\hat{P}(Q) = Q \cdot F(U(Q)) - P(U(Q)). \quad (15)$$

Then, we use Definition 1.2 of the entropy (more precisely the relation (8)) to get

$$\nabla_Q \hat{P}(Q) = F(U). \quad (16)$$

Hence,

$$\nabla_U^2 E \nabla_U F(U) = \nabla_U^2 E \nabla_Q^2 \hat{P}(Q) \nabla_U Q = \nabla_U^2 E \nabla_Q^2 \hat{P}(Q) \nabla_U^2 E,$$

is symmetric.

Moreover, gathering (14) and (16), we remark that (5) is equivalent to

$$\partial_t (\nabla_Q E^*(Q)) + \partial_x (\nabla_Q \hat{P}(Q)) = 0.$$

This is to say that (5) is a Godunov system [13] under the variable  $Q$ . This structure is important because it is equivalent to the existence of entropy equality. ■

Assuming that System (5) admits a strictly convex entropy  $E$ , there is a change of variable  $U \mapsto V$  which is particularly interesting. It is based on the decomposition  $(U_1, U_2)$  of the unknown  $U$  and is obtained by the partial gradient of the entropy  $E$ . More precisely, it is given by

$$V = (U_1, \nabla_{U_2} E(U_1, U_2))^1.$$

The main advantage of this change of variable is the fact that  $A_0(V)$  is block diagonal [18]. This fact is a consequence of the expression (9) of  $A_0(V)$ .

Let us now consider the Saint-Venant system which is nothing but the Green–Naghdi equation with  $\alpha = 0$ . We endow the system with the unknown  $U = (h, hu)$  and the entropy  $E$  given by (3). Then, we take the change of variable  $Q = \nabla_U E(U)$ . This is to say that  $Q = (gh - u^2/2, u)$ . We can check that the system is equivalent to

$$A_0(Q) \partial_t Q + A_1(Q) \partial_x Q = 0,$$

where

$$A_0(Q) = \frac{1}{g} \begin{pmatrix} 1 & u \\ u & gh + u^2 \end{pmatrix}, \quad \text{and} \quad A_1(Q) = \frac{1}{g} \begin{pmatrix} u & gh + u^2 \\ gh + u^2 & 3ghu + u^3 \end{pmatrix}.$$

We can also consider the change of variable  $U \mapsto V = (h, \nabla_{hu} E)$  to get  $V = (h, u)$  and write the system as

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0,$$

with

$$A_0(V) = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}, \quad \text{and} \quad A_1(V) = g \begin{pmatrix} u & h \\ h & hu \end{pmatrix}.$$

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<sup>1</sup>This change of variable can be considered only if the application  $\phi \mapsto \nabla_{U_1} E(U_1, \phi)$  is invertible.

## 1.2 Hamiltonian structure of the Green–Naghdi equations

In [20], Li shows that System (1) admits a Hamiltonian structure under the unknown  $U = (h, m)$  defined by

$$m = \mathcal{L}_h(u) = hu - \alpha (h^3 u_x)_x. \quad (17)$$

The change of variable  $(h, u) \mapsto (h, m)$  is valid since the Sturm–Liouville operator  $\mathcal{L}_h$  is a diffeomorphism from  $\mathbb{H}^{s+2}(\mathbb{R})$  to  $\mathbb{H}^s(\mathbb{R})$  due to the fact that  $h$  is positively bounded by below. Let us also mention that the variable  $m$  has been used in [11] to define the generalized velocity  $k = \frac{m}{h}$ . We now recall the definition of the variational derivative in order to understand the Hamiltonian structure.

**Definition 1.4** [12, 23] *Let us also consider a smooth application  $G$  and define  $\mathcal{G}$  by*

$$\mathcal{G}(U) = \int_{\mathbb{R}^d} G(U).$$

*Then, the variational derivative at  $U$  of  $\mathcal{G}$  is the element  $\delta\mathcal{G}(U)$  such that we have for all  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{G}(U + \epsilon\phi) = \int_{\mathbb{R}^d} \delta\mathcal{G}(U) \cdot \phi.$$

*Moreover, the second variation at  $U$  of  $\mathcal{G}$  is the symmetric operator  $\delta^2\mathcal{G}(U)$  such that we have for all  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,*

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \delta\mathcal{G}(U + \epsilon\phi) = \delta^2\mathcal{G}(U)\phi.$$

Let us now fix a strictly positive real  $\bar{h}$ . We now illustrate in the following proposition the Hamiltonian structure of the Green–Naghdi equations inherited from the free-surface Euler equations' structure.

**Proposition 1.5** [20] *Let  $\bar{h} > 0$  be a real constant. System (1) is equivalent to*

$$\partial_t U = \mathcal{J}(U) \delta\mathcal{H}_{\bar{h}}(U), \quad (18)$$

where

$$U = (h, m) = (h, \mathcal{L}_h(u)),$$

$$\mathcal{H}_{\bar{h}}(h, u) = \int_{\mathbb{R}} gh(h - \bar{h})/2 + hu^2/2 + \alpha h^3(u_x)^2/2, \quad (19)$$

and

$$\mathcal{J}(U) = - \begin{pmatrix} 0 & \partial_x(h(\cdot)) \\ h\partial_x & \partial_x(m(\cdot)) + m\partial_x \end{pmatrix}. \quad (20)$$

*This is to say that we have for all test functions  $\phi, \psi$*

$$\mathcal{J}(U) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = - \begin{pmatrix} \partial_x(h\psi) \\ h\partial_x\phi + \partial_x(m\psi) + m\partial_x\psi \end{pmatrix}. \quad (21)$$

Let us note that by Definition 1.4 of the variational derivative and the expression (19) of  $\mathcal{H}_{\bar{h}}(U)$ , we have

$$\delta\mathcal{H}_{\bar{h}}(U) = (\sigma, u),$$

with

$$\sigma = gh - g\bar{h}/2 - u^2/2 - \frac{3}{2}\alpha h^2(u_x)^2. \quad (22)$$

The variable  $\sigma$  has also been used in [10] for the canonical representation of the Green–Naghdi equations.

We also remark that  $\mathcal{H}_{\bar{h}}$ , which is called the Hamiltonian with respect to  $\bar{h}$ , can be seen as the integral of an energy of the system. In fact, we have

$$\mathcal{H}_{\bar{h}} = \int_{\mathbb{R}} E_{\bar{h}},$$

where

$$E_{\bar{h}} = gh(h - \bar{h})/2 + hu^2/2 + \alpha h^3(u_x)^2/2. \quad (23)$$

is an energy of the system. This fact is developed in the following proposition.

**Proposition 1.6** *System (1) is endowed with the following conservation law,*

$$\partial_t E_{\bar{h}} + \partial_x(u(E_{\bar{h}} + p)) = 0, \quad (24)$$

where

$$p(h, u) = gh^2/2 + \alpha h^2\ddot{h},$$

and  $E_{\bar{h}}$  is given by (23).

**Proof** *Let us just consider (2) and remark that  $E = E_{\bar{h}} + gh\bar{h}/2$ . Injecting this equality into (2), we get*

$$\partial_t(E_{\bar{h}} + gh\bar{h}/2) + \partial_x(u(E_{\bar{h}} + gh\bar{h}/2 + p)) = 0.$$

Hence,

$$\partial_t E_{\bar{h}} + \partial_x(u(E_{\bar{h}} + p)) = -\frac{g\bar{h}}{2}(\partial_t h + \partial_x(hu)).$$

*Then using the mass conservation equality, we see that the right hand side term vanishes.*

■

Let us now remark that the energy  $E_{\bar{h}}$  of the conservation law (24) represents the physical energy of the system. Thus, the Hamiltonian (19) is the whole space integral of the physical energy and it can be interpreted as the total energy of the system. Therefore, as one may expect, it is conserved over time. Indeed, we just need to integrate (2) on  $\mathbb{R}$  and observe that the integral of the second term of the energy conservation vanishes

if  $u$  vanishes at infinity for all time <sup>2</sup>. This result can also be obtained using the Hamiltonian structure (18) of the system and the fact that  $\mathcal{J}(U)$  is a skew-symmetric operator acting on the space of vector-valued functions whose second component converges to 0 at infinity. Hence,

$$\frac{d}{dt}\mathcal{H}(U(t)) = \int_{\mathbb{R}} \delta\mathcal{H}(U) \cdot \partial_t U = \int_{\mathbb{R}} \delta\mathcal{H}(U) \cdot \mathcal{J}(U)\delta\mathcal{H}(U) = 0.$$

### 1.3 General idea

We insist in this introduction on the notion of symmetrizability because it turns out to be useful to prove the local well-posedness of hyperbolic systems (see [3] for instance) as well as the stability of constant solutions of hyperbolic systems with dissipative terms [15, 25, 18, 24].

