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THE LIQUIDITY REGIMES AND THE PREPAYMENT OPTION OF A CORPORATE LOAN IN THE FINITE HORIZON CASE

TIMOTHEE PAPIN* AND GABRIEL TURINICI†

Abstract. We investigate the prepayment option related to a corporate loan. The default intensity of the firm is supposed to follow a Cox-Ingersoll-Ross (CIR) process and the short interest rate is assumed constant. A liquidity term that represents the funding costs of the bank is introduced and modeled as a continuous time discrete state Markov jump process. The prepayment option is an American option with the payoff being an implicit function of the parameters of the problem. We give a verification result that allows to compute the price of the option. Numerical results are completely consistent with the theory; it is seen that the exercise domain may entirely disappear during such a liquidity crisis meaning that it is not optimal for the borrower to prepay. The method allows to quantify and interpret these findings.

Key words. liquidity regime, loan prepayment, mortgage option, American option, option pricing, prepayment option, CIR process, switching regimes, Markov modulated dynamics.

AMS subject classifications. 91G20, 91G30, 91G40, 91G50, 91G60, 91G80, 93E20

1. Introduction. A loan contract issued by a bank for its corporate clients often includes a prepayment option which entitles the client to pay all or a fraction of its loan earlier than the maturity. The principle of such an option is very close to the embedded option of a callable bond, although the technical part is different.

We focus here on loans with continuous dividend payment and final down-payment of the principal; the loans are affected by borrower's default in which case interests stop at the time of default and only a fraction of the principal is recovered.

When the interest rates are constant and the borrower cannot default, a simple procedure allows the borrower to decide whether the exercise of the option is worthwhile: he compares the actualized value of the remaining payments with the nominal (outstanding) value of the loan, denoted by K . If the former is larger than the latter then it may be optimal to prepay.

When the interest rates are not constant or borrower is subject to default, the computation of the actualization is less straightforward. It involves all possible scenarios of the interest rate and default intensity in a risk-neutral framework in order to compute the average value of remaining payments (including the final payment of the principal if applicable); this quantity will be called "PVRP" and denoted ξ and is the *present value of the remaining payments*, i.e. the cash amount equivalent of the value of remaining payments.

To continue the evaluation of the prepayment option, the PVRP is compared with the nominal value K : if $\xi \geq K$ then the borrower could prepay; note that it is not always optimal to immediately prepay when $\xi = K$ because in some situations waiting longer can be even more worthwhile for the borrower.

The bank, that proposes the loan, finances it through a bond program (possibly mutualized for several loans) at some spread depending on its own credit profile and market conditions. In order for the corporate loan to be profitable the rate of the bond, that is also indexed on LIBOR or EURIBOR, has to be lower than the rate of the loan. This condition is easy to check at the origination of both contracts and is

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always enforced by the bank. However if the client prepays the bank finds itself in a risky situation: the periodic interest payments from client are terminated but the bank still has to pay the interests (indexed on a floating rate) and principal of its own bond; the bond does not have a prepayment option or such an option is costly. The risk is that the amount K received from the client at prepayment time cannot be invested in another product with interest rate superior to that of the bond. Therefore the prepayment option is essentially a reinvestment risk and the longer the maturity of the loan, the riskier the option.

According to the normal market practice, the borrower rarely pays a premium or a penalty to prepay his/her loan. However, the prepayment risk is not negligible and the bank needs to assess it to be protected against the liquidity risk.

Thus a first question is what provisions need to be made (at the bank level) in order to handle the prepayment risk. This is an option pricing problem that can be modeled as an embedded compound American option on a risky debt owned by the borrower. As Monte-Carlo simulations are slow to converge to assess accurately the continuation value of the option during the life of the loan and that the binomial tree techniques are time-consuming for long-term loans (cf. works by D. Cossin et al. (Cossin, 2002)), we decided to focus, in this paper, on PDE version instead of binomial trees and Monte Carlo techniques.

Another crucial question is whether it is possible that many clients decide to prepay simultaneously. This may occur in a post-crisis era when clients can borrow again at lower rates. We address this question by introducing liquidity regimes to model funding costs.

Probably the most characteristic of the liquidity is that it oscillates between distinct regimes following the state of the economic environment. Between two crisis, markets are confident and it is easier for the banks to issue bonds and the liquidity market is stable. However, during a crisis, liquidity is rare and the transition between these two distinct behaviors is rarely smooth and often sudden.

In order to model the presence of distinct liquidity regimes we simulate the liquidity cost by a continuous time observable Markov chain that can have a finite list of possible values, one for each liquidity regime (see equations (3.5) and (3.6) below). It was seen in Papin and Turinici (2013, 2014) that considering several liquidity regimes explains better clients' prepayment behavior than a constant liquidity model.

Specific technical circumstances make the evaluation of the prepayment option less straightforward: although the goal is to value an American option the payoff of the option is highly non-standard (is dependent on the $PV RP$) and is close to a compound option. The characterization of the exercise region is not standard and technical conditions have to be met. Furthermore our focus here is on a specific type of dynamics (of CIR type) with even more specific interest on the situation when several regimes are present.

The most important variable throughout the paper is the borrower credit risk defined by his/her default intensity (called in the following simply "intensity"); it follows a CIR stochastic process and the liquidity cost of the bank, defined as the cost of the lender to access the cash on the market, has several distinct regimes that we model by a Markov chain. We prove a verification theorem that allows to find the price of the option and a theoretical result that allows to use rigorously an algorithm to identify the boundary of the exercise region; numerical examples are also presented.

The plan of the paper is as follows: we discuss in Section 2 related works; then we introduce the model in Section 3, followed in Sections 4 and 5 by technical properties

of the *PVRP*. In Section 6 it is explained the term structure of the liquidity specific to this model. The theoretical result concerning the price of the prepayment option is given in Section 7 and the numerical results in Section 8.

2. Literature review. Historically works that involved a prepayment option appeared in the study of mortgages, see for instance (Hillard and Slawson, 1998) and (Chen et al., 2009) for recent contributions to that literature. In that view, the prepayment option is a function depending on the interest rate and house price. Their approach is based on a binomial tree with interest rate and a house value following a CIR process.

There also exist mortgage prepayment models that use the statistical data to infer information about the prepayment, see (Schwartz and Torous, 1993). However the volume and history of data in the corporate loan market are not compatible with these methods.

Another contribution by D. Cossin et al. (Cossin, 2002) uses a binomial tree framework and is applied to corporate loans. But the approach seems to be best adapted for short maturity loans due to the inherent computational difficulties in the numerical implementation of the binomial trees.

These papers considered the practical algorithm to find a solution and did not have to consider the geometry of the exercise region because it is explicitly given by the numerical algorithm. On the contrary we have to take this into account to check the optimality of the solution and furthermore, to the best of our knowledge, none of these approaches were meant to treat several liquidity regimes. In particular the technical hypothesis (7.10) in Lemma 7.1 was never invoked in previous works on finite maturity prepayment options.

The analysis of Markov regimes has been investigated in the literature when the underlying(s) follow the Black& Scholes dynamics with drift and volatility having Markov jumps: Guo and Zhang (Zhang and Guo, 2004) derived the closed-form solutions for vanilla American put; Guo treats in (Guo, 2001) Russian options and derives explicit solutions for the optimal stopping time; Mamon and Rodrigo find in (Mamon and Rodrigo, 2005) explicit solutions to vanilla European options. Buffington and Elliott address in (Buffington and Elliott, 2009) European and American options. Other contributions include Jobert and Rogers (2006), Yao et al. (2006), Siu et al. (2008).

A different class of contributions discuss the liquidity; among them several contributions point out that the liquidity displays "regimes" that is a finite list of distinctive macro-economic circumstances, see for instance (Dionne et al., 2011, Liang and Wang, 2012) and references within.

Works involving Markov switching regimes and CIR dynamics appears in Elliott and Siu (2009) where the bond valuation problem is considered but in their model only the mean reverting level is subject to Markov jumps. Moreover the techniques there do not apply to an American option which is our situation here. Or, as detailed below, the American option evaluation is very different from an European option and requires to check a particular hypothesis in order for the valuation to be carried over. In addition we propose a numerical method to solve the variational PDE associated to the American option.

