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► **To cite this version:**

Cyril Imbert, Régis Monneau. Quasi-convex Hamilton-Jacobi equations posed on junctions: the multi-dimensional case. 2014. hal-01073954v1

**HAL Id: hal-01073954**

**<https://hal.science/hal-01073954v1>**

Preprint submitted on 10 Oct 2014 (v1), last revised 25 Aug 2017 (v3)

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# Quasi-convex Hamilton-Jacobi equations posed on junctions: the multi-dimensional case

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October 10, 2014

## Abstract

A *multi-dimensional junction* is the singular  $d$ -manifold obtained by gluing through their boundaries a finite number of copies of a half-hyperplane of  $\mathbb{R}^{d+1}$ . We show that the general theory developed by the authors (2013) for the network setting can be easily adapted to this multi-dimensional case. In particular, we prove that general junction conditions reduce to flux-limited ones and that uniqueness holds true when flux limiters are quasi-convex and continuous. The proof of the comparison principle relies on the construction of a (multi-dimensional) vertex test function.

**AMS Classification:** 35F21, 49L25, 35B51.

**Keywords:** Quasi-convex Hamilton-Jacobi equations, multi-dimensional junctions, flux-limited solutions, flux limiters, comparison principle, multi-dimensional vertex test function, discontinuous Hamiltonians

## 1 Introduction

This paper is concerned with extending the theory developed for Hamilton-Jacobi (HJ) equations posed on junctions in [3] to the multi-dimensional setting.

A *multi-dimensional junction* is made of  $N$  copies of  $\mathbb{R}_+^{d+1}$  glued through their boundaries.

$$J = \bigcup_{i=1, \dots, N} J_i \quad \text{with} \quad \begin{cases} J_i = \{X = (x', x_i) : x' \in \mathbb{R}^d, x_i \geq 0\} \simeq \mathbb{R}_+^{d+1} \\ J_i \cap J_j = \Gamma \simeq \mathbb{R}^d \quad \text{for } i \neq j. \end{cases} \quad (1.1)$$

We emphasize that the common boundary of the hyperspaces  $J_i$  is denoted by  $\Gamma$ . For points  $X, Y \in J$ ,  $d(X, Y)$  denotes  $|x' - y'| + d(x, y)$  with

$$d(x, y) = \begin{cases} x + y & \text{if } X \in J_i, Y \in J_j, i \neq j \\ |x - y| & \text{if } X, Y \in J_i. \end{cases}$$

For a smooth real-valued function  $u$  defined on  $J$ ,  $\partial_i u(X)$  denotes the (spatial) derivative of  $u$  with respect to  $x_i$  at  $X = (x', x_i) \in J_i$  and  $D'u(X)$  denotes the (spatial) gradient of  $u$  with respect to  $x'$ . The “gradient” of  $u$  is defined as follows,

$$Du(X) := \begin{cases} (D'u(X), \partial_i u(X)) & \text{if } X \in J_i \setminus \Gamma, \\ (D'u(x', 0), \partial_1 u(x', 0), \dots, \partial_N u(x', 0)) & \text{if } X \in \Gamma. \end{cases} \quad (1.2)$$

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With such a notation in hand, we consider the following Hamilton-Jacobi equation posed on the multi-dimensional junction  $J$

$$\begin{cases} u_t + H_i(Du) = 0 & t > 0, X \in J_i \setminus \Gamma, \\ u_t + F(Du) = 0 & t > 0, X \in \Gamma \end{cases} \quad (1.3)$$

submitted to the initial condition

$$u(0, X) = u^0(X) \quad \text{for } X \in J. \quad (1.4)$$

The second equation in (1.3) is referred to as *the junction condition*.

The Hamiltonians are supposed to satisfy the following conditions:

$$\begin{cases} \text{(Continuity)} & H_i \in C(\mathbb{R}^{d+1}) \\ \text{(Quasi-convexity)} & \forall \lambda, \{H_i \leq \lambda\} \text{ is convex} \\ \text{(Coercivity)} & \lim_{|P| \rightarrow +\infty} H_i(P) = +\infty. \end{cases} \quad (1.5)$$

We next define the  $A$ -limited flux function  $F_A$  associated with the multi-dimensional junction  $J$ . In order to do so, we first consider  $\pi_i^0(p') \in \mathbb{R}$  such that  $p_i \mapsto H_i(p', p_i)$  reaches its minimum at  $p_i = \pi_i^0(p')$  and  $H_i^-$  is defined by

$$H_i^-(p', p_i) = \begin{cases} H_i(p', p_i) & \text{if } p_i \leq \pi_i^0(p'), \\ H_i(p', \pi_i^0(p')) & \text{if } p_i > \pi_i^0(p'). \end{cases}$$

So-called *flux-limiter functions*  $A: \mathbb{R}^d \rightarrow \mathbb{R}$  are always assumed to be continuous and, in some important cases, to satisfy the following condition,

$$A: \mathbb{R}^d \rightarrow \mathbb{R} \text{ is continuous and quasi-convex.} \quad (1.6)$$

The function  $F_A$  is defined for  $p = (p_1, \dots, p_N)$  and  $P = (p', p)$  as

$$F_A(P) = \max \left( A(p'), \max_{i=1, \dots, N} H_i^-(p', p_i) \right). \quad (1.7)$$

We now consider the following important special case of (1.3),

$$\begin{cases} u_t + H_i(Du) = 0 & t > 0, X \in J_i \setminus \Gamma, \\ u_t + F_A(Du) = 0 & t > 0, X \in \Gamma. \end{cases} \quad (1.8)$$

We point out that  $A$  could be replaced with  $\max(A, A_0)$  where

$$A_0(p') = \max_{i=1, \dots, N} A_i(p') \quad \text{with} \quad A_i(p') = \min_{p_i \in \mathbb{R}} H_i(p', p_i). \quad (1.9)$$

We notice (see Lemma A.1 in Appendix) that the functions  $A_i$ ,  $i = 0, \dots, N$  are quasi-convex, continuous and coercive.

As far as general junction conditions are concerned, we assume that the junction function  $F: \mathbb{R}^d \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies

$$\begin{cases} \text{(Continuity)} & F \in C(\mathbb{R}^d \times \mathbb{R}^N) \\ \text{(Monotonicity)} & \forall i, p_i \mapsto F(p', p_1, \dots, p_N) \text{ is non-increasing} \end{cases} \quad (1.10)$$

and, in some important cases,

$$\text{(Quasi-convexity)} \quad \forall \lambda, \{F \leq \lambda\} \text{ convex.} \quad (1.11)$$

## 1.1 Main results

**Theorem 1.1** (General junction conditions reduce to  $F_A$ ). *Let the Hamiltonians satisfy (1.5) and let  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  satisfy (1.10). There exists a unique coercive continuous function  $A_F : \mathbb{R}^d \rightarrow \mathbb{R}$  such that every relaxed viscosity sub-solution (resp. super-solution) of (1.3) is a  $A_F$ -flux limited sub-solution (resp. super-solution) of (1.8). Moreover, if  $F$  is quasi-convex, so is  $A_F$ .*

*Remark 1.2.* Let  $p_i^0 \geq \pi_i^0(p')$  be minimal such that  $H_i(p', p_i) = A_0$  and let  $p^0$  denote  $(p_1^0, \dots, p_N^0)$ . The function  $A_F$  is defined as follows: for all  $p' \in \mathbb{R}^d$ , if  $F(p', p^0) < A_0(p')$ , then  $A_F(p') = A_0(p')$ , else  $A_F(p')$  is the only  $\lambda \in \mathbb{R}$  such that  $\lambda \geq A_0(p') = \max_i A_i(p')$  and there exists  $p_i^+ \geq p_i^0$  such that

$$H_i(p', p_i^+) = F(p', p^+) = \lambda$$

where  $p^+ = (p_1^+, \dots, p_N^+)$ .