The generalization of the notion of symmetrizability to dispersive perturbations of hyperbolic systems has been the goal of some authors. Especially, Gavriluk and Gouin in [9] (see also [2]) put Euler–Korteweg models and some  $p$ -systems under a quasi symmetric structure. Similar ideas can be partially adapted to some generalized  $p$ -systems like bubbly fluid equations and to modified Lagrangian Green–Naghdi [8].

These generalizations are usually made in the hope of extending the hyperbolic results on well-posedness to their dispersive perturbations. In a very recent work (see [19]), we use the generalized symmetric structure presented in this work (more precisely in Section 3) to prove the asymptotic stability of constant solutions of the Green–Naghdi equations with viscosity. Let us note that the symmetric structure presented here for Green–Naghdi equations holds only in a small enough neighborhood of constant solutions. This is to say that we consider the symmetrizability as a local notion. As we can see in [19], this is not an obstacle to prove the stability of equilibriums since the solution of the viscous Green–Naghdi equations remains close to equilibriums for initial data close enough to these solutions.

Actually, the result presented in this work concerns more general systems written under the following conservative form

$$\partial_t U + \partial_x F(U) = 0, \tag{26}$$

where the unknown  $U$  belongs to  $C([0, T]; \mathcal{A})$  for some  $T > 0$  and some  $\mathcal{A}$  set of continuous functions of  $x$  converging to 0 at infinity<sup>3</sup>. Let us note that  $F$  is not anymore a function of  $\mathbb{R}^N$  but a smooth application defined on  $\mathcal{A}$ . This is actually the case for the Green–Naghdi equations. As we will see in Subsection 3.1, the Green–Naghdi equations

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<sup>2</sup>It has been shown in [20, 17] that the Green–Naghdi equations endowed with the unknown  $(h-\bar{h}, u)$  are well-posed in  $C([0, T]; \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R}))$  for some  $T > 0$  and  $s \geq 2$ . Hence,  $u$  is a continuous function vanishing at infinity using the Sobolev embedding.

<sup>3</sup> $\mathcal{A}$  is also supposed to be a subspace of  $\mathbb{L}^2(\mathbb{R})$ .



under the Hamiltonian variable  $(h, m)$  fits the abstract form (26) with no loss of derivatives through  $F$ .

For sake of simplicity, we mainly consider the one-dimensional problem (26). We are going to see at the end of this section that the result is easily generalized to the multi-dimensional problem of the form

$$\partial_t U + \sum_{i=1}^d \partial_{x_i} F_i(U) = 0. \quad (27)$$

## 2 Weak symmetric structure

In this part, we find a necessary and sufficient condition for the symmetrizability of System (26) under any variable. First, we generalize the notion of symmetrizability, we recall the definition of the differential and we define the Legendre transform of a variational functional. Then, we will see how a convenient strictly convex functional can lead to the symmetrizability.

**Definition 2.1** *Let  $d \in \mathbb{N}$  and consider an operator  $\mathcal{F} : \mathcal{A} \subset \mathbb{L}^2(\mathbb{R}^d) \rightarrow \mathbb{L}^2(\mathbb{R}^d)$  for a vector space  $\mathcal{A}$ . Then, the adjoint operator of  $\mathcal{F}$  is the operator  $\mathcal{F}^T$  which satisfies*

$$\int_{\mathbb{R}^d} \phi \cdot \mathcal{F}(\psi) = \int_{\mathbb{R}^d} \mathcal{F}^T(\phi) \cdot \psi \quad \forall \phi, \psi \in \mathcal{A}.$$

Moreover,  $\mathcal{F}$  is called symmetric if it is equal to its adjoint. In other words,  $\mathcal{F}$  is called symmetric if we have

$$\int_{\mathbb{R}^d} \phi \cdot \mathcal{F}(\psi) = \int_{\mathbb{R}^d} \mathcal{F}(\phi) \cdot \psi \quad \forall \phi, \psi \in \mathcal{A}.$$

Besides,  $\mathcal{F}$  is definite positive if it satisfies for all  $\phi \in \mathcal{A} \setminus \{0\}$ ,

$$\int_{\mathbb{R}^d} \phi \cdot \mathcal{F}\phi > 0.$$

**Definition 2.2 (Weak symmetrizability)** *System (26) is called weakly symmetrizable if there exists a change of unknown  $U \mapsto Q$ <sup>4</sup> such that (26) is equivalent to*

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x(Q) = 0 \quad (28)$$

where  $A_0(Q)$  is a symmetric definite positive operator and  $A_1(Q)$  is a symmetric one in the sense of Definition 2.1 (for  $d = 1$ ).

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<sup>4</sup>A change of unknown  $U \mapsto Q$  is a diffeomorphism defined on the functional space  $U$  belongs to.

**Definition 2.3 (Differentiability)**[12] Let us consider an application  $F : \mathcal{A} \rightarrow \mathcal{B}^5$ . We say that  $F$  is differentiable at  $U \in \mathcal{A}$  if there exists a continuous linear application  $DF(U)$  such that we have,

$$F(U + h) = F(U) + DF(U)(h) + \epsilon(\|h\|),$$

where

$$\lim_{\|h\| \rightarrow 0} \frac{\epsilon(\|h\|)}{\|h\|} = 0.$$

Moreover,  $DF(U)$  is called the differential of  $F$  at  $U$ .

Let us note that during this work,  $\delta_U \mathcal{H}$  represents the variational derivative of the functional  $\mathcal{H}$  with respect to the unknown  $U$ . Likewise,  $D_U F$  represents the differential of the application  $F$  with respect to the unknown  $U$ .

**Definition 2.4 (Legendre Transform)** Let us consider a smooth application  $E : \mathcal{A} \subset \mathbb{L}^2(\mathbb{R}^d) \rightarrow \mathbb{L}^1(\mathbb{R}^d)$  and define the functional  $\mathcal{H}$  by

$$\mathcal{H}(U) = \int_{\mathbb{R}^d} E(U).$$

Assume also that there exists a set  $\mathcal{B} \subset \mathbb{L}^2(\mathbb{R}^d)$  such that the application

$$\delta_U \mathcal{H} : \begin{cases} \mathcal{A} \rightarrow \mathcal{B} \\ U \mapsto \delta_U \mathcal{H}(U) \end{cases}$$

is continuous and invertible i.e  $\delta_U \mathcal{H}$  is a diffeomorphism and the change of unknown induced by  $\delta_U \mathcal{H}$  is valid. Then, the Legendre transform  $\mathcal{H}^*$  of  $\mathcal{H}$  is defined on  $\mathcal{B}$  by

$$\mathcal{H}^*(Q) = \int_{\mathbb{R}^d} Q \cdot (\delta_U \mathcal{H})^{-1}(Q) - E[(\delta_U \mathcal{H})^{-1}(Q)]. \quad (29)$$

We now see in the following proposition some properties of the Legendre Transform of a strictly convex functional.

**Proposition 2.5** The Legendre Transform  $\mathcal{H}^*$  of a strictly convex functional  $\mathcal{H}$  is well defined and is strictly convex.

**Proof** The Legendre transform is well defined since the variational derivative of a smooth strictly convex functional is invertible and continuous. On the other hand, considering the expression (29) of the Legendre transform, we remark that

$$\delta_Q \mathcal{H}^*(Q) = (\delta_U \mathcal{H})^{-1}(Q).$$

In other words,

$$Q = \delta_U \mathcal{H}(U) \iff U = \delta_Q \mathcal{H}^*(Q).$$

Hence, the definite positivity of the second variation of  $\mathcal{H}$  implies the definite positivity of the second variation of  $\mathcal{H}^*$ . This is to say that the strict convexity of  $\mathcal{H}$  implies the strict convexity of  $\mathcal{H}^*$ . ■

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<sup>5</sup>Spaces  $\mathcal{A}$  and  $\mathcal{B}$  are supposed to be normed vector spaces.