Another related work is Zhou and Mamon (2011) where the term structure of the interest rates is analysed. A relevant connected work is Siu (2010) where the bond price is obtained when the short rate process is governed by a Markovian regime-switching jump-diffusion version of the Vasicek model; the authors provide in addition the suitable mathematical arguments to study piecewise Vasicek dynamics (here the

dynamics is still piecewise but CIR).

On the other hand numerical methods are proposed in Huang et al. (2011) for the prepayment option where it is found that a fixed point policy iteration coupled with a direct control formulation seems to perform best.

Finally, we refer to Jaillet et al. (1990) for theoretical results concerning the pricing of American options in general.

The pricing of a simplified one-dimensional model was proposed in an infinite horizon (perpetual) setting in Papin and Turinici (2013). In Papin and Turinici (2014) it is investigated in addition a non-trivial dynamics of the interest rate and a numerical algorithm is proposed together with the introduction of adapted functional spaces. This paper on the contrary addresses the finite horizon case, more close to the actual practice. We also discuss the term structure of liquidity costs which is not relevant in the perpetual setting.

3. Default intensity and theoretical regime switching framework. The prepayment option is an option on the credit risk, intensity and the liquidity cost. The intensity follows a Cox-Ingersoll-Ross process (see Cox et al. (1985), Alfonsi (2005), Lambertson and Lapeyre (2008) for theoretical and numerical aspects of CIR processes and the situations where the CIR process has been used in finance and Duffie et al. (2007), Duan et al. (2012) for some alternatives):

$$d\lambda_t = \gamma(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t, \quad \lambda_0 = \bar{\lambda}_0. \quad (3.1)$$

It is known (see references) that if $2\gamma\theta \geq \sigma^2$ the intensity is always strictly positive.

Denote by \mathcal{A} the characteristic operator (cf. (Øksendal, 2007, Chapter 7.5)) of the CIR process i.e. the operator that acts on any C^2 class function v by

$$(\mathcal{A}v)(t, \lambda) = \partial_t v(t, \lambda) + \gamma(\theta - \lambda)\partial_\lambda v(t, \lambda) + \frac{1}{2}\sigma^2\lambda\partial_{\lambda\lambda}v(t, \lambda). \quad (3.2)$$

The liquidity cost is defined as the specific cost of a bank to access the cash in a money market. This cost is modeled by a continuous time discrete space Markov chain depending on the states of the economy. We assume the interbank offered rate IBOR r to be constant. We assume the economic state of the market is described by a continuous time finite state Markov chain $\mathbf{X} = \{X_t, t \geq 0\}$. The state space of \mathbf{X} is the set of unit vectors $E = \{e_1, e_2, \dots, e_N\}$, $e_i = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^N$. Here T is the transposition operator.

Assuming the process X_t is homogeneous in time and has a rate matrix A , then if $p_t = \mathbb{E}[X_t] \in \mathbb{R}^N$:

$$\frac{dp_t}{dt} = Ap_t, \quad (3.3)$$

$$X_t = X_0 + \int_0^t AX_u du + M_t, \quad (3.4)$$

where $\mathbf{M} = \{M_t, t \geq 0\}$ is a martingale with respect to the filtration generated by \mathbf{X} . Denote by $a_{k,j}$ the entry on the line k and the column j of the $N \times N$ matrix A with $a_{k,j} \geq 0$ for $j \neq k$ and $\sum_{j=1}^N a_{k,j} = 0$ for any k .

In differential form

$$dX_t = AX_t dt + dM_t, \quad X_0 = \bar{X}_0. \quad (3.5)$$

We denote by $l_t \in \mathbb{R}$ the instantaneous liquidity cost of the bank when the economy is in state X_t . To ease notations we introduce the vector \mathcal{L} (containing the liquidity values for each regime) such that the liquidity values l_t are formally a stochastic process obtained by the scalar product:

$$l_t = \langle \mathcal{L}, X_t \rangle. \quad (3.6)$$

We consider that there is no correlation between the dynamics of the credit risk, i.e., the intensity λ_t , of the borrower and the cost to access the cash in a money market, i.e. the liquidity cost l_t , of the lender. Of course, this is a simplification and more general models can be considered. However this is reasonable to the first order because the two effects originate from different reasons: the liquidity is a macro-economic variable depending on the general state of the economy; on the contrary, the default risk depends on the specific characteristics of the borrower and its business plan. See Remarks 2 and 6 for additional comments.

4. Analytical formulas for the PVRP. At the origination, the payments correspond to an interest rate, the sum of the short term interest rate (e.g., LIBOR or EURIBOR) and the contractual margin $\bar{\rho}_0$ chosen such that $\xi = K$. The loan has a fixed coupon corresponding to the sum of the interest rate r and the initial contractual margin $\bar{\rho}_0$. Let $\xi(t, \lambda, X)$ be the present value of the remaining payments at time t of a corporate loan where λ is the intensity at time t , T is the contractual maturity, K is the nominal amount, δ is the recovery rate (in the eventuality of a default) and X_t is the state of the economy at time t . As the values K , δ and T are fixed, we omit them from all subsequent notations.

The loan value $LV(t, \lambda, X)$ is equal to the present value of the remaining payments $\xi(t, \lambda, X)$, minus the prepayment option value denoted by $P(t, \lambda, X)$:

$$LV(t, \lambda, X) = \xi(t, \lambda, X) - P(t, \lambda, X) \quad (4.1)$$

The PVRP ξ is the present value of the cash flows discounted at the risky rate, where the risky rate at time t is the constant risk-free rate r plus the liquidity cost l_t and the intensity λ_t . To describe the cash flows in a term loan we add for all t all coupons $K(r + \bar{\rho}_0)\Delta t + O(\Delta t)$ perceived between t and $t + \Delta t$ and the nominal K reimbursed at the end if no default occurred (and otherwise the portion of nominal recovered $\delta \cdot K$). The term $O(\Delta t)$ arises from the discretization error. We obtain (see also Lando (2004)):

$$\xi(t, \lambda, X) := \mathbb{E} \left[\int_t^T (K(r + \bar{\rho}_0) + \delta \cdot K\lambda_{\tilde{t}}) e^{-\int_t^{\tilde{t}} r + l_u + \lambda_u du} d\tilde{t} + K e^{-\int_t^T r + l_u + \lambda_u du} \right]_{\lambda_t = \lambda, X_t = X} \quad (4.2)$$

REMARK 1. *The exponential weighting term in the formula above expresses, as it is classical, the fact that, for the scenarios when default occurs, the payment of the interest stops at the time of default and is replaced by a one-time payment of the recovery amount δK .*

Since the dynamics of λ_t and l_t are independent we have,

$$\begin{aligned}\xi(t, \lambda, X) &= K(r + \bar{\rho}_0) \int_t^T e^{-r(\tilde{t}-t)} \mathbb{E} \left[e^{-\int_t^{\tilde{t}} \lambda_u du} \middle| \lambda_t = \lambda \right] \mathbb{E} \left[e^{-\int_t^{\tilde{t}} l_u du} \middle| X_t = X \right] d\tilde{t} \\ &\quad + \delta \cdot K \int_t^T e^{-r(\tilde{t}-t)} \mathbb{E} \left[\lambda_{\tilde{t}} e^{-\int_t^{\tilde{t}} \lambda_u du} \middle| \lambda_t = \lambda \right] \mathbb{E} \left[e^{-\int_t^{\tilde{t}} l_u du} \middle| X_t = X \right] d\tilde{t} \\ &\quad + K e^{-r(T-t)} \mathbb{E} \left[e^{-\int_t^T \lambda_u du} \middle| \lambda_t = \lambda \right] \mathbb{E} \left[e^{-\int_t^T l_u du} \middle| X_t = X \right]\end{aligned}\quad (4.3)$$

REMARK 2. *The coefficients γ , θ , σ of the CIR process do not depend on the regime X_t . Thus it is possible to separate the CIR dynamics and the Markov dynamics. A different approach can extend this result by using the properties of the PVRP as explained in Remark 6.*

