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# On the values of logarithmic residues along curves

Delphine POL

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## Abstract

We consider the germ of a reduced curve, possibly reducible. F.Delgado de la Mata proved that such a curve is Gorenstein if and only if its semigroup of values is symmetrical. We extend here this symmetry property to any fractional ideal of a Gorenstein curve. We then focus on the set of values of the module of logarithmic residues along plane curves, which determines the values of the Jacobian ideal thanks to our symmetry Theorem. Moreover, we give the relation with Kähler differentials, which are used in the analytic classification of plane branches. We also study the behaviour of logarithmic residues in an equisingular deformation of a plane curve.

## 1 Introduction

Let  $(D, 0)$  be the germ of a reduced hypersurface in  $(\mathbb{C}^n, 0)$  defined by  $f \in \mathbb{C}\{\underline{x}\} := \mathbb{C}\{x_1, \dots, x_n\}$  and with ring  $\mathcal{O}_D = \mathbb{C}\{\underline{x}\}/(f)$ . In his fundamental paper [Sai80], K.Saito introduces the notions of logarithmic vector fields, logarithmic differential forms and their residues. A logarithmic differential form is a meromorphic form with simple poles along  $(D, 0)$  such that its differential has also simple poles along  $(D, 0)$ . A logarithmic  $q$ -form  $\omega$  may be written as:

$$g\omega = \frac{df}{f} \wedge \xi + \eta$$

where  $g \in \mathbb{C}\{\underline{x}\}$  does not induce a zero divisor in  $\mathcal{O}_D$ ,  $\xi$  is a holomorphic  $(q-1)$ -form and  $\eta$  is a holomorphic  $q$ -form. Then, the logarithmic residue  $\text{res}^q(\omega)$  of  $\omega$  is defined as the coefficient of  $\frac{df}{f}$ , that is to say:

$$\text{res}^q(\omega) = \frac{\xi}{g} \in \Omega_D^{q-1} \otimes_{\mathcal{O}_D} Q(\mathcal{O}_D)$$

with  $\Omega_D^{q-1}$  the module of Kähler differentials on  $D$  and  $Q(\mathcal{O}_D)$  the total ring of fractions of  $\mathcal{O}_D$ . We denote by  $\mathcal{R}_D$  the module of logarithmic residues of logarithmic 1-forms.

In [GS14], M.Granger and M.Schulze prove that the dual over  $\mathcal{O}_D$  of the Jacobian ideal is  $\mathcal{R}_D$ . If moreover  $(D, 0)$  is free, that is to say, if the module of logarithmic differential 1-forms is free, the converse also holds: the dual of  $\mathcal{R}_D$  is the Jacobian ideal. They use this duality to prove a characterization of normal crossing divisors in terms of logarithmic residues: if the module  $\mathcal{R}_D$  is equal to the module of weakly holomorphic functions on  $D$  then  $D$  is normal crossing in codimension 1, and the converse implication was already proved in [Sai80].

The purpose of this paper is to investigate the module of logarithmic residues along plane curves. Plane curves are always free divisors, and they are the only singular free divisors with isolated singularities. Indeed, A.G.Aleksandrov proves in [Ale88, §2] that the singular locus of a singular free divisor is of codimension 2 in the ambient space. We will also study the case of complete intersection curves, for which a notion of multi-residues has been introduced by A.G.Aleksandrov and A.Tsikh in [AT01].

Let  $(D, 0)$  be the germ of a reduced Gorenstein curve in  $(\mathbb{C}^m, 0)$ . In particular,  $(D, 0)$  may be a plane curve or a complete intersection curve. A parametrization of  $(D, 0)$  is given by the normalization of the local ring  $\mathcal{O}_D$ , and induces a map  $\text{val} : Q(\mathcal{O}_D) \rightarrow (\mathbb{Z} \cup (\infty))^p$  called the value map, which associates to a fraction  $g \in Q(\mathcal{O}_D)$  the  $p$ -uple of the valuations of  $g$  along each irreducible components of  $(D, 0)$ . For a fractional ideal  $I \subset Q(\mathcal{O}_D)$ , we denote by  $\text{val}(I)$  the set of values of the non-zero divisors of  $I$ .

We first show in section 2 that the values of the Jacobian ideal and the values of the module of logarithmic residues determine each other. The statement is easy for irreducible curves (see Remark 2.10):

$$v \in \text{val}(\mathcal{R}_D) \iff c - v - 1 \notin \text{val}(\mathcal{J}_D)$$

where  $c \in \mathbb{N}$  is the conductor of the curve, i.e.  $c = \min \{c \in \mathbb{N}; c + \mathbb{N} \subseteq \text{val}(\mathcal{O}_D)\}$ . It is a generalization of the well-known symmetry of the semigroup for Gorenstein curves (see for example [Kun70]). Nevertheless, the semigroup of a reducible curve also satisfies a certain symmetry property which has been proved by F.Delgado de la Mata in [DdlM88]. The statement of this symmetry needs more notations and the proof is more difficult than in the irreducible case.

We prove here that there is also a symmetry between the values of the module of logarithmic residues and the values of the Jacobian ideal, which is in fact satisfied by any fractional ideal and its dual. Whereas the symmetry is immediate for irreducible curves, the proof of this generalization of Delgado's Theorem is much more subtle. It leads to the main result of this section, namely Theorem 2.4, which generalizes to any Gorenstein curve and any fractional ideal the Theorem 2.4 of [Pol15].

In section 3, we give some properties of the set of values of the module of logarithmic residues and of the Jacobian ideal for plane curves. We show how to determine the set of values of the logarithmic residues of a reunion of branches of  $D$  from the set  $\text{val}(\mathcal{R}_D)$ . We also prove that the modules of logarithmic vector fields of the branches determine the set of values of the zero divisors of  $\mathcal{J}_D$ . Moreover, we give the relation between the Jacobian ideal and the Kähler differentials:

$$\text{val}(\mathcal{J}_D) = \gamma + \text{val}(\Omega_D^1) - \underline{1}$$

with  $\underline{1} = (1, \dots, 1)$  and  $\gamma \in \mathbb{N}^p$  the conductor of the curve, that is to say the minimal  $\gamma \in \mathbb{N}^p$  such that  $\gamma + \mathbb{N}^p \subseteq \text{val}(\mathcal{O}_D)$ . The set of values of Kähler differentials is a major ingredient used in [HH11] and [HHH15] to study the problem of the analytic classification of plane curves with one or two branches.

A.G.Aleksandrov and A.Tsikh develop in [AT01] the theory of multi-logarithmic differential forms and multi-residues along reduced complete intersections. Since our symmetry Theorem is true for any Gorenstein curve, it is in particular true for complete intersections. We mention in section 3.4 several properties of reduced complete intersection curves which generalize the properties of plane curves.

The last section is devoted to the study of the behaviour of logarithmic residues in an equisingular deformation of a plane curve. In particular, we define a stratification by the values of the logarithmic residues, which is the same thanks to section 3 as the stratification by Kähler differentials. We prove that this stratification is finite and constructible, but it does not satisfy the frontier condition.

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## 2 The symmetry of values

This section is devoted to the main Theorem 2.4, which is a generalization of the symmetry Theorem 2.8 of [DdlM88]. The statement is true for any fractional ideal of  $Q(\mathcal{O}_D)$ , so that it is in particular true for the Jacobian ideal and the module of logarithmic residues, which are studied in part 3 and 4.

We first introduce some definitions and notations inspired by [DdlM88], and then give the statement of the theorem. We give here a detailed proof of this theorem, and then a consequence on the Poincaré series associated to a fractional ideal of a Gorenstein curve (see Proposition 2.25).

### 2.1 Preliminaries

We recall here some results and notations of [DdlM88].

Let  $(D, 0) \subset (\mathbb{C}^m, 0)$  be the germ of a reduced analytic curve, with  $p$  irreducible components  $D_1, \dots, D_p$ . The ring  $\mathcal{O}_{D_i}$  of the branch  $D_i$  is a one-dimensional integral domain, so that its normalization  $\mathcal{O}_{\tilde{D}_i}$  is isomorphic to  $\mathbb{C}\{t_i\}$  (see for example [dJP00, Corollary 4.4.10]). By the splitting of normalization (see [dJP00, Theorem 1.5.20]), the ring  $\mathcal{O}_{\tilde{D}}$  of the normalization of  $D$  is  $\mathcal{O}_{\tilde{D}} = \bigoplus_{i=1}^p \mathbb{C}\{t_i\}$ . Moreover, the total rings of fraction of  $\mathcal{O}_D$  and  $\mathcal{O}_{\tilde{D}}$  are equal. We denote it by  $Q(\mathcal{O}_D)$ . We then have :

$$Q(\mathcal{O}_{\tilde{D}}) = Q(\mathcal{O}_D) = \bigoplus_{i=1}^p \mathbb{C}\{t_i\} \begin{bmatrix} 1 \\ t_i \end{bmatrix}$$

The normalization of  $D_i$  gives a parametrization of the branch  $D_i$ , which is denoted by

$$\begin{aligned} \varphi_i : \mathbb{C}\{t_i\} &\rightarrow D_i \\ t_i &\mapsto (x_{i,1}(t_i), \dots, x_{i,m}(t_i)) \end{aligned}$$

**Definition 2.1.** Let  $g \in Q(\mathcal{O}_D)$ . We define the valuation of  $g$  along the branch  $D_i$  as the order in  $t_i$  of  $g \circ \varphi_i(t_i)$ . We denote it by  $\text{val}_i(g) \in \mathbb{Z} \cup \{\infty\}$ , with the convention  $\text{val}_i(0) = \infty$ .

We then define the value of  $g$  by  $\text{val}(g) = (\text{val}_1(g), \dots, \text{val}_p(g)) \in (\mathbb{Z} \cup \{\infty\})^p$ .

**Definition 2.2.** Let  $I \subset Q(\mathcal{O}_D)$  be a  $\mathcal{O}_D$ -module. It is called a fractional ideal if it is of finite type over  $\mathcal{O}_D$  and if it contains a non-zero divisor of  $Q(\mathcal{O}_D)$ .

For a fractional ideal  $I$ , we define  $\text{val}(I) = \{\text{val}(g); g \in I \text{ non-zero divisor}\} \subset \mathbb{Z}^p$ . We also set  $\overline{\text{val}(I)} = \{\text{val}(g); g \in I\} \subset (\mathbb{Z} \cup \{\infty\})^p$ .

**Remark 2.3.** We will prove in part 3 that the set  $\text{val}(I)$  determines the set  $\overline{\text{val}(I)}$  (see Proposition 3.7).

Let  $I \subset Q(\mathcal{O}_D)$  be a fractional ideal. We denote by  $I^\vee$  the dual of  $I$ , which is by definition  $I^\vee = \text{Hom}_{\mathcal{O}_D}(I, \mathcal{O}_D)$ . Let  $g \in I$  be a non-zero divisor. The morphism given by

$$\varphi \in \text{Hom}_{\mathcal{O}_D}(I, \mathcal{O}_D) \mapsto \frac{\varphi(g)}{g} \in Q(\mathcal{O}_D)$$

induces an isomorphism between  $\text{Hom}_{\mathcal{O}_D}(I, \mathcal{O}_D)$  and  $\{h \in Q(\mathcal{O}_D); hI \subseteq \mathcal{O}_D\}$  (see [dJP00, Proof of Lemma 1.5.14]). We can therefore consider  $I^\vee$  as a subset of  $Q(\mathcal{O}_D)$ .

We define the conductor ideal of the curve by  $\mathcal{C}_D := \text{Ann}_{\mathcal{O}_D}(\mathcal{O}_{\tilde{D}}/\mathcal{O}_D)$ . It is a fractional ideal of  $Q(\mathcal{O}_D)$ . Moreover, it is also an ideal in  $\mathcal{O}_{\tilde{D}}$ , so that there exists  $\gamma = (\gamma_1, \dots, \gamma_p) \in \mathbb{N}^p$ , called

the conductor, such that  $\mathcal{C}_D = t^\gamma \mathcal{O}_{\tilde{D}}$ , where for  $\alpha \in \mathbb{Z}^p$ , we set  $t^\alpha = (t_1^{\alpha_1}, \dots, t_p^{\alpha_p})$ . By definition,  $\mathcal{C}_D = \mathcal{O}_{\tilde{D}}^\vee$ .

We also introduce the following notations, which are analogous to the notations of [DdlM88].

Let  $\mathcal{M} \subseteq \mathbb{Z}^p$  and  $v \in \mathbb{Z}^p$ . For  $i \in \{1, \dots, p\}$ , we define:

$$\Delta_i(v, \mathcal{M}) = \{\alpha \in \mathcal{M} ; \alpha_i = v_i \text{ and } \forall j \neq i, \alpha_j > v_j\}$$

and  $\Delta(v, \mathcal{M}) = \bigcup_{i=1}^p \Delta_i(v, \mathcal{M})$ . For a fractional ideal  $I \subset Q(\mathcal{O}_D)$ , we write  $\Delta(v, I)$  instead of  $\Delta(v, \text{val}(I))$ . We consider the product order on  $\mathbb{Z}^p$ , so that for  $\alpha, \beta \in \mathbb{Z}^p$ ,  $\alpha \leq \beta$  means that for all  $i \in \{1, \dots, p\}$ ,  $\alpha_i \leq \beta_i$  and  $\inf(\alpha, \beta) = (\min(\alpha_1, \beta_1), \dots, \min(\alpha_p, \beta_p))$ .

We set  $\alpha - \underline{1} = (\alpha_1 - 1, \dots, \alpha_p - 1)$ .

We can now state the main Theorem of this section, which generalizes [Pol15, Theorem 2.4]:

**THEOREM 2.4.** *Let  $D$  be the germ of a reduced curve with ring  $\mathcal{O}_D$ . Then,  $\mathcal{O}_D$  is a Gorenstein ring if and only if for all fractional ideal  $I \subset Q(\mathcal{O}_D)$  and for all  $v \in \mathbb{Z}^p$  the following property is satisfied:*

$$v \in \text{val}(I^\vee) \iff \Delta(\gamma - v - \underline{1}, I) = \emptyset \quad (1)$$

We notice that if we suppose the condition (1) satisfied for all fractional ideals  $I \subset Q(\mathcal{O}_D)$ , it is in particular satisfied by the ring  $\mathcal{O}_D$ , whose dual is  $\mathcal{O}_D^\vee = \mathcal{O}_D$ , so that we recognize Theorem 2.8 of [DdlM88]. Therefore, the curve is indeed a Gorenstein curve, which proves one of the implications of Theorem 2.4.