The following theorem gives us a condition which leads to the symmetrizability of (26) under any variable. For sake of simplicity we first consider the one-dimensional case where  $d = 1$ .

**Theorem 2.6** *Let us consider an open convex subset  $\Omega$  of the normed vector space  $\mathcal{A}$  and assume that there exists a functional  $\mathcal{H}(U) := \int_{\mathbb{R}} E(U)$  defined on  $\mathcal{A}$  and strictly convex on  $\Omega$ , such that  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is a symmetric operator for all  $U \in \Omega$ . Then, (26) has a general Godunov structure (introduced in [10]), i.e. there exist a change of unknown  $U \mapsto Q$  defined on  $\Omega$ , as well as a functional  $\mathcal{R}(Q) = \int_{\mathbb{R}} R(Q)$  such that as long as the solution  $U$  remains in  $\Omega$ , the system is equivalent to*

$$\partial_t (\delta_Q \mathcal{H}^*(Q)) + \partial_x (\delta_Q \mathcal{R}(Q)) = 0, \quad (30)$$

where  $\mathcal{H}^*$  is the Legendre transform of  $\mathcal{H}$ .

**Proof** *Let us first consider the change of variable  $U \mapsto Q$  defined by*

$$Q = \delta_U \mathcal{H}(U). \quad (31)$$

or equivalently by

$$U = \delta_Q \mathcal{H}^*(Q). \quad (32)$$

Considering the fact that  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is symmetric on the open convex set  $\Omega$ , there exists, by Poincaré theorem [5], a differentiable application  $\mathcal{N} : \Omega \rightarrow \mathbb{R}$  such that

$$D_U \mathcal{N}(U) \phi = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \phi \quad \forall \phi \in \mathcal{A}. \quad (33)$$

We now define the functional  $\mathcal{R}$  by

$$\mathcal{R}(Q) = \int_{\mathbb{R}} Q \cdot F(U(Q)) - \mathcal{N}(U(Q)). \quad (34)$$

We differentiate (34) and take the action on a test function  $\psi$ . This leads to

$$D_Q \mathcal{R}(Q) \psi = \int_{\mathbb{R}} F(U(Q)) \cdot \psi + Q \cdot D_U F(U) D_Q U(\psi) - D_U \mathcal{N}(U) D_Q U(\psi).$$

Then, we have by (31),

$$D_Q \mathcal{R}(Q) \psi = \int_{\mathbb{R}} F(U(Q)) \cdot \psi + \delta_U \mathcal{H}(U) \cdot D_U F(U) D_Q U(\psi) - D_U \mathcal{N}(U) D_Q U(\psi).$$

Finally, using (33), we find

$$D_Q \mathcal{R}(Q) \psi = \int_{\mathbb{R}} F(U(Q)) \cdot \psi,$$

or equivalently

$$\delta_Q \mathcal{R}(Q) = F(U(Q)). \quad (35)$$

Considering, (26) together with (32) and (35), we can rewrite (26) under

$$\partial_t (\delta_Q \mathcal{H}^*(Q)) + \partial_x (\delta_Q \mathcal{R}(Q)) = 0.$$

Hence, the system owns a general Godunov structure.

■

Intuitively, the general Godunov structure (30) implies the symmetrizability of the system. Indeed, the symmetrizability under the unknown  $Q$  is quite easy to see. In fact, (30) can be written as

$$\delta_Q^2 \mathcal{H}^*(Q) \partial_t Q + \delta_Q^2 \mathcal{R}(Q) \partial_x Q = 0.$$

Setting  $A_0(Q) = \delta_Q^2 \mathcal{H}^*(Q)$  and  $A_1(Q) = \delta_Q^2 \mathcal{R}(Q)$ , the symmetric structure under  $Q$  is highlighted since  $\mathcal{H}^*(Q)$  is strictly convex as a Legendre Transform of a strictly convex functional. We are going to see in the following theorem that the general Godunov structure of the system implies its symmetrizability under any change of unknown.

**Theorem 2.7** *Let  $\Omega$  be an open convex subset of the normed vector space  $\mathcal{A}$ . Let us also assume that there exists a functional  $\mathcal{H}(U) := \int_{\mathbb{R}} E(U)$  strictly convex on  $\Omega$  such that System (26) is of the form (30) with  $Q = \delta_U \mathcal{H}(U)$ . Then, (26) is weakly symmetrizable on  $\Omega$  for any change of unknown  $U \mapsto V$ . This is to say that there exists a symmetric definite positive operator  $A_0(V)$  and a symmetric one  $A_1(V)$  such that as long as the solution  $U$  remains in  $\Omega$ , the system is equivalent to*

$$A_0(V) \partial_t V + A_1(V) \partial_x V = 0.$$

Moreover, the expressions of  $A_0(V)$  and  $A_1(V)$  are given by

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U, \quad (36)$$

and

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F(U) D_V U. \quad (37)$$

**Proof** *Let us consider (26) and a change of unknown  $U \mapsto V$ . Therefore, the system is equivalent to*

$$D_V U \partial_t V + D_U F(U) D_V U \partial_x V = 0. \quad (38)$$

*We are now taking the left side action of  $(D_V U)^T \delta_U^2 \mathcal{H}(U)$  on (38). This leads us to*

$$(D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U \partial_t V + (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F(U) D_V U \partial_x V = 0. \quad (39)$$

*Hence, the theorem is proved if we show that  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is symmetric. To do so, let us consider the application*

$$\mathcal{N}(U) := \int_{\mathbb{R}} Q \cdot F(U) - \mathcal{R}(Q(U)).$$

This leads to

$$D_U \mathcal{N}(U)\phi = \int_{\mathbb{R}} -\delta_Q \mathcal{R}(Q) \cdot D_U Q \phi + F(U) \cdot D_U Q \phi + Q \cdot D_U F(U)\phi \quad \forall \phi \in \mathcal{A}.$$

Considering the fact that we have by assumption  $\delta_Q \mathcal{R}(Q) = F(U)$  within a constant, together with the fact that  $Q = \delta_U \mathcal{H}(U)$ , we find

$$D_U \mathcal{N}(U)\phi = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U)\phi \quad \forall \phi \in \mathcal{A}.$$

The symmetry of  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is just a consequence of the integrability of  $\phi \mapsto \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U)\phi$ . ■

Gathering Theorems (2.6) and (2.7), we remark that the fact that (26) owns a general Godunov structure through a strictly convex functional  $\mathcal{H}$ , is equivalent to the symmetry of the operator  $\delta_U^2 \mathcal{H}(U) D_U F(U)$ . Moreover, both of these assertions are equivalent to the weak symmetrizability of the system under any change of unknown.

**Corollary 2.8** *The three following statements are equivalent:*

1. System (26) owns a general Godunov structure through a strictly convex functional  $\mathcal{H}^*$ .
2. There exists a strictly convex functional  $\mathcal{H}$  such that the operator  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is symmetric.
3. System (26) is weakly symmetrizable under any change of unknown  $U \mapsto V$  of the form  $A_0(V) \partial_t V + A_1(V) \partial_x V = 0$  where the expressions of  $A_0(V)$  and  $A_1(V)$  are given by (36) and (37).

As mentioned previously, It is well-known that in the case of hyperbolic systems, the Godunov structure and the existence of an entropy equality are equivalent. We may wonder if such an equivalence holds true for the abstract system (26). In fact, as illustrated in the following proposition, the general Godunov structure of the system leads to a conserved quantity.