We obtain furthermore:

$$\begin{aligned}\xi(t, \lambda, X) &= K(r + \bar{\rho}_0) \int_t^T e^{-r(\tilde{t}-t)} B(t, \tilde{t}, \lambda) \mathbb{E} \left[e^{-\int_t^{\tilde{t}} l_u du} \middle| X_t = X \right] d\tilde{t} \\ &\quad - \delta \cdot K \int_t^T e^{-r(\tilde{t}-t)} \partial_{\tilde{t}} B(t, \tilde{t}, \lambda) \mathbb{E} \left[e^{-\int_t^{\tilde{t}} l_u du} \middle| X_t = X \right] d\tilde{t} \\ &\quad + K e^{-r(T-t)} B(t, T, \lambda) \mathbb{E} \left[e^{-\int_t^T l_u du} \middle| X_t = X \right]\end{aligned}\quad (4.4)$$

where for general t, \tilde{t} we use the notation:

$$B(t, \tilde{t}, \lambda) = \mathbb{E} \left[e^{-\int_t^{\tilde{t}} \lambda_u du} \middle| \lambda_t = \lambda \right]. \quad (4.5)$$

The quantity $B(t, \tilde{t}, \lambda)$ shares the same mathematical expression with the price of a zero-coupon in a model where the short interest rates follow a CIR process. This remark allows to write (see (Cox et al., 1985, Lamberton and Lapeyre, 2008)):

$$B(t, \tilde{t}, \lambda) = \alpha(t, \tilde{t}) e^{-\beta(t, \tilde{t})\lambda}, \quad (4.6)$$

where,

$$\begin{aligned}\alpha(t, \tilde{t}) &= \left(\frac{2h e^{(\gamma+h)\frac{\tilde{t}-t}{2}}}{2h + (\gamma+h)(e^{(\tilde{t}-t)h} - 1)} \right)^{\frac{2\gamma\theta}{\sigma^2}} \\ \beta(t, \tilde{t}) &= \frac{2(e^{(\tilde{t}-t)h} - 1)}{2h + (\gamma+h)(e^{(\tilde{t}-t)h} - 1)}, \text{ where } h = \sqrt{\gamma^2 + 2\sigma^2}.\end{aligned}\quad (4.7)$$

Obviously $B(t, \tilde{t}, \lambda)$ is monotonic with respect to λ .

In order to compute $\mathbb{E} \left[e^{-\int_t^{\tilde{t}} l_u du} \middle| X_t = X \right]$, we invoke the time homogeneity of the Markov chain and obtain:

$$\mathbb{E} \left[e^{-\int_t^{\tilde{t}} l_u du} \middle| X_{\tilde{t}} = \langle \mathbf{X}, e_k \rangle \right] = \mathbb{E} \left[e^{-\int_0^{t-\tilde{t}} l_u du} \middle| X_{\tilde{t}} = \langle \mathbf{X}, e_k \rangle \right]. \quad (4.8)$$

Let $f_k(t)$ be defined by:

$$f_k(t) = \mathbb{E} \left[e^{-\int_0^t l_u du} \middle| X_0 = \langle \mathbf{X}, e_k \rangle \right]. \quad (4.9)$$

Therefore one can write:

$$\begin{aligned}
\xi(t, \lambda, e_k) &= K(r + \bar{\rho}_0) \int_t^T e^{-r(\tilde{t}-t)} B(t, \tilde{t}, \lambda) f_k(\tilde{t} - t) d\tilde{t} \\
&\quad - \delta \cdot K \int_t^T e^{-r(\tilde{t}-t)} \partial_t B(t, \tilde{t}, \lambda) f_k(\tilde{t} - t) d\tilde{t} \\
&\quad + K e^{-r(T-t)} B(t, T, \lambda) f_k(T - t).
\end{aligned} \tag{4.10}$$

Let τ be the time of the first jump from $X_0 = \langle \mathbf{X}, e_k \rangle$ to some other state. Recalling the previous definition of the parameters $a_{k,j}$ (elements of the matrix A) we know (see Lando (Lando, 2004) paragraph 7.7 p 211) that τ is a random variable following an exponential distribution of parameter α_k with,

$$\alpha_k = \sum_{j \neq k} a_{k,j}. \tag{4.11}$$

We also know that conditional to the fact that a jump has occurred at time τ the probability that the jump is from state e_k to state e_j is $p_{k,j}$, where

$$p_{k,j} = \frac{a_{k,j}}{\alpha_k}. \tag{4.12}$$

Thus,

$$\begin{aligned}
f_k(t) &= \mathbb{P}(\tau > t) e^{-\mathcal{L}_k t} + \mathbb{E} \left[\mathbb{P}(\tau \leq t) e^{-\mathcal{L}_k \tau} \sum_{j \neq k} \mathbb{P}(l_\tau = \mathcal{L}_j) \mathbb{E} \left[e^{-\int_\tau^t l_u du} \middle| X_\tau = \langle \mathbf{X}, e_j \rangle \right] \right] \\
&= e^{-(\mathcal{L}_k + \alpha_k)t} + \alpha_k \int_0^t e^{-(\mathcal{L}_k + \alpha_k)\tau} \sum_{j \neq k} p_{k,j} f_j(t - \tau) d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
e^{(\mathcal{L}_k + \alpha_k)t} f_k(t) &= 1 + \alpha_k \int_0^t e^{(\mathcal{L}_k + \alpha_k)(t-\tau)} \sum_{j \neq k} p_{k,j} f_j(t - \tau) d\tau \\
&= 1 + \alpha_k \int_0^t e^{(\mathcal{L}_k + \alpha_k)s} \sum_{j \neq k} p_{k,j} f_j(s) ds.
\end{aligned}$$

By differentiation with respect to t :

$$\frac{d}{dt} \left[e^{(\mathcal{L}_k + \alpha_k)t} f_k(t) \right] = \alpha_k e^{(\mathcal{L}_k + \alpha_k)t} \sum_{j \neq k} p_{k,j} f_j(t).$$

Then

$$\frac{df_k(t)}{dt} + (\mathcal{L}_k + \alpha_k) f_k(t) = \alpha_k \sum_{j \neq k} p_{k,j} f_j(t).$$

Thus,

$$\frac{df_k(t)}{dt} = \left[\sum_{j \neq k} \alpha_k p_{k,j} f_j(t) \right] - (\mathcal{L}_k + \alpha_k) f_k(t). \tag{4.13}$$

Denote $F(t) = (f_1(t), f_2(t), \dots, f_N(t))^T$ and introduce the $N \times N$ matrix B ,

$$B_{i,j} = \begin{cases} \alpha_i p_{i,j} & \text{if } i \neq j \\ -(\alpha_i + \mathcal{L}_i) & \text{if } i = j \end{cases}. \tag{4.14}$$

From equation (4.13) we obtain,

$$\frac{dF(t)}{dt} = BF(t) \text{ thus } F(t) = e^{Bt}F(0), \quad (4.15)$$

$$F(0) = \left(f_k(0) \right)_{k=1}^N = (1, 1, \dots, 1)^T \in \mathbb{R}^N. \quad (4.16)$$

We have therefore analytical formulas for the PVRP $\xi(t, \lambda, X)$. We refer the reader to Elliott and Siu (2009) for similar considerations on a related CIR switching dynamics.

REMARK 3. *When all liquidity parameters \mathcal{L}_k are equal (to some quantity l) then $B = A - l \cdot Id$ (here Id is the identity matrix) and then we obtain (after some computations) that $f_k(t) = e^{-lt}$ thus the payoff is equal to that of a one-regime dynamics with interest rate $r+l$, which is consistent with intuitive image we may have. Another limiting case is when the switching is very fast.*

The margin $\bar{\rho}_0$ is set to satisfy the equilibrium equation:

$$\xi(0, \bar{\lambda}_0, \bar{X}_0) = K, \quad (4.17)$$

which expresses the fact that the present value of the future cash flows (weighted consistently with the probability of survival) is equal to the nominal K . Therefore according to (4.10) if $\bar{X}_0 = \langle \mathbf{X}, e_{k_0} \rangle$,

$$\bar{\rho}_0 = \frac{1 + \delta \int_t^T e^{-r(\tilde{t}-t)} \partial_{\tilde{t}} B(t, \tilde{t}, \lambda) f_{k_0}(\tilde{t}-t) d\tilde{t} - e^{-r(T-t)} B(t, T, \lambda) f_{k_0}(T-t)}{\int_t^T e^{-r(\tilde{t}-t)} B(t, \tilde{t}, \lambda) f_{k_0}(\tilde{t}-t) d\tilde{t}} - r. \quad (4.18)$$