**Theorem 1.3** (Comparison principle on a multi-dimensional junction). *Assume that the Hamiltonians satisfy (1.5), the junction function satisfies (1.10) and (1.11) and that the initial datum  $u_0$  is uniformly continuous. Then for all (relaxed) sub-solution  $u$  and (relaxed) super-solution  $v$  of (1.3)-(1.4) satisfying for some  $T > 0$  and  $C_T > 0$ ,*

$$u(t, X) \leq C_T(1 + d(0, X)), \quad v(t, X) \geq -C_T(1 + d(0, X)), \quad \text{for all } (t, X) \in [0, T) \times J, \quad (1.12)$$

we have

$$u \leq v \quad \text{in } [0, T) \times J.$$

## 1.2 Comparison with known results

Our results are related to [1, 2] where an optimal control problem in a two-domain setting is studied. The state of the system evolves according to two different dynamics on each side of an hypersurface. Moreover, the two dynamics at the interface corresponding to the maximal and minimal Ishii's discontinuous solutions of the associated Hamilton-Jacobi equation are identified. One of the two value functions is characterized in terms of partial differential equations. We showed in [3] that, in the one-dimensional setting, both can be characterized by using the notion of flux-limited solutions introduced in [3]. The result of the present paper indicates that such a connexion holds in the general two-domain setting. Moreover, we can deal with quasi-convex Hamiltonians instead of convex ones.

The reader is also referred to [5, 4] for optimal control problems in multi-domains. In particular, the authors impose some transmission conditions. As we already mentioned it in [3], Definition 2.4 is strongly related to these works.

**Organization of the article.** The paper is organized as follows. In Section 2, the notion of viscosity solution in the setting of multi-dimensional junction is introduced. Section 3 is devoted to the construction of the vertex test function. The proof of a technical lemma is presented in an appendix.

**Notation.** For a function  $f : D \rightarrow \mathbb{R}$ ,  $\text{epi } f$  denotes its epigraph  $\{(X, r) \in D \times \mathbb{R} : r \geq f(X)\}$  and  $\text{hypo } f$  denotes its hypograph  $\{(X, r) \in D \times \mathbb{R} : r \leq f(X)\}$ .

# 2 Viscosity solutions on a multi-dimensional junction

## 2.1 Definitions

### 2.1.1 Class of test functions

For  $T > 0$ , set  $J_T = (0, T) \times J$ . The class of test functions on  $J_T$  is chosen as follows,

$$C^1(J_T) = \{\varphi \in C(J_T), \varphi \text{ restricted to } (0, T) \times J_i \text{ is } C^1 \text{ for } i = 1, \dots, N\}.$$

### 2.1.2 Classical viscosity solutions

In order to define classical viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes  $u^*$  and  $u_*$  of a (locally bounded) function  $u$  defined on  $[0, T) \times J$ :

$$u^*(t, x) = \limsup_{(s, y) \rightarrow (t, x)} u(s, y) \quad \text{and} \quad u_*(t, x) = \liminf_{(s, y) \rightarrow (t, x)} u(s, y).$$

**Definition 2.1** (Classical viscosity solutions). Assume the Hamiltonians satisfy (1.5) and the flux function  $F$  satisfies (1.10). Let  $u : [0, T) \times J \rightarrow \mathbb{R}$  be locally bounded.

- i) We say that  $u$  is a (classical viscosity) *sub-solution* (resp. *super-solution*) of (1.3) in  $J_T$  if for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$  for some  $t_0 > 0$ , we have

$$\begin{aligned} \varphi_t + H_i(D\varphi) &\leq 0 \quad (\text{resp.} \quad \geq 0) \quad \text{at } (t_0, x_0) && \text{if } x_0 \in J_i \setminus \Gamma \\ \varphi_t + F(D\varphi) &\leq 0 \quad (\text{resp.} \quad \geq 0) \quad \text{at } (t_0, x_0) && \text{if } x_0 \in \Gamma. \end{aligned} \quad (2.1)$$

- ii) We say that  $u$  is a (classical viscosity) *sub-solution* (resp. *super-solution*) of (1.3)-(1.4) on  $[0, T) \times J$  if additionally

$$u^*(0, x) \leq u_0(x) \quad (\text{resp.} \quad u_*(0, x) \geq u_0(x)) \quad \text{for all } x \in J.$$

- iii) We say that  $u$  is a (classical viscosity) *solution* if  $u$  is both a sub-solution and a super-solution.

**Definition 2.2** (Flux-limited solutions). Consider a continuous flux-limiter function  $A : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then  $u$  is a  $A$ -flux limited sub-solution (resp. super-solution, solution) of (1.8) if it is a classical sub-solution (resp. super-solution, solution) of (1.3) with  $F = F_A$ .

### 2.1.3 Relaxed viscosity solutions

We next introduce relaxed viscosity solutions.

**Definition 2.3** (Relaxed viscosity solutions). Assume the Hamiltonians satisfy (1.5) and the flux function  $F$  satisfies (1.10). Let  $u : [0, T) \times J \rightarrow \mathbb{R}$  be locally bounded.

- i) We say that  $u$  is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (1.3) in  $J_T$  if for all test function  $\varphi \in C^1(J_T)$  such that

$$u^* \leq \varphi \quad (\text{resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, x_0) \in J_T$$

with equality at  $(t_0, x_0)$  for some  $t_0 > 0$ , we have

$$\varphi_t + H_i(D\varphi) \leq 0 \quad (\text{resp.} \quad \geq 0) \quad \text{at } (t_0, x_0)$$

if  $x_0 \notin \Gamma$ , and

$$\begin{array}{l} \text{either} \quad \varphi_t + F(D\varphi) \leq 0 \quad (\text{resp.} \quad \geq 0) \\ \text{or} \quad \varphi_t + H_i(D\varphi) \leq 0 \quad (\text{resp.} \quad \geq 0) \quad \text{for some } i \end{array} \quad \Bigg| \quad \text{at } (t_0, x_0)$$

if  $x_0 \in \Gamma$ .

- ii) We say that  $u$  is a *relaxed (viscosity) solution* if  $u$  is both a sub-solution and a super-solution.

### 2.1.4 A reduced set of test functions

Let  $\pi_i^\pm : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$  be defined as follows

$$\begin{aligned}\pi_i^+(p', \lambda) &= \inf\{p_i : H_i(p', p_i) = H_i^+(p', p_i) = \lambda\} \\ \pi_i^-(p', \lambda) &= \sup\{p_i : H_i(p', p_i) = H_i^-(p', p_i) = \lambda\}.\end{aligned}$$

**Definition 2.4** (Flux-limited viscosity solutions – again). Assume the Hamiltonians satisfy (1.5) and consider a continuous flux-limiter function  $A : \mathbb{R}^d \rightarrow \mathbb{R}$  such that for all  $p' \in \mathbb{R}^d$ ,  $A(p') \geq A_0(p')$ . Given  $u : [0, T) \times J \rightarrow \mathbb{R}$  locally bounded, the function  $u$  is a *reduced sub-solution* (resp. *super-solution*) of (1.3) with  $F = F_A$  in  $J_T$  if and only if  $u$  is a sub-solution (resp. super-solution) outside  $\Gamma$  and for all test function  $\varphi \in C^1(J_T)$  touching  $u$  from above at  $(t_0, X_0) \in (0, +\infty) \times \Gamma$ , of the following form

$$\varphi(t, x', x) = \phi(t, x') + \phi_0(x)$$

with

$$\begin{cases} \phi \in C^1((0, +\infty) \times \mathbb{R}^d) \\ D'\phi(t_0, x'_0) = p'_0 \end{cases} \quad \begin{cases} \phi_0 \in C^1(\mathbb{R}) \\ \partial_i \phi_0(0) = \pi_i^+(p'_0, A(p'_0)) \end{cases}$$

we have

$$\varphi_t + F_A(D\varphi) \leq 0 \quad (\text{resp.} \quad \geq 0).$$

**Proposition 2.5** (Equivalence of Definitions 2.2 and 2.4). *The definitions 2.2 and 2.4 are equivalent.*

*Proof.* It is clear that flux-limited sub-solutions (resp. super-solutions) are reduced sub-solutions (resp. reduced super-solutions). To prove that the converse holds true, we proceed as in [3] by considering critical slopes in  $x$ . Precisely, it is enough to prove the following lemmas.