**Proposition 2.9** *Let us assume that (26) is a general Godunov system on an open convex subset  $\Omega$  of  $\mathcal{A}$ . i.e. there exists a strictly convex functional  $\mathcal{H} = \int_{\mathbb{R}} E(U)$  defined on  $\Omega$  such that, as long as  $U$  remains in  $\Omega$ , the system is equivalent to (30) for a functional  $\mathcal{R}(Q) = \int_{\mathbb{R}} R(Q)$  defined on  $\delta_U \mathcal{H}(\Omega)$ . Then, the solution  $U$  satisfies*

$$\int_{\mathbb{R}} \partial_t E(U) + \partial_x N(U) \, dx = 0,$$

where

$$N(U) = Q(U) \cdot F(U) - R(Q(U)).$$

**Proof** Let us first write (26) as

$$\partial_t U + D_U F(U) \partial_x U = 0. \quad (40)$$

We now take the left side action of  $D_U E(U)$  on (40) and find

$$D_U E(U) \partial_t U + D_U E(U) D_U F(U) \partial_x U = 0. \quad (41)$$

We then take the integral on  $\mathbb{R}$  and use the definition of the variational derivative to get

$$\int_{\mathbb{R}} D_U E(U) \partial_t U + \delta_U \mathcal{H}(U) \cdot D_U F(U) \partial_x U = 0. \quad (42)$$

On the other hand, as illustrated in the proof of Theorem (2.7), we have

$$\int_{\mathbb{R}} D_U N(U) \phi = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \phi \quad \forall \phi \in \mathcal{A}. \quad (43)$$

Therefore,

$$\int_{\mathbb{R}} D_U N(U) \partial_x U = \int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \partial_x U \quad (44)$$

Hence, we can write (42) as

$$\int_{\mathbb{R}} D_U E(U) \partial_t U + D_U N(U) \partial_x U = 0. \quad (45)$$

Or

$$\int_{\mathbb{R}} \partial_t E(U) + \partial_x N(U) = 0. \quad (46)$$

■

Let us remark that contrary to the case of hyperbolic systems, the reciprocal of Proposition 2.9 is false. In fact, (43) and (44) are not any more equivalent. This is due to the fact that, contrary to the hyperbolic system case,  $\delta_U \mathcal{H}(U)$  and  $\delta_U \mathcal{N}(U)$  depend not only on  $U$  but also on its derivatives.

Let us also remark that the notion of symmetry introduced for (26) corresponds to the symmetry for the  $\mathbb{L}^2$  scalar product and is a weak notion while the symmetry of hyperbolic system is a strong one. This is due to the fact that the assertion

$$\int_{\mathbb{R}} \phi \cdot \mathcal{F} \psi = \int_{\mathbb{R}} \mathcal{F} \psi \cdot \phi \quad \forall \phi, \psi \text{ test functions}, \quad (47)$$

does not imply

$$\phi \cdot \mathcal{F} \psi = \mathcal{F} \psi \cdot \phi \quad \forall \phi, \psi \text{ test functions}. \quad (48)$$

Therefore, the weak symmetry of the system does not lead to a conservation law but to an equality of the form (46). However, as we can see in [19], this definition is strong enough

to allow us to generalize the hyperbolic techniques to the Green–Naghdi equations. In fact, if we considered a stronger definition like the one deduced by (48) for the symmetric operator and a stronger condition such as the symmetry of  $D_U^2 E(U) D_U F(U)$  for Theorem 2.6, we would obtain a conservation law in addition to similar theorems. However, less equations would be covered (i.e. the result would be less general). Moreover, the strong symmetry of  $D_U^2 E(U) D_U F(U)$  is more tedious to be checked than the weak symmetry of  $\delta_U^2 \mathcal{H}(U) D_U F(U)$ .

We end this section with two final remarks. The first one is about an interesting change of variable while the second is on the multi-dimensional generalization.

**Remark 2.10** *Let us consider  $\mathcal{H}$  such that  $\delta_U^2 \mathcal{H}(U) D_U F(U)$  is symmetric in the sense of Definition 2.1. There is an interesting change of unknown for (26) based on the decomposition  $(U_1, U_2)$  of the unknown  $U$ . This is  $U \mapsto (V_1, V_2)$  with*

$$(V_1, V_2) = (U_1, \delta_{U_1} \mathcal{H}(U_1, U_2)) \quad (49)$$

*if it is well defined. The main advantage of this change of variable is the block diagonal structure of the matrix operator  $A_0(V)$  defined by (36). Using this expression, we have*

$$A_0(V) = \begin{pmatrix} A_0^{11} & A_0^{12} \\ A_0^{21} & A_0^{22} \end{pmatrix},$$

where

$$\begin{aligned} A_0^{11} &= \delta_{U_1}^2 \mathcal{H}(U) + \delta_{U_2 U_1}^2 \mathcal{H}(U) D_{V_1} U_2 + (D_{V_1} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_1} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2, \\ A_0^{12} &= \delta_{U_2 U_1}^2 \mathcal{H}(U) D_{V_2} U_2 + (D_{V_1} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_2} U_2, \\ A_0^{21} &= (A_0^{12})^T = (D_{V_2} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2, \\ A_0^{22} &= (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_2} U_2. \end{aligned}$$

Therefore,  $A_0(V)$  is block diagonal since

$$A_0^{21} = (A_0^{12})^T = (D_{V_2} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2 = 0.$$

This is due to the fact that (49) implies that

$$\begin{aligned} (D_{V_2} U_2)^T \delta_{U_1 U_2}^2 \mathcal{H}(U) + (D_{V_2} U_2)^T \delta_{U_2}^2 \mathcal{H}(U) D_{V_1} U_2 \\ &= (D_{V_2} U_2)^T D_{U_1} V_2 + (D_{V_2} U_2)^T D_{U_2} V_2 D_{V_1} U_2 \\ &= (D_{V_2} U_2)^T D_{U_1} V_2 D_{V_1} U_1 + (D_{V_2} U_2)^T D_{U_2} V_2 D_{V_1} U_2 \\ &= (D_{V_2} U_2)^T (D_{U_1} V_2 D_{V_1} U_1 + D_{U_2} V_2 D_{V_1} U_2) \\ &= (D_{V_2} U_2)^T D_{V_1} V_2 = 0. \end{aligned}$$

**Remark 2.11** *Considering the multi-dimensional generalization*

$$\partial_t U + \sum_{i=1}^n \partial_{x_i} F_i(U) = 0 \quad (50)$$

*of (26), we can easily extend the results presented in Theorems 2.6 and 2.7. In fact, we have the equivalence between the following three statements*

1. There exists a strictly convex functional  $\mathcal{H}(U) = \int_{\mathbb{R}} E(U)$  such that  $\delta_U^2 \mathcal{H}(U) D_U F_i(U)$  is symmetric for all  $i \in \{1, \dots, n\}$ .
2. System (50) is a general Godunov system. i.e. it is equivalent to

$$\partial_t(\delta_Q \mathcal{H}^*(Q)) + \sum_{i=1}^n \partial_{x_i}(\delta_Q \mathcal{R}_i(Q)) = 0,$$

for some functionals  $\mathcal{R}_i(Q) = \int_{\mathbb{R}} R_i(Q)$  with  $i \in \{1, \dots, N\}$  and a strictly convex functional  $\mathcal{H}^*$ .

3. System (50) is symmetrizable under any change of unknown  $U \mapsto V$  i.e the system is equivalent to

$$A_0(V) \partial_t V + \sum_{i=1}^n A_i(V) \partial_{x_i} V = 0,$$

where the symmetric definite operator  $A_0(V)$  is given by

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U, \quad (51)$$

and the symmetric operators  $A_i(V)$  by

$$A_i(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F_i(U) D_V U. \quad (52)$$

Moreover, if one of these statements is satisfied, the solution of the equation satisfies

$$\int_{\mathbb{R}} \partial_t E(U) + \sum_{i=1}^n \partial_{x_i} (Q \cdot F_i(U) - R_i(Q)) = 0.$$

## 3 Application to Green–Naghdi type equations

### 3.1 Symmetrization of the Green–Naghdi system

In this part, we are going to apply the result of the previous section to the Green–Naghdi type system (1). In fact, we will symmetrize the system around constant solutions. First, we show that System (1) is of the form (26) under convenient variables.