**Remark 2.5.** It is not sufficient to check if (1) is satisfied for one fractional ideal  $I$  to prove that the curve is Gorenstein. Indeed, by definition, for every curve, the equivalence (1) is satisfied by  $I = \mathcal{O}_{\tilde{D}}$  and  $I^\vee = \mathcal{C}_D$ .

## 2.2 Proof of Theorem 2.4

From now on, we assume that  $\mathcal{O}_D$  is a Gorenstein ring. For example,  $D$  can be a plane curve or a complete intersection curve. We recall here a property of Gorenstein curves:

**Proposition 2.6** ([Eis95, Theorem 21.21], [dJP00, lemma 5.2.8]). *Let  $I \subset Q(\mathcal{O}_D)$  be a fractional ideal of a Gorenstein curve. Then it is a maximal Cohen-Macaulay module over  $\mathcal{O}_D$  and:*

- The dual  $I^\vee$  of  $I$  is also a fractional ideal, and  $I^{\vee\vee} = I$ .
- If  $I \subset J$  are fractional ideals,  $J^\vee \subseteq I^\vee$  and moreover,  $\dim_{\mathbb{C}} J/I = \dim_{\mathbb{C}} I^\vee/J^\vee$ .

In particular, for a Gorenstein curve,  $\mathcal{C}_D^\vee = \mathcal{O}_{\tilde{D}}$ , so that for all  $\alpha \in \mathbb{Z}^p$ ,  $(t^\alpha \mathcal{O}_{\tilde{D}})^\vee = t^{-\alpha} \mathcal{C}_D = t^{\gamma-\alpha} \mathcal{O}_{\tilde{D}}$ .

The following proposition comes from the definition of a fractional ideal, and will be very useful:

**Proposition 2.7.** *Let  $I$  be a fractional ideal. Then there exist  $\nu$  and  $\lambda$  in  $\mathbb{Z}^p$  such that*

$$t^\nu \mathcal{O}_{\tilde{D}} \subseteq I \subseteq t^\lambda \mathcal{O}_{\tilde{D}} \quad (2)$$

*In particular, it implies that  $\nu + \mathbb{N}^p \subseteq \text{val}(I) \subseteq \lambda + \mathbb{N}^p$ . Moreover, we can replace in (2.7)  $\lambda$  by any element of  $\mathbb{Z}^p$  lower than  $\lambda$ , and  $\nu$  by any greater element of  $\mathbb{Z}^p$ .*

For a Gorenstein curve, by Proposition 2.6, we have the following sequence of inclusions:

$$t^{\gamma-\lambda} \mathcal{O}_{\tilde{D}} \subseteq I^\vee \subseteq t^{\gamma-\nu} \mathcal{O}_{\tilde{D}} \quad (3)$$

Let  $I \subset Q(\mathcal{O}_D)$  be a fractional ideal. We first prove the implication  $\Rightarrow$  of (1).

**Proposition 2.8.** *Let  $v \in \mathbb{Z}^p$ . Then:*

$$v \in \text{val}(I^\vee) \Rightarrow \Delta(\gamma - v - \underline{1}, I) = \emptyset$$

**Proof.** Let  $v = (v_1, \dots, v_p) \in \text{val}(I^\vee)$ , and  $g \in I^\vee$  with  $v = \text{val}(g)$ . We assume  $\Delta(\gamma - v - \underline{1}, I) \neq \emptyset$ . For the sake of simplicity, we may assume that  $\Delta_1(\gamma - v - \underline{1}, I) \neq \emptyset$ , which means that there exists  $h \in I$  with  $\text{val}(h) = (\gamma_1 - v_1 - 1, w_2, \dots, w_p) \in I$  and for all  $j \geq 2$ ,  $w_j > \gamma_j - v_j - 1$ . Since  $gh \in \mathcal{O}_D$ , we have  $(\gamma_1 - 1, w_2 + v_2, \dots, w_p + v_p) \in \text{val}(\mathcal{O}_D)$ , with  $w_j + v_j \geq \gamma_j$ . Therefore,  $\Delta_1(\gamma - \underline{1}, \mathcal{O}_D) \neq \emptyset$ .

Nevertheless, from Corollary 1.9 of [DdlM88],  $\Delta(\gamma - \underline{1}, \mathcal{O}_D) = \emptyset$ , which leads to a contradiction. Therefore,  $\Delta(\gamma - v - \underline{1}, I) = \emptyset$ .  $\square$

**Notation 2.9.** We set  $\mathcal{V} = \{v \in \mathbb{Z}^p; \Delta(\gamma - v - \underline{1}, I) = \emptyset\}$ .

The set  $\mathcal{V}$  contains the values of  $I^\vee$ , but *a priori* it may be bigger. In particular, it is not obvious that  $\mathcal{V}$  is the set of values of a  $\mathcal{O}_D$ -module. Our purpose here is to prove that  $\mathcal{V}$  is indeed equal to  $\text{val}(I^\vee)$ .

**Remark 2.10.** For irreducible curves, the statement of Theorem 2.4 can be rephrased as follows: for all  $v \in \mathbb{Z}$ ,  $v \in \text{val}(I^\vee)$  if and only if  $\gamma - v - 1 \notin \text{val}(I)$ , which is a generalization of the Theorem of E.Kunz (see [Kun70]). The proof is easy for irreducible curves. Indeed, we have:

$$\begin{aligned} \dim_{\mathbb{C}} I^\vee / t^{\gamma-\lambda} \mathbb{C} \{t\} &= \text{Card}(\text{val}(I^\vee) \cap (\gamma - \lambda + \mathbb{N})^c) \\ \dim_{\mathbb{C}} t^\lambda \mathbb{C} \{t\} / I &= \text{Card}((\lambda + \mathbb{N}) \cap (\text{val}(I))^c) \end{aligned}$$

Since by Proposition 2.6,  $\dim_{\mathbb{C}} I^\nu / t^{\gamma-\lambda} \mathbb{C} \{t\} = \dim_{\mathbb{C}} t^\lambda \mathbb{C} \{t\} / I$ , we have the result.

The proof for a reducible curve is also based on a dimension argument, but which is much more subtle than in the irreducible case. First of all, we need the following properties, which should be compared with 1.1.2 and 1.1.3 in [DdlM88].

**Proposition 2.11.** *For a fractional ideal  $I \subset \overline{Q(\mathcal{O}_D)}$ , if  $v, v' \in \text{val}(I)$ , then  $\inf(v, v') \in \text{val}(I)$ . Similarly, if  $v, v' \in \text{val}(I)$ , then  $\inf(v, v') \in \text{val}(I)$ .*

**Remark 2.12.** If  $v \in \text{val}(I)$  and  $v' \in \overline{\text{val}(I)}$ , then  $\inf(v, v') \in \text{val}(I)$ .

**Proposition 2.13.** *Let  $v \neq v' \in \text{val}(I)$ . If there exists  $i \in \{1, \dots, p\}$  such that  $v_i = v'_i$ , then there exists  $v'' \in \text{val}(I)$  such that:*

1.  $v''_i > v_i$
2.  $v''_j \geq \min(v_j, v'_j)$
3. If moreover  $v_j \neq v'_j$ , then  $v''_j = \min(v_j, v'_j)$

The proposition 2.11 is a consequence of the fact that the valuation of a general linear combination of two elements is the lowest one. The proposition 2.13 comes from the fact that a convenient linear combination will increase the valuation on the component  $D_i$ , but we cannot say what happens on the other components where the equality holds.

### 2.2.1 Dimension and values

Let  $v \in \mathbb{Z}^p$ . We set  $I_v = \{g \in I; \text{val}(g) \geq v\}$  and  $\ell(v, I) = \dim_{\mathbb{C}} I/I_v$ . Then  $\ell(v, I) < \infty$ . We denote by  $(e_1, \dots, e_p)$  the canonical basis of  $\mathbb{Z}^p$ . For  $\mathcal{M} \subseteq \mathbb{Z}^p$  and  $v \in \mathbb{Z}^p$ , let

$$\Lambda_i(v, \mathcal{M}) = \{\alpha \in \mathcal{M} ; \alpha_i = v_i \text{ and } \alpha \geq v\}$$

We then have (see [DdlM88, Proposition 1.11]):

**Proposition 2.14.** *For all  $v \in \mathbb{Z}^p$ ,  $\ell(v + e_i, I) - \ell(v, I) \in \{0, 1\}$  and  $\ell(v + e_i, I) = \ell(v, I) + 1$  if and only if  $\Lambda_i(v, \text{val}(I)) \neq \emptyset$ .*

Thanks to this proposition we can compute some dimensions from the set of values:

**Corollary 2.15.** *Let  $\nu, \lambda \in \mathbb{Z}^p$  such that  $\nu + \mathbb{N}^p \subseteq \text{val}(I) \subseteq \lambda + \mathbb{N}^p$  (see Proposition 2.7). Let  $(\alpha^{(j)})_{0 \leq j \leq M+1}$  be a finite sequence of elements of  $\mathbb{Z}^p$  satisfying:*

- $\alpha^{(0)} = \lambda$  and  $\alpha^{(M+1)} = \nu$
- For all  $j \in \{0, \dots, M\}$ , there exists  $i(j) \in \{1, \dots, p\}$  such that  $\alpha^{(j+1)} = \alpha^{(j)} + e_{i(j)}$

Then:

$$\dim_{\mathbb{C}} I/t^\nu \mathcal{O}_{\bar{D}} = \ell(\nu, I) = \text{Card} \left\{ j \in \{0, \dots, M\}; \Lambda_{i(j)}(\alpha^{(j)}, \text{val}(I)) \neq \emptyset \right\} \quad (4)$$

### 2.2.2 Preliminary steps

We recall that  $\mathcal{V} = \{v \in \mathbb{Z}^p; \Delta(\gamma - v - \underline{1}, I) = \emptyset\}$ , and this set contains  $\text{val}(I^\vee)$ . The purpose of this section is to show that if the inclusion  $\text{val}(I^\vee) \subseteq \mathcal{V}$  is strict, then it has some combinatorial and numerical consequences. We then prove that it leads to a contradiction, which finishes the proof of Theorem 2.4.

*First step*

We first show that if  $\mathcal{V} \neq \text{val}(I^\vee)$ , then there is an element  $w \in \mathcal{V} \setminus \text{val}(I^\vee)$  which satisfies some properties which will be used in the next steps.

Let us assume that  $\mathcal{V} \neq \text{val}(I^\vee)$ , and let  $w^{(0)} \in \mathcal{V} \setminus \text{val}(I^\vee)$  be "an intruder". By Proposition 2.7, there exist  $\lambda, \nu \in \mathbb{Z}^p$  satisfying  $\nu + \mathbb{N}^p \subseteq \text{val}(I) \subseteq \lambda + \mathbb{N}^p$  and  $\gamma - \nu \leq w^{(0)} \leq \gamma - \lambda$ . For the remaining of the proof, we fix such  $\lambda, \nu$ .

The following Proposition gives an essential property of  $w^{(0)}$ :

**Proposition 2.16.** *There exists  $j \in \{1, \dots, p\}$  such that  $\Lambda_j(w^{(0)}, \text{val}(I^\vee)) = \emptyset$ . Moreover, the corresponding coordinate satisfies  $w_j^{(0)} < \gamma_j - \lambda_j$ .*

**Proof.** If for all  $i \in \{1, \dots, p\}$ ,  $\Lambda_i(w^{(0)}, \text{val}(I^\vee)) \neq \emptyset$ , then for all  $i \in \{1, \dots, p\}$  there exists  $\alpha^{(i)} \in \text{val}(I^\vee)$  such that  $\alpha_i^{(i)} = w_i^{(0)}$  and  $\alpha_j^{(i)} \geq w_j^{(0)}$ . As a consequence, by Proposition 2.11,  $\inf(\alpha^{(1)}, \dots, \alpha^{(p)}) = w^{(0)} \in \text{val}(I^\vee)$ , which is a contradiction. It gives the existence of a  $j \in \{1, \dots, p\}$  such that  $\Lambda_j(w^{(0)}, \text{val}(I^\vee)) = \emptyset$ . It is immediate to see that  $w_j^{(0)} < \gamma_j - \lambda_j$  since if  $w_j^{(0)} = \gamma_j - \lambda_j$ , then  $\gamma - \lambda \in \Lambda_j(w^{(0)}, \text{val}(I^\vee))$ , which contradicts the emptiness.  $\square$

*Second step*

For the sake of simplicity, we assume that  $\Lambda_p(w^{(0)}, \text{val}(I^\vee)) = \emptyset$ . The Corollary 2.15 together with a convenient finite sequence  $(\alpha^{(j)})$  can be used to compute the dimension of the quotient  $I^\vee/t^{\gamma-\lambda} \mathcal{O}_{\bar{D}}$ . We define a number  $\ell'$  as the cardinality of the set obtained by replacing  $\text{val}(I^\vee)$  by  $\mathcal{V}$  in (4). This number may *a priori* depend on the chosen sequence  $(\alpha^{(j)})$ .

In order to compute  $\dim_{\mathbb{C}} I^\vee/t^{\gamma-\lambda} \mathcal{O}_{\bar{D}}$ , we consider a sequence  $(\alpha^{(j)})_{0 \leq j \leq n_0}$  with  $n_0 = \sum_{i=1}^p \nu_i - \lambda_i$  satisfying:

- $\alpha^{(0)} = \gamma - \nu$  and  $\alpha^{(n_0)} = \gamma - \lambda$
- for all  $j \in \{0, \dots, n_0 - 1\}$ , there exists  $i(j) \in \{1, \dots, p\}$  such that  $\alpha^{(j+1)} = \alpha^{(j)} + e_{i(j)}$
- there exists  $j_0 \in \{0, \dots, n_0 - 1\}$  such that  $\alpha^{(j_0)} = w^{(0)}$  and  $\alpha^{(j_0+1)} = w^{(0)} + e_p$

The existence of such a sequence follows from Proposition 2.16. Moreover, this sequence satisfies the required properties of Corollary 2.15.