**Lemma 2.6** (Critical slopes for super-solutions). *Let  $u$  be a super-solution of (1.8) away from  $\Gamma$  and let  $\varphi$  touch  $u_*$  from below at  $P_0 = (t_0, X_0)$  with  $X_0 \in \Gamma$ . Then the “critical slopes” defined as follows*

$$\bar{p}_i = \sup\{\bar{p} \in \mathbb{R} : \exists r > 0, \varphi(t, X) + \bar{p}x \leq u(t, X) \text{ for } (t, X) \in B_r(P_0) \cap (0, +\infty) \times J_i\}$$

satisfy for all  $i = 1, \dots, N$ ,

$$\varphi_t(P_0) + H_i(D'\varphi(P_0), \partial_i \varphi(P_0) + \bar{p}_i) \geq 0.$$

**Lemma 2.7** (Critical slopes for sub-solutions). *Let  $u$  be a sub-solution of (1.8) away from  $\Gamma$  and let  $\varphi$  touch  $u_*$  from above at  $P_0 = (t_0, X_0)$  with  $X_0 \in \Gamma$ . Then the “critical slopes” defined as follows*

$$\bar{p}_i = \inf\{\bar{p} \in \mathbb{R} : \exists r > 0, \varphi(t, X) + \bar{p}x \geq u(t, X) \text{ for } (t, X) \in B_r(P_0) \cap (0, +\infty) \times J_i\}$$

satisfy for all  $i = 1, \dots, N$ ,

$$\varphi_t(P_0) + H_i(D'\varphi(P_0), \partial_i \varphi(P_0) + \bar{p}_i) \leq 0.$$

The proofs of these lemmas are straightforward adaptations of the corresponding ones in [3] so we skip them. The remaining of the proof is also analogous but we give some details in the sub-solution case for the reader's convenience.

Let  $\varphi$  touch  $u_*$  from below at  $P_0 = (t_0, X_0)$  with  $X_0 \in \Gamma$  and let  $\lambda$  denote  $\varphi_t(P_0)$  and  $P = (p', p_1, \dots, p_N)$  denote  $D\varphi(P_0)$ . We want to prove

$$F_A(p', p_i) \leq \lambda. \tag{2.2}$$

We know from Lemma 2.6 that for all  $i = 1, \dots, N$ ,

$$H_i(p', p_i + \bar{p}_i) \leq \lambda \tag{2.3}$$

for some  $\bar{p}_i \leq 0$ . In particular,

$$A_0(p') \leq \lambda.$$

We write next

$$\begin{aligned} F_A(p', p_i) &= \max(A(p'), H_i^-(p', p_i)) \\ &\leq \max(A(p'), H_i^-(p', p_i + \bar{p}_i)) \\ &\leq \max(A(p'), H_i(p', p_i + \bar{p}_i)) \\ &\leq \max(A(p'), \lambda). \end{aligned}$$

If (2.2) does not hold true, then

$$A_0(p') \leq \lambda < A(p').$$

Moreover, we have from (2.3) that

$$p_i + \bar{p}_i < \pi_i^+(p', A(p')).$$

Hence, we can consider the following test function

$$\phi(t, x', x) = \varphi(t, x', 0) + \phi_0(x)$$

with  $\partial_i \phi_0(0) = \pi_i^+(p', A(p'))$ . From the definition of reduced sub-solutions, we thus get

$$A(p') = F_A(p', \pi_i^+(p', A(p'))) \leq \lambda$$

which is the desired contradiction.  $\square$

## 2.2 Stability

In the following proposition, we assert that, for the special junction functions  $F_A$ , the junction condition is in fact always satisfied *in the classical sense*, that is to say in the sense of Definition 2.1.

**Proposition 2.8** ( $F_A$  junction conditions are always satisfied in the classical sense). *Assume the Hamiltonians satisfy (1.5) and consider a continuous flux-limiter function  $A$ . If  $F = F_A$ , then relaxed viscosity solutions in the sense of Definition 2.3 coincide with viscosity solutions in the sense of Definition 2.1.*

*Proof.* We treat successively the super-solution case and the sub-solution case.

**Case 1: the super-solution case.** Let  $u$  be a relaxed super-solution and let us assume by contradiction that there exists a test function  $\varphi$  touching  $u$  from below at  $P_0 = (t_0, X_0)$  for some  $t_0 \in (0, T)$  and  $X_0 \in \Gamma$ , such that

$$\varphi_t + F_A(D\varphi) < 0 \quad \text{at } P_0. \quad (2.4)$$

Consider next the test function  $\tilde{\varphi}$  satisfying  $\tilde{\varphi} \leq \varphi$  in a neighborhood of  $P_0$ , with equality at  $P_0$  such that

$$\begin{aligned} \tilde{\varphi}_t(P_0) &= \varphi_t(P_0) \\ D'\tilde{\varphi}(P_0) &= D'\varphi(P_0) \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \min(\pi_i^0(D'\varphi(P_0)), \partial_i \varphi(P_0)) \quad \text{for } i = 1, \dots, N. \end{aligned}$$

Using the fact that  $F_A(D\varphi) = F_A(D\tilde{\varphi}) \geq H_i^-(D'\varphi, \partial_i \tilde{\varphi}) = H_i(D'\varphi, \partial_i \tilde{\varphi})$  at  $P_0$ , we deduce a contradiction with (2.4) using the viscosity inequality satisfied by  $\varphi$  for some  $i \in \{1, \dots, N\}$ .

**Case 2: the sub-solution case.** Let now  $u$  be a relaxed sub-solution and let us assume by contradiction that there exists a test function  $\varphi$  touching  $u$  from above at  $P_0 = (t_0, X_0)$  for some  $t_0 \in (0, T)$  and  $X_0 \in \Gamma$ , such that

$$\varphi_t + F_A(D\varphi) > 0 \quad \text{at } P_0. \quad (2.5)$$

Let us define

$$I = \{i \in \{1, \dots, N\}, \quad H_i^-(D'\varphi, \partial_i\varphi) < F_A(D\varphi) \quad \text{at } P_0\}$$

and for  $i \in I$ , let  $q_i \geq \pi_i^0(D'\varphi(P_0))$  be such that

$$H_i(D'\varphi(P_0), q_i) = F_A(D\varphi(P_0))$$

where we have used the fact that  $H_i(D'\varphi(P_0), +\infty) = +\infty$ . Then we can construct a test function  $\tilde{\varphi}$  satisfying  $\tilde{\varphi} \geq \varphi$  in a neighborhood of  $P_0$ , with equality at  $P_0$ , such that

$$\begin{aligned} \tilde{\varphi}_t(P_0) &= \varphi_t(P_0) & \text{and } \partial_i \tilde{\varphi}(P_0) &= \begin{cases} \max(q_i, \partial_i \varphi(P_0)) & \text{if } i \in I, \\ \partial_i \varphi(P_0) & \text{if } i \notin I. \end{cases} \\ D'\tilde{\varphi}(P_0) &= D'\varphi(P_0) \end{aligned}$$

Using the fact that  $F_A(D\varphi) = F_A(D\tilde{\varphi}) \leq H_i(D'\varphi, \partial_i\varphi)$  at  $P_0$  for all  $i$ , we deduce a contradiction with (2.5) using the viscosity inequality for  $\varphi$  for some  $i \in \{1, \dots, N\}$ .  $\square$

### 2.3 General junction conditions reduce to flux-limited ones

We first prove Theorem 1.1.

*Proof of Theorem 1.1.* With the notation of Remark 1.2 in hand, we first remark that there exists only one  $\lambda \geq A_0(p')$  such that there exists  $p^+ = (p_1^+, \dots, p_N^+)$  with  $p_i^+ \geq p_i^0$  such that

$$H_i(p', p_i^+) = F(p', p^+) = \lambda.$$

The coercivity of  $A_F$  is a direct consequence of the fact that  $A_F \geq A_0$ . We thus prove next that  $A_F$  is continuous. Consider a sequence  $(p'_n)_n$  converging towards  $p'$  and let  $p_n^+ = (p_{1,n}^+, \dots, p_{N,n}^+)$  with  $p_{i,n}^+ \geq p_i^0$  be such that

$$H_i(p'_n, p_{i,n}^+) = F(p'_n, p_n^+) = A_n \quad (2.6)$$

with  $A_n = A_F(p'_n)$ . We first claim that  $(p_{i,n}^+)_n$  is bounded. Indeed, if not, then  $A_n \rightarrow +\infty$  and, for  $n$  large enough,

$$F(p'_n, p^0) \geq A_n$$

which is impossible. The claim also implies that  $(A_n)_n$  is also bounded. Consider now to converging subsequences, still denoted by  $(p_n^+)_n$  and  $(A_n)_n$ , and let  $p^+$  and  $A$  be their limits. We can pass to the limit in (2.6) and get

$$H_i(p', p_i^+) = F(p', p^+) = A$$

which implies that  $A = A_F(p')$ .