**Proposition 3.1** *Let  $s \geq 2$  be an integer and set  $\mathcal{A} = \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ . Then, System (1) is of the form (26) where  $F : \mathcal{A} \rightarrow \mathcal{A}$  is differentiable.*

**Proof** Let  $\bar{h} > 0$  and  $\bar{u} \in \mathbb{R}$  be constants and let us consider the unknown

$$U = (\eta, w)$$

where

$$\eta = h - \bar{h}, \quad \text{and} \quad w = \mathcal{L}_h(u) - \bar{h}\bar{u}.$$



We denote  $\mathcal{L}_h(u)$  by  $m$  and  $\bar{h}\bar{u}$  by  $\bar{m}$ . Let us also remark that System (1) can be written as

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) = 0, \\ \frac{\partial w}{\partial t} + \frac{\partial}{\partial x}(mu) + \frac{\partial}{\partial x}(-2\alpha h^3(\partial_x u)^2 + \frac{g}{2}h^2) = 0 \end{cases} \quad (53)$$

This is a consequence of the Hamiltonian structure (18) of the system. In fact, developing the first line of (18) we find easily the first equation of (53). Then, we develop its second line to get

$$\partial_t m + h\partial_x \sigma + \partial_x(mu) + m\partial_x u = 0.$$

Now, using the expression (22) of  $\sigma$  together with the fact that  $m = \mathcal{L}_h(u)$ , we find the second equation of (53).

Hence, a possible choice for  $F$  is

$$F(U) = \begin{pmatrix} (\eta + \bar{h})\mathcal{L}_h^{-1}(w + \bar{m}) - \bar{h}\bar{u} \\ (w + \bar{m})\mathcal{L}_h^{-1}(w + \bar{m}) - 2\alpha(\eta + \bar{h})^3(\partial_x \mathcal{L}_h^{-1}(w + \bar{m}))^2 + \frac{g}{2}(\eta + \bar{h})^2 - \frac{g}{2}\bar{h}^2 - \bar{m}\bar{u} \end{pmatrix}$$

On the other hand, assuming that  $h \in \mathbb{H}^s(\mathbb{R}) + \bar{h}$  is positively bounded by below,  $\mathcal{L}_h$  is a diffeomorphism from  $\mathbb{H}^{s+1}(\mathbb{R}) + \bar{u}$  to  $\mathbb{H}^{s-1}(\mathbb{R}) + \bar{h}\bar{u}$ . This together with the fact that  $\mathbb{H}^{s-1}(\mathbb{R})$  is an algebra for  $s \geq 2$  leads to the fact that  $F$  can be seen as an application from  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$  to  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ . To see better, let us consider the first component of  $F(U)$ . Starting from  $w \in \mathbb{H}^{s-1}(\mathbb{R})$ , we receive  $\mathcal{L}_h^{-1}(w + \bar{m}) \in \mathbb{H}^{s+1}(\mathbb{R}) + \bar{u}$ . Therefore,  $\mathcal{L}_h^{-1}(w + \bar{m}) \in \mathbb{H}^s(\mathbb{R}) + \bar{u}$ . On the other hand,  $\eta + \bar{h} \in \mathbb{H}^s(\mathbb{R}) + \bar{h}$ . Hence, the product is in  $\mathbb{H}^s(\mathbb{R}) + \bar{h}\bar{u}$ .

The differentiability of  $F$  is due to the fact that it is a composition of differentiable applications. ■

We are now going to see that System (1) satisfies the assumptions of the theorems presented in the previous section 2.

**Proposition 3.2** *Let us consider a constant solution  $\bar{V} = (\bar{h}, \bar{u})$  with  $\bar{h} > 0$ . Then, there exists a neighborhood in  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$  of  $\bar{V}$ , such that as long as the solution  $V = (h, u)$  remains in this neighborhood, System (1) is symmetrizable under any change of variable defined on this neighborhood.*

*In other words, (1) is locally symmetrizable around constant solutions.*

**Proof** We endow (1) with the unknown  $U = (\eta, w)$  such that  $\eta = h - \bar{h}$  and  $w = \mathcal{L}_h(u) - \bar{h}\bar{u}$ . Then, we show that the system is a general Godunov system using the functional

$$\mathcal{H}_{\bar{h}, \bar{u}}(U) = \int_{\mathbb{R}} \frac{gh(h - \bar{h})}{2} + \frac{h(u - \bar{u})^2}{2} + \frac{\alpha h^3(u_x)^2}{2}.$$

Let us first remark that  $\mathcal{H}_{\bar{h}, \bar{u}}(U)$  is strictly convex in a small neighborhood<sup>6</sup> of  $\bar{V}$ . This is due to the fact that the second variation of  $\mathcal{H}_{\bar{h}, \bar{u}}$  satisfies (see Appendix A for details)

<sup>6</sup>for the classical norm of  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ .

$$\delta^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) = \begin{pmatrix} g - 3\alpha h(u_x)^2 - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1}(-u) + 3\alpha \partial_x(h^2 u_x) & - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1} \\ \mathcal{L}_h^{-1}(-u) + 3\alpha \partial_x(h^2 u_x) & \mathcal{L}_h^{-1} \end{pmatrix}. \quad (54)$$

In other words, we have for all test functions  $\phi_1, \phi_2$ ,

$$\delta^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} g - 3\alpha h(u_x)^2 - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1}(-u\phi_1 + 3\alpha \partial_x(h^2 u_x \phi_1)) - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1}(\phi_2) \\ \mathcal{L}_h^{-1}(-u\phi_1 + 3\alpha \partial_x(h^2 u_x \phi_1)) + \mathcal{L}_h^{-1}(\phi_2) \end{pmatrix}. \quad (55)$$

We now take the  $\mathbb{L}^2$  scalar product of (55) with  $(\phi_1, \phi_2)$  and we find <sup>7</sup>

$$\int_{\mathbb{R}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \cdot \delta^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \int_{\mathbb{R}} (g - 3\alpha h(u_x)^2)(\phi_1)^2 + \left( \mathcal{L}_h^{-\frac{1}{2}}(-u\phi_1 + 3\alpha \partial_x(h^2 u_x \phi_1)) + \mathcal{L}_h^{-\frac{1}{2}}(\phi_2) \right)^2.$$

Now, considering the fact that  $g - 3\alpha h(u_x)^2$  is bounded positively by below for  $(h, u)$  close enough to  $\bar{V}$ , the strict convexity of  $\mathcal{H}_{\bar{h}, \bar{u}}$  on the small neighborhood of  $\bar{V}$  is concluded. Remaking that  $\bar{U} = U(\bar{V}) = (0, 0)$ , we can formulate this conclusion as following:

There exists a neighborhood in  $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$  of  $\bar{U}$  such that as long as the solution  $U = (\eta, w)$  is in this neighborhood,  $\delta_U^2 \mathcal{H}_{\bar{h}}$  is definite positive. Especially we have on  $\bar{U}$ ,

$$\delta^2 \mathcal{H}_{\bar{h}}(\bar{U}) = \begin{pmatrix} g & 0 \\ 0 & \mathcal{L}_h^{-1} \end{pmatrix}. \quad (56)$$

We now consider the Legendre transform  $\mathcal{H}_{\bar{h}, \bar{u}}^*$  which is defined by

$$\mathcal{H}_{\bar{h}, \bar{u}}^*(Q) = \int_{\mathbb{R}} Q \cdot U - E_{\bar{h}, \bar{u}}, \quad (57)$$

where

$$E_{\bar{h}, \bar{u}} = \frac{gh(h - \bar{h})}{2} + \frac{h(u - \bar{u})^2}{2} + \alpha h^3 (u_x)^2.$$

and

$$Q = \delta_U \mathcal{H}_{\bar{h}, \bar{u}}(U). \quad (58)$$

This is to say that

$$Q = (\sigma, u - \bar{u}),$$

---

<sup>7</sup>  $\mathcal{L}_h^{-\frac{1}{2}}$  is the symmetric operator such that  $\mathcal{L}_h^{-\frac{1}{2}} \circ \mathcal{L}_h^{-\frac{1}{2}} = \mathcal{L}_h^{-1}$ . The existence of this operator is guaranteed by the symmetry definite positivity of  $\mathcal{L}_h^{-1}$ .

with

$$\sigma = gh - g\bar{h}/2 - u^2/2 + \bar{u}^2/2 - \frac{3}{2}\alpha h^2(u_x)^2.$$

This leads to the following expression for  $\mathcal{H}_{\bar{h},\bar{u}}^*$

$$\mathcal{H}_{\bar{h},\bar{u}}^*(Q) = \int_{\mathbb{R}} \frac{g(h - \bar{h})^2}{2} + \frac{\bar{h}(u - \bar{u})^2}{2} - \alpha h^3(u_x)^2 + \frac{3}{2}\alpha h^2\bar{h}(u_x)^2.$$