For example, when  $p = 2$ , we can choose a sequence  $\alpha$  as follows:

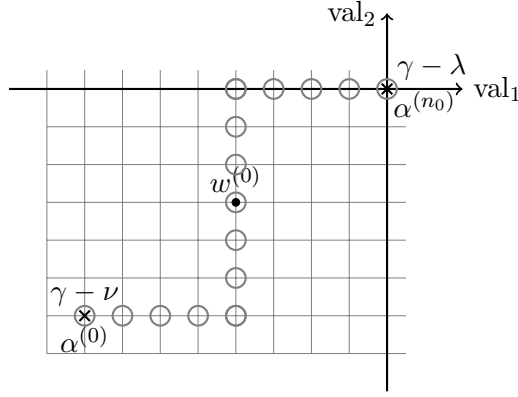


Figure 1

**Remark 2.17.** In [Pol15] we choose a sort of "canonical" sequence, but in order to be less technical, we impose less conditions on  $\alpha$ .

From Corollary 2.15, we have:

$$\dim_{\mathbb{C}} I^{\vee} / t^{\gamma-\lambda} \mathcal{O}_{\tilde{D}} = \text{Card} \left\{ j \in \{0, \dots, n_0 - 1\} ; \Lambda_{i(j)}(\alpha^{(j)}, \text{val}(I^{\vee})) \neq \emptyset \right\} \quad (5)$$

We want to compare this dimension with the number  $\ell'$  previously announced:

$$\ell' = \text{Card} \left\{ j \in \{0, \dots, n_0 - 1\} ; \Lambda_{i(j)}(\alpha^{(j)}, \mathcal{V}) \neq \emptyset \right\} \quad (6)$$

**Lemma 2.18.** *For the sequence  $\alpha$  defined above, we have:*

$$\ell' \geq 1 + \dim_{\mathbb{C}} I^{\vee} / t^{\gamma-\lambda} \mathcal{O}_{\tilde{D}} \quad (7)$$

**Proof.** It is clear that  $\Lambda_{i(j)}(\alpha^{(j)}, \text{val}(I^{\vee})) \neq \emptyset \Rightarrow \Lambda_{i(j)}(\alpha^{(j)}, \mathcal{V}) \neq \emptyset$ . Moreover, since there exists  $j_0$  such that  $\alpha^{(j_0)} = w^{(0)}$  and  $\alpha^{(j_0+1)} = \alpha^{(j_0)} + e_p$ ,  $\Lambda_p(\alpha^{(j_0)}, \mathcal{V}) \neq \emptyset$ , but the assumptions on  $w^{(0)}$  implies  $\Lambda_p(\alpha^{(j_0)}, \text{val}(I^{\vee})) = \emptyset$ . Hence the inequality.  $\square$

From now on, our sequence  $\alpha$  is fixed.

*Third step*

The purpose of this third step is to compare this number  $\ell'$  to  $\dim_{\mathbb{C}} I / t^{\nu} \mathcal{O}_{\tilde{D}}$ .

For  $i \in \{0, \dots, n_0\}$  we set  $\beta^{(i)} = \gamma - \alpha^{(n_0-i)}$ . The sequence  $\beta$  satisfies the properties of Corollary 2.15, so that it can be used to compute the dimension  $\dim_{\mathbb{C}} I / t^{\nu} \mathcal{O}_{\tilde{D}}$ .

For the sequence  $\alpha$  of Figure 1, we represent the sequence  $\beta$  on the following diagram, where  $v^{(0)} = \gamma - w^{(0)} - \underline{1}$ :



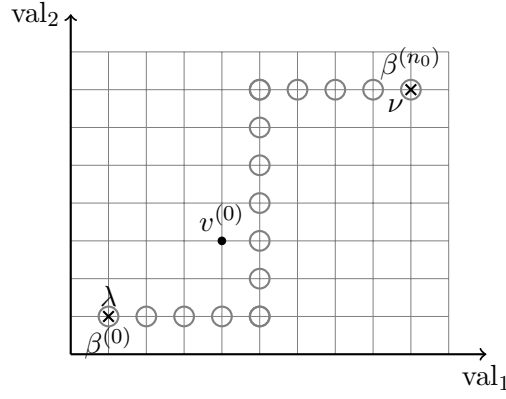


Figure 2

The following proposition gives a relation between  $\ell'$  and  $\dim_{\mathbb{C}} I/t^{\nu} \mathcal{O}_{\tilde{D}}$ :

**Proposition 2.19.** *With the above notations we have:*

$$\dim_{\mathbb{C}} I/t^{\nu} \mathcal{O}_{\tilde{D}} \leq \sum_{i=1}^p (\nu_i - \lambda_i) - \ell'$$

To prove this proposition, we need the following lemma:

**Lemma 2.20.** *Let  $w \in \mathbb{Z}^p$  and  $i \in \{1, \dots, p\}$ . Then:*

$$\Lambda_i(w, \mathcal{V}) \neq \emptyset \Rightarrow \Lambda_i(\gamma - w - e_i, \text{val}(I)) = \emptyset$$

**Proof.** Let  $w' \in \Lambda_i(w, \mathcal{V})$ . By the definition of  $\mathcal{V}$ , we have  $\Delta(\gamma - w' - \underline{1}, \text{val}(I)) = \emptyset$ . Moreover,  $(\gamma - w' - e_i)_i = \gamma_i - w'_i - 1$  and for  $j \neq i$ ,  $(\gamma - w' - e_i)_j = \gamma_j - w'_j \leq \gamma_j - w_j$ . Thus  $\Lambda_i(\gamma - w' - e_i, \text{val}(I)) = \Delta_i(\gamma - w' - \underline{1}, \text{val}(I)) = \emptyset$ .  $\square$

**Proof of Proposition 2.19.** We first notice that the two sequences  $\alpha^{(j)}$  and  $\beta^{(j)}$  has the same number of terms, namely  $n_0 + 1 = \sum_{i=1}^p (\nu_i - \lambda_i) + 1$ .

By Corollary 2.15, we have:

$$\dim_{\mathbb{C}} I/t^{\nu} \mathcal{O}_{\tilde{D}} = \text{Card} \left\{ j \in \{0, \dots, n_0 - 1\} ; \Lambda_{i(n_0-j-1)}(\beta^{(j)}, \text{val}(I)) \neq \emptyset \right\} \quad (8)$$

We notice that for all  $j \in \{0, \dots, n_0 - 1\}$ ,  $\gamma - \alpha^{(j)} - e_{i(j)} = \gamma - \alpha^{(j+1)} = \beta^{(n_0-(j+1))}$ .

Therefore, by the previous Lemma, if  $\Lambda_i(\alpha^{(j)}, \mathcal{V}) \neq \emptyset$  then  $\Lambda_i(\beta^{(n_0-(j+1))}, \text{val}(I)) = \emptyset$ . We then obtain the result by comparing (8) and (6).  $\square$

### 2.2.3 End of the proof of Theorem 2.4

We can now finish the proof of Theorem 2.4.

The inclusion  $\text{val}(I^{\vee}) \subseteq \mathcal{V}$  is given by Proposition 2.8. It remains to prove that this inclusion cannot be strict.

The Proposition 2.19 gives:

$$\dim_{\mathbb{C}} I/t^{\nu} \mathcal{O}_{\tilde{D}} \leq \sum_{i=1}^p (\nu_i - \lambda_i) - \ell'$$

Moreover, since  $\sum_{i=1}^p (\nu_i - \lambda_i) = \dim_{\mathbb{C}} t^\lambda \mathcal{O}_{\tilde{D}} / t^\nu \mathcal{O}_{\tilde{D}} = \dim_{\mathbb{C}} t^\lambda \mathcal{O}_{\tilde{D}} / I + \dim_{\mathbb{C}} I / t^\nu \mathcal{O}_{\tilde{D}}$ , we have by Proposition 2.6:

$$\dim_{\mathbb{C}} I^\vee / t^{\gamma-\lambda} \mathcal{O}_{\tilde{D}} = \sum_{i=1}^p (\nu_i - \lambda_i) - \dim_{\mathbb{C}} I / t^\nu \mathcal{O}_{\tilde{D}}$$

Hence the inequality:

$$\ell' \leq \dim_{\mathbb{C}} I^\vee / t^{\gamma-\lambda} \mathcal{O}_{\tilde{D}} \quad (9)$$

However, by Lemma 2.18, if  $\mathcal{V} \neq \text{val}(I^\vee)$ , then  $\ell' \geq 1 + \dim_{\mathbb{C}} I^\vee / t^{\gamma-\lambda} \mathcal{O}_{\tilde{D}}$ , which contradicts (9). Therefore, we have  $\mathcal{V} = \text{val}(I^\vee)$ , that is to say:  $v \in \text{val}(I^\vee) \iff \Delta(\gamma - v - 1, I) = \emptyset$ .  $\square$

**Remark 2.21.** Another consequence of the equality  $\mathcal{V} = \text{val}(I^\vee)$  is that the number  $\ell'$  is equal to the dimension of  $I^\vee / t^{\gamma-\lambda} \mathcal{O}_{\tilde{D}}$ . Therefore, the inequality in Proposition 2.19 is in fact an equality. Moreover, since for all  $w \in \mathbb{Z}^p$ , there exist  $\lambda', \nu' \in \mathbb{Z}^p$  such that  $\gamma - \lambda' + \mathbb{N}^p \subseteq \text{val}(I^\vee) \subseteq \gamma - \nu' + \mathbb{N}^p$  and  $\gamma - \nu' \leq w \leq \gamma - \lambda'$ , it also has the following consequence:

$$\Lambda_i(w, \text{val}(I^\vee)) \neq \emptyset \iff \Lambda_i(\gamma - w - e_i, \text{val}(I)) = \emptyset \quad (10)$$

### 2.3 Poincaré series of a fractional ideal

This section follows a suggestion of Antonio Campillo. Let  $(D = D_1 \cup \dots \cup D_p, 0)$  be the germ of a reduced reducible curve, and  $I \subset Q(\mathcal{O}_D)$  be a fractional ideal.

The following definitions are inspired by [CDGZ03]. We recall that  $I_v = \{g \in I; \text{val}(g) \geq v\}$ .

We consider the set of formal Laurent series  $\mathcal{L} = \mathbb{Z}[[t_1^{-1}, \dots, t_p^{-1}, t_1, \dots, t_p]]$ . This set is not a ring, but it is a  $\mathbb{Z}[t_1^{-1}, \dots, t_p^{-1}, t_1, \dots, t_p]$ -module.

We set:

$$L_I(t_1, \dots, t_p) = \sum_{v \in \mathbb{Z}^p} c_I(v) t^v \quad (11)$$

with  $c_I(v) = \dim_{\mathbb{C}} I_v / I_{v+1}$  and

$$P_I(t) = L_I(t) \prod_{i=1}^p (t_i - 1) \quad (12)$$

**Remark 2.22.** In [CDGZ03], the authors study the case  $I = \mathcal{O}_D$  with  $D$  a plane curve. They prove that  $P_{\mathcal{O}_D}$  is in fact a polynomial, and for plane curves with at least two components,  $\frac{P_{\mathcal{O}_D}(t)}{t_1 \dots t_p - 1}$  is the Alexander polynomial of the curve (see [CDGZ03, Theorem 1]).

Our purpose here is to deduce from Theorem 2.4 a relation between  $P_I(t)$  and  $P_{I^\vee}(t)$ .

The following lemma is a direct consequence of the definition (12) of  $P_I$ :

**Lemma 2.23.** *We define for  $v \in \mathbb{Z}^p$ ,*

$$\alpha_I(v) = \sum_{J \subseteq \{1, \dots, p\}} (-1)^{\text{Card}(J^c)} c_I(v - e_J) \quad (13)$$

where we denote for  $J = \{j_1, \dots, j_k\}$ ,  $e_J = e_{j_1} + \dots + e_{j_k}$ . Then

$$P_I(t) = \sum_{v \in \mathbb{Z}^p} \alpha_I(v) t^v$$

We use the previous lemma to prove the following property:

**Lemma 2.24.** *The formal Laurent series  $P_I(t)$  is a polynomial.*

**Proof.** Let  $\lambda, \nu \in \mathbb{Z}^p$  be such that  $\nu + \mathbb{N}^p \subseteq \text{val}(I) \subseteq \lambda + \mathbb{N}^p$ . The only possibly non-zero  $\alpha_I(v)$  are those such that  $\lambda \leq v \leq \nu$ . Indeed, let us assume for example that  $v_p < \lambda_p$  or  $v_p > \nu_p$ . We can then prove thanks to Corollary 2.15 that for all  $J \subset \{1, \dots, p\}$  such that  $p \notin J$ ,  $c_I(v - e_{J \cup \{p\}}) = c_I(v - e_J)$ . By the definition (13), it gives us the result.  $\square$

The symmetry of Theorem 2.4 has the following consequence:

**Proposition 2.25.** *With the same notations,*

$$P_{I^\vee}(t) = (-1)^{p+1} t^\gamma P_I\left(\frac{1}{t_1}, \dots, \frac{1}{t_p}\right) \quad (14)$$

**Proof.** The property (14) is in fact equivalent to the following property:

$$\forall v \in \mathbb{Z}^p, \alpha_{I^\vee}(v) = (-1)^{p+1} \alpha_I(\gamma - v) \quad (15)$$

This property is obvious if  $v \notin \{\omega \in \mathbb{Z}^p; \gamma - \nu \leq \omega \leq \gamma - \lambda\}$  since both  $\alpha_{I^\vee}(v)$  and  $\alpha_I(\gamma - v)$  are zero.

