



**HAL**  
open science

# What is known about the Value 1 Problem for Probabilistic Automata?

Nathanaël Fijalkow

► **To cite this version:**

Nathanaël Fijalkow. What is known about the Value 1 Problem for Probabilistic Automata?. 2014. hal-01074111

**HAL Id: hal-01074111**

**<https://hal.science/hal-01074111>**

Preprint submitted on 13 Oct 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# What is known about the Value 1 Problem for Probabilistic Automata?

Nathanaël Fijalkow

LIAFA, Paris 7,  
University of Warsaw

**Abstract.** The value 1 problem is a decision problem for probabilistic automata over finite words: are there words accepted by the automaton with arbitrarily high probability? Although undecidable, this problem attracted a lot of attention over the last few years. The aim of this paper is to review and relate the results pertaining to the value 1 problem.

In particular, several algorithms have been proposed to partially solve this problem. We show the relations between them, leading to the following conclusion: the Markov Monoid Algorithm is the *most correct* algorithm known to (partially) solve the value 1 problem.

## 1 Introduction

In 1963 Rabin [Rab63] introduced the notion of probabilistic automata, which are finite automata with randomized transitions. This powerful model has been widely studied and has applications in many fields like image processing [CK97], computational biology [DEKM99] and speech processing [Moh97]. Several algorithmic properties of probabilistic automata have been considered in the literature. For instance, Schützenberger [Sch61] proved in 1961 that *functional equivalence* is decidable in polynomial time (see also [Tze92]), and even faster with randomized algorithms, which led to applications in software verification [KMO<sup>+</sup>11].

However, many natural decision problems are undecidable, and part of the literature on probabilistic automata is about *undecidability results*. For example the *emptiness*, the *isolation* and the *value 1* problems are undecidable, as shown in [Paz71,BMT77,GO10]. To overcome untractability results, a lot of effort went into finding subclasses of probabilistic automata for which natural decision problems become decidable. For instance, the papers [KVAK10,CKV<sup>+</sup>11] look at restrictions implying a decidable model-checking problem against  $\omega$ -regular specifications, and the paper [CSV13] investigates whether assuming isolated cut-points leads to decidability for the emptiness problem.

We focus here on the efforts made to understand the value 1 problem. The aim of this paper is to review and relate the attempts made in this direction over the last few years [GO10,CSV11,FGO12,CT12,BBG12,FGKO14].

## 2 Definitions

Let  $Q$  be a finite set of states. A probability distribution over  $Q$  is a function  $\delta : Q \rightarrow [0, 1]$  such that  $\sum_{q \in Q} \delta(q) = 1$ .

Let  $A$  be a finite alphabet. The transitions of a probabilistic automaton are given by a function  $\Delta : Q \times A \rightarrow \mathcal{D}(Q)$ ; equivalently, for each letter  $a \in A$  we consider a probabilistic transition matrix  $M_a$ , which is a square matrix in  $[0, 1]^{Q \times Q}$  such that every row of  $M_a$  is a probability distribution over  $Q$ . The value of  $M_a(s, t)$  is the probability to go from state  $s$  to state  $t$  when reading the letter  $a$ .

Given an input word  $w \in A^*$ , we denote  $\mathbb{P}_{\mathcal{A}}(s \xrightarrow{w} t)$  the probability to go from state  $s$  to state  $t$  when reading the word  $w$ . Formally, if  $w = a_1 a_2 \cdots a_n$  then  $\mathbb{P}_{\mathcal{A}}(s \xrightarrow{w} t) = (M_{a_1} M_{a_2} \cdots M_{a_n})(s, t)$ .

**Definition 1 (Probabilistic automaton).** A tuple  $\mathcal{A} = (Q, A, q_0, \Delta, F)$  represents a probabilistic automaton, where  $Q$  is a finite set of states,  $A$  is the finite input alphabet,  $q_0 \in Q$  is the initial state,  $\Delta$  define the transitions and  $F \subseteq Q$  is the set of accepting states.

**Definition 2 (Acceptance probability).** The acceptance probability of a word  $w \in A^*$  by  $\mathcal{A}$  is  $\sum_{f \in F} \mathbb{P}_{\mathcal{A}}(q_0 \xrightarrow{w} f)$ , denoted  $\mathbb{P}_{\mathcal{A}}(w)$ .

**Definition 3 (Value).** The value of  $\mathcal{A}$ , denoted  $\text{val}(\mathcal{A})$ , is the supremum acceptance probability over all possible input words:

$$\text{val}(\mathcal{A}) = \sup_{w \in A^*} \mathbb{P}_{\mathcal{A}}(w) . \quad (1)$$

We are interested in the following decision problem:

Given a probabilistic automaton  $\mathcal{A}$ , decide whether  $\text{val}(\mathcal{A}) = 1$ .