We only do the proof for super-solutions since the proof for sub-solutions follows along the same lines. Let  $\varphi$  be a test function touching  $u_*$  from below at  $P_0 = (t_0, X_0)$ . We only need to consider the case where  $X_0 \in \Gamma$ . We can also assume that

$$\varphi(t, X) = \phi(t, x') + \phi_0(x)$$

with

$$D'\phi(t_0, x'_0) = p'_0 \quad \text{and} \quad \partial_i \phi_0(0) = \pi_i^+(p'_0, A_F(p'_0)).$$

We have

$$\varphi_t(P_0) + \max(F(D\varphi(P_0)), \max_i H_i(D\varphi(P_0))) \geq 0$$

which yields

$$\varphi_t(P_0) + \max(F(p'_0, \pi^+(p'_0, A_F(p'_0))), A_F(p'_0)) \geq 0.$$

In view of the definition of  $A_F$ , we get

$$\varphi_t(P_0) + A_F(p'_0) \geq 0.$$

Now compute

$$F_{A_F}(D\varphi(P_0)) = \max(A_F(p'_0), \max_i H_i^-(p'_0, \pi_i^+(p'_0, A_F(p'_0))) = A_F(p'_0).$$

The proof is now complete.  $\square$

We now turn to the following useful proposition.

**Proposition 2.9** (Quasi-convex Hamiltonians and flux functions generate quasi-convex flux limiters). *If the Hamiltonians  $H_i$  satisfy (1.5) and the flux function  $F$  satisfies (1.10)-(1.11), then  $A_F$  is continuous, quasi-convex and coercive.*

The proof of this proposition is postponed and can be found in Appendix.

## 2.4 Existence

Using Perron's method as in [3], we easily get existence of relaxed viscosity solutions for general junction functions  $F$  satisfying (1.10).

**Theorem 2.10** (Existence). *Let  $T > 0$ . Assume that Hamiltonians satisfy (1.5), that the junction function  $F$  satisfies (1.10) and that the initial datum  $u^0$  is uniformly continuous. Then there exists a relaxed viscosity solution  $u$  of (1.3)-(1.4) in  $[0, T) \times J$  and a constant  $C_T > 0$  such that*

$$|u(t, X) - u^0(X)| \leq C_T \quad \text{for all } (t, X) \in [0, T) \times J.$$

## 3 Vertex test function

This section is devoted to the construction of the vertex test function to be used in the proof of the comparison principle.

We will use below the following shorthand notation

$$H(X, p', p) = \begin{cases} H_i(p', p) & \text{for } p = p_i & \text{if } X \in J_i \setminus \Gamma, \\ F_A(p', p) & \text{for } p = (p_1, \dots, p_N) & \text{if } X \in \Gamma. \end{cases} \quad (3.1)$$

In particular, keeping in mind the definition of  $Du$  (see (1.2)), Problem (1.8) on the junction can be rewritten as follows

$$u_t + H(X, Du) = 0 \quad \text{for all } (t, X) \in (0, +\infty) \times J.$$

Then our key result is the following one.

**Theorem 3.1** (The vertex test function). *Let  $A : \mathbb{R}^d \rightarrow \mathbb{R}$  be quasi-convex and  $\gamma > 0$ . Assume the Hamiltonians satisfy (1.5). Then there exists a function  $G : J^2 \rightarrow \mathbb{R}$  enjoying the following properties.*

*i)* (Regularity)

$$G \in C(J^2) \quad \text{and} \quad \begin{cases} G(X, \cdot) \in C^1(J) & \text{for all } X \in J, \\ G(\cdot, Y) \in C^1(J) & \text{for all } Y \in J. \end{cases}$$

*ii)* (Bound from below)  $G \geq 0 = G(0, 0)$ .

iii) (Compatibility condition on the diagonal) For all  $X \in J$ ,

$$0 \leq G(X, X) - G(0, 0) \leq \gamma. \quad (3.2)$$

iv) (Compatibility condition on the gradients) For all  $(X, Y) \in J^2$ ,

$$H(Y, -D_Y G(X, Y)) - H(X, D_X G(X, Y)) \leq \gamma \quad (3.3)$$

where notation introduced in (1.2) and (3.1) are used.

v) (Superlinearity) There exists  $g : [0, +\infty) \rightarrow \mathbb{R}$  nondecreasing and s.t. for  $(X, Y) \in J^2$

$$g(d(X, Y)) \leq G(X, Y) \quad \text{and} \quad \lim_{a \rightarrow +\infty} \frac{g(a)}{a} = +\infty. \quad (3.4)$$

vi) (Gradient bounds) For all  $K > 0$ , there exists  $C_K > 0$  such that for all  $(X, Y) \in J^2$ ,

$$d(X, Y) \leq K \quad \implies \quad |G_x(X, Y)| + |G_y(X, Y)| \leq C_K. \quad (3.5)$$

We now assert that Theorem 1.3 is a direct consequence of Theorem 3.1.

*Proof of Theorem 1.3.* Use Theorem 3.1 and proceed as in [3]. □

### 3.1 The case of smooth convex Hamiltonians

Assume that the Hamiltonians  $H_i$  satisfy the following assumptions for  $i = 1, \dots, N$ ,

$$\begin{cases} H_i \in C^2(\mathbb{R}^{d+1}) \text{ with } D^2 H_i > 0 \text{ in } \mathbb{R}^{d+1}, \\ \lim_{|P| \rightarrow +\infty} \frac{H_i(P)}{|P|} = +\infty \end{cases} \quad (3.6)$$

and the flux limiter

$$A \in C^2(\mathbb{R}^d) \quad \text{and} \quad D^2 A > 0 \text{ in } \mathbb{R}^{d+1}. \quad (3.7)$$

It is useful to associate with each  $H_i$  satisfying (3.6) its partial inverse functions  $\pi_i^\pm$ :

$$\text{for } \lambda \geq A_i(p'), \quad H_i(p', \pi_i^\pm(p', \lambda)) = \lambda \quad \text{such that} \quad \pi_i^-(p', \lambda) \leq \pi_i^0(p') \leq \pi_i^+(p', \lambda) \quad (3.8)$$

where we recall that  $A_i(p') = \min_{p_i} H_i(p', p_i)$ .

**Lemma 3.2** (Properties of  $\pi_i^\pm$ ). *Assume (3.6). Then  $\pi_i^\pm(p', \cdot) \in C^2(A_i(p'), +\infty)$  and  $\pi_i^\pm \in C(\text{epi } A_i)$ . Moreover,  $\pi_i^\pm$  is concave w.r.t.  $(p', \lambda)$  in  $\text{epi } A_i$  and  $\pm \pi_i^\pm$  is non-decreasing w.r.t.  $\lambda$ .*

*Proof.* The regularity of  $\pi^\pm$  can be derived thanks to the inverse function theorem. As far as the concavity of  $\pi_i^\pm$  is concerned, we can drop the subscript  $i$  and we do so for clarity. let  $(p', \lambda), (q', \mu) \in \text{epi } A$  and  $t \in (0, 1)$ . Then

$$\begin{aligned} t\lambda + (1-t)\mu &= tH(p', \pi^+(p', \lambda)) + (1-t)H(q', \pi^+(q', \mu)) \\ &\geq H(tp' + (1-t)q', t\pi^+(p', \lambda) + (1-t)\pi^+(q', \mu)). \end{aligned}$$

Hence

$$\pi^+(tp' + (1-t)q', t\lambda + (1-t)\mu) \geq t\pi^+(p', \lambda) + (1-t)\pi^+(q', \mu)$$

which is the desired result. The monotonicity of  $\pi^+$  is easy to derive from the convexity of  $H$ . The proof of the lemma is now complete. □

We next define the function  $G^0$  for  $X \in J_i, Y \in J_j, i, j = 1, \dots, N$ , as follows,

$$G^0(X, Y) = \sup_{(P, \lambda) \in \mathcal{G}_A^{ij}} (p' \cdot (x' - y') + p_i x - p_j y - \lambda) \quad (3.9)$$

where

$$\mathcal{G}_A^{ij} = \begin{cases} \{(P, \lambda) \in \mathbb{R}^{d+2} \times \mathbb{R} : P = (p', p_i, p_j), H_i(p', p_i) = H_j(p', p_j) \geq A(p')\} & \text{if } i \neq j \\ \{(P, \lambda) \in \mathbb{R}^{d+1} \times \mathbb{R} : P = (p', p_i), H_i(p', p_i) \geq A(p')\} & \text{if } i = j \end{cases} \quad (3.10)$$

with  $A \geq A_0$  and  $A$  quasi-convex.

