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The symmetric structure of the Green-Naghdi type equations

Dena Kazerani¹²

Abstract

In this study, we generalize the notion of the symmetry of the hyperbolic systems of conservation law to the conservative evolution equations of conservation law. In particular, we apply our general result to the Green-Naghdi type equations which is a dispersive perturbation of the Saint-Venant system which is hyperbolic.

Keywords : Symmetric systems, Conservation law, Strict convexity, variational derivative, Green-Naghdi equations.

1 Introduction

Incompressible Euler equations and water waves problem model free surface incompressible fluids under the influence of the gravity. The complexity of these systems leads us to consider reduced complex geophysical models to describe coastal oceanic flows. In this work, we focus on a particular type of these reduced models called the Green-Naghdi type model:

$$\begin{cases} \partial_t h + \partial_x hu = 0, \\ \partial_t hu + \partial_x hu^2 + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = 0 \end{cases} \quad (1)$$

under the hypothesis that $h(x, t) \geq 0$. Here, α is a positive real number, g is the gravity constant, $h \in \mathbb{H}^s(\mathbb{R}) + h_e$ with $h_e > 0$ (i.e $h - h_e \in \mathbb{H}^s(\mathbb{R})$) and $u \in \mathbb{H}^{s+1}(\mathbb{R})$ for $s \geq 2$. The unknown h represents the fluid height and u its average horizontal speed. Moreover, the material derivative $\dot{()}$ is defined by $\dot{()} = \partial_t() + u\partial_x()$.

For $\alpha = 0$, the system is hyperbolic and equivalent to Saint-Venant or isentropic Euler equations. Hence, for $\alpha \neq 0$, the system (1) can be seen as a dispersive perturbation of the Saint-Venant system. The $\alpha = \frac{1}{3}$ case has been derived from water waves problem for irrotational flows by Li [17] and by Alvarez & Lannes [1]. In [14], Ionescu has derived the same system by a variational method considering the Lagrangian formulation of the

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irrotational incompressible Euler equations. In [4], the authors obtain (1) for $\alpha = \frac{1}{4}$ by a different but a formal method without any hypothesis on the irrationality of the fluid.

It is also worth remarking that (1) admits the following conservation law ([9], [10], [4]),

$$\partial_t E + \partial_x (u(E + p)) = 0, \quad (2)$$

where

$$E(h, u) = gh^2/2 + hu^2/2 + \alpha h^3(u_x)^2/2,$$

and

$$p(h, u) = gh^2/2 + \alpha h^2 \ddot{h}.$$

1.1 Symmetric structure

Let us now recall some classical notions on hyperbolic $N \times N$ systems of conservation laws:

$$\partial_t U + \partial_x F(U) = 0 \quad (3)$$

where the unknown U belongs to $C([0, T]; (\mathbb{H}^s(\mathbb{R}))^N)$ for $s \geq 2$ and the flux $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a smooth function.

Definition 1.1 *The hyperbolic system (3) is called symmetrizable if there exists a change of variable $U \rightarrow Q$ such that (3) is equivalent to*

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x(Q) = 0, \quad (4)$$

where $A_0(Q)$ is a symmetric definite positive matrix and $A_1(Q)$ is a symmetric one.

Definition 1.2 *The pair (E, P) with $\nabla_U^2 E$ definite positive, is an entropy pair for the system (3) if the solution U of the system satisfies*

$$\partial_t E(U) + \partial_x P(U) = 0. \quad (5)$$

or equivalently,

$$(\nabla_U F(U))^T \nabla_U E(U) = \nabla_U P(U). \quad (6)$$

Proposition 1.3 ([6], [12], [19], [3], [18].) *Assume that the system (3) admits the entropy pair (E, P) . Then, it is symmetrizable. This symmetry can be obtained using the Legendre transform E^* of the entropy.*

Proof (E, P) is an entropy pair of the system. Hence, (5) and (6) are satisfied. Let us now consider the change of variable $U \rightarrow Q$ such that

$$Q = \nabla_U E(U), \quad (7)$$

or equivalently,

$$U = \nabla_Q E^*(Q), \quad (8)$$

where E^* is the Legendre transform of E [5]. Therefore, it is defined by

$$E^*(Q) = Q \cdot (\nabla_U E)^{-1}(Q) - E((\nabla_U E)^{-1}(Q)). \quad (9)$$

In fact, the strict convexity of E implies the invertibility of $\nabla_U E$. Applying the change of variable (7), the equality (6) becomes

$$(\nabla_U F(U))^T Q(U) = \nabla_U P(U). \quad (10)$$

Let us now define

$$\hat{P}(Q) = Q \cdot F(U(Q)) - P(U(Q)). \quad (11)$$

Then,

$$F(U) = \nabla_Q \hat{P}(Q). \quad (12)$$

Gathering (7) and (10), (3) is written

$$\partial_t (\nabla_Q E^*(Q)) + \partial_x (\nabla_Q \hat{P}(Q)) = 0. \quad (13)$$

Hence, (3) written under the unknown Q is symmetric of the form (4) with

$$A_0(Q) = \nabla_Q^2 E^*(Q),$$

and

$$A_1(Q) = \nabla_Q^2 \hat{P}(Q).$$

■

Proposition 1.4 [16]. *Let us consider the hyperbolic system of conservation laws (3) with the entropy pair (E, P) . For any change of variable $U \rightarrow V$, the system is symmetrizable under V with*

$$A_0(V) = (\nabla_V U)^T \nabla_U^2 E \nabla_V U,$$

and

$$A_1(V) = (\nabla_V U)^T \nabla_U^2 E \nabla_U F(U) \nabla_V U.$$

Proof Under the change of variable, (3) becomes

$$\nabla_V U \partial_t V + \nabla_U F(U) \nabla_V U \partial_x V = 0. \quad (14)$$