By (13), it is sufficient to prove that for all  $v \in \mathbb{Z}^p$ ,  $c_{I^\vee}(v) = p - c_I(\gamma - v - 1)$ . We have:

$$c_{I^\vee}(v) = \text{Card}\{i \in \{1, \dots, p\}; \Lambda_i(v + e_1 + \dots + e_{i-1}, \text{val}(I^\vee)) \neq \emptyset\}$$

$$c_I(\gamma - v - 1) = \text{Card}\{i \in \{1, \dots, p\}; \Lambda_i(\gamma - v - e_1 - \dots - e_i, \text{val}(I)) \neq \emptyset\}$$

The result follows from the equivalence (10).  $\square$

### 3 On the structure of the set of values of logarithmic residues

In this part we give several properties of the module of logarithmic residues along plane curves or complete intersection curves. We first recall some definitions. We then study the structure of the set of values of logarithmic residues along plane curves, and give the relation with the set of values of Kähler differentials, which is used in the analytic classification of branches proposed in [HH11], and [HHH15] for two branches. We extend some of these properties to the set of values of multi-residues along complete intersection curves.

#### 3.1 Preliminaries on logarithmic residues

We recall here some definitions and results about logarithmic vector fields, logarithmic differential forms and their residues, which can be found in [Sai80].

Let us consider a reduced hypersurface germ  $(D, 0) \subset (\mathbb{C}^n, 0)$  defined by  $f \in \mathbb{C}\{x_1, \dots, x_n\}$ . We denote by  $\Theta_n$  the module of germs of holomorphic vector fields on  $(\mathbb{C}^n, 0)$ , and  $\mathbb{C}\{\underline{x}\} = \mathbb{C}\{x_1, \dots, x_n\}$ .

**Definition 3.1.** *A germ of vector field  $\delta \in \Theta_n$  is called logarithmic along  $D$  if  $\delta(f) = \alpha f$  with  $\alpha \in \mathbb{C}\{\underline{x}\}$ . We denote by  $\text{Der}(-\log D)$  the module of logarithmic vector fields along  $(D, 0)$ .*

*A germ of meromorphic  $q$ -form  $\omega \in \frac{1}{f} \Omega^q$  with simple poles along  $D$  is called logarithmic if  $f d\omega$  is holomorphic. We denote by  $\Omega^q(\log D)$  the module of logarithmic  $q$ -forms on  $(D, 0)$ .*

Moreover, these two modules are reflexive and each is the dual  $\mathbb{C}\{\underline{x}\}$ -module of the other (see [Sai80, Lemma 1.6]).

**Definition 3.2.** *If  $\text{Der}(-\log D)$  (or equivalently  $\Omega^1(\log D)$ ) is a free  $\mathbb{C}\{\underline{x}\}$ -module, we call  $(D, 0)$  a germ of free divisor.*

In particular, plane curves are free divisors (see [Sai80, 1.7]).

**Proposition 3.3 (Saito criterion).** *The germ  $(D, 0)$  is a free divisor at 0 if and only if there exists  $(\delta_1, \dots, \delta_n)$  in  $\text{Der}(-\log D)$  such that  $\delta_j = \sum a_{ij}\partial_{x_i}$  with  $\det((a_{ij})_{1 \leq i, j \leq n}) = uf$ , where  $u$  is invertible.*

To define the notion of logarithmic residues, we need the following characterization of logarithmic differential forms:

**Proposition 3.4.** *A meromorphic  $q$ -form  $\omega$  with simple poles along  $D$  is logarithmic if and only if there exist  $g \in \mathbb{C}\{\underline{x}\}$ , which does not induce a zero divisor in  $\mathcal{O}_D = \mathbb{C}\{\underline{x}\}/(f)$ , a holomorphic  $(q-1)$ -form  $\xi$  and a holomorphic  $q$ -form  $\eta$  such that:*

$$g\omega = \frac{df}{f} \wedge \xi + \eta \quad (16)$$

**Definition 3.5.** *The residue  $\text{res}^q(\omega)$  of  $\omega \in \Omega^q(\log D)$  is defined by*

$$\text{res}^q(\omega) := \frac{\xi}{g} \in Q(\mathcal{O}_D) \otimes_{\mathcal{O}_D} \Omega_D^{q-1}$$

where  $\xi$  and  $g$  are given by (16), and  $\Omega_D^{q-1} = \frac{\Omega_{\mathbb{C}^n}^{q-1}}{df \wedge \Omega_{\mathbb{C}^n}^{q-2} + f\Omega_{\mathbb{C}^n}^{q-1}}$  is the module of Kähler differentials on  $D$ .

If  $q = 1$ , we write  $\text{res}(\omega)$  instead of  $\text{res}^1(\omega)$ . We define  $\mathcal{R}_D := \{\text{res}(\omega); \omega \in \Omega^1(\log D)\} \subseteq Q(\mathcal{O}_D)$ , and we call  $\mathcal{R}_D$  the module of logarithmic residues of  $D$ . In particular,  $\mathcal{R}_D$  is a finite type  $\mathcal{O}_D$ -module. Moreover, we have the inclusion  $\mathcal{O}_{\tilde{D}} \subseteq \mathcal{R}_D$  (see [Sai80, Lemma 2.8]).

We denote by  $\mathcal{J}_D \subseteq \mathcal{O}_D$  the Jacobian ideal of  $D$ , that is to say the ideal of  $\mathcal{O}_D$  generated by the partial derivatives of  $f$ .

The following result gives the relation between the module of logarithmic residues and the Jacobian ideal:

**Proposition 3.6 ([GS14, Proposition 3.4]).** *Let  $(D, 0)$  be the germ of a reduced divisor. Then  $\mathcal{J}_D^\vee = \mathcal{R}_D$ . If moreover  $D$  is free,  $\mathcal{R}_D^\vee = \mathcal{J}_D$ .*

## 3.2 Some basic properties of the values of logarithmic residues

We give here some properties of the set of values of logarithmic residues and of the Jacobian ideal of plane curves.

Since plane curves are free divisors, the module  $\text{Der}(-\log D)$  is free. Let us assume that  $\delta_i = \alpha_i\partial_x + \beta_i\partial_y$ ,  $i = 1, 2$  is a basis of  $\text{Der}(-\log D)$ . Then by duality, a basis of  $\Omega^1(\log D)$  is:

$$\begin{aligned} \omega_1 &= \frac{\beta_2 dx - \alpha_2 dy}{f} \\ \omega_2 &= \frac{-\beta_1 dx + \alpha_1 dy}{f} \end{aligned}$$

If  $g = c_1 \cdot f'_x + c_2 \cdot f'_y$  with  $c_1, c_2 \in \mathbb{C}$  induces a non zero divisor in  $\mathcal{O}_D$ , then the module of residues is generated by  $\text{res}(\omega_1) = \frac{c_1 \cdot \beta_2 - c_2 \cdot \alpha_2}{g}$  and  $\text{res}(\omega_2) = \frac{-c_1 \cdot \beta_1 + c_2 \alpha_1}{g}$ . Thus, the module of logarithmic residues can be generated by two elements.

Let  $D = D_1 \cup \dots \cup D_p$  be the germ of a reduced plane curve defined by  $f = f_1 \cdots f_p$  where for all  $i \in \{1, \dots, p\}$ ,  $f_i$  is irreducible. We first want to prove that the negative values of  $\mathcal{R}_D$  determine all the values of  $\mathcal{R}_D$ . Since we will need a similar result for the Jacobian ideal, we state it for an arbitrary fractional ideal.

We recall that for  $g \in Q(\mathcal{O}_D)$ ,  $\text{val}_i(g) = \infty$  means that the restriction of  $g$  on  $D_i$  is zero.

The following proposition shows that the values of the zero divisors are determined by the faces of the negative quadrant with origin  $\nu$ .

**Proposition 3.7.** *Let  $I \subset Q(\mathcal{O}_D)$  be a fractional ideal and let  $\nu \in \mathbb{Z}^p$  be such that  $t^\nu \mathcal{O}_{\tilde{D}} \subseteq \text{val}(I)$ . The values of the zero divisors of  $I$  are determined by*

$$\mathcal{M}_I = \bigcup_{q=1}^p \bigcup_{\sigma \in \mathfrak{S}_q} \{w \in \text{val}(I); \forall i \in \{1, \dots, q\}, w_{\sigma(i)} = \nu_{\sigma(i)}\}$$

where  $\mathfrak{S}_q$  is the group of permutations of  $q$  elements. More precisely, the values of the zero divisors are exactly the values obtained by replacing for  $v \in \mathcal{M}_I$  the coordinates which satisfies  $v_j = \nu_j$  by  $\infty$ .

**Proof.** Let  $g \in I$  be a zero divisor. Then by Proposition 2.11,  $\inf(\text{val}(g), \nu) \in \text{val}(I)$ . It is then obvious that  $\inf(\text{val}(g), \nu) \in \mathcal{M}_I$ .

Conversely, let  $v \in \mathcal{M}_I$ . There exists  $q \in \{1, \dots, p\}$  and  $\sigma \in \mathfrak{S}_q$  such that for all  $i \in \{1, \dots, p\}$ ,  $v_i = \nu_i$ . Let  $h \in I$  be such that  $\text{val}(h) = v$ . Since  $t^\nu \mathcal{O}_{\tilde{D}} \subseteq I$ , there exists  $g \in I$  satisfying for all  $i \in \{1, \dots, q\}$ ,  $g|_{D_{\sigma(i)}} = h|_{D_{\sigma(i)}}$  and for all  $j \notin \sigma(\{1, \dots, q\})$ ,  $\text{val}_j(g) > v_j$ . Then  $h - g$  is a zero divisor whose value satisfies for  $i \in \{1, \dots, q\}$ ,  $\text{val}_{\sigma(i)}(h - g) = \infty$  and for all  $j \notin \sigma(\{1, \dots, q\})$ ,  $\text{val}_j(h - g) = v_j$ .  $\square$

**Corollary 3.8.** *Let  $I \subset Q(\mathcal{O}_D)$  be a fractional ideal and  $\nu \in \mathbb{Z}^p$ . Assume that  $\nu + \mathbb{N}^p \subseteq \text{val}(I)$ . Let  $v \in \mathbb{Z}^p$ . Then*

$$v \in \text{val}(I) \iff \inf(v, \nu) \in \text{val}(I)$$

In particular, it means that the set

$$\text{val}(I) \cap \{v \in \mathbb{Z}^p; v \leq \nu\}$$

determines the set  $\text{val}(I)$ .

**Proof.** The implication  $\Rightarrow$  comes from Proposition 2.11. For the implication  $\Leftarrow$ , let  $v \in \mathbb{Z}^p$  be such that  $\inf(v, \nu) \in \text{val}(I)$ . If  $v \leq \nu$ , then  $v = \inf(v, \nu) \in \text{val}(I)$ . If there exists  $j$  such that  $v_j > \nu_j$ , then  $\inf(v, \nu) \in \mathcal{M}_I$  where  $\mathcal{M}_I$  is defined in Proposition 3.7. Therefore, since  $t^\nu \mathcal{O}_{\tilde{D}} \subseteq I$ , there exists a zero divisor  $g \in I$ ,  $q \in \{1, \dots, p\}$  and  $\sigma \in \mathfrak{S}_q$  which satisfy for all  $i \in \{1, \dots, q\}$ ,  $\text{val}_{\sigma(i)}(g) = \infty$  if  $v_{\sigma(i)} = \nu_{\sigma(i)}$  and for all  $j \notin \sigma(\{1, \dots, q\})$ ,  $\text{val}_j(g) = v_j < \nu_j$ . Let  $w = \max(v, \nu)$ . Then  $w \in \text{val}(I)$  and  $v = \inf(w, \text{val}(g)) \in \text{val}(I)$ .  $\square$

**Remark 3.9.** By Propositions 3.7 and 3.8, the set  $\text{val}(I) \cap \{v \in \mathbb{Z}^p; v \leq \nu\}$  also determines  $\overline{\text{val}(I)}$ .

The inclusion  $\mathcal{O}_{\tilde{D}} \subseteq \mathcal{R}_D$  gives the following corollary:

**Corollary 3.10.** *The set of values of  $\mathcal{R}_D$  is determined by the set*

$$\{v \in \text{val}(\mathcal{R}_D); v \leq 0\}$$

More precisely, we have:

$$\text{val}(\mathcal{R}_D) = \{v \in \text{val}(\mathcal{R}_D); v \leq 0\} \cup \{v \in \mathbb{Z}; \inf(v, 0) \in \text{val}(\mathcal{R}_D)\}$$

Let us determine the values of  $\mathcal{R}_D$  which come from the branches or union of branches.

**Proposition 3.11.** *Let  $q \in \{1, \dots, p\}$  and  $D' = D_{i_1} \cup \dots \cup D_{i_q}$  be the union of  $q$  branches of  $D$ . Then*

$$\Omega^1(\log D') \subseteq \Omega^1(\log D)$$

*Moreover,  $v \in \{w \in \text{val}(\mathcal{R}_D); \forall j \in \{1, \dots, q\}, v_{i_j} = 0\}$  if and only if there exists a logarithmic 1-form  $\omega \in \Omega^1(\log D')$  such that for all  $j \notin \{i_1, \dots, i_q\}$ ,  $\text{val}_j(\text{res}(\omega)) = v_j$ , and for all  $j \in \{1, \dots, q\}$ ,  $\text{val}_{i_j}(\text{res}(\omega)) = \infty$ .*

**Proof.** For the first part of the statement, we set  $F_1$  the equation of  $D'$ . Let  $\omega$  be a logarithmic 1-form along  $D'$ . Then,  $F_1\omega$  and  $F_1d\omega$  are holomorphic, so that  $f\omega$  and  $fd\omega$  are holomorphic.

The second part of the statement comes from this inclusion and Proposition 3.7.  $\square$

In particular, the logarithmic residues of the irreducible components satisfy the following property:

**Corollary 3.12.** *We have the following inclusion:*

$$\mathcal{R}_{D_1} \oplus \dots \oplus \mathcal{R}_{D_p} \hookrightarrow \mathcal{R}_D$$

*Therefore,  $\text{val}_1(\mathcal{R}_{D_1}) \times \dots \times \text{val}_p(\mathcal{R}_{D_p}) \subseteq \text{val}(\mathcal{R}_D)$ .*

**Remark 3.13.** If  $D = D_1 \cup D_2$  is a plane curve satisfying  $\mathcal{R}_D = \mathcal{R}_{D_1} \oplus \mathcal{R}_{D_2}$ , then by [Sch13], it is a splayed divisor, and in fact it is even a normal crossing plane curve.