We just now need to remark that there exists a functional  $\mathcal{R}$  of  $Q$  such that

$$F(U) = \delta_Q \mathcal{R}(Q).$$

We can get to this equality setting

$$\mathcal{R}(Q) = \int_{\mathbb{R}} gu \left( \frac{h^2 - \bar{h}^2}{2} \right) - \alpha h^3 u(u_x)^2 - \bar{h}\bar{u}\sigma - \bar{h}\bar{u}^2(u - \bar{u}) + g\bar{h}^2\bar{u}/2. \quad (59)$$

Hence, the system is equivalent (on the domain of strict convexity of  $\mathcal{H}_{\bar{h},\bar{u}}$ ) to

$$\partial_t \left( \delta_Q \mathcal{H}_{\bar{h},\bar{u}}^*(Q) \right) + \partial_x \left( \delta_Q \mathcal{R}(Q) \right) = 0. \quad (60)$$

■

Let us remark that the quantity  $E_{\bar{h},\bar{u}}$  introduced in the proof of Proposition 3.2, is actually an energy for the system. Indeed, we can check that the solution of (1) satisfies

$$\partial_t E_{\bar{h},\bar{u}} + \partial_x \left( u E_{\bar{h},\bar{u}} + (u - \bar{u})p \right) = 0. \quad (61)$$

where  $p$  is defined by (4).

Proposition 3.2 together with Theorem 2.7 implies the symmetrizability of the system under any variable around constant solutions. We now provide in the two following propositions some explicit symmetric forms of System (1).

**Proposition 3.3** *The Green–Naghdi type system (1) is written under the symmetric form*

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x(Q) = 0, \quad (62)$$

where  $Q = (\sigma, u - \bar{u})$  is defined by (58) and

$$A_0(Q) = \left( \begin{array}{c} \frac{1}{g-3\alpha h(u_x)^2} \\ \frac{u}{g-3\alpha h(u_x)^2} - 3\alpha \partial_x \left( \frac{h^2 u_x}{g-3\alpha h(u_x)^2}(\cdot) \right) \end{array} \right) \mathcal{L}_h + \left( \begin{array}{c} \frac{u+3\alpha h^2 u_x \partial_x}{g-3\alpha h(u_x)^2} \\ \frac{u}{g-3\alpha h(u_x)^2} \end{array} \right) \left( u + 3\alpha h^2(u_x)\partial_x \right) - 3\alpha \partial_x \left( h^2 u_x \frac{u(\cdot)+3\alpha h^2 u_x \partial_x(\cdot)}{g-3\alpha h(u_x)^2} \right) \quad (63)$$

and

$$A_1(Q) = \left( \begin{array}{c} \frac{u}{g-3\alpha h(u_x)^2} \\ h + \frac{u^2}{g-3\alpha h(u_x)^2} - 3\alpha \partial_x \left( \frac{h^2 u(u_x)}{g-3\alpha h(u_x)^2}(\cdot) \right) \end{array} \right) 3hu + \left( \begin{array}{c} h + \frac{u^2+3\alpha h^2 u_x \partial_x}{g-3\alpha h(u_x)^2} \\ \frac{u^3+3\alpha h^2 u^2 u_x \partial_x}{g-3\alpha h(u_x)^2} - \alpha \partial_x \left( h^3 u_x(\cdot) \right) - \alpha u \partial_x \left( h^3 \partial_x(\cdot) \right) \\ -3\alpha \partial_x \left( \frac{h^2 u^2 u_x + 3\alpha h^4 u(u_x)^2 \partial_x(\cdot)}{g-3\alpha h(u_x)^2} \right) \end{array} \right). \quad (64)$$

**Proof** This is a consequence of the general Godunov structure (60) of the system.

We just need to set  $A_0(Q) = \delta_Q^2 \mathcal{H}_{\bar{h}, \bar{u}}^*(Q)$  and  $A_1(Q) = \delta_Q^2 \mathcal{R}(Q)$  to get the result. ■

Let us remark that the operators  $A_0(Q)$  and  $A_1(Q)$  defined by (63) and (64) are linear second order differential operators.

**Proposition 3.4** *The Green–Naghdi type system (1) is symmetric under the unknown  $V = (h, u)$  of the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x(V) = 0, \quad (65)$$

with

$$A_0(V) = \begin{pmatrix} g - 3\alpha h(u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix}, \quad (66)$$

and

$$A_1(V) = \begin{pmatrix} gu - 3\alpha hu(u_x)^2 & gh - 3\alpha h^2(u_x)^2 \\ gh - 3\alpha h^2(u_x)^2 & hu + 2\alpha\partial_x(h^3u_x) - \alpha h^3u_x\partial_x - \alpha u\partial_x(h^3\partial_x()) \end{pmatrix}. \quad (67)$$

**Proof** This proposition is just a consequence of Theorem 2.7 and Proposition 3.2. In fact, we check that the change of unknown  $U \mapsto V$  such that

$$\begin{cases} U = (\eta, w), \\ V = (\eta, \delta_w \mathcal{H}_{\bar{h}, \bar{u}}(U)), \end{cases}$$

leads to  $V = (h - \bar{h}, u - \bar{u})$ . This fact is true since  $\delta_w \mathcal{H}_{\bar{h}, \bar{u}}(U) = \mathcal{L}_h^{-1}(m) - \bar{u}$ . This change of variable is valid by the properties of the Sturm–Liouville operator  $\mathcal{L}_h$  while  $h$  is positively bounded by below. Hence, the system is symmetric with

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) D_V U,$$

and

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{h}, \bar{u}}(U) \nabla_U F(U) D_V U.$$

Basic computations (similar to those presented in Appendix A) show that their analytic expressions are given by (66) and (67). ■

Let us remark that similarly to Proposition 3.3, the operators  $A_0(V)$  and  $A_1(V)$  are second order differential operators. However, the analytic expressions of these operators are much simpler than the expressions of  $A_0(Q)$  and  $A_1(Q)$  in Proposition 3.3. In fact, as we expected, the symmetric definite positive operator of Proposition 3.4 is diagonal. This fact is explained in Remark 2.10.

**Remark 3.5** *A similar structure to (65) (but non symmetric) is used in [17] to study the linearized Green–Naghdi system in order to prove the local well-posedness.*

Let us now apply Proposition 2.9 to the Green–Naghdi type equations to get a conserved quantity. According to this proposition, as long as the solution  $U$  remains in the domain of strict convexity of  $\mathcal{H}_{\bar{h}, \bar{u}}$ , it satisfies

$$\int_{\mathbb{R}} \partial_t E_{\bar{h}, \bar{u}}(U) + \partial_x N(U) = 0, \quad (68)$$

where

$$N(U) = Q \cdot F(U) - R(U),$$

is such that

$$R(U) = gu \left( \frac{h^2 - \bar{h}^2}{2} \right) - \alpha h^3 u (u_x)^2 - \bar{h} \bar{u} \sigma - \bar{h} \bar{u}^2 (u - \bar{u}) + g \bar{h}^2 \bar{u} / 2$$

is given by (59). Now, we use the expressions of  $Q$ ,  $F(U)$  and  $R(U)$  and we find

$$N(U) = \frac{ghu(h - \bar{h})}{2} + \frac{g}{2} \bar{u} \bar{h}^2 + \left( \frac{gh^2 + hu^2}{2} + 3\alpha h^3 (u_x)^2 \right) (u - \bar{u}) + \frac{\alpha}{2} h^3 u (u_x)^2 - \frac{g}{2} \bar{u} \bar{h}^2.$$

Then considering the fact that  $(h - \bar{h}, u - \bar{u}) \in \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ , we remark that

$$\lim_{x \rightarrow \pm\infty} N(U) = 0.$$

Therefore,

$$\int_{\mathbb{R}} \partial_x N(U) = 0.$$

This is to say that the equality (68) is equal to

$$\int_{\mathbb{R}} \partial_t E_{\bar{h}, \bar{u}}(U) = 0.$$

In other words,

$$\frac{d}{dt} \int_{\mathbb{R}} E_{\bar{h}, \bar{u}}(U) = 0.$$

Hence, we conclude the conservation of the energy integral  $\mathcal{H}_{\bar{h}, \bar{u}}(U)$  from the general Godunov structure of the system. Let us note that we could get the conservation of the energy integral simply by integrating the energy conservation law (61).

