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OGABI CHOKRI

Academie de Grenoble, 38000. Grenoble. France

Abstract. In this article we study the asymptotic behavior,as $\epsilon \to 0$, of the solution of a nonlinear elliptic, anisotropic singular perturbations problem in cylindrical domain, the limit problem is given and strong convergences are proved, we also give an application to intergo-differential problems.

1. Description of the problem and main theorems

The aim of this manuscript is to analyze nonlinear diffusion problems when the diffusion coefficients in certain directions are going towards zero. We consider a general nonlinear elliptic singularly perturbed problem which can be considered as a generalization to some class of integro-differential problem (see [3]), let us begin by describing the linear part of the problem as given in [2] and [3]. For $\Omega = \omega_1 \times \omega_2$ a bounded cylindrical domain of $\mathbb{R}^N$ $(N \geq 2)$ where ω_1, ω_2 are Lipschitz domains of $\mathbb{R}^p$ and $\mathbb{R}^{N-p}$ respectively, we denote by $x = (x_1, ..., x_N) = (X_1, X_2)$ the points in $\mathbb{R}^N$ where

$$X_1 = (x_1, ..., x_p) \ \in \omega_1 \text{and } X_2 = (x_{p+1}, ..., x_N) \in \omega_2,$$

i.e. we split the coordinates into two parts. With this notation we set

$$\nabla = (\partial_{x_1}, ..., \partial_{x_N})^T = \begin{pmatrix} \nabla_{X_1} \\ \nabla_{X_2} \end{pmatrix},$$

where

$$\nabla_{X_1} = (\partial_{x_1}, ..., \partial_{x_p})^T \text{ and } \nabla_{X_2} = (\partial_{x_{p+1}}, ..., \partial_{x_N})^T$$

To make it simple we use this abuse of notation

$$\nabla_{X_i} u \in L^2(\Omega) \text{ instead of } \nabla_{X_i} u \in \left[L^2(\Omega)\right]^{p;N-p} \text{ for a function } u$$

Let $A = (a_{ij}(x))$ be a $N \times N$ symmetric matrix which satisfies the ellipticity assumption

$$\exists \lambda > 0 : A\xi \cdot \xi \geq \lambda \left|\xi\right|^2 \ \forall \xi \in R^N \text{ for a.e } x \in \Omega,$$

and

$$a_{ij}(x) \in L^\infty(\Omega), \forall i, j = 1, 2, ..., N, \tag{1}$$

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where " $\cdot$ " is the canonical scalar product on R^N. We decompose A into four blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are respectively $p \times p$ and $(N-p) \times (N-p)$ matrices. For $0 < \epsilon \leq 1$ we set

$$A_\epsilon = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix},$$

then we have therefore, for a.e. $x \in \Omega$ and every $\xi \in R^N$

$$\begin{aligned} A_\epsilon \xi \cdot \xi &\geq \lambda \left(\epsilon^2 \left| \overline{\xi_1} \right|^2 + \left| \overline{\xi_2} \right|^2 \right) \geq \lambda \epsilon^2 \left| \xi \right|^2 \ \forall \xi \in R^N \ , \\ \text{and } A_{22} \overline{\xi_2} \cdot \overline{\xi_2} &\geq \lambda \left| \overline{\xi_2} \right|^2, \end{aligned} \tag{2}$$

where we have set

$$\xi = \begin{pmatrix} \overline{\xi_1} \\ \overline{\xi_2} \end{pmatrix},$$

with,

$$\overline{\xi_1} = (\xi_1, ..., \xi_p)^T \text{ and } \overline{\xi_2} = (\xi_{p+1}, ..., \xi_N)^T$$

And finally let $B : L^2(\Omega) \to L^2(\Omega)$ be a nonlinear locally-Liptchitz operator i.e, for every bounded set $E \subset L^2(\Omega)$ there exists $K_E \geq 0$ such that

$$\forall u, v \in E : \left\| B(u) - B(v) \right\|_{L^2(\Omega)} \leq K_E \left\| u - v \right\|_{L^2(\Omega)}, \tag{3}$$

,and B satisfies the growth condition

$$\exists r > 2, \ M \geq 0, \ \forall u \in L^2(\Omega) : \left\| B(u) \right\|_{L^r(\Omega)} \leq M \left(1 + \left\| u \right\|_{L^2(\Omega)} \right), \tag{4}$$

We define the space

$$V = \left\{ u \in L^2(\Omega) : \nabla_{X_1} u \in L^2(\Omega) \right\}$$

Moreover we suppose that for every $E \subset V$ bounded in $L^2(\Omega)$ we have

$$\overline{conv} \left\{ B(E) \right\} \subset V, \tag{5}$$

where $\overline{conv} \left\{ B(E) \right\}$ is the closed convex hull of $B(E)$ in $L^2(\Omega)$.This last condition is the most crucial, it will be used in the proof of the interior estimates and the convergence theorem.

For $\beta > M \left| \Omega \right|^{\frac{1}{2} - \frac{1}{r}}$ we consider the problem

$$\begin{cases} \int\limits_\Omega A_\epsilon \nabla u_\epsilon . \nabla \varphi dx + \beta \int\limits_\Omega u_\epsilon \varphi dx = \int\limits_\Omega B(u_\epsilon) \varphi dx, \ \forall \varphi \in \mathcal{D}(\Omega) \\ u_\epsilon \in H_0^1(\Omega) \end{cases} \tag{6}$$

The existence of u_ϵ will be proved in the next section, Now, passing to the limit $\epsilon \to 0$ formally in (6) we obtain the limit problem

$$\int\limits_\Omega A_{22} \nabla_{X_2} u_0 . \nabla_{X_2} \varphi dx + \beta \int\limits_\Omega u_0 \varphi dx = \int\limits_\Omega B(u_0) \varphi dx, \ \forall \varphi \in \mathcal{D}(\Omega) \tag{7}$$

Our goal is to prove that u_0 exists and it satisfies (7), and give a sense to the formal convergence $u_\epsilon \rightsquigarrow u_0$, actually we would like to obtain convergence in $L^2(\Omega)$. We refer to [2] for more details about the linear theory of problem (6). However the nonlinear theory is poorly known, a monotone problem has been solved in [4] (using monotonicity argument), and also a case where B is represented by an integral operator has been studied in [3] (in the last section of this paper, we shall give an application to integro-differential problems). Generally, in singular perturbation problems for PDEs, a simple analysis of the problem gives only weak convergences, and often it is difficult to prove strong convergence, the principal hardness is the passage to the limit in the nonlinear term. In this article we expose a resolution method based on the use of several approximated problems involving regularization with compact operators and truncations. Let us give the main results.

Theorem 1. *(Existence and L^r-regularity of solutions) Assume (1), (2), (4), and that B is continuous on $L^2(\Omega)$ (not necessarily locally-Lipschitz) then (6) has at least a solution $u_\epsilon \in H_0^1(\Omega)$. Moreover, if $u_\epsilon \in H_0^1(\Omega)$ is a solution to (6) then $\|u_\epsilon\|_{L^r(\Omega)} \leq \frac{M}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}$ for every $\epsilon > 0$.*

For the convergence theorem and the interior estimates we need the following assumption

$$\partial_k A_{22},\ \partial_i a_{ij},\ \partial_j a_{ij} \in L^\infty(\Omega) \quad k = 1, ..., p, \quad i = 1, ..., p, \quad j = p+1, ..., N \tag{8}$$

Theorem 2. *(Interior estimates) Assume (1), (2), (3), (4), (5), (8). Let $(u_\epsilon) \subset H_0^1(\Omega)$ be a sequence of solutions to (6) then for every open set $\Omega' \subset\subset \Omega$ (i.e $\overline{\Omega'} \subset \Omega$) there exists $C_{\Omega'} \geq 0$ (independent of ϵ) such that*

$$\forall \epsilon : \|u_\epsilon\|_{H^1(\Omega')} \leq C_{\Omega'}$$

Theorem 3. *(The convergence theorem) Assume (1), (2), (3), (4), (5), (8). Let $(u_\epsilon) \subset H_0^1(\Omega)$ be a sequence of solutions to (6) then there exists a subsequence (u_{ϵ_k}) and $u_0 \in H_{loc}^1(\Omega) \cap L^2(\Omega)$ such that : $\nabla_{X_2} u_0 \in L^2(\Omega)$ and*

$$u_{\epsilon_k} \to u_0, \nabla_{X_2} u_{\epsilon_k} \to \nabla_{X_2} u_0 \text{ in } L^2(\Omega) \text{ strongly as } \epsilon_k \to 0$$

and for a.e X_1 we have $u_0(X_1, .) \in H_0^1(\omega_2)$,and

$$\int_{\omega_2} A_{22} \nabla_{X_2} u_0(X_1, .) \cdot \nabla_{X_2} \varphi dX_2 + \beta \int_{\omega_2} u_0(X_1, .) \varphi dX_2$$

$$= \int_{\omega_2} B(u_0)(X_1, .) \varphi dX_2, \quad \forall \varphi \in \mathcal{D}(\omega_2) \tag{9}$$

Corollary 1. *If problem (9) has a unique solution (in the sense of **theorem 3**) then the convergences given in the previous theorem hold for the whole sequence (u_ϵ).*

Proof. The proof is direct, let (u_ϵ) be a sequence of solutions to (6) and suppose that u_ϵ does not converge to u_0 (as $\epsilon \to 0$) then there exists a subsequence (u_{ϵ_k}) and $\delta > 0$ such that $\forall \epsilon_k$, $\|u_{\epsilon_k} - u_0\|_{L^2(\Omega)} > \delta$ or $\|\nabla_{X_2}(u_{\epsilon_k} - u_0)\|_{L^2(\Omega)} > \delta$. By **theorem 3** one can extract a subsequence of (u_{ϵ_k}) which converges to some u_1 in the sense of **theorem 3**, assume that (9) has a unique solution then $u_1 = u_0$.and this contradicts the previous inequalities. □

In the case of non-uniqueness we can reformulate the convergences ,given in the previous theorem, using $\epsilon - nets$ like in [3]. Let us recall the definition of $\epsilon - nets$ ([3])

Definition 1. *Let (X, d) be metric space, Y, Y' two subsets of X, then we say that Y is an $\epsilon -$ net of Y', if for every $x \in Y'$ there exists an $a \in Y$ such that*

$$d(x, a) < \epsilon$$

We define the following space introduced in [3]

$$W = \left\{ u \in L^2(\Omega) : \nabla_{X_2} u \in L^2(\Omega), \text{ and for a.e } X_1,\ u(X_1, .) \in H_0^1(\omega_2) \right\},$$

equipped with the Hilbertian norm (see [3])

$$\|u\|_W^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla_{X_2} u\|_{L^2(\Omega)}^2$$