We now study the set of values of the dual of  $\mathcal{R}_D$ , namely, the Jacobian ideal  $\mathcal{J}_D$ . We show that the modules of logarithmic vector fields  $\text{Der}(-\log D_i)$  for  $i \in \{1, \dots, p\}$  give informations on the structure of the set of values of the Jacobian ideal.

**Proposition 3.14.** *Let  $v \in \mathbb{Z}^p$ . With the notation of Proposition 3.7,  $v \in \mathcal{M}_{\mathcal{J}_D}$  if and only if there exists  $D' = D_{i_1} \cup \dots \cup D_{i_q}$  and  $\delta \in \text{Der}(-\log D') = \text{Der}(-\log D_{i_1}) \cap \dots \cap \text{Der}(-\log D_{i_q})$  such that for  $j \notin \{i_1, \dots, i_q\}$ ,  $\text{val}_j(\delta(f_j)) = v_j - \sum_{i \neq j} \text{val}_i(f_i)$ . In particular, the set of zero divisors of  $\mathcal{J}_D$  are determined by the family of modules  $\bigcap_{i \in I} \text{Der}(-\log D_i)$  for  $I \subset \{1, \dots, p\}$ .*

**Proof.** We first notice that for all  $g \in \mathcal{J}_D$ , there exists  $\delta \in \Theta_2$  such that  $\delta(f) = g$  in  $\mathcal{O}_D$ . Moreover,  $\delta(f)$  induces in  $\mathcal{O}_{\tilde{D}} = \prod_{i=1}^p \mathcal{O}_{\tilde{D}_i}$  the element:

$$\delta(f) = (f_2 \cdots f_p \delta(f_1), \dots, f_1 \cdots f_{p-1} \delta(f_p))$$

By Proposition 3.7,  $v \in \mathcal{M}_{\mathcal{J}_D}$  if and only if there is  $\delta \in \Theta_2$  such that

$$\text{val}(\delta(f)) = (\infty, \dots, \infty, v_k, \dots, v_p)$$

which is equivalent to:

$$\forall i \in \{1, \dots, k-1\}, \delta(f_i) \in (f_i) \text{ and } \forall i \in \{k, \dots, p\}, \text{val}_i(\delta(f)) = v_i$$

Hence the result.  $\square$

### 3.3 The relation with Kähler differentials

We give here the relation between logarithmic residues and Kähler differentials. More precisely, we show that the values of the Jacobian ideal can be obtained from the values of the Kähler differentials, and the relation with the values of logarithmic residues comes from Theorem 2.4.

We first recall the definition of the module of Kähler differentials:

$$\Omega_D^1 = \frac{\Omega_{\mathbb{C}^2}^1}{\mathcal{O}_{\mathbb{C}^2} df + f \Omega_{\mathbb{C}^2}^1}$$

**Definition 3.15.** Let  $D = D_1 \cup \dots \cup D_p$  be a plane curve germ with irreducible components  $D_i$ , and  $\varphi_i(t_i) = (x_i(t_i), y_i(t_i))$  be a parametrization of  $D_i$ . Let  $\omega = a dx + b dy \in \Omega_D^1$ . We denote by  $\varphi_i^*(\omega)$  the 1-form defined on  $D_i$  by:

$$\varphi_i^*(\omega) = (a \circ \varphi_i(t_i) x_i'(t_i) + b \circ \varphi_i(t_i) y_i'(t_i)) dt_i$$

We denote  $\varphi^*(\omega) = (\varphi_1^*(\omega), \dots, \varphi_p^*(\omega))$ .

We then define the valuation of  $\omega$  along  $D_i$  by:

$$\text{val}_i(\omega) = \text{val}_i(\varphi_i^*(\omega)) = \text{val}_i(a \circ \varphi_i(t_i) x_i'(t_i) + b \circ \varphi_i(t_i) y_i'(t_i)) + 1$$

The value of  $\omega$  is then  $\text{val}(\omega) = (\text{val}_1(\omega), \dots, \text{val}_p(\omega))$ .

**Proposition 3.16.** We have the following equality:  $\text{val}(\mathcal{J}_D) = \gamma + \text{val}(\Omega_D^1) - \underline{1}$ . To be more precise, there exists  $g \in \mathcal{C}_D$  of value  $\gamma$  such that  $\mathcal{J}_D = g \cdot \frac{\varphi^*(\Omega_D^1)}{dt}$ , where we denote by  $\frac{\varphi^*(\Omega_D^1)}{dt}$  the fractional ideal of  $\mathcal{O}_{\bar{D}}$  generated by  $(x_1'(t_1), \dots, x_p'(t_p))$  and  $(y_1'(t_1), \dots, y_p'(t_p))$ .

To prove this equality, we need the following propositions:

**Proposition 3.17 ([DdlM87, Theorem 2.7]).** Let  $f = f_1 \cdots f_p$  be a reduced equation of a plane curve germ. We assume that for all  $i \in \{1, \dots, p\}$ ,  $f_i$  is irreducible. We denote by  $D_i$  the branch defined by  $f_i$ , and by  $c_i$  its conductor. The conductor of  $D$  is given by

$$\gamma = \left( c_1 + \sum_{i=2}^p \text{val}_1(f_i), \dots, c_p + \sum_{i=1}^{p-1} \text{val}_p(f_i) \right)$$

**Lemma 3.18.** Let  $D = D_1 \cup \dots \cup D_p$  be a plane curve germ defined by a reduced equation  $f \in \mathbb{C}\{x, y\}$ . Then:

$$\begin{aligned} \text{val}(f'_x) &= \gamma + \text{val}(y) - \underline{1} \\ \text{val}(f'_y) &= \gamma + \text{val}(x) - \underline{1} \end{aligned}$$

**Proof.** If  $f$  is irreducible, the result is given by Teissier's lemma (see [CNP11, lemma 2.3]). If  $f$  is reducible, we use Proposition 3.17, the equality  $\text{val}_j \left( \frac{\partial f}{\partial x} \right) = \sum_{i \neq j} \text{val}_j(f_i) + \text{val}_j \left( \frac{\partial f_j}{\partial x} \right)$  and Teissier's lemma to obtain the result.  $\square$

**Proof of Proposition 3.16.** For lack of reference, we give here the proof. Let  $i \in \{1, \dots, p\}$ . Then, since  $\varphi_i$  is a parametrization of  $D_i$ , we have  $f'_x(\varphi_i(t_i)) \cdot x'_i(t_i) + f'_y(\varphi_i(t_i)) \cdot y'_i(t_i) = 0$ . We first assume that  $x'_i(t_i) \cdot y'_i(t_i) \neq 0$ . Then, there exists  $g_i(t_i) \in \mathbb{C}\{t_i\}$  such that

$$\frac{f'_x(\varphi_i(t_i))}{y'_i(t_i)} = -\frac{f'_y(\varphi_i(t_i))}{x'_i(t_i)} = g_i(t_i)$$

By lemma 3.18 we have  $\text{val}_i(f'_x) = \gamma_i + \text{val}_i(y) - 1$ , so that  $\text{val}_i(g_i) = \gamma_i$ . Since  $\gamma$  is the conductor, there exists  $\tilde{g} \in \mathbb{C}\{x, y\}$  such that for all  $i \in \{1, \dots, p\}$ ,  $g_i(t_i) = \tilde{g}(\varphi_i(t_i))$ , which gives us the result.

If for example  $x'_i(t_i) = 0$ , then  $y'_i(t_i) \neq 0$  and  $f'_y(t_i) = 0$ . We set again  $g_i(t_i) = \frac{f'_x(\varphi_i(t_i))}{y'_i(t_i)}$ , which is of valuation  $\gamma_i$ . The relation  $f'_y(t_i) = g_i(t_i) \cdot x'_i(t_i)$  is of course true. The end of the proof is the same as for the case  $x'_i(t_i) \cdot y'_i(t_i) \neq 0$ .  $\square$

**Corollary 3.19.** *We have the following equivalence:*

$$v \in \text{val}(\mathcal{R}_D) \iff \Delta(-v, \text{val}(\Omega_D^1)) = \emptyset$$

$$\text{Moreover, } \mathcal{R}_D = \frac{1}{g} \cdot \left( \frac{\varphi^*(\Omega_D^1)}{\underline{dt}} \right)^\vee$$

**Proof.** It is a consequence of both Proposition 3.16 and Theorem 2.4.  $\square$

**Remark 3.20.** The latter corollary gives also the relation between meromorphic regular forms as defined in [Bar78] and Kähler differentials. Indeed, by [Ale90, §4], the module  $\mathcal{R}_D$  of logarithmic residues is isomorphic to the module of regular meromorphic forms  $\omega_D$ , which can be defined as

$$\omega_D = \text{Ext}_{\mathcal{O}_S}^1(\Omega_D^1, \Omega_S^2). \text{ In particular, } \omega_D \simeq \frac{1}{g} \cdot \left( \frac{\varphi^*(\Omega_D^1)}{\underline{dt}} \right)^\vee.$$

**Remark 3.21.** Another consequence of Proposition 3.16 is the following inclusion:

$$\gamma + (\text{val}(\mathcal{O}_D) \setminus \{0\}) - \underline{1} \subseteq \text{val}(\mathcal{J}_D)$$

Indeed, if  $h \in \mathfrak{m}\mathcal{O}_D$ , with  $\mathfrak{m}$  the maximal ideal of  $\mathcal{O}_D$ , then  $\text{val}(dh) = \text{val}(h)$ , which gives us the inclusion  $\text{val}(\mathcal{O}_D) \setminus \{0\} \subseteq \text{val}(\Omega_D^1)$ .

We mention here the relation between logarithmic forms and the torsion of Kähler differentials. It leads to a determination of the dimension of the torsion of  $\Omega_D^1$  when  $D$  is a plane curve which is slightly different from the proofs of O.Zariski (see [Zar66]) and R.Michler (see [Mic95]).

We first assume that  $(D, 0)$  is a germ of reduced hypersurface of  $(\mathbb{C}^n, 0)$ . The following property has been proved by A.G.Aleksandrov:

**Proposition 3.22 ([Ale05, 3.1]).** *For all  $1 \leq q \leq n$ , the following map:*

$$\begin{aligned} \frac{\Omega^q(\log D)}{\frac{df}{f} \wedge \Omega_{\mathbb{C}^n}^{q-1} + \Omega_{\mathbb{C}^n}^q} &\rightarrow \text{Tors}(\Omega_D^q) \\ [\omega] &\mapsto [f\omega] \end{aligned}$$

*is an isomorphism of  $\mathcal{O}_D$ -modules.*

**Proof.** It is a consequence of the characterization (16) of logarithmic forms.  $\square$

**Corollary 3.23.** *The map  $\text{res}^q$  induces an isomorphism of  $\mathcal{O}_D$ -modules:*

$$\frac{\text{res}^q(\Omega^q(\log D))}{\Omega_D^{q-1}} \simeq \text{Tors}(\Omega_D^q)$$



**Proof.** We have the following exact sequences:

$$\begin{aligned} 0 &\rightarrow \Omega_{\mathbb{C}^n}^q \rightarrow \Omega^q(\log D) \rightarrow \text{res}^q(\Omega^q(\log D)) \rightarrow 0 \\ 0 &\rightarrow \Omega_{\mathbb{C}^n}^q \rightarrow \frac{df}{f} \wedge \Omega_{\mathbb{C}^n}^{q-1} + \Omega_{\mathbb{C}^n}^q \xrightarrow{\text{res}^q} \Omega_D^{q-1} \rightarrow 0 \end{aligned}$$

Therefore,  $\frac{\text{res}^q(\Omega^q(\log D))}{\Omega_D^{q-1}} \simeq \frac{\Omega^q(\log D)}{\frac{df}{f} \wedge \Omega_{\mathbb{C}^n}^{q-1} + \Omega_{\mathbb{C}^n}^q} \simeq \text{Tors}(\Omega_D^q)$ .  $\square$

**Corollary 3.24.** *Let  $(D, 0) \subseteq (\mathbb{C}^2, 0)$  be a plane curve germ. We denote by  $\tau = \dim_{\mathbb{C}} \mathcal{O}_D / \mathcal{I}_D$  the Tjurina number. Then  $\frac{\mathcal{R}_D}{\mathcal{O}_D} \simeq \text{Tors}(\Omega_D^1)$  and  $\dim_{\mathbb{C}} \text{Tors}(\Omega_D^1) = \tau$ .*

**Proof.** We use Propositions 2.6 and 3.6 to prove that  $\dim_{\mathbb{C}} \mathcal{R}_D / \mathcal{O}_D = \dim_{\mathbb{C}} \mathcal{O}_D / \mathcal{I}_D = \tau$ .  $\square$

### 3.4 Complete intersection curves

This section is devoted to the study of complete intersection curves, which are a particular case of Gorenstein curves.

We begin with the definition of multi-logarithmic forms along a reduced complete intersection  $C$  defined by a regular sequence  $(h_1, \dots, h_k)$  in  $(\mathbb{C}^m, 0)$  given in [Ale12]. We then focus on the case of complete intersection curves for which we have natural extensions of several results of the plane curve case.

**Definition 3.25 ([Ale12]).** *Let  $\omega \in \frac{1}{h_1 \dots h_k} \Omega^q$  with  $q \in \mathbb{N}$ . Then  $\omega$  is called a multi-logarithmic differential  $q$ -form along the complete intersection  $C$  if*

$$\forall i \in \{1, \dots, k\}, dh_i \wedge \omega \in \sum_{j=1}^k \frac{1}{h_j} \Omega^{q+1}$$

We denote by  $\Omega^q(\log C)$  the  $\mathbb{C}\{\underline{x}\}$ -module of multi-logarithmic  $q$ -forms along  $C$ .

To simplify the notations, we set  $\tilde{\Omega}^q := \sum_{j=1}^k \frac{1}{h_j} \Omega^q$ .