We multiply (14) by $(\nabla_V U)^T \nabla_U^2 E$ to obtain

$$(\nabla_V U)^T \nabla_U^2 E \nabla_V U \partial_t V + (\nabla_V U)^T \nabla_U^2 E \nabla_U F(U) \nabla_V U \partial_x V = 0. \quad (15)$$

Let us note that $\nabla_U^2 E \nabla_U F(U)$ is symmetric. In fact, we have by (12) that

$$\nabla_U^2 E \nabla_U F(U) = \nabla_U^2 E \nabla_U (\nabla_Q \hat{P}(Q)) = \nabla_U^2 E \nabla_Q^2 \hat{P}(Q) \nabla_U Q. \quad (16)$$

where Q is defined by (7). Therefore,

$$\nabla_U^2 E \nabla_U F(U) = \nabla_U^2 E \nabla_Q^2 \hat{P}(Q) \nabla_U Q. \quad (17)$$

■

Gathering the propositions 1.3 and 1.4, we notice that the hyperbolic entropic system (3) is symmetrizable in two different ways under the change of variable (7).

There is an interesting change of variable for the hyperbolic system (3) based on the decomposition (U_1, U_2) of the unknown U . It is $U \rightarrow V$ with $V = (U_1, \nabla_{U_1} E(U_1, U_2))$ (assuming that it is well defined, see [16]).

Remark 1.5 *The notion of symmetrizability can be seen as a local notion. It means that if the entropy of the system satisfies the assumptions of the propositions 1.3 and 1.4 on a neighborhood of a solution, then the system is symmetrizable for the solutions which remain in this neighborhood.*

Corollary 1.6 *The Saint-Venant system is symmetrizable under at least two change of variables.*

Proof *The Saint-Venant system satisfies the conservation law (2) for $\alpha = 0$ with the entropy $E = gh^2/2 + hu^2/2$. It is easy to check that the Hessian $\nabla_U^2 E$ is definite positive for the unknown $U = (h, m)$ with $m = hu$ and $h > 0$.*

- **First change of variable.** *We consider the change of variable*

$$Q = \nabla E(U).$$

That is to say that $Q = (gh - u^2/2, u)$,

$$E^*(Q) = gh^2/2,$$

and

$$\hat{P}(Q) = guh^2/2.$$

Hence,

$$A_0(Q) = \frac{1}{g} \begin{pmatrix} 1 & u \\ u & gh + u^2 \end{pmatrix},$$

and

$$A_1(Q) = \frac{1}{g} \begin{pmatrix} u & gh + u^2 \\ gh + u^2 & 3ghu + u^3 \end{pmatrix}.$$

Remark 1.7 *We have $\hat{P}(Q) = (uE_{h_e}^*)(Q)$.*

- **Second change of variable.** *Let us note that $U \rightarrow V$ with $U = (h, m)$ and $V = (h, \partial_m E(U))$ is a change of variable for the system, since $h > 0$. Moreover, we have $V = (h, u)$.*

Then, using the proposition 1.4, we obtain a symmetric form with

$$A_0(V) = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix},$$

and

$$A_1(V) = g \begin{pmatrix} u & h \\ h & u \end{pmatrix}.$$

■

1.2 Hamiltonian structure

Definition 1.8 [20] Assume that X is a functional space. A system of partial derivative equations of unknown $U \in C([0, T]; X)$ is called *Hamiltonian* if there exists a skew-symmetric Hamiltonian operator¹ \mathcal{J} and a functional $\mathcal{H} : X \rightarrow \mathbb{R}$ called the *Hamiltonian* such that the equation can be written under the following form:

$$\partial_t U = -\mathcal{J}(U)\delta\mathcal{H}(U), \quad (18)$$

where $\delta\mathcal{H}(U(t))$ denotes the variational derivative² of the functional \mathcal{H} at $U(t)$.

Proposition 1.9 [20] Let us consider the solution U of the Hamiltonian system (18). Then, the quantity $\mathcal{H}(U(t))$ does not depend on time. In other words, we have

$$\frac{d}{dt}(\mathcal{H}(U(t))) = 0. \quad (19)$$

Hence, $\mathcal{H}(U)$ is a conserved quantity of the system.

Proof We can get the result using the equation and the skew-symmetry of \mathcal{J} . In fact,

$$\frac{d}{dt}\mathcal{H}(U(t)) = \int_{\mathbb{R}} \delta\mathcal{H}(U)\partial_t U = - \int_{\mathbb{R}} \delta\mathcal{H}(U)\mathcal{J}(U)\delta\mathcal{H}(U) = 0.$$

■

Hamiltonian structure of Green-Naghdi type equations

In [17], Li shows that the system (1) admits a Hamiltonian structure using the unknown $U = (h, m)$ for

$$m = \mathcal{L}_h(u) = hu - \alpha(h^3 u_x)_x.$$

The change of unknown $(h, u) \rightarrow (h, m)$ is valid since the Sturm-Liouville operator \mathcal{L}_h is a continuous isomorphism from $\mathbb{H}^s(\mathbb{R})$ to $\mathbb{H}^{s+2}(\mathbb{R})$. More precisely, the following proposition holds.