Now we can give **Theorem 3** in the following form

Theorem 4. *Under assumptions of theorem 3 then Ξ ,the set of solutions of (9) in W, is non empty and we have $\Xi \cap H_{loc}^1(\Omega) \neq \varnothing$, and moreover for every $\eta > 0$, there exists $\epsilon_0 > 0$ such that Ξ is an $\eta -$ net of Ξ_{ϵ_0} in W where*

$$\Xi_{\epsilon_0} = \{u_\epsilon \text{ solution to (6) for } 0 < \epsilon < \epsilon_0\}$$

Proof. **Theorem 1** and **3** ensure that $\Xi \cap H_{loc}^1(\Omega) \neq \varnothing$. For the $\eta -$*net* convergence, let us reasoning by contradiction, then there exists $\eta > 0$ and a sequence $\epsilon_k \to 0$ such that Ξ is not an $\eta - net$ of Ξ_{ϵ_k} in W for every k (remark that $\Xi_{\epsilon_k} \neq \varnothing$ by **Theorem 1)** in other words there exists a sequence $(u_{\epsilon'_k})$ with $\epsilon'_k \to 0$ such that for every $u_0 \in \Xi$ we have $\left\| u_{\epsilon'_k} - u_0 \right\|_W \geq \eta$, according to theorem 3 there exists a subsequence of $(u_{\epsilon'_k})$ which converges to some $u_0 \in \Xi$ in$W$ and this contradicts the previous inequality. □

2. Existence and $L^r - regularity$ for the solutions and weak convergences

2.1. **Existence and $L^r -$*regularity*.** In this subsection we prove **Theorem 1,** we start by the following result on the L^r- regularity for the solutions

Proposition 1. *Assume (1), (2), (4) then if $u_\epsilon \in H_0^1(\Omega)$ is a solution to (6) then $u_\epsilon \in L^r(\Omega)$ and $\|u_\epsilon\|_{L^r(\Omega)} \leq \frac{M}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}$ for every $\epsilon > 0$*

Proof. We will proceed as in [1]. Let $u_\epsilon \in H_0^1(\Omega)$ be a solution to (6), given $g \in \mathcal{D}(\Omega)$ and let $w_\epsilon \in H_0^1(\Omega)$ be the unique solution to the linear problem

$$\int_\Omega A_\epsilon \nabla w_\epsilon \cdot \nabla \varphi dx + \beta \int_\Omega w_\epsilon \varphi dx = \int_\Omega g\varphi dx, \ \ \forall \varphi \in \mathcal{D}(\Omega), \tag{10}$$

the existence of w_ϵ follows by the Lax-Milgram theorem (thanks to assumptions (1), (2)).

Take u_ϵ as a test function and using the symmetry of A_ϵ we get

$$\begin{aligned}\int_\Omega u_\epsilon g dx &= \int_\Omega A_\epsilon \nabla w_\epsilon \cdot \nabla u_\epsilon dx + \beta \int_\Omega w_\epsilon u_\epsilon dx \\ &= \int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla w_\epsilon dx + \beta \int_\Omega w_\epsilon u_\epsilon dx \\ &= \int_\Omega B(u_\epsilon) w_\epsilon dx.\end{aligned}$$

Given s such that $\frac{1}{r} + \frac{1}{s} = 1$, then by (4) we obtain

$$\left| \int_\Omega u_\epsilon g dx \right| \leq M(1 + \|u_\epsilon\|_{L^2(\Omega)}) \|w_\epsilon\|_{L^s(\Omega)} \tag{11}$$

Now we have to estimate $\|w_\epsilon\|_{L^s(\Omega)}$. Let $\rho \in C^1(\mathbb{R}, \mathbb{R})$, such that $\rho(0) = 0$ and $\rho' \geq 0$ and $\rho' \in L^\infty$ then $\rho(w_\epsilon) \in H_0^1(\Omega)$, take $\rho(w_\epsilon)$ as a test function in (10) we get

$$\int_\Omega \rho'(w_\epsilon) A_\epsilon \nabla w_\epsilon \cdot \nabla w_\epsilon dx + \beta \int_\Omega w_\epsilon \rho(w_\epsilon) dx = \int_\Omega g \rho(w_\epsilon) dx.$$

Now, using ellipticity assumption (2) we derive

$$\lambda \left(\int_\Omega \rho'(w_\epsilon) |\epsilon \nabla_{X_1} w_\epsilon|^2 dx + \int_\Omega \rho'(w_\epsilon) |\nabla_{X_2} w_\epsilon|^2 dx \right) + \beta \int_\Omega w_\epsilon \rho(w_\epsilon) dx \leq \int_\Omega g \rho(w_\epsilon) dx$$

Thus

$$\beta \int_\Omega w_\epsilon \rho(w_\epsilon) dx \leq \int_\Omega g \rho(w_\epsilon) dx$$

Assume that $\forall x \in \mathbb{R} : |\rho(x)| \leq |x|^{\frac{1}{r-1}}$, so that $|\rho(x)|^r \leq |x| |\rho(x)| = x\rho(x)$ then, we obtain

$$\begin{aligned}\beta \int_\Omega w_\epsilon \rho(w_\epsilon) dx &\leq \|g\|_{L^s(\Omega)} \left(\int_\Omega |\rho(w_\epsilon)|^r dx \right)^{\frac{1}{r}} \\ &\leq \|g\|_{L^s(\Omega)} \left(\int_\Omega w_\epsilon \rho(w_\epsilon) dx \right)^{\frac{1}{r}},\end{aligned}$$

then

$$\beta \left(\int_\Omega w_\epsilon \rho(w_\epsilon) \right)^{\frac{1}{s}} \leq \|g\|_{L^s(\Omega)}$$

Now, for $\delta > 0$ taking $\rho(x) = x(x^2+\delta)^{\frac{s-2}{2}}$ we show easily that ρ satisfies the above assumptions, so we obtain

$$\beta \left(\int_\Omega w_\epsilon^2 (w_\epsilon^2+\delta)^{\frac{s-2}{2}} \right)^{\frac{1}{s}} \leq \|g\|_{L^s(\Omega)},$$

let $\delta \to 0$ by Fatou's lemma we get

$$\beta \|w_\epsilon\|_{L^s(\Omega)} \leq \|g\|_{L^s(\Omega)}$$

Finally by (11) we get

$$\left| \int_\Omega u_\epsilon g dx \right| \leq \frac{M(1+\|u_\epsilon\|_{L^2(\Omega)})}{\beta} \|g\|_{L^s(\Omega)}$$

By density we can take $g \in L^s(\Omega)$ and therefore by duality we get

$$\|u_\epsilon\|_{L^r(\Omega)} \leq \frac{M(1+\|u_\epsilon\|_{L^2(\Omega)})}{\beta},$$

hence by Holder's inequality we obtain

$$\|u_\epsilon\|_{L^r(\Omega)} \leq \frac{M}{\beta} + \frac{M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}{\beta} \|u_\epsilon\|_{L^r(\Omega)},$$

then

$$\|u_\epsilon\|_{L^r(\Omega)} \leq \frac{M}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}$$

□

Now, it remains to prove the existence of u_ϵ, the proof is based on the Schauder fixed point theorem. Let $v \in L^2(\Omega)$ and $v_\epsilon \in H_0^1(\Omega)$ be the unique solution to the linearized problem

$$\int_\Omega A_\epsilon \nabla v_\epsilon \cdot \nabla \varphi dx + \beta \int_\Omega v_\epsilon \varphi dx = \int_\Omega B(v)\varphi dx, \ \ \forall \varphi \in \mathcal{D}(\Omega) \tag{12}$$

The existence of v_ϵ follows by the Lax-Milgram theorem (thanks to assumptions (1), (2)). Let $\Gamma : L^2(\Omega) \to L^2(\Omega)$ be the mapping defined by $\Gamma(v) = v_\epsilon$.We prove that Γ is continuous, fix $v \in L^2(\Omega)$ and let $v_n \to v$ in $L^2(\Omega)$, we note $v_\epsilon^n = \Gamma(v_n)$ then we have

$$\int_\Omega A_\epsilon \nabla (v_\epsilon^n - v_\epsilon) \cdot \nabla \varphi dx + \beta \int_\Omega (v_\epsilon^n - v_\epsilon)\varphi dx = \int_\Omega (B(v_n) - B(v))\varphi dx, \ \ \forall \varphi \in \mathcal{D}(\Omega)$$

Take $(v_\epsilon^n - v_\epsilon)$ as a test function, estimating using ellipticity assumption (2) and Holder's inequality we get

$$\beta \|v_\epsilon^n - v_\epsilon\|_{L^2(\Omega)} \leq \|B(v_n) - B(v)\|_{L^2(\Omega)}.$$

Passing to the limit as $n \to \infty$ and assume that B is continuous, then the continuity of Γ follows. Now, we define the set

$$S=\left\{v\in H_0^1(\Omega):\|\nabla v\|_{L^2(\Omega)}\leq\frac{\sqrt{\beta}}{\epsilon\sqrt{2\lambda}}\left(\frac{M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\right)\text{ and }\|v\|_{L^2(\Omega)}\leq\frac{M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\right\}$$

It is clear that S is a convex bounded set in $H_0^1(\Omega)$ and it is closed in $L^2(\Omega)$, then S is compact in $L^2(\Omega)$ (thanks to the compact Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$). Let us check that S is stable by Γ. For $v\in S$, taking $\varphi=v_\epsilon$ in (12) and estimating using ellipticity assumption (2) and Hölder's inequality we get

$$\lambda\epsilon^2\left\|\nabla v_\epsilon\right\|_{L^2(\Omega)}^2+\beta\left\|v_\epsilon\right\|_{L^2(\Omega)}^2\leq\left\|B(v)\right\|_{L^2(\Omega)}\left\|v_\epsilon\right\|_{L^2(\Omega)},$$

then by Young's inequality we derive

$$\lambda\epsilon^2\left\|\nabla v_\epsilon\right\|_{L^2(\Omega)}^2+\beta\left\|v_\epsilon\right\|_{L^2(\Omega)}^2\leq\frac{1}{2\beta}\left\|B(v)\right\|_{L^2(\Omega)}^2+\frac{\beta}{2}\left\|v_\epsilon\right\|_{L^2(\Omega)}^2,$$

and (4) gives

$$\begin{aligned}\lambda\epsilon^2\left\|\nabla v_\epsilon\right\|_{L^2(\Omega)}^2+\frac{\beta}{2}\left\|v_\epsilon\right\|_{L^2(\Omega)}^2 &\leq \frac{\left|\Omega\right|^{1-\frac{2}{r}}\left(M+M\left\|v\right\|_{L^2(\Omega)}\right)^2}{2\beta}\\ &\leq \frac{\left|\Omega\right|^{1-\frac{2}{r}}\left(M+\frac{M^2\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\right)^2}{2\beta}\leq\frac{\beta}{2}\left(\frac{M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\right)^2,\end{aligned}$$

hence

$$\begin{cases}\left\|v_\epsilon\right\|_{L^2(\Omega)}\leq\frac{M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\\ \left\|\nabla v_\epsilon\right\|_{L^2(\Omega)}\leq\frac{\sqrt{\beta}}{\epsilon\sqrt{2\lambda}}\left(\frac{M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\right)\end{cases}$$

And therefore $v_\epsilon=\Gamma(v)\in S$.Whence, there exists at least a fixed point $u_\epsilon\in S$ for Γ, in other words u_ϵ is a solution to (6).