REMARK 4. *Note that we assume no additional commercial margin. If an additional commercial margin μ_0 is to be considered then $\bar{\rho}_0$ is first computed as above and then replaced by $\bar{\rho}_0 + \mu_0$ in Equation (4.2). Equation (4.17) still holds with some adapted value instead of $\bar{\lambda}_0$. With these changes all results in the paper are valid, with the provision that the price of the prepayment option is $P(0, \bar{\lambda}_0, \bar{X}_0)$.*

We will also need to introduce for any $k = 1, \dots, N$ the (parity boundary) function $\bar{\Lambda}_k^0(t)$ such that

$$\xi(t, \bar{\Lambda}_k^0(t), e_k) = K, \quad \forall t \in [0, T[. \quad (4.19)$$

Of course, $\bar{\Lambda}_{\bar{X}_0}^0(0) = \bar{\lambda}_0$. Moreover $\forall t \in [0, T]$, $\xi(t, \lambda, e_k)$ is decreasing with respect to λ ; when $\xi(t, 0, e_k) < K$ there is no solution to eqn. (4.17) and we will chose by convention $\bar{\Lambda}_k^0(t) = 0$. This can happen for a regime different from \bar{X}_0 (the initial regime) when the PVRP value is always less than the parity, see also latter the numerical application. In this case there is never a parity boundary in this regime and no exercise. We put $\bar{\Lambda}_k^0(t) = 0$ just as a mathematical convention (which allows to unify the mathematical formulas and not distinguish such a case separately).

REMARK 5. *The value $\bar{\Lambda}_k^0(T) = \bar{\Lambda}_k^0(t) \Big|_{t=T}$ is not defined because*

$$\xi(T, \lambda, e_k) = K, \quad \forall \lambda \in \mathbb{R}^+, \forall k. \quad (4.20)$$

5. Further properties of the PVRP ξ . We also introduce below a PDE formulation for ξ . To ease the notations we introduce the operator $\mathcal{A}^{\mathcal{R}}$ that acts on functions $v(t, \lambda, X)$ as follows:

$$(\mathcal{A}^{\mathcal{R}}v)(t, \lambda, e_k) = (\mathcal{A}v)(t, \lambda, e_k) - (r + \mathcal{L}_k + \lambda)v(t, \lambda, e_k) + \sum_{j=1}^N a_{k,j} \left(v(t, \lambda, e_j) - v(t, \lambda, e_k) \right). \quad (5.1)$$

Having defined the dynamics (3.1) and (3.5) one can use an adapted version of the Feynman-Kac formula (in the same way as in Elliott and Siu (2009), Siu (2010), Zhang and Guo (2004)) in order to conclude that PVRP defined by (4.2) satisfies the equation:

$$\begin{cases} (\mathcal{A}^{\mathcal{R}}\xi)(t, \lambda, e_k) + (\delta \cdot \lambda + r + \bar{\rho}_0)K = 0, \\ \xi(T, \lambda, e_k) = K, \quad \forall \lambda > 0 \text{ and } \forall e_k \in E. \end{cases} \quad (5.2)$$

REMARK 6. *When the coefficients γ , θ , σ of the CIR process depend on the regime (see Remark 2) the analytical formulas in the previous section do not work any more; however the PDE formulation will work with the operator $\mathcal{A}^{\mathcal{R}}$ in Equation (5.2) replaced by the operator $\mathcal{A}_k^{\mathcal{R}}$ defined as:*

$$\begin{aligned} \mathcal{A}_k^{\mathcal{R}}(v)(t, \lambda, e_k) &= \partial_t v(t, \lambda, e_k) + \gamma_k(\theta_k - \lambda)\partial_\lambda v(t, \lambda, e_k) + \frac{1}{2}\sigma_k^2\lambda\partial_{\lambda\lambda}v(t, \lambda, e_k) \\ &- (r + \mathcal{L}_k + \lambda)v(t, \lambda, e_k) + \sum_{j=1}^N a_{k,j} \left(v(t, \lambda, e_j) - v(t, \lambda, e_k) \right). \end{aligned} \quad (5.3)$$

See also the discussion in Remark 8 for the boundary conditions.

6. Term-structure of the liquidity cost. The continuous time Markov chain allows to define the liquidity cost of the bank to access the cash on the market according to several distinct regimes. Therefore in each regime we can build a term-structure of the liquidity cost that represents the cost at different terms (time horizons). The term-structure curve of the liquidity cost can be compared to a term structure curve for the interest rates. In the more stressful regime, the curve will be inverted (convex) and the short maturities have a higher price than medium-term maturities. It is the rarest type of curve and is observed during an economic recession (see Figure 8.1). The liquidity cost $L_{t,T}$ for a contractual maturity T at time t is defined by the following equality:

$$e^{-L_{t,T} \cdot (T-t)} = \mathbb{E} \left[e^{-\int_t^T l_u du} \mid X_t = \langle \mathbf{X}, e_k \rangle \right]. \quad (6.1)$$

Therefore,

$$L_{t,T} = -\frac{\ln(f_k(T-t))}{T-t}. \quad (6.2)$$

7. Valuation of the prepayment option. The valuation problem of the prepayment option is equivalent to that of an American call option on a risky debt owned by the borrower. It is not a vanilla option but rather close in principle to a compound product because the payoff is at its turn a contingent claim. The prepayment option allows borrower to buy back and refinance his debt according to his updated credit profile at any time during the life of the option.

As discussed above, the prepayment exercise results in a pay-off $(\xi(t, \lambda) - K)^+$ for the borrower. The option is therefore an American option with pay-off:

$$\chi(t, \lambda, X) = (\xi(t, \lambda, X) - K)^+. \quad (7.1)$$

The following result allows to compute the price of the prepayment option.

THEOREM 7. *For each function $\Lambda : [0, T] \rightarrow (\mathbb{R}_+)^N$ which is C^2 on $[0, T[$ and such that the domain $\{(t, \lambda) | t \in]0, T[, \lambda > \Lambda_k(t)\}$ is locally Lipschitz for any $k = 1, \dots, N$, define $P_\Lambda(t, \lambda, X)$ such that:*

$$P_\Lambda(\cdot, \cdot, e_k) \text{ is Lipschitz continuous on } [0, T] \times [0, \infty[, \forall k \quad (7.2)$$

$$P_\Lambda(t, \lambda, e_k) = \chi(t, \lambda, e_k), \quad \forall \lambda \in [0, \Lambda_k(t)], \quad t \in [0, T] \quad (7.3)$$

$$(\mathcal{A}^\mathcal{R} P_\Lambda)(t, \lambda, e_k) = 0, \quad \forall \lambda > \Lambda_k(t), \quad t \in]0, T[\quad k = 1, \dots, N \quad (7.4)$$

$$\lim_{\lambda \rightarrow \infty} P_\Lambda(t, \lambda, e_k) = 0, \quad k = 1, \dots, N, \quad t \in [0, T] \quad (7.5)$$

$$P_\Lambda(T, \lambda, e_k) = 0, \quad k = 1, \dots, N \text{ and } \forall \lambda > 0. \quad (7.6)$$

Suppose a function $\Lambda^* : [0, T] \rightarrow (\mathbb{R}_+)^N$ (satisfying same hypotheses as above) exists such that $\Lambda^*(t) \in \prod_{k=1}^N [0, (\bar{\rho}_0 - \mathcal{L}_k)^+ \wedge \bar{\Lambda}_k^0(t)]$ and for all $k = 1, \dots, N$ and $\forall t \in]0, T[$:

$$\left\| \frac{d\Lambda^*(t)}{dt} \right\| \leq \infty, \quad \forall t < T, \quad \forall \lambda, \quad (7.7)$$

$$P_{\Lambda^*}(t, \lambda, X) \geq \chi(t, \lambda, X) \quad \forall t, \lambda, X, \quad (7.8)$$

$$\frac{\partial P_{\Lambda^*}(t, \lambda, e_k)}{\partial \lambda} \Big|_{\lambda=(\Lambda_k^*(t))^+} = \frac{\partial \chi(t, \lambda, e_k)}{\partial \lambda} \Big|_{\lambda=(\Lambda_k^*(t))^-} \quad \text{if } \Lambda_k^*(t) > 0, \quad (7.9)$$

$$\sum_{j=1}^N a_{k,j} \left(P_{\Lambda^*}(t, \lambda, e_j) - \chi(t, \lambda, e_j) \right) + K(\mathcal{L}_k + \lambda(1 - \delta) - \bar{\rho}_0) \leq 0, \quad (7.10)$$

$$\forall \lambda \in] \min_j \Lambda_j^*(t), \Lambda_k^*(t) [.$$

Then $P = P_{\Lambda^*}$.