**Proposition 3.3** (The vertex test function – the smooth convex case). *Let  $A \geq A_0$  with  $A_0$  given by (1.9) and assume that the Hamiltonians satisfy (3.6). Then  $G^0$  satisfies*

i) (Regularity)

$$G^0 \in C(J^2) \quad \text{and} \quad \begin{cases} G^0 \in C^1(\{(X, Y) \in J \times J, x \neq y\}), \\ G^0(0, \cdot) \in C^1(J) \quad \text{and} \quad G^0(\cdot, 0) \in C^1(J); \end{cases}$$

ii) (Bound from below)  $G^0 \geq G^0(0, 0)$ ;

iii) (Compatibility conditions) (3.2) and (3.3) hold with  $\gamma = 0$ ;

iv) (Superlinearity) (3.4) holds for some  $g = g^0$ ;

v) (Gradient bounds) (3.5) holds only for  $(X, Y) \in J^2$  such that  $x \neq y$  or  $(x, y) = (0, 0)$ ;

The proof of this proposition is postponed until Subsection 3.4. With such a result in hand, we can now prove Theorem 3.1 in the case of smooth convex Hamiltonians.

**Lemma 3.4** (The case of smooth convex Hamiltonians). *Assume that the Hamiltonians satisfy (3.6). Then the conclusion of Theorem 3.1 holds true.*

*Proof.* It is enough to regularize  $G_{ii}^0$  in a neighborhood of  $\{x_i = y_i\}$ . Fix  $\varepsilon_0 > 0$  and  $t_0 > 0$  and consider a function  $\varepsilon : [0, +\infty[ \rightarrow [0, \varepsilon_0[$  such that  $\varepsilon(t) = \varepsilon_0$  for  $t \geq t_0$ . Consider also a smooth cut-off function  $\Psi : \mathbb{R} \rightarrow [0, 1]$  such that  $\text{supp } \Psi \subset [-1, 1]$  and a mollifier  $\rho$ . Then define

$$G_{ii}(X, Y) = \left(1 - \Psi\left(\frac{x_i - y_i}{\varepsilon(x_i + y_i)}\right)\right) G_{ii}^0(X, Y) + \Psi\left(\frac{x_i - y_i}{\varepsilon(x_i + y_i)}\right) \rho_{\varepsilon(x_i + y_i)} \star G_{ii}^0(X, Y).$$

This regularization procedure introduces a small error  $\gamma$  in (3.2) and (3.3) but preserves the other desired properties.  $\square$

### 3.2 The vertex test function in $J_i \times J_j$ with $i \neq j$

In order to prove Proposition 3.3, we first need to study  $G^0$  for  $X \in J_i$  and  $Y \in J_j$  with  $i \neq j$ . Then, one can write

$$G_{ij}^0(X, Y) = \mathfrak{G}_{ij}(x' - y', x_i, -y_j)$$

with

$$\mathfrak{G}_{ij}(Z) = \sup_{(P, \lambda) \in \mathcal{G}_A^{ij}} (P \cdot Z - \lambda)$$

with  $\mathcal{G}_A^{ij}$  is defined in (3.10). Remark that for  $X \in J_i$  and  $Y \in J_j$ , we have  $Z = X - Y \in Q$  where

$$Q = \mathbb{R}^d \times [0, +\infty[ \times ]-\infty, 0].$$

We also consider the simplex

$$\mathcal{T} = \{(\alpha_i, \alpha_j, \alpha_0) \in [0, 1]^3 : \alpha_i + \alpha_j + \alpha_0 = 1\}.$$

**Lemma 3.5** (Necessary conditions for the maximiser). *Given  $Z \in Q$ , the supremum defining  $\mathfrak{G}_{ij}$  is reached for some  $(P, \lambda) \in \mathcal{G}_A^{ij}$  and there exists  $(\alpha_i, \alpha_j, \alpha_0) \in \mathcal{T}$  such that*

$$Z = D(\alpha \cdot H)(P)$$

with  $H = (H_i, H_j, A)$ .

*Proof.*  $\mathfrak{G}_{ij}(Z)$  is defined by maximizing a linear function under a equality constraint and an inequality constraint. Constraints are qualified if

$$D(H_i - H_j) \text{ is not colinear with } D(H_i - A).$$

When constraints are qualified, Karush-Kuhn-Tucker theorem asserts that there exists  $\alpha_j \in \mathbb{R}$  and  $\alpha_0 \geq 0$  such that

$$Z = \nabla_P H_i + \alpha_j (\nabla_P H_j - \nabla_P H_i) + \alpha_0 \nabla_P (A - H_i)$$

with

$$\alpha = 0 \quad \text{if } A(p') < H_i(p', p_i).$$

If one sets  $\alpha_i = 1 - \alpha_0 - \alpha_j$ , Equivalently, we have

$$\begin{cases} z_i = \alpha_i \partial_i H_i(p', p_i) \geq 0 \\ z_j = \alpha_j \partial_j H_j(p', p_i) \leq 0 \\ z' = \alpha_i \nabla_{p'} H_i + \alpha_j \nabla_{p'} H_j + \alpha_0 \nabla_{p'} A \end{cases}$$

In particular,  $(\alpha_i, \alpha_j, \alpha_0) \in \mathcal{T}$ . Hence, the result is proved when constraints are qualified. This is in particular true if

$$\partial_i H_i(p', p_i) > 0 \text{ and } \partial_j H_j(p', p_j) < 0.$$

Now assume that  $\partial_i H_i(p', p_i) \leq 0$ . We remark that in all cases,  $\partial_i H_i(p', p_i) \geq 0$  since  $z_i \geq 0$ . Hence,  $\partial_i H_i(p', p_i) = 0$  or, in other words,  $H_i(p', p_i) = A_i(p')$ . But the constraint  $H_i(p', p_i) \geq A(p')$ , the assumption  $A(p') \geq A_0(p')$  and the simple fact that  $A_i(p') \leq A_0(p')$  imply in particular that  $A(p') = A_0(p')$ . We arrive at the same conclusion if  $\partial_j H_j(p', p_j) \geq 0$ . In other words,

$$\text{Constraints are qualified as soon as } \forall p', A(p') > A_0(p'). \quad (3.11)$$

In particular, the result of the lemma holds true under this latter condition:  $A(p') > A_0(p')$  for all  $p' \in \mathbb{R}^d$ . If now there are some  $p'$  such that  $A(p') = A_0(p')$ , we remark that

$$\mathfrak{G}_{ij}(Z) = \lim_{\varepsilon \rightarrow 0} \mathfrak{G}_{ij}^\varepsilon(Z)$$

where  $\mathfrak{G}_{ij}^\varepsilon(Z)$  is associated with  $A^\varepsilon(p') = \varepsilon + A(p')$ . From the previous case, we know that there exists  $P_\varepsilon$  and  $\lambda_\varepsilon$  such that

$$\mathfrak{G}_{ij}^\varepsilon(Z) = P_\varepsilon \cdot Z - \lambda_\varepsilon$$

and  $\alpha^\varepsilon = (\alpha_i^\varepsilon, \alpha_j^\varepsilon, \alpha_0^\varepsilon) \in \mathcal{T}$  such that

$$Z = D(\alpha \cdot H)(P_\varepsilon).$$

We can extract a subsequence such that  $\alpha^\varepsilon \rightarrow \alpha$ . Moreover,  $P_\varepsilon \cdot Z - \lambda_\varepsilon$  is bounded from above and

$$\lambda_\varepsilon = H_i(p'^\varepsilon, p_i^\varepsilon) = H_j(p'^\varepsilon, p_j^\varepsilon).$$