2.2. **Weak convergences as $\epsilon\to 0$.** Throughout this article we use the notations $\rightharpoonup$, $\to$ for weak and strong convergences of sequences respectively. Assume (1), (2), (4) and let (u_ϵ) be a sequence of solutions to (6)**,** . We begin by a simple analysis of the problem, considering problem (6) and taking $\varphi=u_\epsilon\in H_0^1(\Omega)$, by ellipticity assumption (2) we get

$$\lambda\left(\int_\Omega\left|\epsilon\nabla_{X_1}u_\epsilon\right|^2dx+\int_\Omega\left|\nabla_{X_2}u_\epsilon\right|^2dx\right)+\beta\int_\Omega u_\epsilon^2dx\leq\int_\Omega B(u_\epsilon)u_\epsilon dx\ ,$$

and Hölder's inequality gives

$$\lambda\epsilon^2\left\|\nabla_{X_1}u_\epsilon\right\|_{L^2(\Omega)}^2+\lambda\left\|\nabla_{X_2}u_\epsilon\right\|_{L^2(\Omega)}^2+\beta\left\|u_\epsilon\right\|_{L^2(\Omega)}^2\leq\left\|B(u_\epsilon)\right\|_{L^2(\Omega)}\left\|u_\epsilon\right\|_{L^2(\Omega)},$$

and therefore (4) and **Proposition 1** give

$$\lambda\epsilon^2\left\|\nabla_{X_1}u_\epsilon\right\|_{L^2(\Omega)}^2+\lambda\left\|\nabla_{X_2}u_\epsilon\right\|_{L^2(\Omega)}^2+\beta\left\|u_\epsilon\right\|_{L^2(\Omega)}^2\leq\frac{M^2\left|\Omega\right|^{1-\frac{2}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\left(1+\frac{M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}{\beta-M\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}}}\right)$$

Whence

$$\begin{cases} \|\epsilon\nabla_{X_1}u_\epsilon\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\lambda}} \\ \|\nabla_{X_2}u_\epsilon\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\lambda}} \\ \|u_\epsilon\|_{L^2(\Omega)} \leq \frac{M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}} \end{cases} \tag{13}$$

,where $C^2 = \frac{M^2|\Omega|^{1-\frac{2}{r}}}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}\left(1 + \frac{M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}\right)$. Remark that the gradient of u_ϵ is not bounded uniformly in $H_0^1(\Omega)$ so we cannot obtain strong convergence (using Sobolev embedding for example) in $L^2(\Omega)$, however there exists a subsequence (u_{ϵ_k}) and $u_0 \in L^2(\Omega)$ such that: $u_{\epsilon_k} \rightharpoonup u_0$, $\nabla_{X_2}u_{\epsilon_k} \rightharpoonup \nabla_{X_2}u_0$ and $\epsilon_k\nabla_{X_1}u_{\epsilon_k} \rightharpoonup 0$ weakly in $L^2(\Omega)$ (we used weak compacity in $L^2(\Omega)$, and the continuity of the operator of derivation on $\mathcal{D}'(\Omega)$). The function u_0 constructed before represents a good candidate for solution to the limit problems (7),(9).

Corollary 2. *We have $u_0 \in L^r(\Omega)$.*

Proof. Since (u_{ϵ_k}) is bounded in $L^r(\Omega)$ then one can extract a subsequence noted always (u_{ϵ_k}) which converges weakly to some $u_1 \in L^r(\Omega)$ and therefore $u_{\epsilon_k} \rightharpoonup u_1$ in $\mathcal{D}'(\Omega)$, so $u_1 = u_0$ □

3. Interior estimates and $H^1_{loc} - regularity$

For every $g \in V$ consider the linear problem (10), then one can prove the

Theorem 5. *Assume (1), (2), (8) then for every $\Omega' \subset\subset \Omega$ (i.e $\overline{\Omega'} \subset \Omega$) there exists $C_{\Omega',g} \geq 0$ independent of ϵ such that*

$$\forall\epsilon : \|v_\epsilon\|_{H^1(\Omega')} \leq C_{\Omega',g} \tag{14}$$

Proof. The proof is the same as in [2] (see the rate estimations theorem in [2]), remark that the additional term βv_ϵ is uniformly bounded in $L^2(\Omega)$. □

To obtain interior estimates for the nonlinear problem we use the well known Banach-Steinhaus's theorem

Theorem 6. *(see [6]) Let Y and Z be two separated topological vector spaces, and let $(\mathcal{A}_\epsilon)$ be a family of continuous linear mappings from $Y \to Z$, G is convex compact set in Y. Suppose that for each $x \in G$ the orbit $\{\mathcal{A}_\epsilon(x)\}_\epsilon$ is bounded in Z, then $(\mathcal{A}_\epsilon)$ is uniformly bounded on G, i.e. there exists a bounded F set in Z such that $\forall\epsilon$, $\mathcal{A}_\epsilon(G) \subset F$.*

Now, we are ready to prove **Theorem 2.** Let $(\Omega_j)_{j\in\mathbb{N}}$,$(\forall j : \overline{\Omega_j} \subset \Omega_{j+1})$ be an open covering of Ω, so we can define a family $(p_j)_j$ of seminorms on $H^1_{loc}(\Omega)$ by

$$p_j(u) = \|u\|_{H^1(\Omega_j)} \text{ for every } u \in H^1_{loc}(\Omega)$$

Set $Z = (H^1_{loc}(\Omega), (p_j)_j)$, we can check easily that Z is a separated locally convex topological vector space where the topology is generated by the family of seminorms $(p_j)_j$, we also set $Y = L^2(\Omega)$. We define a family $(\mathcal{A}_\epsilon)_\epsilon$ of linear mappings from Y to Z by $\mathcal{A}_\epsilon(g) = v_\epsilon$ where v_ϵ is the unique solution to (10) (existence and uniqueness follows by Lax-Milgram, thanks to (1), (2)). $\forall\epsilon$, $\mathcal{A}_\epsilon : Y \to Z$ is continuous (we can check easily that $\mathcal{A}_\epsilon : Y \to H^1(\Omega)$ and the injection $H^1(\Omega) \hookrightarrow Z$ are continuous).

We note Z_w, Y_w the spaces Z and Y equipped with the weak topology, then for every ϵ, $\mathcal{A}_\epsilon : Y_w \to Z_w$ is still continuous. Let $E = \left\{ u \in V : \|u\|_{L^2(\Omega)} \leq \frac{M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}} \right\}$ and assume (5) then $G = \overline{conv}\left\{B\left(E\right)\right\} \subset V$, it is clear that G is bounded in Y thus G is compact in Y_w. Recall that a set is bounded in a locally convex topological space if and only if the seminorms that generate the topology are bounded on this set, suppose (8) then according to (14) we have, for $g \in G$, $\{\mathcal{A}_\epsilon(g)\}_\epsilon$ is bounded in Z, and therefore $\{\mathcal{A}_\epsilon(g)\}_\epsilon$ is bounded in Z_w so by **Theorem 6** there exists a bounded set F in Z_w (also note that F is also bounded in Z) such that $\forall \epsilon$, $\mathcal{A}_\epsilon(G) \subset F$. Now let (u_ϵ) be a sequence of solutions to (6), and assume in addition (3) and (4) then (13) gives $(u_\epsilon)_\epsilon \subset E$ whence $(B(u_\epsilon))_\epsilon \subset G$, and therefore $\mathcal{A}_\epsilon(B(u_\epsilon)) \subset F$ for every ϵ, in other words we have

$$\forall j \ , \ \exists C_j \geq 0 \text{ such that } \forall \epsilon : p_j(\mathcal{A}_\epsilon(B(u_\epsilon))) \leq C_j,$$

where C_j is independent of ϵ, and therefore

$$\forall \epsilon, \forall j, \|u_\epsilon\|_{H^1(\Omega_j)} \leq C_j$$

Now, given $\Omega' \subset\subset \Omega$ then there exists j such that $\Omega' \subset \Omega_j$ thus

$$\forall \epsilon, \|u_\epsilon\|_{H^1(\Omega')} \leq C_j \tag{15}$$

Corollary 3. *Let $(u_\epsilon) \subset H_0^1(\Omega)$ be a sequence of solutions to (6) such that $u_\epsilon \rightharpoonup u_0$ in $L^2(\Omega)$ weakly, then under assumptions of **Theorem 2** we have, $u_0 \in H^1_{loc}(\Omega)$*

Proof. take $\Omega' \subset\subset \Omega$ an open set, and $\psi \in \mathcal{D}(\Omega')$, $1 \leq i \leq N$ then by (15) we have

$$\left| \int_{\Omega'} u_\epsilon \partial_i \psi dx \right| = \left| \int_{\Omega'} \partial_i u_\epsilon \psi dx \right| \leq C_{\Omega'} \|\psi\|_{L^2(\Omega')}$$

Let $\epsilon \to 0$ and using the week convergence $u_\epsilon \rightharpoonup u_0$ we get:

$$\left| \int_{\Omega'} u_0 \partial_i \psi dx \right| \leq C_{\Omega'} \|\psi\|_{L^2(\Omega')}$$

Hence, $u_0 \in H^1_{loc}(\Omega)$. □

4. Strong convergence and proof of theorem 3

Let us begin by some useful propositions

Proposition 2. *Let (g_n) be a sequence in $H_0^1(\Omega)$ and $g \in L^2(\Omega)$ such that $\nabla_{X_2} g \in L^2(\Omega)$ and $\nabla_{X_2} g_n \to \nabla_{X_2} g$ in $L^2(\Omega)$, then we have:*
$g_n \to g$ in $L^2(\Omega)$ and for a.e. X_1 $g(X_1, .) \in H_0^1(\omega_2)$

Proof. We have for a.e X_1 : $\nabla_{X_2} g_n(X_1, .) \to \nabla_{X_2} g$ $(X_1, .)$ in $L^2(\omega_2)$ (up to a subsequence), and since for a.e X_1 and for every n we have $g_n(X_1, .) \in H_0^1(\omega_2)$ then we have for a.e. X_1, $g(X_1, .) \in H_0^1(\omega_2)$.And finally the convergence $g_n \to g$ in $L^2(\Omega)$ follows by Poincaré's inequality $\int_\Omega |g_n - g|^2 \leq C \int_\Omega |\nabla_{X_2}(g_n - g)|^2$ □