The Theorem can be rephrased as follows: for each candidate for the exercise boundary Λ one can construct a price candidate P_Λ ; the real price will correspond to the optimal boundary Λ^* which is the one that fulfills the conditions listed in the second part of the Theorem. In the exercise region the price has to be equal to the payoff. However in the non-exercise region the price is only given by a PDE since in this region the exercise value is always higher than the payoff because more is gained from waiting than from exercising. The PDE reflects the local nature of the properties of the price.

REMARK 8. *Any American option satisfies a PDE equation in its continuation part, which is here equation (7.4). The equation depends only on the dynamics of the underlying because it uses the generator (here $\mathcal{A}^\mathcal{R}$) of the stochastic process. The Equation (7.4) is to be interpreted in the sense of viscosity solutions, see Crandall et al. (1992) for an introduction, Barles and Burdeau (1995) for a treatment of degenerate PDEs and Achdou and Pironneau (2005) for an explanation of how the introduction of weighted Sobolev spaces can also help to give a meaning to this equation. When some $\Lambda_k^*(t) = 0$ no boundary conditions are needed at $\lambda = 0$ as the solution will select by itself the right value. The requirement (7.2) selects an unique solution to (7.3)-(7.6).*

On the contrary, equations such as (7.9) and (7.10) depend on our proof strategy (see Lemma 7.1).

Proof. The Theorem is a verification result. As such we suppose that a "solution candidate" is presented to us that satisfies all hypothesis and we prove that this is indeed the price of the option. This is proved in the form of two inequalities: first we prove that the candidate is a lower bound for the price, then we prove that it is an upper bound for the price. The first implication is more straightforward because the price is a supremum over all stopping times. The second implication is more involved and requires the use of Lemma 7.1.

The valuation of an American option related the admissible trading and consumptions strategies to the price $P(t, \lambda, X)$ of the prepayment option by computing an optimal stopping time associated to the pay-off. Denote by $\mathcal{T}_{t,T}$ the ensemble of stopping times between t and T , then (see Musiela and Rutkowski (2005, Chapter 5) and also Myneni (1992), Lamberton and Lapeyre (2008)):

$$P(t, \lambda, X) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} \left[e^{-\int_t^\tau r+l_u+\lambda_u du} \chi(t, \lambda_\tau, X_\tau) \middle| \lambda_t = \lambda, X_t = X \right].$$

We note that if τ_Λ is the stopping time associated with exiting the domain

$$\bigcup_{k=1}^N \{(t, \lambda) | \lambda > \Lambda_k(t), t \leq T\},$$

then:

$$P_\Lambda(t, \lambda, X) = \mathbb{E} \left[e^{-\int_0^{\tau_\Lambda} r+l_u+\lambda_u du} \chi(t, \lambda_{\tau_\Lambda}, X_{\tau_\Lambda}) \middle| \lambda_t = \lambda, X_t = X \right].$$

Thus for any Λ we have $P \geq P_\Lambda$; when Λ has some null coordinates the continuity (ensured among others by the Remark 8) shows that we still have $P \geq P_\Lambda$. In particular for Λ^* we obtain $P \geq P_{\Lambda^*}$; all that remains to be proved is the reverse inequality i.e. $P \leq P_{\Lambda^*}$.

Note first that since Λ^* is a C^2 function the curve $\Lambda^*(t)$ is C^2 and thus in particular the tangent is defined at any point. Both P_{Λ^*} and χ are C^2 on each subdomain delimited by $\Lambda^*(t)$ and in particular condition (7.3) implies that the derivatives of $P_{\Lambda^*}(\cdot, \cdot, e_k)$ and $\chi(\cdot, \cdot, e_k)$ match along the tangent to $\Lambda_k^*(t)$ (for any k). The condition (7.7) implies that the direction of the λ axis is linearly independent from the tangent to $\Lambda_k^*(t)$ and since the derivatives of $P_{\Lambda^*}(\cdot, \cdot, e_k)$ and $\chi(\cdot, \cdot, e_k)$ match also along the direction of the λ axis this means that they are equal in any direction. Therefore for any k the function $P_{\Lambda^*}(\cdot, \cdot, e_k)$ is of C^1 class on the whole domain.

Now we follow the technique in Thm. 10.4.1 (Øksendal, 2007, Section 10.4 page 227) (see also Zhang and Guo (2004) for similar considerations). First one can use the same arguments as in Øksendal (2007, Appendix D) and work as if P_{Λ^*} is C^2 (not only C^1 as the hypothesis ensures).

Denote $\mathcal{D}_{\Lambda^*} = \bigcup_{k=1}^N \{(t, \lambda, e_k) | \lambda \in [0, \Lambda_k^*(t)]\}$ (this will be the exercise region) and $\mathcal{C}_{\Lambda^*} = \mathbb{R}_+^2 \times E \setminus \mathcal{D}_{\Lambda^*}$ (this will be the continuation region).

The Lemma 7.1 proves that $\mathcal{A}^{\mathcal{R}} P_{\Lambda^*}$ is non-positive everywhere (and is null on \mathcal{C}_{Λ^*}). The Itô formula allows to write:

$$d \left(e^{-\int_0^t r+l_s+\lambda_s ds} P_{\Lambda^*}(t, \lambda_t, X_t) \right) = e^{-\int_0^t r+l_s+\lambda_s ds} (\mathcal{A}^{\mathcal{R}} P_{\Lambda^*})(t, \lambda_t, X_t) dt + dY_t, \quad (7.11)$$

where Y_t is a martingale. Taking averages and integrating from 0 to a stopping time τ it follows from $\mathcal{A}^{\mathcal{R}}P_{\Lambda^*} \leq 0$ that:

$$\begin{aligned} P_{\Lambda^*}(t, \lambda, X) &\geq \mathbb{E} \left[e^{-\int_0^\tau r+L_u+\lambda_u du} P_{\Lambda^*}(t, \lambda_\tau, X_\tau) \middle| \lambda_0 = \lambda, X_0 = X \right] \\ &\geq \mathbb{E} \left[e^{-\int_0^\tau r+L_u+\lambda_u du} \chi(t, \lambda_\tau, X_\tau) \middle| \lambda_0 = \lambda, X_0 = X \right]. \end{aligned}$$

Since this is true for any stopping time τ the conclusion follows. \square

REMARK 9. *An alternative proof is to recall that the price P is the unique solution of the following quasi-variational inequality, see Bensoussan and Lions (1982), Bardi and Capuzzo-Dolcetta (1997):*

$$\max\{\mathcal{A}^{\mathcal{R}}P, \chi - P\} = 0. \quad (7.12)$$

But Lemma 7.1 and hypothesis (7.8) imply that P_{Λ^*} is also a solution of this problem. By uniqueness $P = P_{\Lambda^*}$. In this approach one has nevertheless to check the technical points related to the viscosity interpretation of the equation for the price: formal definition of the quasi-variational inequality, equivalence with the price, uniqueness of the solution, a priori Lipschitz regularity of the price.

LEMMA 7.1. *Under the hypothesis of the Thm. 7 the following inequality holds (strongly except for the values $(t, \lambda, X) = (t, \Lambda_j^*, e_k)$ where it holds in the viscosity sense):*

$$(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(t, \lambda, X) \leq 0, \quad \forall \lambda > 0, \forall X. \quad (7.13)$$

Proof.

The result is needed for the proof of the previous Theorem. The non-trivial part of this lemma comes from the fact that if for fixed k we have for λ in a neighborhood of some λ_1 : $P_{\Lambda^*}(t, \lambda, e_k) = \chi(t, \lambda, e_k)$ this does not necessarily imply $(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(t, \lambda_1, e_k) = (\mathcal{A}^{\mathcal{R}}\chi)(t, \lambda_1, e_k)$ because $\mathcal{A}^{\mathcal{R}}$ depends on other values $P_{\Lambda^*}(t, \lambda, e_j)$ with $j \neq k$.

From (7.4) the conclusion is immediately verified for $X = e_k$ for any $\lambda \in]\Lambda_k^*(t), \infty[$.