Proposition 1.10 [17] Assume that $V = (h, u)$ is the solution of (1). Then, we have

$$\partial_t U = -\mathcal{J}(U)\delta\mathcal{H}_{h_e}(U), \quad (20)$$

where

$$U = (h, m) = (h, \mathcal{L}_h(u)),$$

$$\mathcal{J}(U) = - \begin{pmatrix} \partial_x m + m\partial_x & h\partial_x \\ \partial_x h & \end{pmatrix}, \quad (21)$$

and

$$\mathcal{H}_{h_e}(h, u) = \int_{\mathbb{R}} gh(h - h_e)/2 + hu^2/2 + \alpha h^3(u_x)^2/2 = \int_{\mathbb{R}} E_{h_e}, \quad (22)$$

with

$$E_{h_e} = gh(h - h_e)/2 + hu^2/2 + \alpha h^3(u_x)^2/2. \quad (23)$$

¹See Chapters 6 and 7 of [20] for the definition of a Hamiltonian operator.

²See Section 5 for the definition of the variational derivative.

Let us mention that the Hamiltonian (22) can be obtained by a second order approximation of the Hamiltonian of the water waves problem.

Proposition 1.11 *The system (1) is endowed by the following conservation law,*

$$\partial_t E_{h_e} + \partial_x(u(E_{h_e} + p)) = 0, \quad (24)$$

where

$$p(h, u) = gh^2/2 + \alpha h^2 \ddot{h},$$

and E_{h_e} is given by (23).

Proof *Let us just consider (2). Simple computations lead us to (24) remarking the fact that $E = E_{h_e} + gh_h/2$. ■*

Remark 1.12 *The energy E_{h_e} of the conservation law (24) represents the physical energy of the system. Moreover, the Hamiltonian (22) is the whole space integral of the physical energy. Therefore, it can be interpreted as the total energy of the system.*

Hence, there is another way to prove the conservation of the Hamiltonian. Indeed, we just need to integrate (2) on \mathbb{R} and observe that the integral of the second term of the energy conservation vanishes³.

Remark 1.13 *The variable $k = \frac{m}{h}$ is the same as the generalized velocity introduced in [10].*

1.3 Outline

We insist in this introduction, on the notion of symmetrizability because this structure is crucial to prove the local well-posedness of the Cauchy problem ([3]). Moreover, Natalini and Hanouzet in [13] and Yong in [21] used this structure to prove the global well-posedness for small data of entropy dissipative hyperbolic systems (see also [16]).

The generalization of the notion of symmetrizability to dispersive perturbations of hyperbolic systems has been the goal of some authors. Especially, Gavriluk and Gouin in [8] (see also [2]) put Euler-Korteweg models and some p-systems under a quasi symmetric structure. Similar ideas can be partially adapted to some generalized p-systems like bubbly fluid equations and to modified Lagrangian Green-Naghdi, see [7] .

These generalizations are usually made in order to enable us to extend the hyperbolic results on well-posedness to their dispersive perturbations. In this work, we generalize the two types of symmetrization previously mentioned, to (1) using the Hamiltonian of the system. This generalization holds only in a small enough neighborhood of constant

³ $u \in \mathbb{H}^{s+1}$ for $s \geq 2$. Hence, u is a continuous function vanishing at infinity using the Sobolev embedding.

solutions.

Actually, our result concerns more general systems written under the following conservative form,

$$\partial_t U + \partial_x F(U) = 0 \quad (26)$$

where the unknown $U \in C([0, T]; \mathcal{A})$ for a Banach space \mathcal{A} and F is a differential application from \mathcal{A} to \mathcal{A} .

2 Generalized symmetric structure

In this part, we are going to study the symmetric structure of the system (26). First, we generalize the notion of symmetrizability and we define the Legendre transform of a variational functional. Then, we will see how the hyperbolic results are generalized.

Definition 2.1 *System (26) is called symmetrizable if there exists an unknown change $U \rightarrow Q$ such that (26) is equivalent to*

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x(Q) = 0 \quad (27)$$

while $A_0(Q)$ is a symmetric definite positive operator and $A_1(Q)$ is a symmetric one.

Definition 2.2 *Let us consider a variational functional $\mathcal{H} : \mathcal{A} \rightarrow \mathbb{R}$ such that*

$$\mathcal{H}(U) = \int_{\mathbb{R}} E(U)$$

where E is a differential application defined on \mathcal{A} . Assume also that the application

$$\delta_U \mathcal{H} : \begin{cases} \mathcal{A} \rightarrow \mathcal{B} \\ U \rightarrow \delta_U \mathcal{H}(U) \end{cases}$$

is invertible. Then, the Legendre transform \mathcal{H}^* of \mathcal{H} (if it exists) is defined on \mathcal{B} by

$$\mathcal{H}^*(Q) = \int_{\mathbb{R}} Q \cdot (\delta_U \mathcal{H})^{-1}(Q) - E[(\delta_U \mathcal{H})^{-1}(Q)] dx \quad (28)$$

Remark 2.3 *Consider a variational function \mathcal{H} with the Legendre transform \mathcal{H}^* . Then, we have*

$$\delta_Q \mathcal{H}^*(Q) = (\delta_U \mathcal{H})^{-1}(Q).$$

In other words,

$$Q = \delta_U \mathcal{H}(U) \iff U = \delta_Q \mathcal{H}^*(Q).$$

Moreover, the definite positivity of the second variation of \mathcal{H} implies the definite positivity of the second variation of \mathcal{H}^* .

Theorem 2.4 *Let us consider the system (26) and assume that*

1. *it admits the following conservation law,*

$$\partial_t E(U) + \partial_x P(U) = 0, \quad (29)$$

where $E(U)$ and $P(U)$ are continuous twice differential applications defined on \mathcal{A} .

2. *The functional $\mathcal{H}(U) = \int_{\mathbb{R}} E(U)$ as well as its Legendre Transform are well defined and the second variation $\delta_U^2 \mathcal{H}(U)$ is a definite positive operator.*
3. *The functional $\mathcal{P}(U) = \int_{\mathbb{R}} P(U)$ and $\hat{\mathcal{P}}(Q) = \int_{\mathbb{R}} Q \cdot F(U(Q)) - P(U(Q))$ are well defined.*

Then , (26) is symmetrizable in the sense of Definition 2.1.