Proposition 3. *Let $f, v \in L^2(\Omega)$ such that $\nabla_{X_2} v \in L^2(\Omega)$ and*

$$\int_\Omega A_{22}\nabla_{X_2} v \cdot \nabla_{X_2}\varphi dx + \beta \int_\Omega v\varphi dx = \int_\Omega f\varphi dx, \ \ \forall \varphi \in \mathcal{D}(\Omega),$$

then we have for a.e X_1

$$\int_{\omega_2} A_{22}\nabla_{X_2} v(X_1,.)\cdot\nabla_{X_2}\varphi dX_2 + \beta \int_{\omega_2} v(X_1,.)\varphi dX_2 = \int_{\omega_2} f(X_1,.)\varphi dX_2, \quad \forall \varphi \in \mathcal{D}(\omega_2)$$

Moreover, if for a.e X_1 we have $v(X_1,.) \in H_0^1(\omega_2)$ then v is the unique function which satisfies the previous equalities

Proof. Same arguments as in [2]. □

4.1. **The cut-off problem:** Let $\phi \in \mathcal{D}(\Omega)$, and let $(u_\epsilon) \subset H_0^1(\Omega)$ be a sequence of solutions to (6) such that u_ϵ converges weakly in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega)$. we define $w_\epsilon \in H_0^1(\Omega)$ to be the unique solution to the cut-off problem (under assumptions (1), (2) existence and uniqueness of w_ϵ follows from the Lax-Milgram theorem)

$$\int_\Omega A_\epsilon \nabla w_\epsilon \cdot \nabla\varphi dx + \beta \int_\Omega w_\epsilon\varphi dx = \int_\Omega B(\phi u_\epsilon)\varphi dx, \ \ \forall \varphi \in \mathcal{D}(\Omega) \tag{16}$$

The following Lemma is fundamental in this paper

Lemma 1. *Assume (1), (2), (3),(4), (5), (8) then there exists $w_0 \in W$ such that $w_\epsilon \to w_0$ in W strongly and*

$$\int_\Omega A_{22}\nabla_{X_2} w_0 \cdot \nabla_{X_2}\varphi dx + \beta \int_\Omega w_0\varphi dx = \int_\Omega B(\phi u_0)\varphi dx, \ \ \forall \varphi \in \mathcal{D}(\Omega),$$

$$\int_{\omega_2} A_{22}\nabla_{X_2} w_0(X_1,.) \cdot \nabla_{X_2}\varphi dX_2 + \beta \int_{\omega_2} w_0(X_1,.)\varphi dX_2 = \int_{\omega_2} B(\phi u_0)(X_1,.)\varphi dX_2, \ \ \forall \varphi \in \mathcal{D}(\omega_2),$$

and w_0 is the unique function which satisfies the two previous weak formulations.

Admit this lemma for the moment then we have the following

Proposition 4. *Assume (1), (2), (3),(4), (5), (8), let (u_ϵ) be a sequence of solutions to (6) such that $u_\epsilon \rightharpoonup u_0$ weakly in $L^2(\Omega)$, then we have $u_\epsilon \to u_0$ in W strongly and*

$$\int_{\omega_2} A_{22}\nabla_{X_2} u_0(X_1,.)\cdot\nabla_{X_2}\varphi dX_2 + \beta \int_{\omega_2} u_0(X_1,.)\varphi dX_2 = \int_{\omega_2} B(u_0)(X_1,.)\varphi dX_2, \quad \forall \varphi \in \mathcal{D}(\omega_2)$$

4.2. **Proof of Proposition 4.**

4.2.1. ***Approximation by truncations***. Let (u_ϵ) be a sequence in $H_0^1(\Omega)$ of solutions to (6), assume (1), (2) and define $w_\epsilon^n \in H_0^1(\Omega)$ the unique solution (by Lax-Milgram theorem) to the problem

$$\int_\Omega A_\epsilon \nabla w_\epsilon^n \cdot \nabla \varphi + \beta \int_\Omega w_\epsilon^n \varphi = \int_\Omega B(\phi_n u_\epsilon)\varphi, \ \ \forall \varphi \in \mathcal{D}(\Omega), \tag{17}$$

where (ϕ_n) is a sequence in $\mathcal{D}(\Omega)$ which converges to 1 in $L^{\frac{2r}{r-2}}(\Omega)$.

Proposition 5. *Suppose (1), (2), (3), (4) then we have*

$$\|\nabla_{X_2} w_\epsilon^n - \nabla_{X_2} u_\epsilon\|_{L^2(\Omega)} \to 0$$

as $n \to \infty$ uniformly on ϵ

Proof. Subtracting (6) from (17) and taking $\varphi = (w_\epsilon^n - u_\epsilon) \in H_0^1(\Omega)$ we get

$$\int_\Omega A_\epsilon \nabla (w_\epsilon^n - u_\epsilon) \cdot \nabla (w_\epsilon^n - u_\epsilon) dx + \beta \int_\Omega (w_\epsilon^n - u_\epsilon)^2 dx$$
$$= \int_\Omega \left(B(\phi_n u_\epsilon) - B(u_\epsilon)\right)(w_\epsilon^n - u_\epsilon) dx$$

By (2) and Hölder's inequality we derive

$$\lambda \|\nabla_{X_2}(w_\epsilon^n - u_\epsilon)\|_{L^2(\Omega)}^2 \leq \|(B(\phi_n u_\epsilon) - B(u_\epsilon))\|_{L^2} \|w_\epsilon^n - u_\epsilon\|_{L^2},$$

and **Proposition 1** gives $\|u_\epsilon\|_{L^2(\Omega)} \leq \frac{M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}$, $\|\phi_n u_\epsilon\|_{L^2(\Omega)} \leq \frac{M}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}} \|\phi_n\|_{L^{\frac{2r}{r-2}}(\Omega)}$, we note K the Lipschitz coefficient of B associated with the bounded set

$$\left\{ u \in L^2(\Omega) : \|u\|_{L^2} \leq \sup_n \left(\frac{M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}, \frac{M}{\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}} \|\phi_n\|_{L^{\frac{2r}{r-2}}} \right) < \infty \right\},$$

,

whence (3) and Hölder's inequality give

$$\|\nabla_{X_2}(w_\epsilon^n - u_\epsilon)\|_{L^2(\Omega)}^2 \leq \frac{K}{\lambda} \|\phi_n - 1\|_{L^{\frac{2r}{r-2}}} \|u_\epsilon\|_{L^r} \|w_\epsilon^n - u_\epsilon\|_{L^2}$$

And finally by **Proposition 1** and Poincaré's inequality in the X_2 direction we get

$$\|\nabla_{X_2}(w_\epsilon^n - u_\epsilon)\|_{L^2(\Omega)} \leq \frac{C'KM}{\lambda(\beta - M|\Omega|^{\frac{1}{2}-\frac{1}{r}})} \|\phi_n - 1\|_{L^{\frac{2r}{r-2}}}$$

Whence $\|\nabla_{X_2}(w_\epsilon^n - u_\epsilon)\|_{L^2(\Omega)} \to 0$ as $n \to \infty$ uniformly in ϵ □

4.2.2. ***The convergence***. Fix n ,under assumptions of **Proposition 4** then it follows by **Lemma 1** that there exists $w_0^n \in W$ such that

$$w_\epsilon^n \rightharpoonup w_0^n \text{ strongly in } W \tag{18}$$

and w_0^n is the unique function in W which satisfies

$$\int_\Omega A_{22}\nabla_{X_2} w_0^n \cdot \nabla_{X_2}\varphi dx + \beta\int_\Omega w_0^n\varphi dx = \int_\Omega B(\phi_n u_0)\varphi dx, \ \ \forall \varphi \in \mathcal{D}(\Omega), \tag{19}$$

and for a.e X_1 we have

$$\begin{aligned} &\int_{\omega_2} A_{22}\nabla_{X_2} w_0^n(X_1,.) \cdot \nabla_{X_2}\varphi dX_2 + \beta\int_{\omega_2} w_0^n(X_1,.)\varphi dX_2 \qquad (20)\\ = &\int_{\omega_2} B(\phi_n u_0)(X_1,.)\varphi dX_2, \ \ \forall \varphi \in \mathcal{D}(\omega_2) \end{aligned}$$

For a.e X_1 taking $\varphi = w_0^n(X_1,.) \in H_0^1(\omega_2)$ in (20), by ellipticity assumption (2), Hölder's inequality we obtain

$$\lambda\int_{\omega_2} \left|\nabla_{X_2} w_0^n(X_1,.)\right|^2 dX_2 \leq \left\|B(\phi_n u_0)(X_1,.)\right\|_{L^2(\omega_2)} \left\|w_0^n(X_1,.)\right\|_{L^2(\omega_2)},$$

and Poincaré's inequality in the X_2 direction gives

$$\begin{aligned} \left\|\nabla_{X_2} w_0^n(X_1,.)\right\|_{L^2(\omega_2)} &\leq \frac{C'}{\lambda}\left\|B(\phi_n u_0)(X_1,.)\right\|_{L^2(\omega_2)}\\ \left\|w_0^n(X_1,.)\right\|_{L^2(\omega_2)} &\leq \frac{C'^2}{\lambda}\left\|B(\phi_n u_0)(X_1,.)\right\|_{L^2(\omega_2)} \end{aligned}$$

integrating over ω_1 yields

$$\begin{aligned} \left\|\nabla_{X_2} w_0^n\right\|_{L^2(\Omega)} &\leq \frac{C'}{\lambda}\left\|B(\phi_n u_0)\right\|_{L^2(\Omega)},\\ \left\|w_0^n\right\|_{L^2(\Omega)} &\leq \frac{C'^2}{\lambda}\left\|B(\phi_n u_0)\right\|_{L^2(\Omega)}, \end{aligned}$$

and by (4) and Hölder's inequality (remark that $u_0 \in L^r(\Omega)$ since (u_ϵ) is bounded in $L^r(\Omega)$ and $u_\epsilon \rightharpoonup u_0$ in $L^2(\Omega)$) we obtain

$$\begin{aligned} \left\|\nabla_{X_2} w_0^n\right\|_{L^2(\Omega)} &\leq \frac{C\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}} M\left(+\left\|\phi_n\right\|_{L^{\frac{2r}{r-2}}}\left\|u_0\right\|_{L^r}\right)}{\lambda},\\ \left\|w_0^n\right\|_{L^2(\Omega)} &\leq \frac{C^2\left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}} M\left(+\left\|\phi_n\right\|_{L^{\frac{2r}{r-2}}}\left\|u_0\right\|_{L^r}\right)}{\lambda}, \end{aligned}$$

(we note that The the right hand sides of the previous inequality is uniformly bounded). Using weak compacity in $L^2(\Omega)$, one can extract a subsequence noted always (w_0^n) which converges weakly to some $w_0 \in L^2(\Omega)$ and such that $\nabla_{X_2} w_0^n \rightharpoonup \nabla_{X_2} w_0$ weakly. Now, passing to the limit as $n \to \infty$ in (19) and using

$$\left\|B(\phi_n u_0) - B(u_0)\right\|_{L^2(\Omega)} \leq K\left\|\phi_n - 1\right\|_{L^{\frac{2r}{r-2}}}\left\|u_0\right\|_{L^r(\Omega)} \tag{21}$$

we get

$$\int_\Omega A_{22}\nabla_{X_2}w_0\cdot\nabla_{X_2}\varphi dx+\beta\int_\Omega w_0\varphi dx=\int_\Omega B(u_0)\varphi dx,\ \ \forall\varphi\in\mathcal{D}(\Omega) \tag{22}$$