We now treat the alternative $\lambda < \min_j \Lambda_j^*(t)$; in particular it follows that $0 \leq \lambda < \min_j \Lambda_j^*(t) \leq \bar{\Lambda}_\ell^0(t)$ for any ℓ thus $\bar{\Lambda}_\ell^0(t) > 0$. Note that $\Lambda_k^*(t) < \bar{\Lambda}_k^0(t)$ implies $\xi(t, \Lambda_k^*(t), e_k) \geq \xi(t, \bar{\Lambda}_k^0(t), e_k) = K$ for any $k = 1, \dots, N$ thus $\chi(t, \lambda, e_k) = \xi(t, \lambda, e_k) - K$ for any $\lambda \in [0, \Lambda_k^*(t)]$ and any k . Furthermore since $\lambda < \min_j \Lambda_j^*(t)$ we obtain $P_{\Lambda^*}(t, \lambda, e_k) = \chi(t, \lambda, e_k) = \xi(t, \lambda, e_k) - K$ for any k . Fix $X = e_k$; then:

$$\begin{aligned} (\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(t, \lambda, e_k) &= (\mathcal{A}^{\mathcal{R}}\chi)(t, \lambda, e_k) \\ &= (\mathcal{A}^{\mathcal{R}}(\xi - K))(t, \lambda, e_k) \\ &= (\mathcal{A}^{\mathcal{R}}\xi)(t, \lambda, e_k) - \mathcal{A}^{\mathcal{R}}(K) \\ &= -(\delta \cdot \lambda + r + \bar{\rho}_0)K + (r + \mathcal{L}_k + \lambda)K \\ &= K(\mathcal{L}_k + (1 - \delta)\lambda - \bar{\rho}_0) \\ &\leq K(\mathcal{L}_k + (1 - \delta)\Lambda_k^*(t) - \bar{\rho}_0) \leq 0 \end{aligned} \quad (7.14)$$

the last inequality being true by hypothesis.

The last alternative is $\lambda \in]\min_j \Lambda_j^*(t), \Lambda_k^*(t)[$; in this case $P_{\Lambda^*}(t, \lambda, e_k) = \chi(t, \lambda, e_k)$ but some terms $P_{\Lambda^*}(t, \lambda, e_j)$ for $j \neq k$ may differ from $\chi(t, \lambda, e_j)$. This point arises

because the payoff χ itself has a complex structure and the technical condition to be satisfied was not treated in previous works (see for instance Zhang and Guo (2004)).

Recalling the properties of ξ one obtains (and since $P_{\Lambda^*}(t, \lambda, e_k) = \chi(t, \lambda, e_k)$):

$$\begin{aligned}
(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(t, \lambda, e_k) &= (\mathcal{A}\chi)(t, \lambda, e_k) - (r + \mathcal{L}_k + \lambda)\chi(t, \lambda, e_k) + \sum_{j=1}^N a_{k,j} \left(P_{\Lambda^*}(t, \lambda, e_j) - \chi(t, \lambda, e_k) \right) \\
&= (\mathcal{A}^{\mathcal{R}}\chi)(t, \lambda, e_k) + \sum_{j=1}^N a_{k,j} \left(P_{\Lambda^*}(t, \lambda, e_j) - \chi(t, \lambda, e_j) \right) \\
&= (\mathcal{A}^{\mathcal{R}}\xi)(t, \lambda, e_k) - \mathcal{A}^{\mathcal{R}}(K) + \sum_{j=1}^N a_{k,j} \left(P_{\Lambda^*}(t, \lambda, e_j) - \chi(t, \lambda, e_j) \right) \\
&= -K(\delta \cdot \lambda + r + \bar{\rho}_0) + (r + \mathcal{L}_k + \lambda)K + \sum_{j=1}^N a_{k,j} \left(P_{\Lambda^*}(t, \lambda, e_j) - \chi(t, \lambda, e_j) \right) \leq 0, \tag{7.15}
\end{aligned}$$

where for the last inequality we use hypothesis (7.10). Finally, since we proved that $(\mathcal{A}^{\mathcal{R}}P_{\Lambda^*})(t, \lambda, X) \leq 0$ strongly except for the values $(t, \lambda, X) = (t, \Lambda_j^*(t), e_k)$ and since P_{Λ^*} is of C^1 class we obtain the conclusion. \square

REMARK 10. *Several remarks are relevant here:*

1. *when $N > 1$ checking (7.10) does not involve any computation of derivatives and is straightforward.*
2. *the Theorem is a verification result i.e., only gives sufficient conditions for a candidate to be the option price. However results in the literature (see Chen and Chadam (2007) and references within) indicate that the boundary will probably be even more regular, C^∞ on $[0, T[$. The behaviour near final time T is not expected to be singular with respect to λ (because there is no singularity in the payoff function there) but we do not exclude that $\lim_{t \uparrow T} \frac{d\Lambda_k^*(t)}{dt} = \infty$ which is equivalent to say that the derivative with respect to λ of the inverse of $t \mapsto \Lambda_k^*(t)$ is null.*
3. *The condition (7.10) is a sufficient technical hypothesis. Its role is to ensure that the solution has local optimality properties, i.e., waiting beyond the optimal exercise time is not optimal.*

8. Numerical Application. The numerical resolution of the partial differential equation (7.4) is required in order to compute the value of the prepayment option. We describe below the use of a finite difference method as discretization choice, but some cases may require different treatment.

To avoid working with an infinite domain we truncate at λ_{max} . Then a boundary condition is imposed on λ_{max} which leads to a numerical problem posed in the finite domain $\cup_{k=1}^N \{(t, \lambda) | \lambda \in [\Lambda_k(t), \lambda_{max}]\}$.

We introduce the time step Δt and space step $\Delta \lambda$ and look for an approximation $P_{\ell, k}^n$ of $P_{\Lambda}(n\Delta t, \ell\Delta \lambda, e_k)$. The first and second derivative are approximated by (centered) finite difference formula and the time propagation by a Crank-Nicholson

scheme:

$$\begin{aligned}
& \frac{P_{\ell,k}^{n+1} - P_{\ell,k}^n}{\Delta t} + \frac{\gamma(\theta - (\ell\Delta\lambda))}{2} \left[\frac{P_{\ell+1,k}^{n+1} - P_{\ell-1,k}^{n+1}}{2\Delta\lambda} + \frac{P_{\ell+1,k}^n - P_{\ell-1,k}^n}{2\Delta\lambda} \right] \\
& + \frac{\sigma^2}{4}(\ell\Delta\lambda) \left[\frac{P_{\ell+1,k}^{n+1} - 2P_{\ell,k}^{n+1} + P_{\ell-1,k}^{n+1}}{\Delta\lambda^2} + \frac{P_{\ell+1,k}^n - 2P_{\ell,k}^n + P_{\ell-1,k}^n}{\Delta\lambda^2} \right] \\
& - (r + \mathcal{L}_k + (\ell\Delta\lambda)) \frac{P_{\ell,k}^{n+1} + P_{\ell,k}^n}{2} + \sum_{j=1}^N a_{k,j} \left[\frac{P_{\ell,j}^{n+1} - P_{\ell,k}^{n+1}}{2} + \frac{P_{\ell,j}^n - P_{\ell,k}^n}{2} \right] = 0
\end{aligned} \tag{8.1}$$

At each time step the advancement of this implicit time stepping scheme required the resolution of one linear system. A standard computation shows that the truncation error of this scheme is $O(\Delta t^2 + \Delta\lambda^2)$, see Press et al. (2007) for details. This truncation arises from the use of time steps Δt and $\Delta\lambda$. Our implementation of this finite difference method is written in *MATLAB*[©]. As the method is implicit it is unconditionally stable for our setting.

See also Remark 8 for the situation when some $\Lambda_k(t)$ is null: there the PDE for regime e_k is defined over the full semi-axis $\lambda > 0$ and it is never optimal to exercise in this regime. The PDE is defined with homogeneous boundary conditions at λ_{max} (or Neumann, see below) and without any boundary conditions at $\lambda = 0$. To ensure the same number of equations and unknowns the equation is discretized at $\lambda = 0$ too but the second order derivative is null there. Only first order terms and a first order derivative remain. The first order derivative is discretized with a lateral second order finite difference formula that involves only the function values at $\lambda = 0, \Delta\lambda, 2\Delta\lambda$ using the identity:

$$f'(x) = \frac{-\frac{3}{2}f(x) + 2f(x+h) - \frac{1}{2}f(x+2h)}{h} + O(h^2). \tag{8.2}$$

We consider a numerical application with $\lambda_{max} = 1000$ bps, $\Delta\lambda = 1/5$ bps and $\Delta t = 1/12$. Two approaches have been considered for imposing a boundary value at λ_{max} : either consider that $P_\Lambda(0, \lambda_{max}, e_k) = 0, \forall k = 1, \dots, N$ (homogeneous Dirichlet boundary condition) or that $\frac{\partial}{\partial\lambda} P_\Lambda(0, \lambda_{max}, e_k) = 0, \forall k = 1, \dots, N$ (homogeneous Neumann boundary condition). Both are correct in the limit $\lambda_{max} \rightarrow \infty$. We tested the precision of the results by comparing with numerical results obtained on a much larger grid (10 times larger) while using same $\Delta\lambda$. The Neumann boundary condition gives much better results for the situations we considered and as such was always chosen (see also Figure 8.5).