Proof *The proof contains 3 steps. All of these three steps are the generalization of the steps of the proof of the proposition (1.3).*

- **Link between \mathcal{H} , \mathcal{P} and F .** *The conservation law (29) of the equation is obviously equivalent to*

$$D_U E(U) \partial_t U + D_U P(U) \partial_x U = 0. \quad (30)$$

On the other hand, the equation (26) and the differentiability of F leads us to

$$\partial_t U + D_U F(U) \partial_x U = 0. \quad (31)$$

We are now taking the action of $D_U E(U)$ from left side on (26). This leads us to

$$D_U E(U) \partial_t U + D_U E(U) D_U F(U) \partial_x U = 0. \quad (32)$$

Comparing equations (30) and (32) together, we have for all $\phi \in (C_c^\infty(\mathbb{R}))^N$

$$D_U E(U) D_U F(U) \phi = D_U P(U) \phi. \quad (33)$$

We are now going to integrate (33) on \mathbb{R} . This leads us to

$$\int_{\mathbb{R}} D_U E(U) D_U F(U) \phi = \int_{\mathbb{R}} D_U P(U) \phi. \quad (34)$$

Hence, we have by Definition 5.4 of the variational derivative,

$$\int_{\mathbb{R}} \delta_U \mathcal{H}(U) \cdot D_U F(U) \phi = \int_{\mathbb{R}} \delta_U \mathcal{P}(U) \cdot \phi. \quad (35)$$

This is equivalent to say that ⁴

$$\int_{\mathbb{R}} (D_U F(U))^T \delta_U \mathcal{H}(U) \cdot \phi = \int_{\mathbb{R}} \delta_U \mathcal{P}(U) \cdot \phi. \quad (36)$$

Finally, we get

$$\delta_U \mathcal{P}(U) = (D_U F(U))^T \delta_U \mathcal{H}(U). \quad (37)$$

⁴The symbol $(D_U E(U))^T$ stands for the adjoint operator of the operator $D_U E(U)$, see Defintiion 5.1.

- **Change of variable.** Let us now consider the change of variable $U \rightarrow Q$ such that

$$Q = \delta_U \mathcal{H}(U). \quad (38)$$

The invertibility of $\delta_U \mathcal{H}$ is a consequence of the monotonicity of $\delta_U^2 \mathcal{H}$ using the inverse function theorem. Let us note that applying the change of variable (38), (37) becomes

$$\delta_U \mathcal{P}(U) = (D_U F(U))^T Q. \quad (39)$$

- **Final step.** We are now defining

$$\hat{\mathcal{P}}(Q) = \int_{\mathbb{R}} Q \cdot F(U(Q)) - P(U(Q)). \quad (40)$$

Basic computations show that we have for all test function $\phi \in (C_c^\infty(\mathbb{R}))^N$,

$$D_Q \hat{\mathcal{P}}(Q) \phi = \int_{\mathbb{R}} F(U(Q)) \cdot \phi + Q \cdot D_U F(U) D_Q U(Q) \phi - D_U P(U) D_Q(U) \phi. \quad (41)$$

Hence, we get by the definition of the variational derivative and the adjoint

$$D_Q \hat{\mathcal{P}}(Q) \phi = \int_{\mathbb{R}} F(U(Q)) \cdot \phi + (D_U F(U))^T Q \cdot D_Q U(Q) \phi - \delta_U \mathcal{P}(U) \cdot D_Q(U) \phi. \quad (42)$$

Using (39) we have

$$D_Q \hat{\mathcal{P}}(Q) \phi = \int_{\mathbb{R}} F(U(Q)) \cdot \phi. \quad (43)$$

Hence,

$$\delta_Q \hat{\mathcal{P}}(Q) = F(U). \quad (44)$$

Gathering (38) and (40), (26) writes

$$\partial_t (\delta_Q \mathcal{H}^*(Q)) + \partial_x (\delta_Q \hat{\mathcal{P}}(Q)) = 0. \quad (45)$$

Hence, (26) written under the unknown Q is symmetric of the form (27) with

$$A_0(Q) = \delta_Q^2 \mathcal{H}^*(Q),$$

and

$$A_1(Q) = \delta_Q^2 \hat{\mathcal{P}}(Q).$$

■

Remark 2.5 The proof of Theorem 2.4 can be seen as another proof of the proposition 1.3 for the case of hyperbolic entropic systems which satisfy the assumptions 2 and 3 of the theorem like Saint-Venant system.

Remark 2.6 The equation (45) is of the form called general Godunov's system in [9].

Theorem 2.7 *Let us consider the system (26) of unknown U satisfying the hypothesis of the theorem (2.7). Then, for any change of variable $U \rightarrow V$, the system (26) is symmetrizable under V with*

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U,$$

and

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F(U) D_V U.$$

Proof *Considering the unknown V , (26) becomes*

$$D_V U \partial_t V + D_U F(U) D_V U \partial_x V = 0. \quad (46)$$

We are now taking the left side action of $(D_V U)^T \delta_U^2 \mathcal{H}(U)$ on (46). This leads us to

$$(D_V U)^T \delta_U^2 \mathcal{H}(U) D_V U \partial_t V + (D_V U)^T \delta_U^2 \mathcal{H}(U) D_U F(U) D_V U \partial_x V = 0. \quad (47)$$

Hence, we just need to prove that $\delta_U^2 \mathcal{H}(U) D_U F(U)$ is symmetric. In fact, we have by (44) that

$$\delta_U^2 \mathcal{H}(U) D_U F(U) = \delta_U^2 \mathcal{H}(U) D_U \delta_Q \hat{\mathcal{P}}(Q) = \delta_U^2 \mathcal{H}(U) D_Q \delta_Q \hat{\mathcal{P}}(Q) D_U Q(U). \quad (48)$$

where Q is defined by (38). Hence,

$$\delta_U^2 \mathcal{H}(U) D_U F(U) = \delta_U^2 \mathcal{H}(U) \delta_Q^2 \hat{\mathcal{P}}(Q) \delta_U^2 \mathcal{H}(U). \quad (49)$$

■

Gathering the theorems 2.4 and 2.7, we notice that the system (26) considered in the theorem 2.4 is symmetrizable in two different ways under the change of variable (38).