If  $k = 1$ , the definition of multi-logarithmic forms coincides with the definition of logarithmic forms 3.1.

Then we have the following characterization which should be compared with [Sai80, 1.1]:

**THEOREM 3.26 ([Ale12, §3, Theorem 1]).** *Let  $\omega \in \frac{1}{h_1 \dots h_k} \Omega^q$ , with  $q \geq k$ . Then  $\omega \in \Omega^q(\log C)$  if and only if there exist a holomorphic function  $g \in \mathbb{C}\{\underline{x}\}$  which does not induce a zero-divisor in  $\mathcal{O}_C$ , a holomorphic differential form  $\xi \in \Omega^{q-k}$  and a meromorphic  $q$ -form  $\eta \in \tilde{\Omega}^q$  such that:*

$$g\omega = \xi \wedge \frac{dh_1 \wedge \dots \wedge dh_k}{h_1 \dots h_k} + \eta \quad (17)$$

**Definition 3.27.** *Let  $\omega \in \Omega^q(\log C)$ ,  $q \geq k$ . Let us assume that  $g, \xi, \eta$  satisfy the properties of Theorem 3.26. Then the multi-residue of  $\omega$  is:*

$$\text{res}_C^q(\omega) := \frac{\xi}{g} \in Q(\mathcal{O}_C) \otimes_{\mathcal{O}_C} \Omega_C^{q-k} = Q(\mathcal{O}_{\tilde{C}}) \otimes_{\mathcal{O}_{\tilde{C}}} \Omega_{\tilde{C}}^{q-k}$$

We define  $\mathcal{R}_C^{q-k} := \text{res}_C^q(\Omega^q(\log C))$ . In particular, if  $q = k$ ,  $\text{res}_C^k(\omega) \in Q(\mathcal{O}_C)$ , and we denote  $\mathcal{R}_C := \text{res}_C^k(\Omega^k(\log C))$ .

It is proved in [Ale12] that for  $\omega \in \Omega^g(\log C)$  the multi-residue  $\text{res}_C^g(\omega)$  is well-defined with respect to the choices of  $\xi, g$  and  $\eta$  in (17).

**Proposition 3.28** ([Sch13, Proposition 4.1]). *Let  $\mathcal{J}_C \subseteq \mathcal{O}_C$  be the Jacobian ideal, that is to say the ideal of  $\mathcal{O}_C$  generated by the  $k \times k$  minors of the Jacobian matrix. Then:*

$$\mathcal{J}_C^\vee = \mathcal{R}_C$$

**Remark 3.29.** In a forthcoming paper, we will give a more direct proof of this duality, which does not rely on the isomorphism between the module of logarithmic multi-residues and the module of regular meromorphic forms given in [AT01, Theorem 3.1].

From now on, we assume that  $C$  is a reduced complete intersection curve defined by a regular sequence  $(h_1, \dots, h_{m-1})$ .

We recall that  $\mathcal{C}_C$  denotes the conductor ideal of  $C$ . The following proposition is a generalization of Proposition 3.16.

**Proposition 3.30.** *Let  $C = C_1 \cup \dots \cup C_p$  be a reduced complete intersection curve defined by a regular sequence  $(h_1, \dots, h_{m-1})$ . Then there exists  $g \in \mathcal{C}_C$  with  $\text{val}(g) = \gamma$  such that  $\mathcal{J}_C = g \cdot \frac{\varphi^*(\Omega_C^1)}{dt}$ . In particular,  $\text{val}(\mathcal{J}_C) = \gamma + \text{val}(\Omega_C^1) - \underline{1}$ .*

**Proof.** Let  $\varphi_i(t_i) = (x_{i,1}(t_i), \dots, x_{i,m}(t_i))$  be a parametrization of  $C_i$ . Let  $J_i$  denote the  $k \times k$  minor of  $\text{Jac}(h_1, \dots, h_{m-1})$  obtained by removing the column  $i$ . Let  $i \in \{1, \dots, p\}$ . Then, for all  $j \in \{1, \dots, m-1\}$  we have  $h_j \circ \varphi_i(t_i) = 0$ , thus:

$$\left( \text{Jac}(h_1, \dots, h_k) \circ \varphi_i(t_i) \right) (x'_{i,1}(t_i), \dots, x'_{i,m}(t_i))^t = (0, \dots, 0)^t$$

We multiply on the left by the adjoint of the matrix obtained by removing the last column of  $\text{Jac}(h_1, \dots, h_k) \circ \varphi_i(t_i)$ , which gives the needed relations: for all  $j \in \{1, \dots, m-1\}$ ,

$$(J_m \circ \varphi_i(t_i)) \cdot x'_{i,j}(t_i) + (-1)^{m-(j-1)} (J_j \circ \varphi_i(t_i)) \cdot x'_{i,m}(t_i) = 0$$

We assume for example that  $x'_{i,m}(t_i) \neq 0$ .

By setting  $g_i(t_i) = \frac{J_m \circ \varphi_i(t_i)}{x'_{i,m}(t_i)}$  one obtains for all  $\ell \in \{1, \dots, m\}$ ,

$$g_i(t_i) \cdot x'_{i,\ell}(t_i) = (-1)^{m-\ell} J_\ell \circ \varphi_i(t_i) \tag{18}$$

It remains to prove that  $\text{val}_i(g) = \gamma_i$ .

Let us denote by  $\Pi_C$  the ramification ideal of the curve  $C$ , which is the  $\mathcal{O}_{\tilde{C}}$ -module generated by

$$(x'_{i,1}(t_1), \dots, x'_{i,p}(t_p))_{1 \leq i \leq m}$$

By [Pie79, Corollary 1, Proposition 1], one has:

$$\mathcal{C}_C \Pi_C = \mathcal{J}_C \mathcal{O}_{\tilde{C}}$$

Thus, we have the equality  $\inf(\text{val}(\Pi_C)) + \gamma = \inf(\text{val}(\mathcal{J}_C))$ .

The equalities (18) imply that for all  $i \in \{1, \dots, p\}$ , and  $j \in \{1, \dots, m\}$  we have  $\text{val}_i(J_j) = \inf(\text{val}_i(\mathcal{J}_C))$  if and only if  $\text{val}_i(x'_{i,j}) = \inf(\text{val}_i(\Pi_C))$ .

Therefore, if  $j$  is such that  $\text{val}_i(J_j) = \inf(\text{val}_i(\mathcal{J}_C))$ , then  $\text{val}_i(J_j) = \gamma_i + \text{val}_i(x'_{i,j})$ , which gives us  $\text{val}_i(g_i) = \gamma_i$ .  $\square$

**Definition 3.31.** Let  $\omega = \sum a_j dx_j \in \Omega_C^1$ . We set

$$\text{val}_i(\omega) = \text{val}_i(\varphi_i^*(\omega)) = 1 + \text{val}_i \left( \sum_{j=1}^m (a_j \circ \varphi_i)(t_i) \cdot x'_{i,j}(t_i) \right)$$

**Corollary 3.32.** The values of  $\mathcal{J}_C$  and  $\Omega_C^1$  satisfy:

$$\text{val}(\mathcal{J}_C) = \gamma + \text{val}(\Omega_C^1) - (1, \dots, 1)$$

Moreover, with the notation of part 2, for all  $v \in \mathbb{Z}^p$ , we have the following equivalence:

$$v \in \text{val}(\mathcal{R}_C) \iff \Delta(-v, \text{val}(\Omega_C^1)) = \emptyset$$

and  $\mathcal{R}_C = \frac{1}{g} \cdot \left( \frac{\varphi^*(\Omega_C^1)}{dt} \right)^\vee$ , where  $g$  is given by Proposition 3.30.

**Proof.** The statement for  $\mathcal{J}_C$  is a direct consequence of Proposition 3.30. The second statement is a consequence of the symmetry Theorem 2.4 together with Proposition 3.28.  $\square$

**Remark 3.33.** As for plane curves, the isomorphism between multi-residues and regular meromorphic differential forms given in [AT01] gives the relation between regular meromorphic forms and Kähler differentials.

## 4 Equisingular deformations of plane curves and the stratification by logarithmic residues

The purpose of this last section is to study the behaviour of the values of logarithmic residues in an equisingular deformation of a plane curve. By the results of part 3, the stratification by the values of logarithmic residues is the same as the stratification by the values of Kähler differentials which is an essential ingredient of the analytic classification of plane curves described in [HH11] and [HHH15] respectively for irreducible curves and for reducible curves with two branches. They prove that in a stratum of the stratification by the values of Kähler differentials, the analytic classification can be expressed in terms of a normal parametrization of the curve. The analytic equivalence is then represented by the group action of the roots of the unity of a certain order determined by  $\text{val}(\Omega_D^1)$  (see [HH11, Theorem 2.1]). We end this section with several algorithms which can be used to compute the set of values of  $\mathcal{R}_D$ , inspired by [BGM88] and [HH07].

We first recall some results on equisingular and admissible deformations, and we then study some properties of the stratification by the values of logarithmic residues.

Let  $D$  be a plane curve defined in a neighbourhood  $U$  of the origin of  $\mathbb{C}^2$  by a reduced equation  $f \in \mathcal{O}_{\mathbb{C}^2}(U)$ . We consider a deformation  $F$  of  $f$  with base space  $(S, 0) = (\mathbb{C}^k, 0)$  with ring  $\mathcal{O}_S$ . It means that  $F(x, y, s) \in \mathcal{O}_{\mathbb{C}^2} \hat{\otimes} \mathcal{O}_S$  satisfies  $F(x, y, 0) = f(x, y)$ . Let  $X = U \times S$ ,  $\mathcal{O}_X = \mathcal{O}_{\mathbb{C}^2} \hat{\otimes} \mathcal{O}_S$ ,  $W = F^{-1}(0) \subseteq U \times S$ . We assume  $F(0, 0, s) = 0$  for all  $s$ . For  $s \in S$ , we set  $D_s = W \cap (\mathbb{C}^2 \times \{s\})$  and  $\mathfrak{m}_{S,s}$  the maximal ideal of  $\mathcal{O}_{S,s}$ , and  $F_s = F(\cdot, \cdot, s)$ . In particular,  $D_0 = D$  and  $F_0 = f$ .

### 4.1 Equisingular and admissible deformations of plane curves

The following numbers are classical invariants of plane curves:

**Definition 4.1.** Let  $D$  be a reduced plane curve defined by  $f \in \mathbb{C}\{x, y\}$ .

- The Milnor number is  $\mu = \mathbb{C}\{x, y\} / (f'_x, f'_y)$

- The Tjurina number is  $\tau = \mathbb{C}\{x, y\}/(f'_x, f'_y, f)$
- The delta-invariant is  $\delta = \dim_{\mathbb{C}} \mathcal{O}_{\tilde{D}}/\mathcal{O}_D$

The following proposition gives the relation between  $\mu$  and  $\delta$ :

**Proposition 4.2** ([Mil68]). *We have the following relation:*

$$\mu = 2\delta - p + 1$$

where  $p$  is the number of irreducible components of  $(D, 0)$ .

Let us assume that  $F$  is an equisingular deformation of  $f$  (i.e. for all  $s \in S$ ,  $\mu(F_s) = \mu(f)$ ). From the equisingularity Theorem for plane curves (see [Tei77, §3.7]), it implies that a parametrization  $\varphi$  of  $(D, 0)$  gives rise to a deformation  $\varphi_s$  of the parametrization. We denote by  $\text{val}_{D_s}(g)$  the value of  $g \in Q(\mathcal{O}_{D_s})$  along  $D_s$ . Another consequence of the equisingularity Theorem for plane curves is:

**Corollary 4.3.** *With the same notations, if  $F$  is an equisingular deformation of  $f$ :*

1. All fibers  $D_s$  have the same conductor  $\gamma$ .
2. Let  $x(t, s) = (x_1(t_1, s), \dots, x_p(t_p, s))$ ,  $y(t, s) = (y_1(t_1, s), \dots, y_p(t_p, s))$  be a parametrization of  $D_s$ . For all  $s \in S$ ,

$$\inf(\text{val}_{D_s}(x(t, s)), \text{val}_{D_s}(y(t, s))) = \inf(\text{val}_D(x(t, 0)), \text{val}_D(y(t, 0))) = (m^{(1)}, \dots, m^{(p)})$$

where  $m^{(j)}$  is the multiplicity of the component  $D_j$  of  $D$ .

**Proof.**

1. It comes from Theorem 3.17, since by the equisingularity Theorem (see [Tei77, §3.7, (10)]), the conductors and the intersection multiplicities  $(D_i \cdot D_j) = \text{val}_i(f_j)$  do not depend on  $s$ .
2. For all  $j \in \{1, \dots, p\}$ ,  $\inf(\text{val}_{D_{s,j}}(x_j(t_j, s), y_j(t_j, s)))$  is the multiplicity of  $D_{s,j}$ , which does not depend on  $s$  by the equisingularity Theorem.  $\square$

The following proposition will be used in the next section, since it gives a common denominator for the logarithmic residues with interesting properties.

**Proposition 4.4.** *There exists  $\alpha, \beta \in \mathbb{C}$  such that for all  $s$  in a neighbourhood of 0,  $\text{val}(\alpha F'_x(s) + \beta F'_y(s)) = \gamma + (m^{(1)}, \dots, m^{(p)}) - \underline{1}$ . In particular,  $\alpha F'_x(s) + \beta F'_y(s)$  induces a non zero divisor in  $\mathcal{O}_D$  whose value does not depend on  $s$ .*

**Proof.** One can prove that thanks to the equisingularity Theorem, there exists a linear change of coordinates  $(u, v)$  such that for all  $s$  in a neighbourhood of  $0 \in S$ ,  $\text{val}_{D_s}(u) = (m^{(1)}, \dots, m^{(p)})$ . The conclusion follows from Corollary 4.3 and Lemma 3.18.  $\square$

We want now to understand the behaviour of a generating family of the module of residues.