### 3.2 Two-dimensional extension

Let us fix  $\bar{V} = (\bar{h}, \bar{u}, \bar{v}) \in \mathbb{R}^3$  with  $\bar{h} > 0$  and consider the 2D Green–Naghdi model

$$\partial_t h + \partial_x hu + \partial_y hv = 0, \quad (69a)$$

$$\partial_t hu + \partial_x hu^2 + \partial_y huv + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = 0, \quad (69b)$$

$$\partial_t hv + \partial_x huv + \partial_y hv^2 + \partial_y (gh^2/2 + \alpha h^2 \ddot{h}) = 0, \quad (69c)$$

where  $\dot{h} = \partial_t h + u\partial_x h + v\partial_y h$ .

This system is equivalent to

$$\partial_t U + \partial_x F_1(U) + \partial_y F_2(U) = 0, \quad (70)$$

where

$$F_1(U) = \begin{pmatrix} hu \\ gh^2/2 + hu^2 - 2\alpha h^3(\operatorname{div}(u, v))^2 - \alpha u\partial_x(h^3\operatorname{div}(u, v)) + \alpha h^3\operatorname{div}(u, v)v_y \\ huv - \alpha\partial_y(h^3u\operatorname{div}(u, v)), \end{pmatrix}$$

and

$$F_2(U) = \begin{pmatrix} hv \\ huv - \alpha\partial_x(h^3v\operatorname{div}(u, v)) \\ gh^2/2 + hv^2 - 2\alpha h^3(\operatorname{div}(u, v))^2 - \alpha v\partial_y(h^3\operatorname{div}(u, v)) + \alpha h^3\operatorname{div}(u, v)u_x \end{pmatrix},$$

for

$$U = (h - \bar{h}, m - \bar{h}\bar{u}, n - \bar{h}\bar{v}),$$

where

$$(m, n) = \mathcal{L}_h(u, v),$$

with

$$\mathcal{L}_h(u, v) = h(u, v) - \alpha\nabla(h^3\operatorname{div}(u, v)).$$

The change of unknown  $(u, v) \mapsto (n, m)$  is valid because as it is mentioned in [1],  $\mathcal{L}_h$  is a diffeomorphism acting on the space

$$\mathbb{H}^{s+1}(\operatorname{div}) = \{(u, v) \in (\mathbb{H}^s(\mathbb{R}^2) + \bar{u}) \times (\mathbb{H}^s(\mathbb{R}^2) + \bar{v}) \text{ such that } \operatorname{div}(u, v) \in \mathbb{H}^s(\mathbb{R}^2)\},$$

while  $s$  is large enough and  $h$  is positively bounded by below. More precisely, this change of variable is valid because we assume that the unknown  $(h, u, v)$  belongs to

$$(\mathbb{H}^s(\mathbb{R}^2) + \bar{h}) \times \mathbb{H}^{s+1}(\operatorname{div}).$$

**Proposition 3.6** *The solution of System (69) satisfies the following conservation law*

$$\partial_t E_{\bar{V}} + \partial_x (uE_{\bar{V}} + (u - \bar{u})p) + \partial_y (vE_{\bar{V}} + (v - \bar{v})p) = 0. \quad (72)$$

where

$$E_{\bar{V}} = gh(h - \bar{h})/2 + h(u - \bar{u})^2/2 + h(v - \bar{v})^2/2 + \alpha h^3(u_x + v_y)^2/2, \quad (73a)$$

and

$$p = gh^2/2 + \alpha h^2\ddot{h}. \quad (73b)$$

Let us consider the space integral  $\mathcal{H}_{\bar{V}}$  of the energy  $E_{\bar{V}}$ ,

$$\mathcal{H}_{\bar{V}}(U) = \int_{\mathbb{R}^2} E_{\bar{V}}(U). \quad (74)$$

Similarly to the one dimensional case, this functional is strictly convex as an application of  $U$  while  $V = (h, u, v)$  is close enough to the equilibrium  $\bar{V} = (\bar{h}, \bar{u}, \bar{v})$ . This is to say that  $\delta_{\bar{V}}^2 \mathcal{H}_{\bar{V}}(U)$  is definite positive for  $U$  close to  $\bar{U} = U(\bar{V}) = (0, 0, 0)$ . Let us now consider the change of variable

$$U \mapsto Q = \delta_U \mathcal{H}_{\bar{V}}(U),$$

defined on the **domain of strict convexity** of  $\mathcal{H}_{\bar{V}}$ .

In other words,

$$Q = \begin{pmatrix} gh - g\bar{h}/2 - (u^2 - \bar{u}^2)/2 - (v^2 - \bar{v}^2)/2 - 3\alpha h^2(\operatorname{div}(u, v))^2/2 \\ u - \bar{u} \\ v - \bar{v} \end{pmatrix}. \quad (75)$$

We are going to see in the following proposition that the 2-dimensional Green–Naghdi equation (69) is a general Godunov system under the variable  $Q$ .

**Proposition 3.7** *There exists a neighborhood for the norm  $\mathbb{H}^s \times \mathbb{H}^s(\operatorname{div})$  of  $\bar{V}$  such that as long as the solution  $V$  of (69) remains in this neighborhood, the system is equivalent to*

$$\partial_t(\delta_Q \mathcal{H}_{\bar{V}}^*(Q)) + \partial_x(\delta_Q \mathcal{R}_1(Q)) + \partial_y(\delta_Q \mathcal{R}_2(Q)) = 0, \quad (76)$$

where  $Q$  is defined by (75) and  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are two functionals defined on a neighborhood of  $\bar{Q} = Q(\bar{V}) = (g\bar{h}/2, 0, 0)$ . A possible expression for these functionals is given by (78).

**Proof** *Let us first remark that the Legendre transform  $\mathcal{H}_{\bar{V}}^*$  of the energy integral  $\mathcal{H}_{\bar{V}}$  is defined by*

$$\begin{aligned} \mathcal{H}_{\bar{V}}^*(Q) &= \int_{\mathbb{R}^2} Q \cdot U - E_{\bar{V}}(U) \\ &= \int_{\mathbb{R}^2} g(h - \bar{h})^2/2 + \bar{h}(u - \bar{u})^2/2 + \bar{h}(v - \bar{v})^2/2 - \alpha h^3(\operatorname{div}(u, v))^2 + \frac{3}{2}\alpha h^2 \bar{h}(\operatorname{div}(u, v))^2. \end{aligned}$$

We know by Definition 2.4 of the Legendre Transform that we have

$$U = \delta_Q \mathcal{H}_{\bar{V}}^*(Q). \quad (77)$$

Let us now consider the variational functionals  $\mathcal{R}_1$  and  $\mathcal{R}_2$  defined by

$$\mathcal{R}_1(Q) = \int_{\mathbb{R}^2} g \left( \frac{uh^2 - \bar{u}\bar{h}^2}{2} \right) + \bar{h}\bar{u}(u^2 - \bar{u}^2) - \alpha h^3 u(\operatorname{div}(u, v))^2, \quad (78a)$$

and

$$\mathcal{R}_2(Q) = \int_{\mathbb{R}^2} g \left( \frac{vh^2 - \bar{v}\bar{h}^2}{2} \right) + \bar{h}\bar{v}(v^2 - \bar{v}^2) - \alpha h^3 v (\operatorname{div}(u, v))^2. \quad (78b)$$

We can easily check that

$$F_1(U) = \delta_Q \mathcal{R}_1(Q), \quad (79)$$

$$F_2(U) = \delta_Q \mathcal{R}_2(Q). \quad (80)$$

Considering (77) together with (79) and (80) we get the result. ■

According to Proposition 3.7, the 2-dimensional Green–Naghdi system (69) is symmetrizable under any change of variable **around any constant solution**  $\bar{V}$ . Especially, the general Godunov structure of the system leads directly to the following symmetric structure under the unknown  $Q$ :

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x Q + A_2(Q)\partial_y Q = 0. \quad (81)$$

where

$$A_0(Q) = \delta_Q^2 \mathcal{H}_{\bar{V}}^*(Q), \quad (82a)$$

$$A_1(Q) = \delta_Q^2 \mathcal{R}_1(Q), \quad (82b)$$

and

$$A_2(Q) = \delta_Q^2 \mathcal{R}_2(Q). \quad (82c)$$

Considering the fact that we can recover the physical variable  $V = (h, u, v)$  using the partial variational derivative of the energy integral, we have the following corollary.