Since  $H_i$  and  $H_j$  are assumed to be superlinear, we conclude that we can also extract a converging subsequence from  $P_\varepsilon$ . This achieves the proof of the lemma.  $\square$

**Lemma 3.6** (Uniqueness of  $P$ ). *Let  $Z \in Q$ . If there exists  $\alpha, P$  and  $\beta, Q$  such that*

$$\begin{cases} \mathfrak{G}_{ij}(Z) = P \cdot Z - \lambda = Q \cdot Z - \mu, \\ Z = D(\alpha \cdot H)(P) = D(\beta \cdot H)(Q). \end{cases}$$

*Then  $P = Q$ .*

*Proof.* We consider the function  $\Psi : \mathbb{R}^{d+2} \times \mathcal{T} \rightarrow \mathbb{R}$  defined as follows

$$\Psi(P, \alpha) = D(\alpha \cdot H)(P).$$

By assumption, we have

$$0 = D(\alpha \cdot H)(P) - D(\beta \cdot H)(Q).$$

If  $\bar{P}$  denotes  $Q - P$  and  $\bar{\alpha}$  denotes  $\beta - \alpha$ , then

$$\begin{aligned} 0 &= \int_0^1 D\Psi(P + \theta\bar{P}, \alpha + \theta\bar{\alpha})d\theta \\ &= \int_0^1 D_P\Psi(P + \theta\bar{P}, \alpha + \theta\bar{\alpha})\bar{P}d\theta + \int_0^1 D_\alpha\Psi(P + \theta\bar{P}, \alpha + \theta\bar{\alpha})\bar{\alpha}d\theta. \end{aligned}$$

Taking the scalar product with  $\bar{P}$  yields

$$\begin{aligned} 0 &= \int_0^1 D_{\bar{P}\bar{P}}^2((\alpha + \theta\bar{\alpha}) \cdot H)(P + \theta\bar{P})\bar{P} \cdot \bar{P}d\theta + \int_0^1 D_P H(P + \theta\bar{P})\bar{\alpha} \cdot \bar{P}d\theta \\ &= T_1 + T_2 \end{aligned}$$

with  $T_i \geq 0$ ,  $i = 1, 2$  and

$$\begin{aligned} T_1 &= \int_0^1 D_{\bar{P}\bar{P}}^2((\alpha + \theta\bar{\alpha}) \cdot H)(P + \theta\bar{P})\bar{P} \cdot \bar{P}d\theta \geq 0 \\ T_2 &= \int_0^1 D_P H(P + \theta\bar{P})\bar{\alpha} \cdot \bar{P}d\theta \geq 0. \end{aligned}$$

Indeed, keeping in mind that

$$\begin{cases} H_i(P) = H_j(P) \\ H_i(Q) = H_j(Q) \end{cases} \quad \text{and} \quad \begin{cases} \alpha_0(A(P) - H_i(P)) = 0 \\ \beta_0(A(Q) - H_i(Q)) = 0 \end{cases}$$

we remark that

$$\begin{aligned} \int_0^1 D_P H(P + \theta\bar{P})\bar{\alpha} \cdot \bar{P}d\theta &= \bar{\alpha} \cdot (H(Q) - H(P)) \\ &= \bar{\alpha}_i(H_i(Q) - H_i(P)) + \bar{\alpha}_j(H_j(Q) - H_j(P)) + \bar{\alpha}_0(A(Q) - A(P)) \\ &= (\beta_0 - \alpha_0)(A(Q) - H_i(Q) - A(P) + H_i(P)) \\ &= \beta_0(H_i(P) - A(P)) + \alpha_0(H_i(Q) - A(Q)) \geq 0. \end{aligned}$$

Hence, we get

$$\begin{aligned} 0 &= \int_0^1 D_{\bar{P}\bar{P}}^2((\alpha + \theta\bar{\alpha}) \cdot H)(P + \theta\bar{P})\bar{P} \cdot \bar{P}d\theta \\ 0 &= \beta_0(H_i(P) - A(P)) \\ 0 &= \alpha_0(H_i(Q) - A(Q)). \end{aligned}$$

We distinguish three cases. We will use several times the fact that  $H_i(p', p_i) = \lambda$  and  $\partial_i H_i(p', p_i) \geq 0$  implies that  $p_i = \pi_i^+(p', \lambda)$ . We will also use the corresponding property for  $p_j$ :  $p_j = \pi_j^-(p', p_j)$ .

- *Case 1.* If there exists  $\theta \in (0, 1)$  such that  $\alpha + \theta\bar{\alpha} \in \text{int } \mathcal{T}$ , then  $P = Q$  and

$$\lambda = P \cdot Z - \mathfrak{G}_{ij}(Z) = \mu.$$

- *Case 2.* If  $\alpha = \beta$  is a vertex of  $\mathcal{T}$ , then either  $\alpha = (1, 0, 0)$  or  $\alpha = (0, 1, 0)$  or  $\alpha = (0, 0, 1)$ .
  - In the first subcase,  $\alpha_i = 1$ , we get  $p' = q'$  and  $p_i = q_i$  and  $Z = \nabla_P H_i(P)$  and

$$0 = (p_j - q_j)z_j = (P - Q) \cdot Z = \lambda - \mu.$$

We conclude by remarking that  $p_j = \pi_j^-(p', \lambda) = q_j$ . The second subcase is similar.

- If now  $\alpha = (0, 0, 1)$ , then  $p' = q'$  and  $Z = \nabla_P A(P)$  and

$$0 = (p_i - q_i)z_i + (p_j - q_j)z_j = P \cdot Z = \lambda - \mu$$

and we conclude as in the two previous subcases.

- *Case 3.* Assume finally that there exists  $\theta \in (0, 1)$  such that  $\alpha + \theta\bar{\alpha} \in \partial\mathcal{T}$  but is not a vertex. In this third case, this implies that two components of  $a = \alpha + \theta\bar{\alpha} = (a_i, a_j, a_0)$  are not 0.
  - If  $a_0 = 0$  then  $p' = q'$  and  $p_i = q_i$  and  $p_j = q_j$ , i.e.  $P = Q$ .
  - If  $a_i = 0$  then  $p' = q'$  and  $p_j = q_j$  and  $z_i = 0$  and  $\lambda = \mu$  and  $p_i = \pi^+(p', \lambda) = q_i$ . The third subcase  $a_j = 0$  is similar to the second one.

The proof of the lemma is now complete. □

The two previous lemmas imply the following one.

**Lemma 3.7** (Gradients of  $G_{ij}^0$ ). *The function  $G_{ij}^0$  is  $C^1$  in  $J_i \times J_j$ , up to the boundary, and*

$$DG_{ij}^0(X, Y) = (p', p_i, p_j), \quad p_i = \pi_i^+(p', \lambda), \quad p_j = \pi_j^-(p', \lambda)$$

where  $(p', \lambda) = (\mathfrak{P}(X, Y), \mathfrak{L}(X, Y))$  are uniquely determined by the relation

$$G_{ij}^0(X, Y) = p' \cdot (x' - y') + p_i x_i - p_j x_j - \lambda.$$

In particular, the maps  $\mathfrak{P}$  and  $\mathfrak{L}$  are continuous in  $J_i \times J_j$ .

The following lemma is elementary but it will be used below.

**Lemma 3.8** ( $G_{ij}^0$  at the boundary). *The restriction of  $\mathfrak{G}_{ij}$  to  $\{z_i = 0\}$  and  $\{z_j = 0\}$  equals respectively  $(H_i \vee A)^*$  and  $(H_j \vee A)^*$ .*

### 3.3 The vertex test function in $J_i \times J_i$

In view of the definition of  $G^0$ , see (3.9), we have

$$G_{ii}^0(X, Y) = (H_i \vee A)^*(X - Y).$$

In particular, we derive from Lemma 3.8 the following one.

**Lemma 3.9** (Continuity of  $G^0$ ). *The function  $G^0$  is continuous in  $J \times J$ .*

We now turn to the regularity of  $G_{ii}^0$ .