Now we will prove that $\nabla_{X_2}w_0^n\to\nabla_{X_2}w_0$ in $L^2(\Omega)$ strongly, using ellipticity assumption (2) we obtain

$$\begin{aligned}&\lambda\left\|\nabla_{X_2}(w_0^n-w_0)\right\|^2_{L^2(\Omega)}\leq \\ &\int_\Omega A_{22}\nabla_{X_2}(w_0^n-w_0)\cdot\nabla_{X_2}(w_0^n-w_0)dx+\beta\left\|w_0^n-w_0\right\|^2_{L^2(\Omega)}\\ \leq\ &\int_\Omega A_{22}\nabla_{X_2}w_0^n\cdot\nabla_{X_2}w_0^n dx-\int_\Omega A_{22}\nabla_{X_2}w_0^n\cdot\nabla_{X_2}w_0 dx-\int_\Omega A_{22}\nabla_{X_2}w_0\cdot\nabla_{X_2}w_0^n dx\\ &+\int_\Omega A_{22}\nabla_{X_2}w_0\cdot\nabla_{X_2}w_0 dx+\beta\left\|w_0^n-w_0\right\|^2_{L^2(\Omega)}\end{aligned} \tag{23}$$

Taking $\varphi=w_\epsilon^n\in H_0^1(\Omega)$ in (19) and (22) and letting $\epsilon\to 0$ we get (thanks to (18))

$$\int_\Omega A_{22}\nabla_{X_2}w_0^n\cdot\nabla_{X_2}w_0^n dx+\beta\int_\Omega\left|w_0^n\right|^2dx=\int_\Omega B(\phi_n u_0)w_0^n dx, \tag{24}$$

and

$$\int_\Omega A_{22}\nabla_{X_2}w_0\cdot\nabla_{X_2}w_0^n dx+\beta\int_\Omega w_0w_0^n dx=\int_\Omega B(u_0)w_0^n dx \tag{25}$$

Replacing (24) and (25) in (23) we get

$$\begin{aligned}&\lambda\left\|\nabla_{X_2}(w_0^n-w_0)\right\|^2_{L^2(\Omega)}\\ \leq\ &\int_\Omega B(\phi_n u_0)w_0^n dx-\int_\Omega B(u_0)w_0^n dx-\int_\Omega A_{22}\nabla_{X_2}w_0^n\cdot\nabla_{X_2}w_0 dx\\ &+\int_\Omega A_{22}\nabla_{X_2}w_0\cdot\nabla_{X_2}w_0 dx+\beta\int_\Omega\left|w_0\right|^2dx-\beta\int_\Omega w_0w_0^n dx\end{aligned} \tag{26}$$

We have $B(\phi_n u_0)\to B(u_0)$ in $L^2(\Omega)$ and since $w_0^n\rightharpoonup w_0$ in $L^2(\Omega)$ then

$$\int_\Omega B(\phi_n u_0)w_0^n dx\to\int_\Omega B(u_0)w_0 dx$$

And since $\nabla_{X_2}w_0^n\rightharpoonup\nabla_{X_2}w_0$ in $L^2(\Omega)$ then $A_{22}\nabla_{X_2}w_0^n\rightharpoonup A_{22}\nabla_{X_2}w_0$ in $L^2(\Omega)$ (since $A_{22}\in L^\infty(\Omega)$). Now, let $n\to\infty$ in (26) we get

$$\left\|\nabla_{X_2}(w_0^n-w_0)\right\|_{L^2(\Omega)}\to 0 \tag{27}$$

Thanks to the uniform convergence proved in **proposition 5**, (27) and (18), we show by the triangular inequality that $\nabla_{X_2}u_\epsilon\to\nabla_{X_2}w_0$ in $L^2(\Omega)$. Now, we must

check that $w_0 = u_0$, according to **Proposition 2,** we have for a.e X_1, $w_0(X_1,.) \in H_0^1(\omega_2)$ and $u_\epsilon \to w_0$ in $L^2(\Omega)$, and therefore $w_0 = u_0$. By (22), we obtain

$$\int_\Omega A_{22}\nabla_{X_2}u_0 \cdot \nabla_{X_2}\varphi dx + \beta\int_\Omega u_0\varphi dx = \int_\Omega B(u_0)\varphi dx, \ \ \forall\varphi \in \mathcal{D}(\Omega),$$

and we finish the proof of **proposition 4** by using **proposition 3.** Finally, if (u_ϵ) is a sequence of solutions to (6) then there exists a subsequence (u_{ϵ_k}) which converges to some u_0 in $L^2(\Omega)$ weakly (see subsection 2.2), whence **Theorem3** follows from **Proposition 4**. Now, it remains to prove **Lemma 1** which will be the subject of the next section.

5. Proof of Lemma 1

Before starting , let us give some tools. For $n \in \mathbb{N}^*$ we note $\Delta_n = (I - n^{-1}\Delta)^{-1}$ the resolvent of the Dirichlet Laplacian on $L^2(\Omega)$, this is a compact operator as well known. Given $f \in L^2(\Omega)$ and we note $U_n = (I - n^{-1}\Delta)^{-1}f$, U_n is the unique weak solution to the singularly perturbed problem:

$$-\frac{1}{n}\Delta U_n + U_n = f,$$

we have the

Theorem 7. *(see [5]): If $f \in H_0^1(\Omega)$ then : $\|U_n - f\|_{L^2(\Omega)} \leq C_\Omega n^{-\frac{1}{4}} \|f\|_{H^1(\Omega)}$*

The following lemma will be used in the approximation

Lemma 2. *For any functions $g \in H^1_{loc}(\Omega) \cap L^2(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ we have : $\phi g \in H_0^1(\Omega)$ and moreover there exists $\Omega' \subset\subset \Omega : \|\phi g\|_{H^1(\Omega)} \leq C_\phi \|g\|_{H^1(\Omega')}$*

Proof. the proof is direct. □

5.1. **Approximation of the cut-off problem by regularization.** Let $(u_\epsilon)\subset H_0^1(\Omega)$ be a sequence of solution to (6) such that $u_\epsilon \rightharpoonup u_0 \in L^2(\Omega)$ weakly, assume (1), (2), (3), (4), (5), (8). For $\phi \in \mathcal{D}(\Omega)$ fixed we note $w_\epsilon^n \in H_0^1(\Omega)$ the unique solution to the following regularized problem (thanks to assumptions (1), (2) and Lax-Milgram theorem).

$$\int_\Omega A_\epsilon\nabla w_\epsilon^n \cdot \nabla\varphi dx + \beta\int_\Omega w_\epsilon^n\varphi dx = \int_\Omega B(\Delta_n(\phi u_\epsilon))\varphi dx, \ \ \forall\varphi \in \mathcal{D}(\Omega) \qquad (28)$$

Proposition 6. *As $n \to \infty$ we have :*

$$\nabla_{X_2}w_\epsilon^n \to \nabla_{X_2}w_\epsilon \text{ in } L^2(\Omega) \text{ uniformly in } \epsilon \ ,$$

where w_ϵ is the solution to the cut-off problem (16)

Proof. Subtracting (16) from (28) and taking $\varphi = (w_\epsilon^n - w_\epsilon) \in H_0^1(\Omega)$ yields

$$\int_\Omega A_\epsilon\nabla(w_\epsilon^n - w_\epsilon) \cdot \nabla(w_\epsilon^n - w_\epsilon)dx + \beta\int_\Omega (w_\epsilon^n - w_\epsilon)^2 dx$$

$$= \int_\Omega \{B(\Delta_n(\phi u_\epsilon)) - B((\phi u_\epsilon))\}(w_\epsilon^n - w_\epsilon)dx$$

Remark that $(\phi u_\epsilon)_\epsilon$ is bounded in $L^2(\Omega)$ (**Proposition 1**) and it is clear that $(\Delta_n(\phi u_\epsilon))_{n,\epsilon}$ is bounded in $L^2(\Omega)$ ($\|\Delta_n(\phi u_\epsilon)\|_{L^2(\Omega)} \leq \|\phi u_\epsilon\|_{L^2(\Omega)}$), then by ellipticity assumption (2) and the local Lipschitzness of B (3) we get

$$\begin{aligned}\lambda \int_\Omega |\nabla_{X_2}(w_\epsilon^n - w_\epsilon)|^2 \, dx &\leq \int_\Omega \{B(\Delta_n(\phi u_\epsilon)) - B(\phi u_\epsilon)\}(w_\epsilon^n - w_\epsilon)dx \\ &\leq K' \left(\int_\Omega |\Delta_n(\phi u_\epsilon) - \phi u_\epsilon|^2 \, dx \right)^{\frac{1}{2}} \|w_\epsilon^n - w_\epsilon\|_{L^2(\Omega)}\end{aligned}$$

Hence, Poincaré's inequality gives

$$\|\nabla_{X_2}(w_\epsilon^n - w_\epsilon)\|_{L^2(\Omega)} \leq \frac{CK'}{\lambda} \left(\int_\Omega |\Delta_n(\phi u_\epsilon) - \phi u_\epsilon|^2 \, dx \right)^{\frac{1}{2}},$$

and by **Theorem 7** we get

$$\|\nabla_{X_2}(w_\epsilon^n - w_\epsilon)\|_{L^2(\Omega)} \leq \frac{C'K}{\lambda} n^{-\frac{1}{4}} \|\phi u_\epsilon\|_{H^1(\Omega)},$$

and **Lemma 2** gives

$$\|\nabla_{X_2}(w_\epsilon^n - w_\epsilon)\|_{L^2(\Omega)} \leq \frac{C'K}{\lambda} C_\phi n^{-\frac{1}{4}} \|u_\epsilon\|_{H^1(\Omega')}$$

So finally by **Theorem 2** we get

$$\|\nabla_{X_2}(w_\epsilon^n - w_\epsilon)\|_{L^2(\Omega)} \leq C" n^{-\frac{1}{4}}$$

where $C" \geq 0$ is independent on ϵ and n □

5.2. **The convergence.**

5.2.1. ***Passage to the limit as*** $\epsilon \to 0$. Let $n \in \mathbb{N}^*$ fixed, taking $\varphi = w_\epsilon^n \in H_0^1(\Omega)$ in (28),and estimating using ellipticity assumption (2) and (4) and **Proposition 1**(as in subsection 2.2) then one can extract a subsequence $(w^n_{\epsilon_k(n)})_k$ which converges (as $\epsilon_k(n) \to 0$) to some w_0^n in the following sense

$$w^n_{\epsilon_k(n)} \rightharpoonup w_0^n \text{ in } L^2(\Omega), \nabla_{X_2} w^n_{\epsilon_k(n)} \rightharpoonup \nabla_{X_2} w_0^n \text{ and } \epsilon_k(n)\nabla_{X_1} w^n_{\epsilon_k(n)} \rightharpoonup 0 \qquad (29)$$

