Orthomartingale-coboundary decomposition for stationary random fields

Mohamed El Machkouri, Davide Giraudo

Orthomartingale-coboundary decomposition for stationary random fields

Mohamed EL MACHKOURI and Davide GIRAUDO

Laboratoire de Mathématiques Raphaël Salem
UMR CNRS 6085, Université de Rouen (France)
mohamed.elmachkouri@univ-rouen.fr
davide.giraudo@etu.univ-rouen.fr

Abstract

We provide a new projective condition for a stationary real random field indexed by the lattice $\mathbb{Z}^d$ to be well approximated by an orthomartingale in the sense of Cairoli (1969). Our main result can be viewed as a multidimensional version of the martingale-coboundary decomposition method which the idea goes back to Gordin (1969). It is a powerfull tool for proving limit theorems or large deviations inequalities for stationary random fields when the corresponding result is valid for orthomartingales.

1 Introduction and notations

In probability theory, a powerfull approach for proving limit theorems for stationary sequences of random variables is to find a way to approximate such sequences by martingales. This idea goes back to Gordin [8] (see Theorem A below). More precisely, let $(X_k)_{k\in\mathbb{Z}}$ be a sequence of real random variables defined on the probability space $(\Omega, \mathcal{F}, \mu)$. We assume that $(X_k)_{k\in\mathbb{Z}}$ is stationary in the sense that its finite-dimensional laws are invariant by translations and we denote by ν the law of $(X_k)_{k\in\mathbb{Z}}$. Let $f : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ be defined by $f(\omega) = \omega_0$ and $T : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ by $(T\omega)_k = \omega_{k+1}$ for any ω in $\mathbb{R}^{\mathbb{Z}}$ and any k in $\mathbb{Z}$. Then the sequence $(f \circ T^k)_{k\in\mathbb{Z}}$ defined on the probability space $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}), \nu)$ is stationary with law ν. So, without loss of generality, we can assume that $X_k = f \circ T^k$ for any k in $\mathbb{Z}$. In 1969, Gordin [8] introduced a powerfull method for proving the central limit theorem (CLT) and the weak invariance principle (WIP) for stationary sequences of dependent random variables satisfying a projective condition (see (1) below). In the sequel, for any $p \geqslant 1$ and any σ-algebra $\mathcal{M} \subset \mathcal{F}$, we denote by $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ the space of p-integrable real random variables defined on $(\Omega, \mathcal{M}, \mu)$ and we consider the norm $\|.\|_p$ defined by $\|Z\|_p^p = \int_\Omega |Z(\omega)|^p d\mu(\omega)$ for any Z in $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$. We denote also by $\mathbb{L}^p(\Omega, \mathcal{F}, \mu) \ominus \mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ the space of all Z in $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ such that $\mathbb{E}(Z \mid \mathcal{M}) = 0$ a.s.

Theorem A (Gordin, 1969) *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T : \Omega \to \Omega$ be a measurable function such that $\mu = T\mu$. Let also $p \geqslant 1$ and $\mathcal{M} \subset \mathcal{F}$ be a σ-algebra such that $\mathcal{M} \subset T^{-1}\mathcal{M}$. If f belongs to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{i\in\mathbb{Z}} T^{-i}\mathcal{M}, \mu)$ such that*

$$\sum_{k\geqslant 0} \left\| \mathbb{E}\left(f \mid T^k\mathcal{M}\right)\right\|_p < \infty \tag{1}$$

then there exist m in $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T\mathcal{M}, \mu)$ and g in $\mathbb{L}^p(\Omega, T\mathcal{M}, \mu)$ such that

$$f = m + g - g \circ T. \tag{2}$$

The term $g - g \circ T$ in (2) is called a coboundary and equation (2) is called the martingale-coboundary decomposition of f. Moreover, the stationary sequence $(m\circ T^i)_{i\in\mathbb{Z}}$ is a martingale-difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}}$ (see Definition 1 below) and for any positive integer n,

$$S_n(f) = S_n(m) + g - g \circ T^n \tag{3}$$

where $S_n(h) = \sum_{i=0}^{n-1} h \circ T^i$ for any function $h : \Omega \to \mathbb{R}$. Combining (3) with the Billingsley-Ibragimov CLT for martingales (see [2] or [11]), one obtain the CLT for the stationary sequence $(f \circ T^k)_{k\in\mathbb{Z}}$ when the projective condition (1) holds. Similarly, combining (3) with the WIP for martingales (see [3]), we derive the WIP for the stationary sequence $(f\circ T^k)_{k\in\mathbb{Z}}$. Thus, Gordin's method provides a sufficient condition for proving limit theorems for stationary sequences when such a limit theorem holds for martingale-difference sequences. Our aim in this work is to provide an extension of Theorem A for random fields indexed by the lattice $\mathbb{Z}^d$ where d is a positive integer (see Theorem 1).