There is an interesting change of variable for the hyperbolic system (26) based on the decomposition (U_1, U_2) of the unknown U . That is $U \rightarrow V$ with $V = (U_1, \delta_{U_1} \mathcal{H}(U_1, U_2))$ assuming that it is well defined.

Remark 2.8 *Considering the multi-variable generalization*

$$\partial_t U + \sum_{i=1}^n \partial_{x_i} F_i[U] = 0 \quad (50)$$

of (26), we can easily extend the results Theorems 2.4 and 2.7. In fact, assume that (50) admits an additional conservation law:

$$\partial_t E[U] + \sum_{i=1}^n \partial_{x_i} P_i[U] = 0 \quad (51)$$

such that $\mathcal{H} = \int_{\mathbb{R}} E$ and its Legendre Transform, $\mathcal{P}_i = \int_{\mathbb{R}} P_i$ and $\hat{\mathcal{P}}_i = \int_{\mathbb{R}} D_U E(U) F_i[U] - P_i[U]$ are well defined with $\delta_U^2 \mathcal{H}(U)$ definite positive. Then, the system is symmetrizable under all valid change of variable. Moreover, it is symmetrizable in two different ways under the change of variable $U \rightarrow Q$ with $Q = \delta_U \mathcal{H}(U)$.

3 Application to Green-Naghdi type equations

3.1 Symmetrization of the Green-Naghdi system

In this part, we are going to see that the Green-Naghdi type system (1) is symmetrizable in different ways in the neighborhood of the equilibrium solution of (1).

Definition 3.1 *The equilibrium solution of (1) is the following global solution*

$$\begin{cases} h(x, t) = h_e \in \mathbb{R}_+^* \\ u(x, t) = 0 \iff m(x, t) = 0 \end{cases} \quad (52)$$

Remark 3.2 *The equilibrium solution (52) is the only constant solution of (1) since we consider the solutions (h, u) which belong to $(\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^{s+1}(\mathbb{R})$.*

Proposition 3.3 *The system (1) is of the form (26) with $\mathcal{A} = \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$.*

Proof *Following Li in [17], we rewrite (1) under the unknown $U = (\eta, m)$ with $\eta = h - h_e$ and $m = \mathcal{L}_h(u)$, where the Sturm-Liouville operator \mathcal{L}_h is defined by $\mathcal{L}_h(u) = hu - \alpha \partial_x (h^3 \partial_x u)$. This leads us to*

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (hu) = 0, \\ \frac{\partial m}{\partial t} + \frac{\partial}{\partial x} (mu) + \frac{\partial}{\partial x} (-2\alpha h^3 (\partial_x u)^2 + \frac{g}{2} h^2) = 0 \end{cases} \quad (53)$$

Hence, a possible choice for F is

$$F(U) = \begin{pmatrix} (\eta + h_e) \mathcal{L}_h^{-1}(m) \\ m \mathcal{L}_h^{-1}(m) - 2\alpha (\eta + h_e)^3 (\partial_x \mathcal{L}_h^{-1}(m))^2 + \frac{g}{2} (\eta + h_e)^2 - \frac{g}{2} h_e^2 \end{pmatrix} \quad (54)$$

Then, using the properties of \mathcal{L}_h ([17], [15]) we notice that

$$(h, u) \in (\mathbb{H}^s(\mathbb{R}) + h_e) \times \mathbb{H}^{s+1}(\mathbb{R}) \Rightarrow (\eta, m) \in \mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$$

Hence, F can be seen as an application from $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ to $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ ⁵.

■

We are now going to see that the system (1) satisfies locally the assumptions of the theorems 2.4 and 2.7. Indeed, the following proposition is a corollary of the local version of the theorems presented in the previous section.

Proposition 3.4 *Consider the equilibrium solution $V_e = (h_e, 0)$. There exists a neighborhood in $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s+1}(\mathbb{R})$ of the equilibrium solution V_e , such that the system (1) is symmetrizable in different ways while the solution (h, u) of (1) remains in this neighborhood.*

In other words, while the solution $V = (h, u)$ of (1) remains close to V_e , the system can be symmetrized.

⁵ $\mathbb{H}^s(\mathbb{R})$ is an algebra for $s \geq 2$.