Now passing the limit (as $\epsilon_k(n) \to 0$) in (28) we get

$$\int_\Omega A_{22}\nabla_{X_2} w_0^n \cdot \nabla_{X_2}\varphi dx + \beta \int_\Omega w_0^n \varphi dx = \lim_{\epsilon_k(n)\to 0} \int_\Omega B(\Delta_n(\phi u_{\epsilon_k(n)}))\varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Since $u_{\epsilon_k(n)} \rightharpoonup u_0$ weakly in $L^2(\Omega)$ then $\phi u_{\epsilon_k(n)} \rightharpoonup \phi u_0$ weakly in $L^2(\Omega)$ so by compacity of Δ_n we get $\Delta_n(\phi u_{\epsilon_k(n)}) \to \Delta_n(\phi u_0)$ in $L^2(\Omega)$ strongly. And therefore, the continuity of B gives $B(\Delta_n(\phi u_{\epsilon_k(n)})) \to B(\Delta_n(\phi u_0))$ in $L^2(\Omega)$ strongly, hence the previous equality becomes

$$\int_\Omega A_{22}\nabla_{X_2} w_0^n \cdot \nabla_{X_2}\varphi dx + \beta \int_\Omega w_0^n \varphi dx = \int_\Omega B(\Delta_n(\phi u_0))\varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

(30)

Take $\varphi = w^n_{\epsilon_k(n)} \in H^1_0(\Omega)$ in (30) and let $\epsilon_k(n) \to 0$ we derive

$$\int_\Omega A_{22}\nabla_{X_2} w^n_0 \cdot \nabla_{X_2} w^n_0 dx + \beta \int_\Omega |w^n_0|^2 \, dx = \int_\Omega B(\Delta_n(\phi u_0)) w^n_0 dx \tag{31}$$

Now, we prove strong convergences for the whole sequence (as $\epsilon \to 0$)

Proposition 7. *As $\epsilon \to 0$ we have $\nabla_{X_2} w^n_\epsilon \to \nabla_{X_2} w^n_0$ strongly in $L^2(\Omega)$*

Proof. Computing

$$\begin{aligned}
I^n_{\epsilon_k(n)} &= \int_\Omega A_\epsilon \begin{pmatrix} \nabla_{X_1} w^n_{\epsilon_k(n)} \\ \nabla_{X_2}(w^n_{\epsilon_k(n)} - w^n_0) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} w^n_{\epsilon_k(n)} \\ \nabla_{X_2}(w^n_{\epsilon_k(n)} - w^n_0) \end{pmatrix} dx + \beta \int_\Omega \left| w^n_{\epsilon_k(n)} - w^n_0 \right|^2 dx \\
&= \int_\Omega B(\Delta_n(\phi u_{\epsilon_k(n)}))\ w^n_{\epsilon_k(n)} dx - \int_\Omega \epsilon_k(n) A_{12} \nabla_{X_2} w^n_0 \cdot \nabla_{X_1} w^n_{\epsilon_k(n)} dx - 2\beta \int_\Omega w^n_{\epsilon_k(n)} w^n_0 dx \\
&- \int_\Omega \epsilon_k(n) A_{21} \nabla_{X_1} w^n_{\epsilon_k(n)} \cdot \nabla_{X_2} w^n_0) dx - \int_\Omega A_{22} \nabla_{X_2} w^n_0 \cdot \nabla_{X_2} w^n_{\epsilon_k(n)} dx \\
&\qquad - \int_\Omega A_{22} \nabla_{X_2} w^n_{\epsilon_k(n)} \cdot \nabla_{X_2} w^n_0) dx + \int_\Omega B(\Delta_n(\phi u_0)) w^n_0 dx
\end{aligned}$$

Let $\epsilon_k(n) \to 0$ and using (29) we get

$$\begin{aligned}
\lim_{\epsilon_k(n)\to 0} I^n_{\epsilon_k(n)} = \lim_{\epsilon_k(n)\to 0} \left(\int_\Omega B(\Delta_n(\phi u_{\epsilon_k(n)}))\ w^n_{\epsilon_k(n)} dx + \int_\Omega B(\Delta_n(\phi u_0)) w^n_0 dx \right) - 2\beta \int_\Omega |w^n_0|^2 \, dx \\
- 2 \int_\Omega A_{22} \nabla_{X_2} w^n_0 \cdot \nabla_{X_2} w^n_0 dx
\end{aligned}$$

Since $B(\Delta_n(\phi u_{\epsilon_k(n)})) \to B(\Delta_n(\phi u_0))$ in $L^2(\Omega)$ strongly and $w^n_{\epsilon_k(n)} \rightharpoonup w^n_0$ weakly then

$$\int_\Omega B(\Delta_n(\phi u_{\epsilon_k(n)}))\ w^n_{\epsilon_k(n)} \to \int_\Omega B(\Delta_n(\phi u_0))\ w^n_0$$

Whence by (31) we get $\lim_{\epsilon_i(n)\to 0} I^n_{\epsilon_i(n)} = 0$. Now using ellipticity assumption (2) we derive

$$\lambda \epsilon_i(n)^2 \int_\Omega \left| \nabla_{X_1} w^n_{\epsilon_i(n)} \right|^2 + \lambda \int_\Omega \left| \nabla_{X_2}(w^n_{\epsilon_i(n)} - w^n_0) \right|^2 \le I^n_{\epsilon_i(n)},$$

and therefore we get

$$\left\| \nabla_{X_2}(w^n_{\epsilon_i(n)} - w^n_0 \right\|_{L^2(\Omega)} \to 0 \text{ as } \epsilon_i(n) \to 0,$$

According to **Proposition 2** we have for a.e X_1, $w_0^n(X_1,.) \in H_0^1(\omega_2)$ and

$$\begin{aligned} \left\|\nabla_{X_2}(w^n_{\epsilon_i(n)} - w_0^n)\right\|_{L^2(\Omega)} &\to 0, \ \left\|w^n_{\epsilon_i(n)} - w_0^n\right\|_{L^2(\Omega)} \to 0 \\ \text{as } \epsilon_i(n) &\to 0, \end{aligned}$$

By (30) and **Proposition 3** we show that for every n fixed, w_0^n is the unique function which satisfies for a.e X_1

$$\int_{\omega_2} A_{22}\nabla_{X_2} w_0^n(X_1,.)\cdot\nabla_{X_2}\varphi dX_2 + \beta\int_{\omega_2} w_0^n(X_1,.)\varphi \, dX_2 = \int_{\omega_2} B(\Delta_n(\phi u_{0)}))(X_1,.)\varphi dX_2, \quad \forall\varphi\in\mathcal{D}(\omega_2)$$

Since the union of zero measure sets is a zero measure set then we have for a.e X_1 and $\forall n \in \mathbb{N}^*$

$$\begin{aligned} &\int_{\omega_2} A_{22}\nabla_{X_2} w_0^n(X_1,.)\cdot\nabla_{X_2}\varphi dX_2 + \beta\int_{\omega_2} w_0^n(X_1,.)\varphi dX_2 \qquad (32)\\ = &\int_{\omega_2} B(\Delta_n(\phi u_{0)}))(X_1,.)\varphi dX_2, \quad \forall\varphi\in\mathcal{D}(\omega_2) \end{aligned}$$

And finally, the uniqueness of w_0^n implies that the whole sequence (w_ϵ^n) converges i.e $\forall n \in \mathbb{N}^*$:

$$\|\nabla_{X_2}(w_\epsilon^n - w_0^n)\|_{L^2(\Omega)} \to 0, \text{ and } \|w_\epsilon^n - w_0^n\|_{L^2(\Omega)} \to 0 \text{ as } \epsilon \to 0$$

□

5.2.2. ***Passage to the limit*** $n \to \infty$. For a.e X_1 and $\forall n \in \mathbb{N}^*$ taking $\varphi = w_0^n(X_1,.) \in H_0^1(\omega_2)$ in (32), using ellipticity assumption (2) and Hölder's inequality we get

$$\lambda\int_{\omega_2} |\nabla_{X_2} w_0^n(X_1,.)|^2 \, dX_2 \le \|B(\Delta_n(\phi u_0))(X_1,.)\|_{L^2(\omega_2)} \|w_0^n(X_1,.)\|_{L^2(\omega_2)}$$

and Poincaré's inequality in the X_2 direction gives

$$\begin{aligned} \|\nabla_{X_2} w_0^n(X_1,.)\|_{L^2(\omega_2)} &\le \frac{C'}{\lambda}\|B(\Delta_n(\phi u_0))(X_1,.)\|_{L^2(\omega_2)} \\ \|w_0^n(X_1,.)\|_{L^2(\omega_2)} &\le \frac{C'^2}{\lambda}\|B(\Delta_n(\phi u_0))(X_1,.)\|_{L^2(\omega_2)} \end{aligned}$$

integrating over ω_1 yields

$$\begin{aligned} \|\nabla_{X_2} w_0^n\|_{L^2(\Omega)} &\le \frac{C'}{\lambda}\|B(\Delta_n(\phi u_0))\|_{L^2(\Omega)}, \\ \|w_0^n\|_{L^2(\Omega)} &\le \frac{C'^2}{\lambda}\|B(\Delta_n(\phi u_0))\|_{L^2(\Omega)}, \end{aligned}$$

and by (4) and Holder's inequality we obtain

$$\begin{aligned}\left\|\nabla_{X_2} w_0^n\right\|_{L^2(\Omega)} &\leq \frac{C' \left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}} M\left(+\left\|\phi\right\|_\infty \left\|u_0\right\|_{L^2}\right)}{\lambda},\\ \left\|w_0^n\right\|_{L^2(\Omega)} &\leq \frac{C'^2 \left|\Omega\right|^{\frac{1}{2}-\frac{1}{r}} M\left(+\left\|\phi\right\|_\infty \left\|u_0\right\|_{L^2}\right)}{\lambda},\end{aligned}$$

(we used the inequality $\left\|\Delta_n(\phi u_0)\right\|_{L^2(\Omega)} \leq \left\|\phi u_0\right\|_{L^2(\Omega)}$ and the notation $\left\|\phi\right\|_\infty = \sup_{x\in\Omega} |\phi(x)|$)

Whence, it follows by weak compacity that there exists $w_0 \in L^2(\Omega)$ and a subsequence noted always (w_0^n) such that

$$\nabla_{X_2} w_0^n \rightharpoonup \nabla_{X_2} w_0 \text{ and } w_0^n \rightharpoonup w_0 \text{ in } L^2(\Omega)$$