**Corollary 3.8** *The two-dimensional Green–Naghdi equation (69) is symmetric under the physical variable  $V = (h, u, v)$  of the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x V + A_2(V)\partial_y V = 0. \quad (83)$$

where

$$A_0 = \begin{pmatrix} g - 3\alpha h (\operatorname{div}(u, v))^2 & 0 & 0 \\ 0 & h - \alpha \partial_x (h^3 \partial_x) & -\alpha \partial_x (h^3 \partial_y) \\ 0 & -\alpha \partial_y (h^3 \partial_x) & h - \alpha \partial_y (h^3 \partial_y) \end{pmatrix}$$

is block diagonal.

**Proof** We first consider the change of variable  $U \mapsto V$  where

$$U = (h, m, n),$$

and

$$V = (h, \delta_{(m,n)} \mathcal{H}_{\bar{V}}(U)).$$

This is to say that

$$V = (h, u - \bar{u}, v - \bar{v}).$$



This change of variable is valid by the invertibility of  $\mathcal{L}_h$  on  $\mathbb{H}^{s+1}(\text{div})$  since  $h$  is positively bounded by below and the physical speed  $(u, v)$  belongs to  $\mathbb{H}^{s+1}(\text{div})$ . We then use Theorem 2.7 to find the following expression for the operators

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{V}}(U) D_V U,$$

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{V}}(U) D_U F_1(U) D_V U \partial_x V,$$

and

$$A_2(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{\bar{V}}(U) D_U F_2(U) D_V U \partial_x V.$$

Using Remark 2.10, we could predict the block diagonal structure of  $A_0(V)$ . ■

Let us mention that similarly to the first dimensional case, the conservation over time of the energy integral  $\mathcal{H}_{\bar{V}}$  can be concluded.

## 4 Conclusion

A generalization of the notion of symmetry for hyperbolic system has been presented. This generalization is essentially made by replacing energy by its integral, matrix by operator, partial derivatives by variational derivatives and matrix transpose by adjoint operators. However, we chose a quite weak definition for symmetric operator. Therefore, contrary to the case of hyperbolic system, the symmetric structure does not lead to a conservation law but to a conserved quantity.

Then, we apply the result to Green–Naghdi equation (1) in a neighborhood of constant solutions of the system. We could notice that this symmetrization are based on the energy integral of a conservation law which represents the total physical energy. This energy integral coincides with the Hamiltonian of the system in the one dimensional case. A generalization to the two-dimensional system is also provided.

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## A Computation of the second variation

In this part, we are going to see how to compute the second variation with respect to  $U = (h, m)$  of

$$\mathcal{H}_{\bar{h}, \bar{u}}(U) = \int_{\mathbb{R}} \frac{gh(h - \bar{h})}{2} + \frac{h(u - \bar{u})^2}{2} + \frac{\alpha h^3 (u_x)^2}{2}.$$

These computations are based on Definition 2.3 of differential and Definition 1.4 of the variational derivative and the second variation. Lets us first compute the variational

derivative with respect to  $h$  of  $\mathcal{H}_{\bar{h},\bar{u}}(U)$ . In fact, fixing the function  $m$ , we have for all test functions  $\phi$ ,

$$\mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) = \int_{\mathbb{R}} E_{\bar{h},\bar{u}}(h + \phi, u). \quad (84)$$

This is to say that

$$\mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) = \int_{\mathbb{R}} E_{\bar{h},\bar{u}}(h, u) + D_h E_{\bar{h},\bar{u}}(h, u)(\phi) + o(\|\phi\|), \quad (85)$$

where

$$\lim_{\phi \rightarrow 0} \frac{o(\|\phi\|)}{\|\phi\|} = 0.$$

Or equivalently

$$\mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) = \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} D_h E_{\bar{h},\bar{u}}(h, u)(\phi) + o(\|\phi\|). \quad (86)$$

Using the definition of  $E_{\bar{h},\bar{u}}$ , we have

$$\begin{aligned} \mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi \\ &+ \int_{\mathbb{R}} h(u - \bar{u})D_h u(\phi) + \frac{3}{2}\alpha h^2(u_x)^2\phi + \alpha h^3 u_x \partial_x D_h u(\phi) + o(\|\phi\|). \end{aligned} \quad (87)$$

In order to compute  $D_h u(\phi)$ , we consider the relation (17):

$$m = hu - \alpha \partial_x (h^3 u_x).$$

We differentiate this relation with respect to  $h$  and take the action on  $\phi$ . We find

$$0 = u\phi + hD_h u(\phi) - \alpha \partial_x (3h^2 \phi u_x) - \alpha \partial_x (h^3 \partial_x (D_h u(\phi))).$$

This leads to

$$D_h u(\phi) = \mathcal{L}_h^{-1} (3\alpha \partial_x (h^2 u_x \phi) - u\phi).$$

Injecting this to (87), we get

$$\begin{aligned} \mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi \\ &+ \int_{\mathbb{R}} \frac{3}{2}\alpha h^2(u_x)^2\phi + (h(u - \bar{u}) + \alpha h^3 u_x \partial_x) \mathcal{L}_h^{-1} (3\alpha \partial_x (h^2 u_x \phi) - u\phi) + o(\|\phi\|). \end{aligned}$$

Hence, we have after an integration by part

$$\begin{aligned} \mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi + \frac{3}{2}\alpha h^2(u_x)^2\phi \\ &+ \int_{\mathbb{R}} (h(u - \bar{u}) - \partial_x (\alpha h^3 u_x)) \mathcal{L}_h^{-1} (3\alpha \partial_x (h^2 u_x \phi) - u\phi) + o(\|\phi\|). \end{aligned}$$

or equivalently

$$\begin{aligned} \mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) &= \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} gh\phi - \frac{g\bar{h}}{2}\phi + \frac{(u - \bar{u})^2}{2}\phi + \frac{3}{2}\alpha h^2(u_x)^2\phi \\ &\int_{\mathbb{R}} + \mathcal{L}_h(u - \bar{u}) \cdot \mathcal{L}_h^{-1}(3\alpha\partial_x(h^2u_x\phi) - u\phi) + o(\|\phi\|). \end{aligned}$$

Now considering the fact that  $\mathcal{L}_h$  is symmetric and using another integration by part, we get

$$\mathcal{H}_{\bar{h},\bar{u}}(h + \phi, m) = \mathcal{H}_{\bar{h},\bar{u}}(h, u) + \int_{\mathbb{R}} \left( gh - g\bar{h}/2 - \frac{u^2}{2} - \frac{\bar{u}}{2} - \frac{3}{2}\alpha h^2(u_x)^2 \right) \phi + o(\|\phi\|).$$

This is to say that

$$\delta_h \mathcal{H}_{\bar{h},\bar{u}}(U) = gh - g\bar{h}/2 - \frac{u^2}{2} - \frac{\bar{u}}{2} - \frac{3}{2}\alpha h^2(u_x)^2,$$

which is nothing but the quantity called  $\sigma$  in Section 1.2.

Using exactly the same type of computations, we find

$$\delta_m \mathcal{H}_{\bar{h},\bar{u}}(U) = u - \bar{u}.$$

On the other hand, we know by the definition of the second variation that

$$\delta_U^2 \mathcal{H}_{\bar{h},\bar{u}}(U) = D_U \delta_U \mathcal{H}_{\bar{h},\bar{u}}(U).$$

This is to say that

$$\delta^2 \mathcal{H}_{\bar{h},\bar{u}}(U) = \begin{pmatrix} D_h \sigma & D_m \sigma \\ D_h u & D_m u \end{pmatrix}.$$

Then, similar computations lead to

$$\delta^2 \mathcal{H}_{\bar{h},\bar{u}}(U) = \begin{pmatrix} g - 3\alpha h(u_x)^2 - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1}(-u) + 3\alpha \partial_x(h^2 u_x) & - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1} \\ \mathcal{L}_h^{-1}(-u) + 3\alpha \partial_x(h^2 u_x) & \mathcal{L}_h^{-1} \end{pmatrix}.$$

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