We recall that plane curves are free divisors. Moreover, they are the only free divisors with isolated singularities, since by [Ale88], the singular locus of a free divisor is of codimension one in the hypersurface. The equisingularity assumption is not sufficient to obtain a deformation  $(\rho_1(s), \rho_2(s))$  of a generating family of  $\mathcal{R}_D$  such that  $(\rho_1(s), \rho_2(s))$  generate  $\mathcal{R}_{D_s}$ : equisingularity is not the "good" functor of deformation for free divisors. A functor of deformation adapted to free divisors is suggested by M.Torielli in [Tor13].

The following definition is equivalent to the definition of M.Torielli (see [Tor13, Definition 3.1]) thanks to both [Tor13, Proposition 3.7] and [GLS07, Theorem 1.91]:

**Definition 4.5.** Let  $D$  be a free divisor defined in a neighbourhood of  $0 \in \mathbb{C}^n$  by a reduced equation  $f$ . An admissible deformation  $X$  of  $D$  with base space  $S$  is a deformation of  $D$  such that the module  $\mathcal{O}_{\mathbb{C}^n \times S, 0} / (F, F'_x, F'_y)$  is a flat  $\mathcal{O}_{S, 0}$ -module.

The following proposition describes an admissible deformation of a plane curve thanks to the Tjurina number.

**Proposition 4.6.** Let  $F$  be a deformation of  $f \in \mathcal{O}_{\mathbb{C}^2}(U)$  with base space  $S$  such that for all  $s \in S$ ,  $\sum_{x_j \in \text{Sing}(D_s)} \tau_{x_j} = \tau_0$  where  $\tau_0$  is the Tjurina number of  $D_0$ . Such a deformation is an admissible deformation.

**Proof.** We use Theorem 1.81 of [GLS07] for  $\mathcal{F} = p_* (\mathcal{O}_{\mathbb{C}^2 \times S} / (F, F'_x, F'_y))$  with  $p : \mathbb{C}^2 \times S \rightarrow S$  the canonical epimorphism (which is finite). The map  $s \mapsto \dim_{\mathbb{C}} \mathcal{F}_s / \mathfrak{m}_{S, s} \mathcal{F}_s$  is then constant, thus,  $\mathcal{F}_0$  is a flat  $\mathcal{O}_{S, 0}$ -module. Since  $\mathcal{F}_0 = \mathcal{O}_{\mathbb{C}^2 \times S, 0} / (F, F'_x, F'_y)$ , the result follows.  $\square$

**Proposition 4.7** ([Tor13, Lemma 3.22]). Let  $F(x, y, s)$  be an equisingular and admissible deformation of the plane curve defined by  $f$  with base space  $S$ . Let  $(\delta_1, \delta_2)$  be a basis of the module of logarithmic vector fields along  $D$ . It induces relations between  $f, f'_x, f'_y$ . By flatness, we can extend them to obtain relative logarithmic vector fields  $\tilde{\delta}_1, \tilde{\delta}_2 \in (\Theta_{U \times S/S}) / (\mathfrak{m}_S \Theta_{U \times S/S})$  of  $F$ . Then, for  $s$  in a neighbourhood of  $0 \in S$ ,  $(\tilde{\delta}_1(s), \tilde{\delta}_2(s))$  is a basis of  $\text{Der}(-\log D_s)$ .

**Corollary 4.8.** If  $\tilde{\delta}_i = A_i(x, y, s)\partial_x + B_i(x, y, s)\partial_y$ , by the duality between the modules  $\text{Der}(-\log D)$  and  $\Omega^1(\log D)$  we find residues:

$$\begin{cases} \text{res}(\tilde{\omega}_1)(s) = \frac{-\beta A_2(s) + \alpha B_2(s)}{\alpha F'_x(s) + \beta F'_y(s)} \\ \text{res}(\tilde{\omega}_2)(s) = \frac{\beta A_1(s) - \alpha B_1(s)}{\alpha F'_x(s) + \beta F'_y(s)} \end{cases}$$

such that they generate the module of residues for all  $s$  in a neighbourhood of  $0 \in S$ , where  $\alpha, \beta \in \mathbb{C}$  are given by Proposition 4.4.

## 4.2 Properties of the stratification by logarithmic residues

We consider an equisingular deformation  $F$  of  $f$  with base space  $(S, 0) \simeq (\mathbb{C}^k, 0)$  for a  $k \in \mathbb{N}$ . We denote by  $\mathcal{R}_s$  the module of logarithmic residues of  $D_s$ . The purpose of this section is to study some properties of the following partition of  $S$ :

**Definition 4.9.** Let  $F(x, y, s)$  be an equisingular deformation of a reduced plane curve  $D$  defined by  $f \in \mathbb{C}\{x, y\}$ , with base space  $S$ . The stratification by logarithmic residues is the partition  $S = \bigcup_{\mathcal{V} \subseteq \mathbb{Z}^p} S_{\mathcal{V}}$  where  $s \in S_{\mathcal{V}}$  if and only if  $\text{val}(\mathcal{R}_s) = \mathcal{V}$ .

The following proposition will be useful to give the relation between the stratification by logarithmic residues and the stratification by the Tjurina number.

**Proposition 4.10.** Let  $(D, 0)$  be a plane curve germ. Then:

$$\dim_{\mathbb{C}} \mathcal{R}_D / \mathcal{O}_{\tilde{D}} = \tau - \delta \tag{19}$$

**Proof.** Thanks to Propositions 2.6 and 3.6 we have:

$$\begin{aligned} \dim_{\mathbb{C}} \mathcal{R}_D / \mathcal{O}_{\tilde{D}} &= \dim_{\mathbb{C}} \mathcal{R}_D / \mathcal{O}_D - \dim_{\mathbb{C}} \mathcal{O}_{\tilde{D}} / \mathcal{O}_D = \dim_{\mathbb{C}} \mathcal{O}_D^{\vee} / \mathcal{R}_D^{\vee} - \delta = \dim_{\mathbb{C}} \mathcal{O}_D / \mathcal{I}_D - \delta \\ &= \tau - \delta \end{aligned} \quad \square$$

**Proposition 4.11.** *The stratification by logarithmic residues satisfies the following properties:*

1. *If  $s, s'$  do not belong to the same stratum of the stratification by  $\tau$ , they do not belong to the same stratum for the stratification by logarithmic residues. In other words, the stratification by logarithmic residues is finer than the stratification by  $\tau$ .*
2. *The stratification by logarithmic residues is finite.*

**Proof.** The first claim is a direct consequence of Proposition 4.10, since the equisingularity condition ensures that  $\delta$  does not depend on the fiber, and the dimension of the quotient  $\mathcal{R}_s/\mathcal{O}_{\widetilde{D}_s}$  can be computed from the values by Corollary 2.15. The second claim comes from both Proposition 4.4, which gives a lower bound  $u$  of the set of values of logarithmic residues which do not depend on  $s$ , and Corollary 3.10. As a consequence, the values of  $\mathcal{R}_s$  are determined by the values  $v$  of  $\mathcal{R}_s$  satisfying  $u \leq v \leq 0$ .  $\square$

The hypothesis of  $D$  being irreducible was forgotten in [Pol15, Proposition 4.2]:

**Proposition 4.12.** *Each stratum  $S_\gamma$  of the stratification by logarithmic residues is constructible. If moreover  $D$  is irreducible, then each stratum is locally closed.*

**Proof.** For lack of reference, we suggest a proof. By the appendix by Teissier in [Zar86], the strata of the stratification by the Tjurina number are locally analytic and locally closed. It is therefore sufficient to consider the behaviour of logarithmic residues in a  $\tau$ -constant stratum  $S_\tau$ . For the sake of simplicity, we denote  $S = S_\tau$ .

By Corollary 4.8, for all  $s$ , the  $\mathcal{O}_S$ -module  $\mathcal{R}_s$  is generated by

$$\begin{cases} \rho_1(s) = \frac{-\beta A_2(s) + \alpha B_2(s)}{\alpha F'_x(s) + \beta F'_y(s)} \\ \rho_2(s) = \frac{\beta A_1(s) - \alpha B_1(s)}{\alpha F'_x(s) + \beta F'_y(s)} \end{cases}$$

where  $\alpha, \beta \in \mathbb{C}$  are given by Proposition 4.4. The value of the common denominator  $\alpha F'_x(s) + \beta F'_y(s)$  does not depend on  $s$ , so that it is sufficient to consider the values of the numerators.

We denote by  $N_1$  and  $N_2$  the numerators of  $\rho_1(s)$  and  $\rho_2(s)$ . We recall that the values  $v$  of  $\mathcal{R}_s$  satisfying  $v \leq 0$  are sufficient to determine  $\text{val}(\mathcal{R}_s)$ , so that it is sufficient to consider the set  $\{X_1, \dots, X_q\} := \{x^i y^j N_k; \text{val}(x^i y^j N_k) \leq u\}$ , where  $u = \text{val}(\alpha F'_x(s) + \beta F'_y(s))$ .

For all  $i \in \{1, \dots, q\}$ , we have  $X_i = \left( \sum_{j \geq 0} a_{i,j,1}(s) t_1^j, \dots, \sum_{j \geq 0} a_{i,j,p}(s) t_p^j \right)$ .

For  $v \in \mathbb{Z}^p$  and  $k \in \{1, \dots, p\}$  we set  $X_{i,k}^v(s) = (a_{i,0,k}(s), \dots, a_{i,v_k,k}(s)) \in \mathcal{O}_S^{v_k+1}$ . For  $v \in \mathbb{Z}^p$  we define the following matrix  $A_v(s) \in \mathcal{M}_{q,\ell_v}(\mathcal{O}_S)$  with  $\ell_v = \sum_{j=1}^p (v_j + 1)$ :

$$A_v(s) = \begin{pmatrix} (X_{1,1}^v(s)) & \dots & (X_{1,p}^v(s)) \\ \vdots & & \vdots \\ (X_{q,1}^v(s)) & \dots & (X_{q,p}^v(s)) \end{pmatrix}$$

We use the rank of the matrices  $A_v(s)$  to characterize the property  $v \in \text{val}(\mathcal{R}_s)$  for  $s \in S$ :

$$v \in \text{val}(\mathcal{R}_s) \iff \forall k \in \{1, \dots, p\}, \text{rank}(A_{v-\underline{1}}(s)) < \text{rank}(A_{v-\underline{1}+e_k}(s))$$

Indeed, if the conditions of the right-hand side are satisfied, then for all  $k \in \{1, \dots, p\}$ , there exists a linear combination  $M_k = \sum_{i=1}^q \lambda_{i,k} X_i(s)$  with  $\lambda_{i,k} \in \mathbb{C}$  such that  $\text{val}(M_k) \geq v$  and  $\text{val}_k(M_k) = v_k$ . We use Proposition 2.11 to conclude.

Therefore, for a given  $\mathcal{V} \subseteq \mathbb{Z}^p$  for which  $S_\mathcal{V} \cap S \neq \emptyset$  and  $\overline{\mathcal{V}} := (\mathcal{V} + u) \cap \{w \in \mathbb{Z}^p; 0 \leq w \leq u\}$ :

$$s \in S_{\mathcal{V}} \iff s \in \bigcap_{v \in \overline{\mathcal{V}}} \left( \bigcup_{1 \leq r \leq M} \left( V(\mathcal{F}_r(A_{v-\underline{1}}(s))) \cap \bigcap_{1 \leq k \leq p} (V(\mathcal{F}_r(A_{v-\underline{1}+e_k}(s)))^c) \right) \right)$$

where  $\mathcal{F}_r(A)$  denotes the ideal generated by the  $r \times r$  minors of the matrix  $A$  and  $M = \min(q, \ell_v + 1)$ . We notice that the elements  $v \notin \mathcal{V}$  can not be reached since otherwise the dimension of  $\mathcal{R}_s/\mathcal{O}_{D_s}$  will be strictly greater than  $\tau - \delta$ , which can be proved by an argument similar to the first two steps of the proof of Theorem 2.4.

Hence the result for reducible curves.

Let us assume now that  $D$  is irreducible. In this case, the rank of the matrix increases exactly by 1 when a valuation is reached. We set  $\overline{\mathcal{V}} = \{v_1 < \dots < v_L\} = (\mathcal{V} + u) \cap \{0, \dots, u\}$ . Then:

$$s \in S_{\mathcal{V}} \iff s \in \bigcap_{\ell=1}^L \bigcap_{j=v_{\ell-1}+1}^{v_{\ell}-1} (V(\mathcal{F}_{\ell}(A_j(s))) \cap (V(\mathcal{F}_{\ell}(A_{v_{\ell}}))^c)$$

Therefore the stratum  $S_{\mathcal{V}}$  is locally closed.  $\square$

We recall here the examples of [Pol15] with more details. The first example shows that the stratification by logarithmic residues may be strictly finer than the stratification by the Tjurina number, whereas the second shows that the stratification by logarithmic residues does not satisfy the frontier condition.

**Example 4.13.** We consider  $f(x, y) = x^5 - y^6$  and the equisingular deformation of  $f$  given by  $F(x, y, s_1, s_2, s_3) = x^5 - y^6 + s_1x^2y^4 + s_2x^3y^3 + s_3x^3y^4$ . The stratification by  $\tau$  is composed of three strata,  $S_{20} = \{0\}$ ,  $S_{19} = \{(0, 0, s_3), s_3 \neq 0\}$  and  $S_{18} = \{(s_1, s_2, s_3), (s_1, s_2) \neq (0, 0)\}$ . The computation of the values of  $\mathcal{J}_{D_s}$  is quite easy in this case and gives thanks to Theorem 2.4, where  $S'_{18} = \{(s_1, s_2, s_3), s_1 \neq 0\}$  and  $S''_{18} = \{(0, s_2, s_3), s_2 \neq 0\}$ :

Stratum	$\dim_{\mathbb{C}} \mathcal{R}_{D_s}/\mathcal{O}_{\widetilde{D}_s}$	negative values
$S_{20}$	10	-1, -2, -3, -4, -7, -8, -9, -13, -14, -19
$S_{19}$	9	-1, -2, -3, -4, -7, -8, -9, -13, -14
$S'_{18}$	8	-1, -2, -3, -4, -7, -8, -9, -14
$S''_{18}$	8	-1, -2, -3, -4, -7, -8, -9, -13

The stratum  $S_{18}$  divides into two strata for the values of  $\mathcal{R}_{D_s}$ , therefore, the stratification by logarithmic residues is strictly finer than the stratification by  $\tau$ .