We consider a loan with a contractual maturity $T = 5$ years, a nominal amount $K = 1$, a recovery rate $\delta = 40\%$. The borrower's default intensity λ_t follows a CIR dynamics with parameters: initial intensity $\bar{\lambda}_0 = 150$ bps, volatility $\sigma = 0.1$, average intensity $\theta = 150$ bps, reversion coefficient $\gamma = 0.5$. We assume a constant interest rate $r = 1\%$ and a liquidity cost defined by a Markov chain of three states $\mathcal{L}_1 = 15$ bps (economic expansion), $\mathcal{L}_2 = 30$ bps (economic stability) and $\mathcal{L}_3 = 250$ bps (recession). The values for the liquidity costs were chosen as follows: we selected a panel of major banks (financing activities in euros) and considered the term structure of the liquidity costs of these banks during the 2007-2008 crisis. The liquidity coefficients \mathcal{L}_k were chosen such that the term structure generated by \mathcal{L}_k match the one observed. For

$N = 3$ the rate 3×3 matrix A is defined as following,

$$A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -2 & 1 \\ 0 & \frac{1}{10} & -\frac{1}{10} \end{pmatrix}. \quad (8.3)$$

This choice is only an example to support the theoretical results, other choices of matrices A are possible and in practice this should be calibrated from macro-economic data (e.g., for instance GDP or unemployment time-series data). The interested reader can consult Sichel (1994), Hamilton (2005) and related literature; the values given here are compatible (as order of magnitude) with the cited works.

The choice was selected in order to have a matrix that displays inversion of the liquidity term structure. On the other hand the matrix has qualitatively some properties that are often seen in practice:

- the rate of transition from recession to expansion (or vice-versa) is null,
- the rate of transition from stability to either of recession or expansion are of the same order of magnitude,
- the rate of transition from expansion to stability (1/10) is lower than from recession to stability (1/2).

We computed the term-structure of the liquidity cost for each state which is plotted in Figure 8.1. At inception, we assume the liquidity cost is in the state e_2 , so $\overline{X}_0 = e_2$. Recall that a basis point, denoted "1 bp" equals 10^{-4} .

In order to find the initial contractual margin we use equation (4.18) and find $\overline{\rho}_0 = 228$ bps at inception in the state e_2 . For information, the contractual margin is $\overline{\rho}_0 = 175$ bps in the lowest state e_1 and $\overline{\rho}_0 = 313$ bps in the highest state e_3 . We recall that the contractual margin takes into account the credit risk (default intensity) and the liquidity cost.

The function $\Lambda_k^*(t)$ is obtained by maximizing $P_\Lambda(t, \lambda, \overline{X}_0)$ backward for all $t \in [0, T[$ and each state k . This is realized numerically in the following way: if the current time step is $(n+1)\Delta t$ then the function $\Lambda_k(n\Delta t) \mapsto P_\Lambda(n\Delta t, \lambda, \overline{X}_0)$ is maximized with respect to the candidate value $\Lambda_k(n\Delta t)$. The maximum will correspond to the optimal boundary $\Lambda_k^*(n\Delta t)$. This is motivated by the stopping time interpretation that appears in the beginning of the proof of the Theorem 7. To accelerate the optimization process, for the initial guess at step at $t = T - \Delta t$ we note that there is little time to switch from the current regime to another. Therefore, we use the optimal boundary for each regime independently (one-regime model), see Figure 8.2, as initial guess. Let $\Upsilon_k^*(T - \Delta t)$ be the optimal boundary for the constant one-regime $X = e_k$ option. We propose as initial guess for $\Lambda^*(T - \Delta t)$ the vector $(\Upsilon_k^*(T - \Delta t))_{k=1}^N$. This initial guess is validated by computing the value $P_\Lambda(t, \overline{\lambda}_0, \overline{X}_0)$ for all neighbors around $(\Upsilon_k^*(T - \Delta t))_{k=1}^N$ in the N-dimensional space where Λ belongs.

Then for each time $t < T - \Delta t$, we search the optimal boundary in the neighborhood of the previous optimal boundary obtained at $t + \Delta t$.

To be accepted, this numerical solution has to verify all conditions of the Theorem 7. The hypothesis (7.8) and (7.10) are satisfied (see Figure 8.5) and the hypothesis (7.10) is accepted after calculation. Moreover $\forall t \in [0, T[, \Lambda_1^*(t) \leq (\overline{\rho}_0 - \mathcal{L}_1) \wedge \overline{\Lambda}_1^0(t)$ and the analogous holds for $\Lambda_2^*(t)$.

In the state $\overline{X}_0 = e_2$ and at inception, the present value of cash flows is at par, so $\xi(0, \overline{\lambda}_0, \overline{X}_0) = 1$. The prepayment option price is $P(0, \overline{\lambda}_0, \overline{X}_0) = 0.0136$. Therefore the loan value equals $\xi(0, \overline{\lambda}_0, \overline{X}_0) - P(0, \overline{\lambda}_0, \overline{X}_0) = 0.9864$.

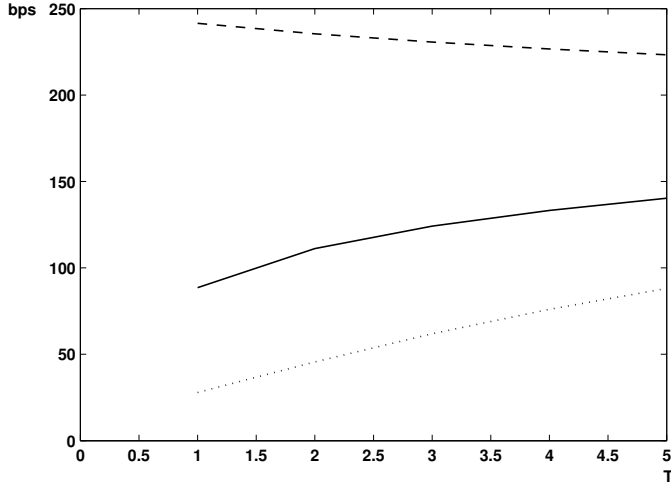


FIG. 8.1. We plot the term-structure of the liquidity cost in bps in the regime $X = e_1$ (dashed, recession), $X = e_2$ (solid, economic stability) and the regime $X = e_3$ (dotted, economic expansion). The curve for the recession regime appears inverted, which transcribes scarcity of short-term money market, in coherence with liquidity prices.

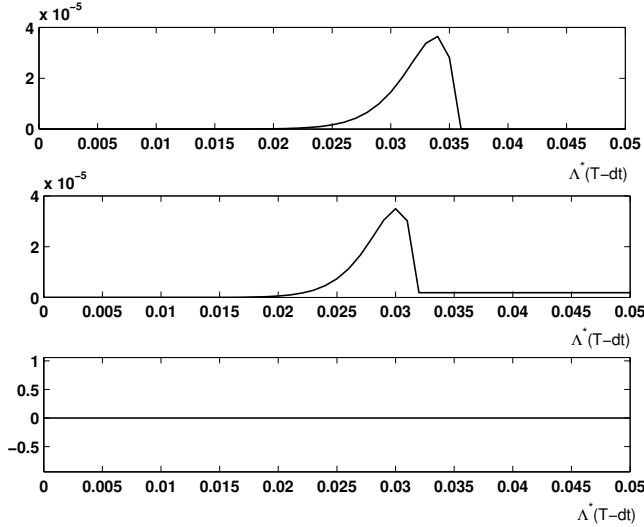


FIG. 8.2. We search for the exercise boundary $\Lambda_k^*(T - dt)$ that maximizes the option price in the state $X = e_1$ (top), $X = e_2$ (middle) and $X = e_3$ (bottom). We obtain $\Lambda_1^*(T - dt) = 339$ bps, $\Lambda_2^*(T - dt) = 301$ bps and $\Lambda_3^*(T - dt) = 0$ bps. Remark: The state $X = e_3$ is a particular case where there exist no exercise boundary since the pay-off is null for all $\lambda > 0$.