**Lemma 3.10** (Regularity of  $G_{ii}^0$ ).  *$G_{ii}^0$  is  $C^1$  in  $J_i \times J_i \setminus \{x_i = y_i\}$ .*

*Proof.* Since  $G_{ii}^0$  is convex, it is  $C^1$  if and only if there exists one and only one subgradient. Consider two distinct subgradients  $(p', p_i)$  and  $(q', q_i)$  of  $(\max(H_i, A))^*$  at some point  $(z', z_i) = (x' - y', x_i - y_i)$ . This implies that

$$(z', z_i) \in \partial \max(H_i, A)(p', p_i) \cap \max(H_i, A)(q', q_i).$$

Since  $H_i$  is strictly convex, this can only happen when  $H_i = A$  and since  $(p', p_i) \neq (q', q_i)$ , this implies  $z_i = 0$ .  $\square$

**Lemma 3.11** (Gradients of  $G_{ii}^0$ ). *For  $(X, Y) \in J_i \times J_i$  such that  $x_i \neq y_i$ , we have*

$$DG_{ii}^0(X, Y) = (p', p_i, -p', -p_i)$$

with  $p_i = \pi_i^\pm(p', \lambda)$  if  $\pm(x_i - y_i) \geq 0$ .  $(p', \lambda) = (\mathfrak{P}(X, Y), \mathfrak{L}(X, Y))$  is uniquely determined by

$$\begin{cases} Z = \alpha_i DH_i(P) + (1 - \alpha_i) DA(P) \\ G_{ii}^0(X, Y) = p' \cdot (x' - y') + p_i(x_i - y_i) - \lambda \end{cases}$$

which holds true for some  $\alpha_i \in [0, 1]$ . In particular, the maps  $\mathfrak{P}$  and  $\mathfrak{L}$  are continuous in  $J_i \times J_i \setminus \{x_i = y_i \neq 0\}$ .

*Proof.* Lemma 3.10 implies that  $P = (p', p_i, -p_i)$  is unique. Hence  $\lambda$  is unique too. Moreover  $p_i = \pi_i^\pm(p', \lambda)$  if  $\pm(x_i - y_i) \geq 0$ . Remark that  $P = D(H_i \vee A)^*(Z)$  with  $Z = X - Y$  is equivalent to

$$Z \in D(H_i \vee A)(P)$$

which implies

$$Z = \alpha_i DH_i(P) + (1 - \alpha_i) DA(P)$$

for some  $\alpha_i \in [0, 1]$ . Lemma 3.6 allows us to conclude.  $\square$

### 3.4 Proof of Proposition 3.3

We now turn to the proof of Proposition 3.3.

*Proof of Proposition 3.3.* The proof proceeds in several steps.

**Step 1: Regularity.** We already noticed in Lemma 3.9 that  $G^0 \in C(J^2)$  and Lemmas 3.7 and 3.10 imply that  $G^0 \in C^1(R)$  for each region  $R$  given by

$$R = \begin{cases} J_i \times J_j & \text{if } i \neq j, \\ T_i^\pm = \{(X, Y) \in J_i \times J_i, \pm(x_i - y_i) \geq 0\} & \text{if } i = j. \end{cases} \quad (3.12)$$

**Step 2: Computation of the gradients.** For each  $R$  given by (3.12) and for all  $(X, Y) \in R \subset J_i \times J_j$ , Lemmas 3.7 and 3.11 imply that

$$G^0(X, Y) = p' \cdot (x' - y') + p_i x_i - p_j y_j - \lambda$$

and

$$(D', \partial_i)G_{|R}^0(X, Y) = (p', p_i) \quad \text{and} \quad -(D', \partial_j)G_{|R}^0(X, Y) = (p', p_j)$$

with  $\lambda = \mathfrak{L}(X, Y)$  and  $p' = \mathfrak{P}(X, Y)$  with

$$(p_i, p_j) = \begin{cases} (\pi_i^+(p', \lambda), \pi_j^-(p', \lambda)) & \text{if } R = J_i \times J_j & \text{with } i \neq j, \\ (\pi_i^\pm(p', \lambda), \pi_i^\pm(p', \lambda)) & \text{if } R = T_i^\pm & \text{with } i = j. \end{cases} \quad (3.13)$$

Notice in particular that  $\mathfrak{P}$  and  $\mathfrak{L}$  are continuous in  $J \times J$  except on  $\cup_{i=1}^N \{x_i = y_i \neq 0\}$ .

**Step 3: Checking the compatibility condition on the gradients.** Let us consider  $(x, y) \in J^2$  with  $x = y = 0$  or  $x \neq y$ . We have

$$\begin{aligned} D_X(G^0(\cdot, Y))(X) &\in \{(p', \pi_i^\pm(\lambda))\} \\ -(D_Y G^0(X, \cdot))(Y) &\in \{(p', \pi_j^\pm(\lambda))\} \end{aligned}$$

with  $\lambda \geq A(p')$ . We claim that

$$H(X, D_X G^0(X, Y)) = \lambda \quad (3.14)$$

and

$$H(Y, -D_Y G^0(X, Y)) = \lambda \quad (3.15)$$

(we use here once again the short hand notation (3.1).

(3.14) is clear except if  $X = 0$ . In this case, if  $Y \neq 0$ , say  $Y \in J_j$ , the desired equality is rewritten as

$$\max(A(p'), \max_i H_i^-(p', p_i)) = \lambda$$

with  $p_i = \pi_i^+(p', \lambda)$  if  $i \neq j$  and  $p_j = \pi_j^-(p', \lambda)$ . Since  $\lambda \geq A(p')$  and  $H_j^-(p', p_j) = \lambda$ , we get the result. If now  $(X, Y) = (0, 0)$ , then  $p_i = \pi_i^+(p', \lambda)$  and  $\lambda = A(p')$ . Hence, we get (3.14) in this case too. One can derive (3.15) in the same way.

**Step 4: Superlinearity.** In view of the definition of  $G^0$ , we deduce from (3.13) that for all  $R > 0$  and  $\lambda > A_i(R(x' - y')/|x' - y'|)$ ,

$$G^0(X, Y) \geq R|x' - y'| + \begin{cases} x\pi_i^+(\widehat{Rx' - y'}, \lambda) - y\pi_j^-(\widehat{Rx' - y'}, \lambda) - \lambda & \text{if } i \neq j, \\ (x - y)\pi_i^\pm(\widehat{Rx' - y'}, \lambda) - \lambda & \text{if } i = j, \pm(x - y) \geq 0 \end{cases}$$

where  $\hat{z} = z/|z|$ . For  $R > 0$ , we define

$$\lambda(R) = \sup_{i=1, \dots, N, |e|=1} A_i(Re),$$

and

$$\pi^0(R) := \min_{\pm, i=1, \dots, N, |e|=1} \pm \pi_i^\pm(Re, \lambda(R)) \geq 0.$$

Hence we get

$$G^0(X, Y) \geq R|x' - y'| + \pi^0(R)d(x, y) - \lambda(R)$$

where

$$d(x, y) = \begin{cases} |x_i - y_i| & \text{if } X, Y \in J_i \\ x_i + y_j & \text{if } X \in J_i, Y \in J_j, i \neq j. \end{cases}$$

From the definition (3.8) of  $\pi_i^\pm$  and the assumption (3.6) on the Hamiltonians, we deduce that

$$\pi^0(R) \rightarrow +\infty \quad \text{as } R \rightarrow +\infty$$

which implies that for any  $K \geq 0$ , there exists a constant  $C_K \geq 0$  such that

$$G^0(X, Y) \geq K(|x' - y'| + d(x, y)) - C_K.$$

Therefore we get (3.4) with

$$g^0(a) = \sup_{K \geq 0} (Ka - C_K).$$

**Step 5: Gradient bounds.** Because each component of the gradients of  $G^0$  are equal to one of the  $\{(p', \pi_k^\pm(p', \lambda))\}_{\pm, k=1, \dots, N}$  with  $\lambda = \mathfrak{L}(X, Y)$  and  $p' = \mathfrak{P}(X, Y)$ , we deduce (3.5) from the continuity of  $\mathfrak{L}$ ,  $\mathfrak{P}$  and  $\pi_k^\pm$ .  $\square$

### 3.5 The general case

Let us consider a slightly stronger assumption than (1.5), namely

$$\begin{cases} H_i \in C^2(\mathbb{R}^{d+1}) & \text{with } \min H_i = H_i(p_i^0) \quad \text{and} \quad D^2 H_i(p_i^0) > 0, \\ D^2 H(I - \widehat{DH} \otimes \widehat{DH}) > 0, \\ \lim_{|q| \rightarrow +\infty} H_i(q) = +\infty. \end{cases} \quad (3.16)$$

Notice that the second line basically says that the sub-level sets are strictly convex. The following technical result will allow us to reduce a large class of quasi-convex Hamiltonians to convex ones.