2 Main results

Definition 1 *We say that a sequence $(X_k)_{k\in\mathbb{Z}}$ of real random variables defined on a probability space $(\Omega, \mathcal{F}, \mu)$ is a martingale-difference (MD) sequence if there exists a filtration $(\mathcal{G}_k)_{k\in\mathbb{Z}}$ such that $\mathcal{G}_k \subset \mathcal{G}_{k+1} \subset \mathcal{F}$ and X_k belongs to $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu) \ominus \mathbb{L}^1(\Omega, \mathcal{G}_{k-1}, \mu)$ for any k in $\mathbb{Z}$.*

The concept of MD sequences can be extended to the random field setting. One can refer for example to Basu and Dorea [1] or Nahapetian [16] where MD random fields are defined in two differents ways and limit theorems are obtained. In this paper, we are interested by orthomartingale-difference random fields in the sense of Cairoli [4]. A good introduction to this concept is done in the book by Khoshnevisan [12]. Let d be a positive integer. We denote by $\langle d\rangle$ the set $\{1, ..., d\}$. For any $s = (s_1, ..., s_d)$ and any $t = (t_1, ..., t_d)$ in $\mathbb{Z}^d$, we write $s \preceq t$ (resp. $s \prec t$, $s \succeq t$ and $s \succ t$) if and only if $s_k \leqslant t_k$ (resp. $s_k < t_k$, $s_k \geqslant t_k$ and $s_k > t_k$) for any k in $\langle d\rangle$ and we denote also $s \wedge t = (s_1 \wedge t_1, ..., s_d \wedge t_d)$.

Definition 2 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A family $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$ of σ-algebras is a commuting filtration if $\mathcal{G}_i \subset \mathcal{G}_j \subset \mathcal{F}$ for any i and j in $\mathbb{Z}^d$ such that $i \preceq j$ and*

$$\mathbb{E}\left(\mathbb{E}\left(Z \mid \mathcal{G}_s\right) \mid \mathcal{G}_t\right) = \mathbb{E}\left(Z \mid \mathcal{G}_{s\wedge t}\right) \quad a.s.$$

for any s and t in $\mathbb{Z}^d$ and any bounded random variable Z.

Definition 2 is known as the "F4 condition".

Definition 3 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. A random field $(X_k)_{k\in\mathbb{Z}^d}$ is an orthomartingale-difference (OMD) random field if there exists a commuting filtration $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$ such that X_k belongs to $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu) \ominus \mathbb{L}^1(\Omega, \mathcal{G}_l, \mu)$ for any $l \not\succeq k$ and k in $\mathbb{Z}^d$.*

Remark 1. Let k be fixed in $\mathbb{Z}^d$ and $S_k = \sum_{0\prec i\preceq k} X_i$ where $(X_i)_{i\in\mathbb{Z}^d}$ is an OMD random field with respect to a commuting filtration $(\mathcal{G}_i)_{i\in\mathbb{Z}^d}$. Then S_k belongs to $\mathbb{L}^1(\Omega, \mathcal{G}_k, \mu)$ and $\mathbb{E}(S_k \mid \mathcal{G}_l) = S_l$ for any $l \preceq k$. We say that $(S_k)_{k\in\mathbb{Z}^d}$ is an orthomartingale (OM) random field.

Arguing as above, without loss of generality, every stationary real random field $(X_k)_{k\in\mathbb{Z}^d}$ can be written as $(f \circ T^k)_{k\in\mathbb{Z}^d}$ where $f : \Omega \to \mathbb{R}$ is a measurable function and for any k in $\mathbb{Z}^d$, $T^k : \Omega \to \Omega$ is a measure-preserving operator satisfying $T^i \circ T^j = T^{i+j}$ for any i and j in $\mathbb{Z}^d$. For any s in $\langle d\rangle$, we denote $T_s = T^{e_s}$ where $e_s = (e_s^{(1)}, \ldots, e_s^{(d)})$ is the unique element of $\mathbb{Z}^d$ such that $e_s^{(s)} = 1$ and $e_s^{(i)} = 0$ for any i in $\langle d\rangle\backslash\{s\}$ and U_s is the operator defined by $U_s h = h \circ T_s$ for any function $h : \Omega \to \mathbb{R}$. We define also U_J as the product operator $\Pi_{s\in J} U_s$ for any $\emptyset \subsetneq J \subset \langle d\rangle$ and we write simply U for $U_{\langle d\rangle} = U_1 \circ U_2 \circ \ldots \circ U_d$. For any $\emptyset \subsetneq J \subset \langle d\rangle$, we denote also by $|J|$ the number of elements in J and by J^c the set $\langle d\rangle\backslash J$. The main result of this paper is the following.