The loan value will be equal to the nominal if the intensity decreases until the exercise region $\lambda \leq \Lambda^*$, see Figure 8.4. The continuation and exercise regions are depicted in Figure 8.5.

The following overall algorithm was implemented:

1. Find the payoff:

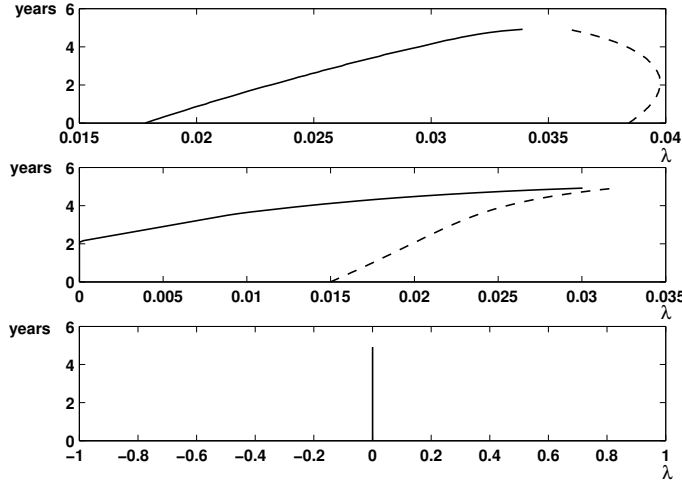


FIG. 8.3. We plot the evolution of the exercise boundary $\Lambda_k^*(t)$ (solid) that maximizes the option price and the par boundary $\bar{\Lambda}_k^0(t)$ (dashed) where $\xi(t, \lambda, X)$ verifies the equation (4.19), for all $t \in [0, T[$ in the state $X = e_1$ (top), $X = e_2$ (middle) and $X = e_3$ (bottom). We obtain $\Lambda_1^*(0) = 178$ bps, $\Lambda_2^*(0) = 0$ and $\Lambda_3^*(0) = 0$ bps. The x axis is λ and the y-axis is the time.

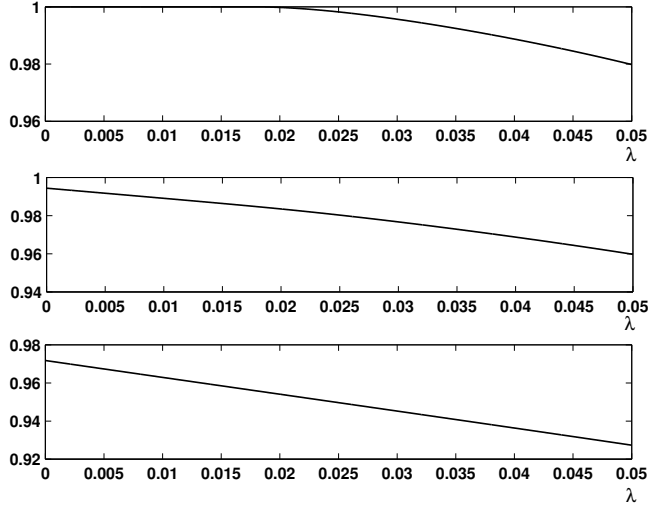


FIG. 8.4. We plot the loan value (at $t = 0$) as a function of the intensity λ in the state $X = e_1$ (top), $X = e_2$ (middle) and $X = e_3$ (bottom). The loan value is decreasing when there is a degradation of the credit quality (that is λ increases) and converges to 0.

- select the option parameters $A, l, K, T, \sigma, \dots$;
- choose the discretization parameters $\lambda_{max}, \Delta t, \Delta \lambda$;
- start an iterative procedure to compute $\bar{\rho}_0$ as the zero of the function of $\bar{\rho}_0 \mapsto \xi - K$; the procedure is a one-dimensional search that uses the function $\bar{\rho}_0 \mapsto \xi$. For any given $\bar{\rho}_0$, ξ is computed by solving equation

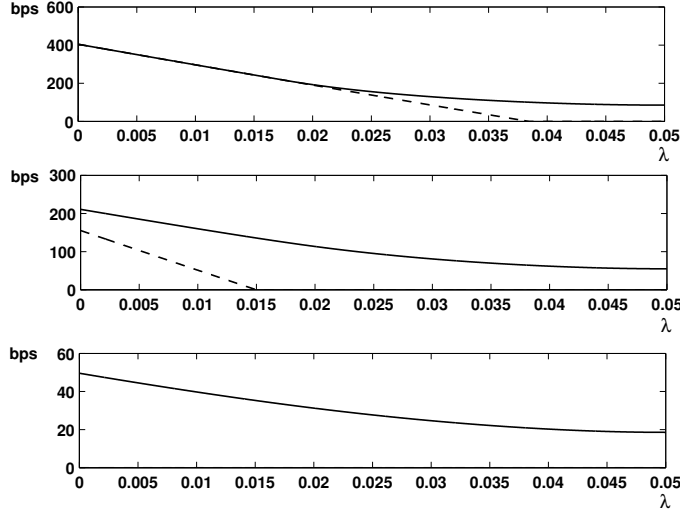


FIG. 8.5. We plot the prepayment option price in bps at inception $P(0, \lambda, X)$ (solid) and pay-off $\chi(0, \lambda, X)$ (dashed) as a function of the intensity λ in the state $X = e_1$ (top), $X = e_2$ (middle) and $X = e_3$ (bottom). Two regions appear: the continuation $\lambda > \Lambda_k^*(t)$ and the exercise region $\lambda < \Lambda_k^*(t)$ except in the third state where there is no exercise region.

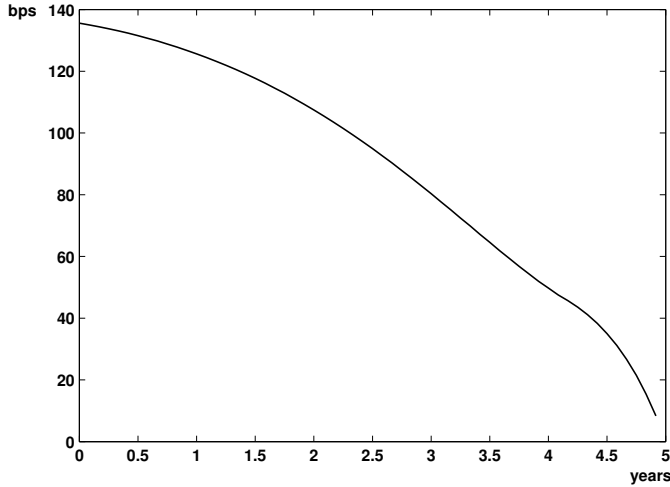


FIG. 8.6. The prepayment option price as a function of the time (the x-axis) in the state $X = e_2$. As expected the option price converges to 0 when the residual maturity of the loan tends to 0.

(5.2) using the discretization (8.1)-(8.2).

(d) compute the payoff χ using the ξ corresponding to $\bar{\rho}_0$ found above;

2. Numerical Solution to Prepayment Option:

(a) find the optimal exercise boundary Λ^* as explained above;

(b) check that Λ^* satisfies the hypothesis of the Theorem 7. In particular the condition (7.9) is checked numerically up to a tolerance depending

on Δx).

3. Find loan value LV .

4. For informative reasons, plot the term structure using equations (4.15) and (6.2).

9. Conclusion. The present work proposes a procedure to compute the price of the prepayment option of a corporate loan. The model takes into account a multi-regime framework for the liquidity price. The numerical illustration shows that the impact of the option may be non-negligible and should be assessed. Moreover it is seen that the borrower may never prepay during recessions but may do so once the economic situation stabilizes. Although the model does not take into account the correlation between borrowers, it may indicate that the end of a recession will trigger simultaneous prepayments of many loans, putting banks at risk. Appropriate provisions have therefore to be set aside to cope with such an alternative and the present model gives quantitative indications on how to evaluate them.

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