Proof We endow (1) with the conservation law (24) and with the unknown $U = (U_1, U_2)$ such that $U_1 = \eta = h - h_e$ and $U_2 = m$. Then, we check that it satisfies the assumptions of Theorem 2.4. Basic computations show that

$$\delta^2 \mathcal{H}_{h_e}(U) = \begin{pmatrix} g - 3\alpha h(u_x)^2 - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1}(-u) + 3\alpha \partial_x(h^2 u_x) & - (u + 3\alpha h^2 u_x \partial_x) \mathcal{L}_h^{-1} \\ \mathcal{L}_h^{-1}(-u) + 3\alpha \partial_x(h^2 u_x) & \mathcal{L}_h^{-1} \end{pmatrix}. \quad (55)$$

We have on the equilibrium solution $U_e = (0, 0)$ that

$$\delta^2 \mathcal{H}_{h_e}(U_e) = \begin{pmatrix} g & 0 \\ 0 & \mathcal{L}_h^{-1} \end{pmatrix}. \quad (56)$$

Indeed, there exists a neighborhood in $\mathbb{H}^s(\mathbb{R}) \times \mathbb{H}^{s-1}(\mathbb{R})$ of U_e such that while the solution $U = (\eta, m)$ is in this neighborhood, $\delta_U^2 \mathcal{H}_{h_e}$ is definite positive. Moreover, its Legendre transform $\mathcal{H}_{h_e}^*$ is defined by

$$\mathcal{H}_{h_e}^*(Q) = \int_{\mathbb{R}} \sigma \eta + u m - H_{h_e}, \quad (57)$$

and

$$\hat{\mathcal{P}}(Q) = \int_{\mathbb{R}} Q \cdot F(U(Q)) - P(U(Q)). \quad (58)$$

where

$$Q = \delta_U \mathcal{H}_{h_e}(U), \quad (59)$$

or equivalently

$$Q = (\sigma, u) \quad \text{with} \quad \sigma = gh - gh_e/2 - u^2/2 - \frac{3}{2}\alpha h^2 (u_x)^2.$$

It is easy to check that $\mathcal{H}_{h_e}, \mathcal{P}$ and $\hat{\mathcal{P}}$ are well defined. Hence, there exists a neighborhood of the equilibrium solution such that the assumptions of theorems are satisfied while the solution remains in this neighborhood. Therefore, the system is locally symmetrizable. ■

Remark 3.5 The variable σ coincides with one of the variables used at the end of [9] in the canonical representation of the Green-Naghdi equation.

We now provide in the two following propositions, some explicit symmetric forms for the system (1).

Proposition 3.6 The Green-Naghdi type system (1) is symmetric under the unknown $Q = (\sigma, u)$, given by (59), of the form

$$A_0(Q) \partial_t Q + A_1(Q) \partial_x(Q) = 0, \quad (60)$$

with

$$A_0(Q) = \left(\frac{1}{g-3\alpha h(u_x)^2} - 3\alpha \partial_x \left(\frac{h^2 u_x}{g-3\alpha h(u_x)^2} \right) \right) \mathcal{L}_h + \left(\frac{u+3\alpha h^2 u_x \partial_x}{g-3\alpha h(u_x)^2} \right) (u + 3\alpha h^2 (u_x) \partial_x) - 3\alpha \partial_x \left(h^2 u_x \frac{u() + 3\alpha h^2 u_x \partial_x ()}{g-3\alpha h(u_x)^2} \right) \quad (61)$$

and

$$A_1(Q) = \left(h + \frac{u^2}{g-3\alpha h(u_x)^2} - 3\alpha \partial_x \left(\frac{h^2 u(u_x)}{g-3\alpha h(u_x)^2} \right) \right) 3hu + \frac{u^3 + 3\alpha h^2 u^2 u_x \partial_x}{g-3\alpha h(u_x)^2} - \alpha \partial_x (h^3 u_x ()) - \alpha u \partial_x (h^3 \partial_x ()) - 3\alpha \partial_x \left(\frac{h^2 u^2 u_x + 3\alpha h^4 u(u_x)^2 \partial_x ()}{g-3\alpha h(u_x)^2} \right) \left(h + \frac{u^2 + 3\alpha h^2 u u_x \partial_x}{g-3\alpha h(u_x)^2} \right). \quad (62)$$

Proof This is an application of the theorem 2.4 and the proposition 3.4. Note that the Legendre transform $\mathcal{H}_{h_e}^*(Q)$ of $\mathcal{H}_{h_e}(Q)$ has the following analytic expression,

$$\mathcal{H}_{h_e}^*(Q) = \int_{\mathbb{R}} \frac{g(h-h_e)^2}{2} + h_e u^2 / 2 - \alpha h^3 (u_x)^2 + \frac{3}{2} \alpha h^2 h_e (u_x)^2.$$

Moreover, the quasi Legendre transform $\hat{\mathcal{P}}$ of the space integral of the energy flux $\mathcal{P} = \int_{\mathbb{R}} P$ is given by (58). In fact, the following lemma holds,

Lemma 3.7 There exists a constant $\beta \in \mathbb{R}$ such that $\hat{\mathcal{P}}$ of (1) defined by

$$\hat{\mathcal{P}}(Q) = \int_{\mathbb{R}} Q \cdot F(U(Q)) - P(U(Q)), \quad (63)$$

satisfies

$$\hat{\mathcal{P}} = \beta + \mathcal{Q}, \quad (64)$$

where

$$\mathcal{Q} = \int_{\mathbb{R}} gu \left(\frac{h^2 - h_e^2}{2} \right) - \alpha h^3 u (u_x)^2. \quad (65)$$

Proof of lemma 3.7. We compute the variational derivative $\delta_Q \mathcal{Q}$ and we observe that

$$\delta_Q \mathcal{Q} = F(U), \quad (66)$$

where $F(U)$ is defined by (54). Hence, \mathcal{P} and \mathcal{Q} have the same variational derivative. Therefore, they are different by a constant. ■

Hence, (1) can be written as

$$\partial_t (\delta_Q \mathcal{H}_{h_e}^*(Q)) + \partial_x (\delta_Q \mathcal{Q}(Q)) = 0. \quad (67)$$

Basic computations show that $\delta_Q^2 \mathcal{H}_{h_e}^*(Q) = A_0(Q)$ and $\delta_Q^2 \mathcal{P}(Q) = \delta_Q^2 \mathcal{Q}(Q) = A_1(Q)$ where the expressions of $A_0(Q)$ and $A_1(Q)$ are given by (61) and (62). ■

Remark 3.8 In the case of Green-Naghdi type equation (1), $A_0(Q)$ and $A_1(Q)$ defined by (61) and (62) are linear second order operators.