## 3 An Equivalent Formulation and the Exact Computational Complexity

The first result about the value 1 problem is its surprising undecidability, obtained with an elementary proof by Hugo Gimbert and Youssouf Oualhadj in [GO10].

In a related yet seemingly different line of work, Christel Baier, Marcus Größer and Nathalie Bertrand undertook a thorough study of probabilistic Büchi automata [BG05,BBG08,BBG09,BBG12]. One of the results obtained there is the undecidability of the emptiness problem for probabilistic Büchi automata with probable semantics. It turns out that the two problems are actually Turing-equivalent:

- the value 1 problem for probabilistic automata over finite words,
- the emptiness problem for probabilistic Büchi automata with probable semantics.

A first (very simple) reduction has been explained in [BBG12]: from a probabilistic automaton  $\mathcal{A}$  over finite words, one can construct a probabilistic Büchi automaton  $\mathcal{A}'$  of linear size, such that  $\text{val}(\mathcal{A}) = 1$  if and only if  $\mathcal{A}'$  is non-empty for the probable semantics. The converse reduction is more involved, and follows from [CSV13], but here the constructed automaton is of exponential size.

Even better, the exact computational complexity has been given in [CSV13]: both problems are  $\Sigma_2^0$ -complete.

**Theorem 1 ([BBG12,CSV13]).** *The value 1 problem for probabilistic automata over finite words and the emptiness problem for probabilistic Büchi automata with probable semantics are Turing-equivalent and  $\Sigma_2^0$ -complete.*

## 4 Decidable Subclasses of Probabilistic Automata

Several subclasses of probabilistic automata were constructed in order to decide the value 1 problem on such instances.

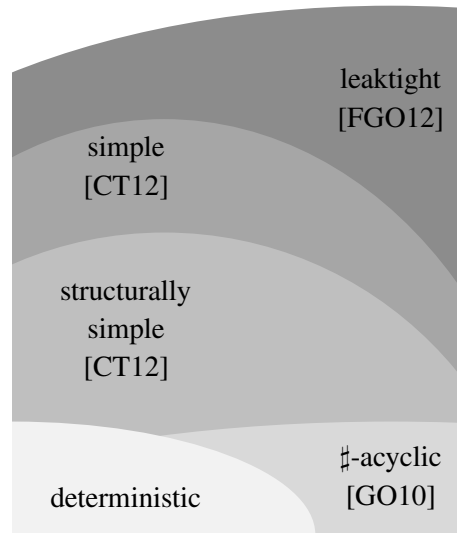
The first class was the #-acyclic automata by Gimbert and Oualhadj [GO10].

Later but concurrently, two different works have been published in the very same conference. The first one introduces simple automata and structurally simple automata, by Krishnendu Chatterjee and Mathieu Tracol [CT12]. The second, by Hugo Gimbert, Youssouf Oualhadj and the author introduces leaktight automata [FGO12].

Although geared towards the same goal (deciding the value 1 problem), the two classes came from different perspectives. The paper of Krishnendu Chatterjee and Mathieu Tracol relies on a theorem from Probability Theory, called the jet decompositions of (infinite) Markov Chains. The paper of Hugo Gimbert, Youssouf Oualhadj and the author relies on a theorem from Algebra, called Simon's theorem, asserting the existence of factorization trees of bounded height.

Subsequent studies [FGKO14] showed that the class of leaktight automata actually strictly contains all the other classes, implying that the Markov Monoid

Algorithm used to decide the value 1 problem for leaktight automata actually decides the value 1 problem for all cases where it is known to be decidable.



## Conclusion and Perspectives

In this paper, we discussed some recent developments about the value 1 problem. We first gathered some results from the literature, explaining that it is actually Turing-equivalent to the emptiness for probabilistic Büchi automata with the probable semantics, and  $\Sigma_2^0$ -complete. Then we presented the different attempts to decide the value 1 problems on subclasses of probabilistic automata. As a conclusion, the Markov Monoid Algorithm introduced in [FGO12], used to decide the value 1 problem for leaktight automata, is actually the *most correct* algorithm known so far, as the class of leaktight automata strictly contains all other classes for which the value 1 problem is known to be decidable.

This motivates a deeper understanding of this algorithm. We know that the Markov Monoid Algorithm cannot solve the value 1 problem, as this problem is undecidable, but then what is the problem solved by this algorithm? In other words, can we characterize for which probabilistic automata the Markov Monoid Algorithm finds a value 1 witness?