**Lemma 3.12** (From quasi-convex to convex Hamiltonians). *Given Hamiltonians  $H_i$  satisfying (3.16), there exists a function  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  such that the functions  $\beta \circ H_i$  satisfy (3.6) for  $i = 1, \dots, N$ . Moreover, we can choose  $\beta$  such that*

$$\beta \text{ is convex, } \beta \in C^2(\mathbb{R}), \quad \beta(0) = 0 \quad \text{and} \quad \beta' \geq \delta > 0. \quad (3.17)$$

*Proof.* In view of (3.16), it is easy to check that  $(\beta \circ H_i)'' > 0$  if and only if we have

$$(\ln \beta')'(\lambda) > -\frac{D^2 H_i}{|DH_i|^2} \circ \pi_i^\pm(p', \lambda) \quad \text{for } \lambda > H_i(p_i^0), \quad p' \in \mathbb{R}^d. \quad (3.18)$$

Because  $D^2 H_i''(p_i^0) > 0$ , we see that the right hand side is negative for  $\lambda$  close enough to  $H_i(p_i^0)$ . Then it is easy to choose a function  $\beta$  satisfying (3.18) and (3.17). Finally, compositing  $\beta$  with another convex increasing function which is superlinear at  $+\infty$  if necessary, we can ensure that  $\beta \circ H_i$  superlinear.  $\square$

**Lemma 3.13** (The case of smooth Hamiltonians). *Theorem 3.1 holds true if the Hamiltonians satisfy (3.16).*

*Proof.* We assume that the Hamiltonians  $H_i$  satisfy (3.16). Let  $\beta$  be the function given by Lemma 3.12. If  $u$  solves (1.8) on  $J_T$ , then  $u$  is also a viscosity solution of

$$\begin{cases} \bar{\beta}(u_t) + \hat{H}_i(Du) = 0 & \text{for } t \in (0, T) \quad \text{and} \quad x \in J_i^*, \\ \bar{\beta}(u_t) + \hat{F}_{\hat{A}}(Du) = 0 & \text{for } t \in (0, T) \quad \text{and} \quad x = 0 \end{cases} \quad (3.19)$$

with  $\hat{F}_{\hat{A}}$  constructed as  $F_A$  where  $H_i$  and  $A$  are replaced with  $\hat{H}_i$  and  $\hat{A}$  defined as follows

$$\hat{H}_i = \beta \circ H_i, \quad \hat{A} = \beta(A)$$

and  $\bar{\beta}(\lambda) = -\beta(-\lambda)$ . We can then apply Theorem 3.1 in the case of smooth convex Hamiltonians to construct a vertex test function  $\hat{G}$  associated to problem (3.19) for every  $\hat{\gamma} > 0$ . This means that we have with  $\hat{H}(x, p) = \beta(H(x, p))$ ,

$$\hat{H}(y, -D_Y G) \leq \hat{H}(x, D_X G) + \hat{\gamma}.$$

This implies

$$H(y, -D_Y G) \leq \beta^{-1}(\beta(H(x, D_X G)) + \hat{\gamma}) \leq H(x, D_X G) + \hat{\gamma} |(\beta^{-1})'|_{L^\infty(\mathbb{R})}.$$

Because of the lower bound on  $\beta'$  given by Lemma 3.12, we get  $|(\beta^{-1})'|_{L^\infty(\mathbb{R})} \leq 1/\delta$  which yields the compatibility condition (3.3) with  $\gamma = \hat{\gamma}/\delta$  arbitrarily small.  $\square$

We are now in position to prove Theorem 3.1 in the general case.

*Proof of Theorem 3.1.* Let us now assume that the Hamiltonians only satisfy (1.5). In this case, we simply approximate the Hamiltonians  $H_i$  by other Hamiltonians  $\tilde{H}_i$  satisfying (3.16) such that

$$|H_i - \tilde{H}_i| \leq \gamma.$$

We then apply Theorem 3.1 to the Hamiltonians  $\tilde{H}_i$  and construct an associated vertex test function  $\tilde{G}$  also for the parameter  $\gamma$ . We deduce that

$$H(y, -\tilde{G}_y) \leq H(x, \tilde{G}_x) + 3\gamma$$

with  $\gamma > 0$  arbitrarily small, which shows again the compatibility condition on the Hamiltonians (3.3) for the Hamiltonians  $H_i$ 's. The proof is now complete in the general case.  $\square$

## A Proof of Proposition 2.9

Before proving Proposition 2.9, we state and prove the following elementary lemma.

**Lemma A.1** (Quasi-convexity of the functions  $A_i$ ). *If the Hamiltonians  $H_i$  are quasi-convex, continuous and coercive, so are the functions  $A_i$ . In particular,  $A_0 = \max A_i$  is quasi-convex, continuous and coercive.*

*Proof.* We only address the question of the quasi-convexity of the functions  $A_i$  since their continuity and coercivity are simpler.

Consider  $p'$  and  $q'$  such that  $A_i(p') \leq \lambda$  and  $A_i(q') \leq \lambda$  for some  $\lambda \in \mathbb{R}$ . There exists  $p_i, q_i \in \mathbb{R}$  such that

$$A_i(p') = H_i(p', p_i) \quad A_i(q') = H_i(q', q_i).$$

Then  $(p', p_i), (q', q_i) \in \{H_i \leq \lambda\}$  and we conclude from the quasi-convexity of  $H_i$  that for  $t, s \geq 0$  with  $t + s = 1$ ,

$$A_i(tp' + sq') \leq H_i(tp' + sq', tp_i + sq_i) \leq tH_i(p', p_i) + sH_i(q', q_i) \leq \lambda.$$

This achieves the proof of the lemma.  $\square$

*Proof of Proposition 2.9.* We assume that the Hamiltonians  $H_i$  are convex,  $\lambda \mapsto H_i(p', \lambda)$  is increasing in  $[A_i(p'), +\infty)$  and decreasing in  $(-\infty, A_i(p')]$  and  $F$  is convex and decreasing in all variables. In particular, the functions  $\pm\pi_i^\pm$  are concave. The general case follows by approximation by remarking that it is enough to find  $\beta$  increasing such that  $\beta \circ F$  and  $\beta \circ H_i$  satisfy the previous assumptions.

We now prove that

$$G(p', \lambda) = F(p', \pi^+(p', \lambda))$$

is convex w.r.t.  $(p', \lambda) \in \text{epi } A_0$ . For  $(p', \lambda), (q', \mu) \in \text{epi } A_0$  and  $t, s \geq 0$  with  $t + s = 1$ , we can use the monotonicity of  $F$  together with the concavity of  $\pi_i^+$  (see Lemma 3.2) to get

$$\begin{aligned} tG(p', \lambda) + sG(q', \mu) &\geq F(tp' + sq', t\pi^+(p', \lambda) + s\pi^+(q', \mu)) \\ &\geq F(tp' + sq', \pi^+(tp' + sq', t\lambda + s\mu)) \\ &= G(tp' + sq', t\lambda + s\mu). \end{aligned}$$

Similarly, we can see that  $G$  is non-increasing with respect to  $\lambda$ .

We next remark that

$$A_F(p') = G(p', A_F(p'))$$

and for  $p', q' \in \mathbb{R}^d$  and  $t, s \geq 0$  with  $t + s = 1$ , we can write

$$\begin{aligned} tA_F(p') + sA_F(q') &= tG(p', A_F(p')) + sG(q', A_F(q')) \\ &\geq G(tp' + sq', tA_F(p') + sA_F(q')) \end{aligned}$$

and

$$A_F(tp' + sq') = G(tp' + sq', A_F(tp' + sq')).$$

We thus deduce from the monotonicity of  $G$  in  $\lambda$  that

$$A_F(tp' + sq') \leq tA_F(p') + sA_F(q').$$

The proof is now complete. □

**Acknowledgements.** This work was partially supported by the ANR-12-BS01-0008-01 HJnet project.

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