Proposition 3.9 The Green-Naghdi type system (1) is symmetric under the unknown $V = (h, u)$ of the form

$$A_0(V)\partial_t V + A_1(V)\partial_x(V) = 0 \quad (68)$$

with

$$A_0(V) = \begin{pmatrix} g - 3\alpha h(u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix}, \quad (69)$$

and

$$A_1(V) = \begin{pmatrix} gu - 3\alpha hu(u_x)^2 & gh - 3\alpha h^2(u_x)^2 \\ gh - 3\alpha h^2(u_x)^2 & hu + 2\alpha\partial_x(h^3u_x) - \alpha h^3u_x\partial_x - \alpha u\partial_x(h^3\partial_x(\cdot)) \end{pmatrix}. \quad (70)$$

Proof This proposition is just a consequence of the theorem 2.7 and the proposition 3.4. In fact, we check that the change of variable $U \rightarrow V$ such that

$$\begin{cases} U = (h, m), \\ V = (h, \delta_m \mathcal{H}_{h_e}(U)), \end{cases}$$

leads us to $V = (h, u)$ since $\delta_m \mathcal{H}_{h_e}(U) = \mathcal{L}_h^{-1}(m)$. This change of variable is valid by the properties of the Sturm-Liouville operator \mathcal{L}_h . Hence, the system is symmetric with $A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{h_e}(U) D_V U$ and $A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{h_e}(U) \nabla_U F(U) D_V U$. Basic computations show that their analytic expressions are given by (69) and (70). ■

Remark 3.10 $A_0(V)$ and $A_1(V)$ given by (69) and (70) are linear second order operators.

Remark 3.11 The analytic expressions of operators are much simpler in the form presented in Proposition (3.9) than in the form of Proposition (3.6). In fact, as we expected, $A_0(V)$ is diagonal in the form considered in Proposition (3.9).

3.2 Two-dimensional extension

Let us now consider the 2D Green-Naghdi model

$$\partial_t h + \partial_x hu + \partial_y hv = 0, \quad (71a)$$

$$\partial_t hu + \partial_x hu^2 + \partial_y huv + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = 0, \quad (71b)$$

$$\partial_t hv + \partial_x huv + \partial_y hv^2 + \partial_y (gh^2/2 + \alpha h^2 \ddot{h}) = 0, \quad (71c)$$

where $\dot{h} = \partial_t h + u\partial_x h + v\partial_y h$.

This system is equivalent to

$$\partial_t U + \partial_x F_1(U) + \partial_y F_2(U) = 0, \quad (72)$$

where

$$F_1(U) = \begin{pmatrix} hu \\ gh^2/2 + hu^2 - 2\alpha h^3(\operatorname{div}(u, v))^2 - \alpha u \partial_x (h^3 \operatorname{div}(u, v)) + \alpha h^3 \operatorname{div}(u, v) v_y \\ huv - \alpha \partial_y (h^3 u \operatorname{div}(u, v)), \end{pmatrix}$$

and

$$F_2(U) = \begin{pmatrix} hv \\ huv - \alpha \partial_x (h^3 v \operatorname{div}(u, v)) \\ gh^2/2 + hv^2 - 2\alpha h^3(\operatorname{div}(u, v))^2 - \alpha v \partial_y (h^3 \operatorname{div}(u, v)) + \alpha h^3 \operatorname{div}(u, v) u_x \end{pmatrix},$$

for $U = (h, m, n)$ where $(m, n) = \mathcal{L}_h(u, v)$ with $\mathcal{L}_h(u, v) = h(u, v) - \alpha \nabla (h^3 \operatorname{div}(u, v))$.

Proposition 3.12 (71) *satisfies the following conservation law*

$$\partial_t E_{h_e} + \partial_x (u(E_{h_e} + p)) + \partial_y (v(E_{h_e} + p)) = 0. \quad (74)$$

where

$$E_{h_e} = gh(h - h_e)/2 + hu^2/2 + hv^2/2 + \alpha h^3(u_x + v_y)^2/2, \quad (75a)$$

and

$$p = gh^2/2 + \alpha h^2 \ddot{h}. \quad (75b)$$

Let us consider the space integral \mathcal{H}_{h_e} of the energy E_{h_e} ,

$$\mathcal{H}_{h_e}(U) = \int_{\mathbb{R}^2} E_{h_e}(U). \quad (76)$$

and consider the change of variable $U \rightarrow Q = \delta_U \mathcal{H}(U)$. In other words,

$$Q = \begin{pmatrix} gh - u^2/2 - v^2/2 - 3\alpha(\operatorname{div}(u, v))^2/2 \\ u \\ v \end{pmatrix}.$$

Let us now consider the Legendre transform \mathcal{H}^* of \mathcal{H} defined by

$$\begin{aligned} \mathcal{H}_{h_e}^*(Q) &= \int_{\mathbb{R}^2} Q \cdot U - E_{h_e}(U) \\ &= \int_{\mathbb{R}^2} g(h - h_e)^2/2 + h_e(u^2 + v^2)/2 - \alpha h^3(\operatorname{div}(u, v))^2 + \frac{3}{2}\alpha h^2 h_e(\operatorname{div}(u, v))^2. \end{aligned}$$

Proposition 3.13 *The two-dimensional Green-Naghdi model (71) is symmetric of the form*

$$A_0(Q)\partial_t Q + A_1(Q)\partial_x Q + A_2(Q)\partial_y Q = 0, \quad (77)$$

assuming that the unknown Q is close enough to the equilibrium solution $(h_e, 0, 0)$.