**Example 4.14.** Let us consider the deformation  $F(x, y, s_1, s_2) = x^{10} + y^8 + s_1x^5y^4 + s_2x^3y^6$  for  $s_1, s_2$  in a neighbourhood of 0 so that the deformation is equisingular. It is given in [BGM92], as an example of the stratification by the  $b$ -function not satisfying the frontier condition.

A stratification  $S = \bigcup_{\alpha} S_{\alpha}$  satisfies the frontier condition if for  $\alpha \neq \beta$ ,  $S_{\alpha} \cap \overline{S_{\beta}} \neq \emptyset$  implies  $S_{\alpha} \subseteq \overline{S_{\beta}}$ , with  $\overline{S_{\beta}}$  the closure of  $S_{\beta}$ .

Contrary to the previous example, this curve is not irreducible. We first give a property of quasi-homogeneous curves:

**Lemma 4.15.** *Let  $D$  be a quasi-homogeneous plane curve germ with  $p$  branches. Then:*

$$\gamma - \underline{1} + (\text{val}(\mathcal{O}_D) \setminus \{0\}) = \text{val}(\mathcal{J}_D)$$

**Proof.** The inclusion  $\subseteq$  is given by Remark 3.21. For the other inclusion, we notice that Remark 3.21 implies  $t^{2\gamma-1}\mathcal{O}_{\tilde{D}} \subseteq \mathcal{I}_D \subseteq \mathcal{C}_D$ . We have the following equality:

$$\dim_{\mathbb{C}} \mathcal{C}_D/t^{2\gamma-1}\mathcal{O}_{\tilde{D}} = \dim_{\mathbb{C}} \mathcal{C}_D/\mathcal{I}_D + \dim_{\mathbb{C}} \mathcal{I}_D/t^{2\gamma-1}\mathcal{O}_{\tilde{D}}$$

By Propositions 4.10 and 2.6, we have  $\dim_{\mathbb{C}} \mathcal{C}_D/\mathcal{I}_D = \tau - \delta$ . Since  $D$  is quasi-homogeneous, we have  $\tau = \mu$  so that by Proposition 4.2 we have  $\dim_{\mathbb{C}} \mathcal{C}_D/\mathcal{I}_D = \delta - p + 1$ . Moreover, since  $\delta = \dim_{\mathbb{C}} \mathcal{O}_{\tilde{D}}/\mathcal{O}_D$ , we have  $\dim_{\mathbb{C}} \mathcal{O}_{\tilde{D}}/\mathcal{C}_D = 2\delta$ , thus  $\dim_{\mathbb{C}} \mathcal{C}_D/t^{2\gamma-1}\mathcal{O}_{\tilde{D}} = 2\delta - p$ . Therefore:

$$\dim_{\mathbb{C}} \mathcal{I}_D/t^{2\gamma-1}\mathcal{O}_{\tilde{D}} = \delta - 1$$

Let  $\mathfrak{m}$  be the maximal ideal of  $\mathcal{O}_D$ . Then  $\text{val}(\mathfrak{m}) = \text{val}(\mathcal{O}_D) \setminus \{0\}$  and the quotient  $\mathfrak{m}/\mathcal{C}_D$  has dimension  $\delta - 1$ . If  $\text{val}(\mathcal{I}_D) \neq \gamma - 1 + (\text{val}(\mathcal{O}_D) \setminus \{0\})$ , by an argument similar to the one of the two first steps of the proof of Theorem 2.4, we have  $\dim_{\mathbb{C}} \mathcal{I}_D/t^{2\gamma-1}\mathcal{O}_{\tilde{D}} > \delta - 1$ , which is a contradiction. Hence the result.  $\square$

Let us come back to our example. We notice that  $F(x, y, s_1, 0)$  is quasi-homogeneous. Therefore, the previous lemma shows that the values of the Jacobian ideal along the quasi-homogeneous stratum does not change. Therefore, the quasi-homogeneous stratum is a stratum of the stratification by logarithmic residues.

Moreover, one can check that there are three strata for the stratification by the Tjurina number: the quasi-homogeneous stratum  $S_1$  defined by  $s_2 = 0$  for which  $\tau = 63$ , the stratum  $S_2$  defined by  $s_1 = 0$  and  $s_2 \neq 0$  for which  $\tau = 54$  and the stratum  $S_3$  defined by  $s_1 s_2 \neq 0$  for which  $\tau = 53$ . Therefore, the stratification by logarithmic residues does not satisfy the frontier condition, since there is a stratum  $S \subseteq S_2$  which contains the origin in its closure, but not the whole quasi-homogeneous stratum.

### 4.3 Algorithms to compute the logarithmic residues along plane curves with one or two components

We suggest here several methods which can be used to compute the values of logarithmic residues.

Thanks to the symmetry Theorem 2.4, computing the values of  $\mathcal{I}_D$  is equivalent to the computation of the values of  $\mathcal{R}_D$ .

#### 4.3.1 Irreducible semi-quasi homogeneous polynomials

This algorithm is useful to study the equisingular deformation of a quasi-homogeneous polynomial of the form  $x^a - y^b$ , with  $\text{gcd}(a, b) = 1$ , and is inspired by [BGM88].

We consider the following equation of an irreducible curve, with  $s_{ij} \in \mathbb{C}$  and  $\text{gcd}(a, b) = 1$ :

$$F(x, y) = x^a - y^b + \sum_{\substack{1 \leq i < a-1 \\ 1 \leq j < b-1 \\ ib+ja > ab}} s_{ij} x^i y^j \quad (20)$$

A parametrization of the curve is given by  $x(t) = t^b + g(t)$ ,  $y(t) = t^a + h(t)$  where  $g, h \in \mathbb{C}\{t\}$  with  $\text{val}(g) > b$ ,  $\text{val}(h) > a$ .

We set for  $i, j \in \mathbb{N}^2$ ,  $\rho(i, j) = ib + ja$ . We define a monomial ordering by:  $(i, j) < (i', j')$  if and only if  $\rho(i, j) < \rho(i', j')$  or  $(\rho(i, j) = \rho(i', j')$  and  $i < i')$ . If  $H = \sum_{i,j} a_{i,j} x^i y^j \in \mathbb{C}\{x, y\}$  is non zero, we set  $\text{exp}(H) = \min((i, j), a_{i,j} \neq 0)$  and  $\rho(H) := \rho(\text{exp}(H))$ .

Polynomials of the form (20) are studied in [BGM88]. The authors give an algorithm to compute the "escalier" of the curve, which is by definition the complement in  $\mathbb{N}^2$  of the set

$$E = \{\text{exp}(g); g \in (F, F'_x, F'_y) \subseteq \mathbb{C}\{x, y\}\}$$



More precisely, they give the explicit computation of a finite family  $(A_j)_{-1 \leq j \leq K}$  of points of  $\mathbb{N}^2$  such that  $E = \bigcup_{j=-1}^K A_j + \mathbb{N}^2$ , and none of the  $A_j$ 's can be removed. Then it is possible to prove:

**Proposition 4.16.** *We have the following equality:*

$$\text{val}(\mathcal{I}_D) = \bigcup_{i=-1}^K \left( \rho(A_i) + \text{val}(\mathcal{O}_D) \right)$$

### 4.3.2 Irreducible plane curve

In [HH07], an algorithm is proposed to compute the set of values of Kähler differentials of an irreducible plane curve. By Proposition 3.19, it gives also the values of  $\mathcal{R}_D$ . In fact, one can see that the algorithm of [BGM88] corresponds to the algorithm of [HH07] by Proposition 3.16. Moreover, if a generating family of  $\mathcal{R}_D$  is known, the algorithm of [HH07, Theorem 2.4] can be used directly on this family to compute the values of  $\mathcal{R}_D$ .

### 4.3.3 Plane curves with two branches

Let  $D = D_1 \cup D_2$  be a plane curve germ with two irreducible components. We suggest here an algorithm to compute the set of negative values of  $\mathcal{R}_D$ . It is more technical than in the irreducible case, and cannot be generalized to plane curves with three or more branches. It can be compared to the fact that the analytic classification proposed in [HHH15] for two branches is also more complicated than in the irreducible case, and can not be easily extended to plane curves with three or more branches.

The algorithm in [HH07] is given for irreducible curves, for which the set of valuations is totally ordered, so that we cannot apply it directly to reducible plane curves. Nevertheless, we can use it if we consider only one of the components.

*First step*

First of all, we set  $g \in (f'_x, f'_y)$  a non-zero divisor of  $\mathcal{O}_D$ , and we fix it as the common denominator of all residues of  $D_1$ ,  $D_2$  and  $D$ , so that we can consider only the numerators to compute the set of values in each case.

Let  $i \in \{1, 2\}$ . We consider only the branch  $D_i$ . By applying the algorithm of [HH07, Theorem 2.4] we compute a standard basis  $G_i$  of  $\mathcal{O}_{D_i}$ . We use it to compute also a standard basis  $R_i$  of  $\mathcal{R}_{D_i}$ , thanks to a generating family of  $\mathcal{R}_{D_i}$ . As in [HH07], a  $G_i$ -product is an element of the form  $\prod_{j=1}^q h_j^{\alpha_j}$  where  $h_j \in G_i$ ,  $\alpha_j \in \mathbb{N}$  and  $q \in \mathbb{N}$ .

To determine the missing values, we first compute the projection  $\text{val}_1(\mathcal{R}_D)$  of  $\text{val}(\mathcal{R}_D)$ . To do this, we apply the algorithm of [HH07, Theorem 2.4] to a generating family of  $\mathcal{R}_D$ , but by considering only the valuation along  $D_1$ . We deduce a family  $R$  of elements of  $\mathcal{R}_D$  such that for all  $v_1 \in \text{val}_1(\mathcal{R}_D) \cap (\text{val}_1(\mathcal{R}_{D_1}))^c$ , there exists  $\rho \in R$  such that  $\text{val}_1(\rho) = v_1$ .

*Second step*

We set  $\mathcal{M}_0 = \{(0, v_2); v_2 \in \text{val}_2(\mathcal{R}_{D_2}) \cap \mathbb{Z}_{\leq 0}\}$  and  $H_0 = R_2$ . By Proposition 3.7, it gives all the values of  $\text{val}(\mathcal{R}_D) \cap (\{0\} \times \mathbb{Z}_{\leq 0})$ .

Let us assume that for a  $k \in \mathbb{N}^*$  we have constructed sets  $\mathcal{M}_{k-1}$  and  $H_{k-1} \subseteq \mathcal{R}_D$  such that

$$\mathcal{M}_{k-1} = \{(v_1, v_2) \in \text{val}(\mathcal{R}_D); -k + 1 \leq v_1 \leq 0 \text{ and } v_2 \leq 0\}$$

and for all  $v_2 \in \text{val}_2(\mathcal{M}_{k-1})$ , there exists  $\rho \in H_{k-1}$  and a  $G_2$ -product  $h$  with  $\text{val}_2(h \cdot \rho) = v_2$  and  $\text{val}_1(h \cdot \rho) \geq -k + 1$ .

Let us compute  $\mathcal{M}_k$  and  $H_k$ . If  $-k \notin \text{val}_1(\mathcal{R}_D)$ ,  $\mathcal{M}_k = \mathcal{M}_{k-1}$  and  $H_k = H_{k-1}$ . Otherwise, there are several cases to consider.

*First case:*  $-k \in \text{val}_1(\mathcal{R}_D) \cap \text{val}_1(\mathcal{R}_{D_1})$ . By Proposition 2.11, one can see that

$$\mathcal{M}_k \supseteq \mathcal{M}_{k-1} \cup \{(-k, v_2), v_2 \in \text{val}_2(\mathcal{M}_{k-1})\} \quad (21)$$

Moreover, by Proposition 2.13, if  $(-k, v_2) \in \text{val}(\mathcal{R}_D)$  with  $v_2 \leq 0$ , then  $v_2 \in \text{val}_2(\mathcal{M}_{k-1})$ . Therefore the inclusion in (21) is an equality and  $H_k = H_{k-1}$ .

*Second case:*  $-k \in \text{val}_1(\mathcal{R}_D)$  but  $-k \notin \text{val}_1(\mathcal{R}_{D_1})$ . There exists  $\rho_0 \in R$  such that  $\text{val}_1(\rho_0) = -k$ . Let  $w_2 = \text{val}_2(\rho_0)$ . We may assume by Proposition 2.11 that  $w_2 \leq 0$  since  $0 \in \text{val}(\mathcal{R}_D)$ .

*First sub-case:*  $w_2 \notin \text{val}_2(\mathcal{M}_{k-1})$ . Thanks to Propositions 2.11 and 2.13, one can check that:

$$\mathcal{M}_k = \mathcal{M}_{k-1} \cup \{(-k, w_2)\} \cup \{(-k, v_2); v_2 \in \text{val}_2(\mathcal{M}_{k-1}) \text{ and } v_2 \leq w_2\}$$

and  $H_k = H_{k-1} \cup \{\rho_0\}$

*Second sub-case:*  $w_2 \in \text{val}_2(\mathcal{M}_{k-1})$ . Thanks to propositions 2.11 and 2.13, one can check that by a convenient linear combination of  $\rho_0$  and elements of form  $h \cdot \rho$  with  $h$  a  $G_2$ -product and  $\rho \in H_{k-1}$ , there exists  $\rho'_0 \in \mathcal{R}_D$  with  $\text{val}(\rho'_0) = (-k, w'_2)$  and  $w'_2 \notin \text{val}_2(\mathcal{M}_{k-1})$ . We then recognize the previous sub-case, and we have  $H_k = H_{k-1} \cup \{\rho'_0\}$ .

We can stop when the minimal value  $-q$  of  $\text{val}_1(\mathcal{R}_D)$  is reached. Then, by Proposition 3.10:

$$\text{val}(\mathcal{R}_D) = \mathcal{M}_q \cup \{v \in \mathbb{Z}^p; \inf(v, 0) \in \mathcal{M}_q\}$$

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