## References

- [BBG08] Christel Baier, Nathalie Bertrand, and Marcus Größer. On decision problems for probabilistic büchi automata. In Roberto M. Amadio, editor, *FoSSaCS*, volume 4962 of *Lecture Notes in Computer Science*, pages 287–301. Springer, 2008.
- [BBG09] Christel Baier, Nathalie Bertrand, and Marcus Größer. Probabilistic acceptors for languages over infinite words. In Mogens Nielsen, Antonín Kucera, Peter Bro Miltersen, Catuscia Palamidessi, Petr Tuma, and Frank D. Valencia, editors, *SOFSEM*, volume 5404 of *Lecture Notes in Computer Science*, pages 19–33. Springer, 2009.
- [BBG12] Christel Baier, Nathalie Bertrand, and Marcus Größer. Probabilistic  $\omega$ -automata. *Journal of the ACM*, 59(1):1, 2012.
- [BG05] Christel Baier and Marcus Größer. Recognizing omega-regular languages with probabilistic automata. In *LICS*, pages 137–146. IEEE Computer Society, 2005.
- [BMT77] Alberto Bertoni, Giancarlo Mauri, and Mauro Torelli. Some recursive unsolvable problems relating to isolated cutpoints in probabilistic automata. In *International Colloquium on Automata, Languages and Programming*, pages 87–94, 1977.
- [CK97] Karel Culik and Jarkko Kari. *Digital images and formal languages*, pages 599–616. Springer-Verlag New York, Inc., 1997.
- [CKV<sup>+</sup>11] Rohit Chadha, Vijay Anand Korthikanti, Mahesh Viswanathan, Gul Agha, and YoungMin Kwon. Model checking MDPs with a unique compact invariant set of distributions. In *QEST*, pages 121–130. IEEE Computer Society, 2011.
- [CSV11] Rohit Chadha, A. Prasad Sistla, and Mahesh Viswanathan. Power of randomization in automata on infinite strings. *Logical Methods in Computer Science*, 7(3), 2011.
- [CSV13] Rohit Chadha, A. Prasad Sistla, and Mahesh Viswanathan. Probabilistic automata with isolated cut-points. In Krishnendu Chatterjee and Jiri Sgall, editors, *MFCS*, volume 8087 of *Lecture Notes in Computer Science*, pages 254–265. Springer, 2013.
- [CT12] Krishnendu Chatterjee and Mathieu Tracol. Decidable problems for probabilistic automata on infinite words. In *Logics in Computer Science*, 2012.
- [DEKM99] Richard Durbin, Sean R. Eddy, Anders Krogh, and Graeme Mitchison. *Biological Sequence Analysis: Probabilistic Models of Proteins and Nucleic Acids*. Cambridge University Press, July 1999.
- [FGKO14] Nathanaël Fijalkow, Hugo Gimbert, Edon Kelmendi, and Youssef Oualhadj. Deciding the value 1 problem for probabilistic leaktight automata. 2014.
- [FGO12] Nathanaël Fijalkow, Hugo Gimbert, and Youssef Oualhadj. Deciding the value 1 problem for probabilistic leaktight automata. In *Logics in Computer Science*, pages 295–304, 2012.
- [GO10] Hugo Gimbert and Youssef Oualhadj. Probabilistic automata on finite words: Decidable and undecidable problems. In *International Colloquium on Automata, Languages and Programming*, pages 527–538, 2010.
- [KMO<sup>+</sup>11] Stefan Kiefer, Andrzej S. Murawski, Joël Ouaknine, Björn Wachter, and James Worrell. Language equivalence for probabilistic automata. In Ganesh Gopalakrishnan and Shaz Qadeer, editors, *CAV*, volume 6806 of *Lecture Notes in Computer Science*, pages 526–540. Springer, 2011.
- [KVAK10] Vijay Anand Korthikanti, Mahesh Viswanathan, Gul Agha, and YoungMin Kwon. Reasoning about MDPs as transformers of probability distributions. In *QEST*, pages 199–208. IEEE Computer Society, 2010.
- [Moh97] Mehryar Mohri. Finite-state transducers in language and speech processing. *Computational Linguistics*, 23:269–311, June 1997.
- [Paz71] Azaria Paz. *Introduction to probabilistic automata*. Academic Press, 1971.

- [Rab63] Michael O. Rabin. Probabilistic automata. *Information and Control*, 6(3):230–245, 1963.
- [Sch61] Marcel-Paul Schützenberger. On the definition of a family of automata. *Information and Control*, 4, 1961.
- [Tze92] Wen-Guey Tzeng. A polynomial-time algorithm for the equivalence of probabilistic automata. *SIAM Journal on Computing*, 21(2):216–227, 1992.