Proof Let us now consider the variational functionals S_1 and S_2 defined by

$$S_1(Q) = \int_{\mathbb{R}^2} u E^*,$$

and

$$S_2(Q) = \int_{\mathbb{R}^2} v E^*,$$

with

$$E^* = gh^2/2 - \alpha h^3(\operatorname{div}(u, v))^2. \quad (79)$$

We can easily observe that

$$U = \delta_Q \mathcal{H}^*(Q), \quad (80)$$

$$F_1(U) = \delta_Q S_1(Q), \quad (81)$$

$$F_2(U) = \delta_Q S_2(Q). \quad (82)$$

Hence, setting

$$A_0(Q) = \delta_Q^2 \mathcal{H}^*(Q),$$

$$A_1(Q) = \delta_Q^2 S_1(Q),$$

and

$$A_2(Q) = \delta_Q^2 S_2(Q),$$

we get the result. Especially, we note that $A_0(Q)$ is symmetric definite positive in a neighborhood of the constant solution and $A_1(Q)$ as well as $A_2(Q)$ are symmetric. ■

Proposition 3.14 *The two-dimensional Green-Naghdi equation (71) is symmetric of the form*

$$A_0(V)\partial_t V + A_1(V)\partial_x V + A_2(V)\partial_y V = 0. \quad (83)$$

where $V = (h, u, v)$.

Proof Indeed, the change of variable $U \rightarrow V$ transforms (71) to

$$D_V U \partial_t V + D_U F_1(U) D_V U \partial_x V + D_U F_2(U) D_V U \partial_y V = 0, \quad (84)$$

Taking the action of $(D_V U)^T \delta_U^2 \mathcal{H}_{h_e}(U)$ from the left side on (84) we get the symmetry with

$$A_0(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{h_e}(U) D_V U,$$

$$A_1(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{h_e}(U) D_U F_1(U) D_V U \partial_x V,$$

and

$$A_2(V) = (D_V U)^T \delta_U^2 \mathcal{H}_{h_e}(U) D_U F_2(U) D_V U \partial_y V.$$

■

4 Conclusion

A generalization of the notion of symmetry for hyperbolic system has been presented. This generalization is essentially made by replacing energy by its integral, matrix by operator, partial derivatives by variational derivatives and matrix transpose by adjoint operators.

Then, we apply the result to Green-Naghdi equation (1) in a neighborhood of the equilibrium solution of the system. Two type of symmetrizations are presented. We could notice that these symmetrization are based on a conservation law which is directly related to the Hamiltonian of the system which represents the total physical energy. In fact, this Hamiltonian is the space integral of the energy of the conservation law. the generalization to the two-dimensional system is also provided.

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5 Appendix

Definition 5.1 *Let us consider the operator $\mathcal{A} : (C_c^\infty(\mathbb{R}))^N \rightarrow (C_c^\infty(\mathbb{R}))^N$. Then, the adjoint \mathcal{A}^T of \mathcal{A} is a $(C_c^\infty(\mathbb{R}))^N \rightarrow (C_c^\infty(\mathbb{R}))^N$ operator such that we have for all $\phi, \psi \in (C_c^\infty(\mathbb{R}))^N$,*

$$\int_{\mathbb{R}} \phi \cdot \mathcal{A}(\psi) = \int_{\mathbb{R}} \psi \cdot \mathcal{A}^T(\phi).$$

Definition 5.2 *Let us consider the operator $\mathcal{A} : (C_c^\infty(\mathbb{R}))^N \rightarrow (C_c^\infty(\mathbb{R}))^N$. Then \mathcal{A} is called symmetric if we have for all $\phi, \psi \in (C_c^\infty(\mathbb{R}))^N$,*

$$\int_{\mathbb{R}} \phi \cdot \mathcal{A}(\psi) = \int_{\mathbb{R}} \psi \cdot \mathcal{A}(\phi).$$

In other words, a symmetric operator is equal to its adjoint.

Notation 5.3 *Let us consider a differential application \mathcal{G} .*

1. $D\mathcal{G}(U)$ (respectively $D^2\mathcal{G}(U)$) denotes its Frechet differential (respectively its second Frechet differential) at point U .
2. $D_{U_i}\mathcal{G}(U)$ denotes its Gateau differential in the direction U_i at point U

Definition 5.4 [11] *Let us consider the functional Banach space \mathcal{A} containing $(C_c^\infty(\mathbb{R}))^N$ (for an integer N) and a variational functional \mathcal{G} defined on \mathcal{A} by*

$$\mathcal{G}(U) = \int_{\mathbb{R}} G(U).$$

Then, the variational derivative at U of \mathcal{G} is $\delta\mathcal{G}(U) \in \mathcal{A}$ such that we have $\forall\phi \in (C_c^\infty(\mathbb{R}))^N$,

$$D\mathcal{G}(U)(\phi) = \int_{\mathbb{R}} \delta\mathcal{G}(U) \cdot \phi.$$

Moreover, the second variation at U of \mathcal{G} is the bilinear operator $\delta^2\mathcal{G}(U)$ acting on $(C_c^\infty(\mathbb{R}))^N$ such that we have $\forall\phi, \psi \in (C_c^\infty(\mathbb{R}))^N$,

$$D^2\mathcal{G}(U)(\phi, \psi) = \int_{\mathbb{R}} \phi \cdot \delta^2 G(U) \psi.$$

Remark 5.5 $\delta^2\mathcal{G}(U)$ is a symmetric operator. In fact $\delta^2\mathcal{G}(U) = D\delta\mathcal{G}(U)$.

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