Remark that $\phi u_0 \in H_0^1(\Omega)$ by **Lemma2**, then by **Theorem 7** $\Delta_n(\phi u_0) \to \phi u_0$ in $L^2(\Omega)$ and therefore, continuity of B gives $B(\Delta_n(\phi u_0)) \to B(\phi u_0)$ in $L^2(\Omega)$. Now, let $n \to \infty$ in (30) yields

$$\int_\Omega A_{22} \nabla_{X_2} w_0 \cdot \nabla_{X_2} \varphi dx + \beta \int_\Omega w_0 \varphi dx = \int_\Omega B(\phi u_0)\varphi dx, \quad \forall \varphi \in \mathcal{D}(\Omega) \tag{33}$$

Take $\varphi = w_\epsilon^n \in H_0^1(\Omega)$ in (33) and let $\epsilon \to 0$ we obtain (by **Proposition 7)**

$$\int_\Omega A_{22} \nabla_{X_2} w_0 \cdot \nabla_{X_2} w_0^n dx + \beta \int_\Omega w_0 w_0^n dx = \int_\Omega B(\phi u_0) w_0^n dx,$$

and as $n \to \infty$ we derive

$$\int_\Omega A_{22} \nabla_{X_2} w_0 \cdot \nabla_{X_2} w_0 dx + \beta \int_\Omega \left|w_0\right|^2 dx = \int_\Omega B(\phi u_0) w_0 dx \tag{34}$$

Now, we prove the strong convergences of w_0^n and $\nabla_{X_2} w_0^n$, by ellipticity assumption (2), (31) and (34) we get

$$\begin{aligned}&\lambda \int_\Omega \left|\nabla_{X_2}(w_0^n - w_0)\right|^2 dx + \beta \int_\Omega \left|w_0^n - w_0\right|^2 dx\\ &\leq \int_\Omega A_{22} \nabla_{X_2}(w_0^n - w_0) \cdot \nabla_{X_2}(w_0^n - w_0) dx + \beta \int_\Omega \left|w_0^n - w_0\right|^2 dx\\ &= \int_\Omega B(\Delta_n(\phi u_0)) w_0^n dx \; - \int_\Omega A_{22} \nabla_{X_2} w_0 \cdot \nabla_{X_2} w_0^n dx - 2\beta \int_\Omega w_0^n w_0 dx\\ &\qquad\qquad - \int_\Omega A_{22} \nabla_{X_2} w_0^n \cdot \nabla_{X_2} w_0 dx + \int_\Omega B(\phi u_0) w_0 dx\end{aligned}$$

Since $B(\Delta_n(\phi u_0)) \to B(\phi u_0)$ in $L^2(\Omega)$ and $w_0^n \rightharpoonup w_0$ in $L^2(\Omega)$ then

$$\int_\Omega B(\Delta_n(\phi u_0)) w_0^n \to \int_\Omega B(\phi u_0) w_0$$

Let $n \to \infty$ in the previous inequality we get

$$\nabla_{X_2} w_0^n \to \nabla_{X_2} w_0 \text{ in } L^2(\Omega) \tag{35}$$

Finally by (35), **Proposition 6** and **Proposition 7** and the triangular inequality we get $\nabla_{X_2} w_\epsilon \to \nabla_{X_2} w_0$ in $L^2(\Omega)$ and therefore (34), **Proposition 2** and **3** complete the proof.

Remark 1. *In addition to convergences given in **Theorem 3** we also have $\epsilon_k u_{\epsilon_k} \to 0$ in $L^2(\Omega)$ strongly, indeed ellipticity assumption gives*

$$\begin{aligned}
&\lambda {\epsilon_k}^2 \int_\Omega |\nabla_{X_1} u_{\epsilon_k}|^2 + \lambda \int_\Omega |\nabla_{X_2}(u_{\epsilon_k} - u_0)|^2 \\
&\quad \le \int_\Omega A_\epsilon \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2}(u_{\epsilon_k} - u_0) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2}(u_{\epsilon_k} - u_0) \end{pmatrix} dx + \beta \int_\Omega |u_{\epsilon_k} - u_0|^2 \, dx,
\end{aligned}$$

and we can prove easily that the right-hand side of this inequality converges to 0.

6. Some Applications

6.1. **A regularity result and rate of convergence.** In this subsection we make some additional assumptions, suppose that for every $u \in L^2(\Omega)$,

$$\nabla_{X_1} B(u) \in L^2(\Omega), \tag{36}$$

and for every $\rho \in \mathcal{D}(\omega_1)$ and $u, v \in L^2(\Omega)$ we have

$$\|\rho B(u) - \rho B(v)\|_{L^2(\Omega)} \le \|B(\rho u) - B(\rho v)\|_{L^2(\Omega)} \tag{37}$$

Remark that **Theorem 3** of section 1 gives only $H^1_{loc}-$ regularity for u_0, however we have the following

Proposition 8. *Under assumptions of **Theorem 3** and (36) we have $u_0 \in H^1(\Omega)$,*

Proof. We will proceed as in [2], let $\omega_1' \subset\subset \omega_1$, for $0 < h < d(\omega_1', \omega_1)$, $X_1 \in \omega_1'$ we set $\tau_h^i u_0(X_1, X_2) = u_0(X_1 + he_i, X_2)$ $i = 1, ..., p$. From (9) we have

$$\begin{aligned}
&\int_{\omega_2} \tau_h^i A_{22} \nabla_{X_2}(\tau_h^i u_0 - u_0) \nabla_{X_2} \varphi dX_2 + \int_{\omega_2} (\tau_h^i A_{22} - A_{22}) \nabla_{X_2} u_0 \nabla_{X_2} \varphi dX_2 \\
&\qquad + \beta \int_{\omega_2} (\tau_h^i u_0 - u_0) \varphi dX_2 = \int_{\omega_2} \left\{ \tau_h^i B(u_0) - B(u_0) \right\} \varphi dX_2
\end{aligned}$$

Taking $\varphi = \frac{\tau_h^i u_0 - u_0}{h^2}$ as a test function, using ellipticity assumption (2) and Hölder's inequality we derive

$$\begin{aligned}
&\lambda \left\| \nabla_{X_2} \left(\frac{\tau_h^i u_0 - u_0}{h} \right) \right\|^2_{L^2(\omega_2)} \le \\
&\left\| \left(\frac{\tau_h^i A_{22} - A_{22}}{h} \right) \nabla_{X_2} u_0 \right\|_{L^2(\omega_2)} \left\| \nabla_{X_2} \left(\frac{\tau_h^i u_0 - u_0}{h} \right) \right\|_{L^2(\omega_2)} \\
&\qquad + \left\| \left(\frac{\tau_h^i B(u_0) - B(u_0)}{h} \right) \right\|_{L^2(\omega_2)} \left\| \left(\frac{\tau_h^i u_0 - u_0}{h} \right) \right\|_{L^2(\omega_2)}
\end{aligned}$$

Using Poincaré's inequality we deduce

$$\left\|\frac{\tau_h^i u_0 - u_0}{h}\right\|_{L^2(\omega_2)} \leq \frac{C}{\lambda}\left\{\begin{array}{c} \left\|\frac{\tau_h^i A_{22}-A_{22}}{h}\right\|_{L^\infty(\omega_2)} \|\nabla_{X_2} u_0\|_{L^2(\omega_2)} \\ + \left\|\frac{\tau_h^i B(u_0)-B(u_0)}{h}\right\|_{L^2(\omega_2)} \end{array}\right\}$$

Using regularity assumption (8) and integrating over ω_1' (we use only the assumption $\partial_k A_{22} \in L^2(\Omega)$) we deduce

$$\left\|\frac{\tau_h^i u_0 - u_0}{h}\right\|_{L(\omega_1'\times\omega_2)} \leq C' + \left\|\left(\frac{\tau_h^i B(u_0) - B(u_0)}{h}\right)\right\|_{L^2(\omega_1'\times\omega_2)}$$

Thanks to regularity of $B(u_0)$ in the X_1 direction (assumption (36)) we get

$$\left\|\frac{\tau_h^i u_0 - u_0}{h}\right\|_{L^2(\omega_1'\times\omega_2)} \leq C'',$$

where C'' is independent on h , whence $\nabla_{X_1} u_0 \in L^2(\Omega)$ and the proof is finished. □

Now, we give a result on the rate of convergence

Proposition 9. *Under assumptions of **Theorem 3** and (36), (37), for $\beta > \max(K, \beta_0)$ (where $\beta_0 > M\,|\Omega|^{\frac{1}{2}-\frac{1}{r}}$ (fixed), and K is the Lipschitz constant of B associated with the bounded set $\left\{\|u\|_{L^2} \leq \frac{M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}{\beta_0 - M|\Omega|^{\frac{1}{2}-\frac{1}{r}}}\right\}$), we have $u_\epsilon \to u_0$ in W and*

$$\|u_\epsilon - u_0\|_{L^2(\omega_1'\times\omega_2)}\,;\ \ \|\nabla_{X_1}(u_\epsilon - u_0)\|_{L^2(\omega_1'\times\omega_2)} \leq C'\epsilon$$

where $C \geq 0$ is independent of ϵ.

Proof. To make calculus easier we suppose that $A_{12}, A_{21} = 0,\ A_{11}, A_{22} = I$. According to **Theorem 3** the set of solutions to (9) is non empty, and we show easily that (9) has a unique solution (thanks to assumption $\beta > \max(K, \beta_0)$), consequently **Corollary 1** implies $u_\epsilon \to u_0$ in W.

From (6) and (9) we have

$$\epsilon^2 \int_\Omega \nabla_{X_1} u_\epsilon \nabla_{X_1} \varphi dx + \int_\Omega \nabla_{X_2}(u_\epsilon - u_0)\nabla_{X_2}\varphi dx + \beta \int_\Omega (u_\epsilon - u_0)\varphi dx = \int_\Omega (B(u_\epsilon) - B(u_0))\varphi dx$$

Given $\omega_1' \subset\subset \omega_1^{''} \subset\subset \omega_1$, and let ρ be a cut-off function with $Supp(\rho) \subset \omega_1^{''}$ and $\rho = 1$ on ω_1'(we can choose $0 \leq \rho \leq 1$). We introduce the test function used by M.Chipot and S.Guesmia in [2], $\varphi = \rho^2(u_\epsilon - u_0) \in H_0^1(\Omega)$ (thanks to the previous proposition). Testing with φ we obtain

$$\begin{aligned} &\epsilon^2 \int_\Omega \nabla_{X_1} u_\epsilon \nabla_{X_1} \rho^2 (u_\epsilon - u_0) dx \\ &+ \int_\Omega \nabla_{X_2}(u_\epsilon - u_0)\nabla_{X_2}\rho^2(u_\epsilon - u_0)dx + \beta \int_\Omega \rho^2 (u_\epsilon - u_0)^2 dx \\ &\qquad = \int_\Omega (B(u_\epsilon) - B(u_0))\rho^2(u_\epsilon - u_0)dx \end{aligned}$$

we deduce

$$\epsilon^2\int_\Omega |\rho\nabla_{X_1}(u_\epsilon-u_0)|^2\,dx+\int_\Omega |\rho\nabla_{X_2}(u_\epsilon-u_0)|^2\,dx$$
$$+\beta\int_\Omega \rho^2(u_\epsilon-u_0)^2dx=-\epsilon^2\int_\Omega \rho^2\nabla_{X_1}u_0\nabla_{X_1}(u_\epsilon-u_0)dx-2\epsilon^2\int_\Omega (u_\epsilon-u_0)\rho\nabla_{X_1}\rho\nabla_{X_1}u_0dx$$
$$-2\epsilon^2\int_\Omega \rho(u_\epsilon-u_0)\nabla_{X_1}(u_\epsilon-u_0)\nabla_{X_1}\rho dx+\int_\Omega (B(u_\epsilon)-B(u_0))\rho^2(u_\epsilon-u_0)dx$$

Using Hölder's inequality for the first three term in the right-hand side, and assumptions (37), (3) and Hölder's inequality for the last one, we obtain

$$\epsilon^2\left\|\rho\nabla_{X_1}(u_\epsilon-u_0)\right\|^2_{L^2(\omega_1''\times\omega_2)}+\left\|\rho\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^2(\omega_1''\times\omega_2)}+$$
$$\beta\left\|\rho(u_\epsilon-u_0)\right\|^2_{L^2(\omega_1''\times\omega_2)}\le\epsilon^2\left\|\rho\nabla_{X_1}u_0\right\|_{L^2(\omega_1''\times\omega_2)}\left\|\rho\nabla_{X_1}(u_\epsilon-u_0)\right\|_{L^2(\omega_1''\times\omega_2)}$$
$$+2\epsilon^2\left\|(u_\epsilon-u_0)\nabla_{X_1}\rho\right\|_{L^2(\omega_1''\times\omega_2)}\left\|\rho\nabla_{X_1}u_0\right\|_{L^2(\omega_1''\times\omega_2)}$$
$$+\epsilon^2\left\|(u_\epsilon-u_0)\nabla_{X_1}\rho\right\|_{L^2(\omega_1''\times\omega_2)}\left\|\rho(u_\epsilon-u_0)\right\|_{L^2(\omega_1''\times\omega_2)}$$
$$+K\left\|\rho(u_\epsilon-u_0)\right\|^2_{L^2(\omega_1''\times\omega_2)},$$

(thanks to **Proposition 1,** we remark that ρu_ϵ, $\rho u_0\in\left\{\|u\|_{L^2}\le\frac{M|\Omega|^{\frac12-\frac1r}}{\beta_0-M|\Omega|^{\frac12-\frac1r}}\right\}$).

Using Young's inequality for ,the first term in the right-hand side of the previous inequality, and boundedness of (u_ϵ) for the rest, we deduce

$$\frac{\epsilon^2}{2}\left\|\rho\nabla_{X_1}(u_\epsilon-u_0)\right\|^2_{L^2(\omega_1''\times\omega_2)}+\left\|\rho\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^2(\omega_1''\times\omega_2)}$$
$$+(\beta-K)\left\|\rho(u_\epsilon-u_0)\right\|^2_{L^2(\omega_1''\times\omega_2)}\le C\epsilon^2$$

whence

$$\|u_\epsilon-u_0\|_{L^2(\omega_1'\times\omega_2)};\ \ \|\nabla_{X_2}(u_\epsilon-u_0)\|_{L^2(\omega_1'\times\omega_2)}\le C'\epsilon,$$

where C' is independent of ϵ. □

6.2. **Application to integro-differential problem.** In this section we provide some concrete examples. In [3] M. Chipot and S. Guesmia studied problem (6) with the following integral operator

$$B(u)=a\left(\int_{\omega_1}h(X_1,X_1',X_2)u(X_1',X_2)dX_1'\right)\tag{38}$$

To prove the convergence theorem the authors based their arguments on the compacity of the operator $u\to\int_{\omega_1}h(X_1,X_1',X_2)u(X_1',X_2)dX_1'$. Indeed, for a sequence $u_n\rightharpoonup u_0$ in $L^2(\Omega)$ we have $\int_{\omega_1}h(X_1,X_1',X_2)u_n(X_1',X_2)dX_1'\to\int_{\omega_1}h(X_1,X_1',X_2)u_0(X_1',X_2)dX_1'$ in $L^2(\Omega)$ (by compacity) and we use the continuity of a and Lebesgue's theorem

(under additional assumption on a) to get $a\left(\int_{\omega_1} h(X_1,X_1',X_2)u_n(X_1',X_2)dX_1'\right)$ $\to a\left(\int_{\omega_1} h(X_1,X_1',X_2)u_0(X_1',X_2)dX_1'\right)$ in $L^2(\Omega)$.

We can give another operator based on the aforementioned one

$$B(u)=\int_{\omega_1} h(X_1,X_1',X_2)a(u(X_1',X_2))dX_1', \tag{39}$$

For $a:\mathbb{R}\to\mathbb{R}$ we note a Liptchitz function i.e there exists $K\geq 0$ such that

$$\forall x,y\in\mathbb{R}: |a(x)-a(y)|\leq K|x-y| \tag{40}$$

In addition, we suppose that a satisfies the growth condition

$$\exists q\in[0,1[\,,\ M\geq 0,\ \forall x\in\mathbb{R}: |a(x)|\leq M(1+|x|^q), \tag{41}$$

and we suppose that

$$h\in L^\infty(\omega_1\times\Omega),\ \nabla_{X_1}h\in L^{\frac{2}{1-q}}(\omega_1\times\Omega) \tag{42}$$

Theorem 8. *Consider problem (6) with B given by (38) or (39). Assume (1), (2), (8), (40), (41) , (42) and for β suitably chosen, then we have the affirmations of* ***theorems 1 , 2*** *and* ***3*** *of section 1 and those of* ***propositions 8, 9***

Proof. Take B as in (39) the proof of this theorem amounts to prove that assumptions (3), (4), (5), (36) and (37) hold. (3) follows directly from (40) and (42), Now assume (41), (42) then we can check easily that (4) holds with $r=\frac{2}{q}$. It remains to prove that (5) holds. For every $u\in V$ (we can also take $u\in L^2(\Omega)$), and $\varphi\in\mathcal{D}(\Omega)$ we have for $1\leq k\leq p$

$$\begin{aligned} I(\varphi)&=\left|\int_\Omega\left(\int_{\omega_1} h(X_1,X_1',X_2)a(u(X_1',X_2))dX_1'\right)\partial_k\varphi(X_1,X_2)dX_1dX_2\right| \\ &=\left|\int_{\omega_1}\left(\int_\Omega h(X_1,X_1',X_2)\partial_k\varphi(X_1,X_2)a(u(X_1',X_2))dX_1dX_2\right)dX_1'\right| \\ &\leq\int_{\omega_1}\left|\left(\int_\Omega h(X_1,X_1',X_2)\partial_k\varphi(X_1,X_2)a(u(X_1',X_2))dX_1dX_2\right)\right|dX_1' \end{aligned}$$

Since $\partial_k h\in L^{\frac{2r}{r-2}}(\omega_1\times\Omega)$ it follows that for a.e $X_1'\in\omega_1:\partial_k\left[a(u(X_1',.))h(.,X_1',.)\right]\in L^{\frac{2r}{r-2}}(\Omega)$, integrating by part we get

$$\begin{aligned} I(\varphi)&\leq\int_{\omega_1}\left|\left(\int_\Omega\partial_k h(X_1,X_1',X_2)\varphi(X_1,X_2)a(u(X_1',X_2))dX_1dX_2\right)\right|dX_1' \\ &\leq\|a(u)\|_{L^r}|\omega_1|^{\frac{1}{2}}\|\partial_k h\|_{L^{\frac{2r}{r-2}}}\|\varphi\|_{L^2(\Omega)} \\ &\leq M'(1+\|u\|_{L^2})\|\varphi\|_{L^2(\Omega)} \end{aligned}$$

And therefore $\partial_k B(u) \in L^2(\Omega)$, whence (36) holds and we have

$$\|\nabla_{X_1} B(u)\|_{L^2} \leq M''(1 + \|u\|_{L^2}),$$

then for every L^2-bounded set $E \subset V$ we have

$$\|\nabla_{X_1} B(u)\|_{L^2} \leq M''', \ u \in E. \tag{43}$$

Now, given a sequence (U_n) in $conv(B(E))$ which converges strongly to some U_0 in $L^2(\Omega)$, by (43) and the convexity of the norm we show that $(\nabla_{X_1} U_n)_n$ is bounded in $L^2(\Omega)$, hence one can extract a subsequence (U_n) such that $(\nabla_{X_1} U_n)$ converges weakly to some c_0 in $L^2(\Omega)$, thanks to the continuity of derivation on $\mathcal{D}'(\Omega)$ which gives $c_0 = \nabla_{X_1} U_0$ and therefore, $U_0 \in V$, whence (5) follows. Finally, one can check easily that (37) holds. Same arguments when B is given by (38) □

6.3. **A generalization.** Consider (38) with

$$h \in L^\infty(\Omega), l \in L^\infty(\omega_1), \nabla_{X_1} l \in L^2(\omega_1), \tag{44}$$

the operator $u \to a\left(l(X_1)\int\limits_{\omega_1} h(X_1', X_2)u(X_1', X_2)dX_1'\right)$ belongs to a class of operators defined by

$$B(u) = a\left(lP(u)\right), \tag{45}$$

where $P : L^2(\Omega) \to L^2(\omega_2)$ is a linear bounded operator (an orthogonal projector for example). The method used by M. Chipot and S. Guesmia is not applicable here, in fact the linear operator P is not necessarily compact, for $u_n \rightharpoonup u_0$ we only have $P(u_n) \rightharpoonup P(u_0)$ weakly and therefore every subsequence $(a\left(lP(u_n)\right))$ is not necessarily convergent in $L^2(\Omega)$ strongly. However we have the following.

Theorem 9. *Consider problem (6) with B given by (45). Assume (1), (2), (8), (40), (41) and (44), then for β suitably chosen, we have affirmations of* ***Theorems 1 , 2*** *and* ***3*** *of section 1 and moreover we have $u_0 \in H^1(\Omega)$*

Proof. The proof of this theorem amounts to prove that assumptions (3), (4), (5) and (36) hold. Since P is Lipschitz then (3) follows by (40). We also can prove (4) using (41) with $r = \frac{2}{q}$.It remains to check that (5), (36) hold, for every $u \in V$ (we can take $u \in L^2(\Omega)$) we have $\nabla_{X_1} a(lP(u)) \in L^2(\Omega)$ and $\nabla_{X_1} a(lPu) = a'(lP(u))P(u)\nabla_{X_1} l$. We can show easily that $\nabla_{X_1} a(lP(E))$ is bounded for any L^2-bounded set $E \subset V$ and we finish the proof as in **Theorem 8**. □

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E-mail address: chokri.ogabi@ac-grenoble.fr