Theorem 1 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T^l : \Omega \to \Omega$ be a measure-preserving operator for any l in $\mathbb{Z}^d$ such that $T^i \circ T^j = T^{i+j}$ for any i and j in $\mathbb{Z}^d$. Let $p \geqslant 1$ and let $\mathcal{M} \subset \mathcal{F}$ be a σ-algebra such that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ is a commuting filtration. If f belongs to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{k\geqslant 1} T_s^k \mathcal{M}, \mu)$ and*

$$\sum_{k\geqslant 1} k^{d-1} \left\| \mathbb{E}\left(f \mid T_s^k \mathcal{M}\right) \right\|_p < \infty \tag{4}$$

for any s in $\langle d\rangle$ then f admits the decomposition

$$f = m + \sum_{\emptyset\subsetneq J\subsetneq\langle d\rangle} \prod_{s\in J} (I - U_s) m_J + \prod_{s=1}^{d} (I - U_s) g, \tag{5}$$

where m, g and $(m_J)_{\emptyset\subsetneq J\subsetneq\langle d\rangle}$ belong to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ and $(U^i m)_{i\in\mathbb{Z}^d}$ and $(U_{J^c}^i m_J)_{i\in\mathbb{Z}^{d-|J|}}$ for all $\emptyset \subsetneq J \subsetneq \langle d\rangle$ are OMD random fields.

Remark 2. If $d = 1$ then Theorem 1 reduces to Theorem A. If $d = 2$ then (5) reduces to

$$f = m + (I - U_1)m_1 + (I - U_2)m_2 + (I - U_1)(I - U_2)g,$$

where m, m_1, m_2 and g belong to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ such that $(U^i m)_{i\in\mathbb{Z}}$ is an OMD random field and $(U_2^k m_1)_{k\in\mathbb{Z}}$ and $(U_1^k m_2)_{k\in\mathbb{Z}}$ are MD sequences. If $d = 3$ then (5) becomes

$$\begin{aligned} f = m &+ (I - U_1)m_1 + (I - U_2)m_2 + (I - U_3)m_3 \\ &+ (I - U_1)(I - U_2)m_{\{1,2\}} + (I - U_1)(I - U_3)m_{\{1,3\}} + (I - U_2)(I - U_3)m_{\{2,3\}} \\ &+ (I - U_1)(I - U_2)(I - U_3)g \end{aligned}$$

where m, m_1, m_2, m_3, $m_{\{1,2\}}$, $m_{\{1,3\}}$, $m_{\{2,3\}}$ and g belong to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ such that $(U^i m)_{i\in\mathbb{Z}^3}$, $(U^i_{\{2,3\}} m_1)_{i\in\mathbb{Z}^2}$, $(U^i_{\{1,3\}} m_2)_{i\in\mathbb{Z}^2}$ and $(U^i_{\{1,2\}} m_3)_{i\in\mathbb{Z}^2}$ are OMD random fields and $(U^k_1 m_{\{2,3\}})_{k\in\mathbb{Z}}$, $(U^k_2 m_{\{1,3\}})_{k\in\mathbb{Z}}$ and $(U^k_3 m_{\{1,2\}})_{k\in\mathbb{Z}}$ are MD sequences.

Remark 3. A decomposition similar to (5) was obtained by Gordin [9] but with reversed orthomartingales and under an assumption on the so-called Poisson equation.

Proposition 1 *Let $(X_i)_{i\in\mathbb{Z}^d}$ be an OMD random field. There exists a positive constant κ such that for any $p > 1$ and any n in $\mathbb{Z}^d_+$,*

$$\left\| \sum_{0\preceq k\preceq n} X_k \right\|_p \leqslant \kappa p^{d/2} \left(\sum_{0\preceq k\preceq n} \|X_k\|_p^2 \right)^{1/2} \tag{6}$$

and the constant $p^{d/2}$ in (6) is optimal in the following sense: there exists a stationary OMD random field $(Z_k)_{k\in\mathbb{Z}^d}$ with $\|Z_0\|_\infty = 1$ and a positive constant κ such that for any $p > 1$

$$\inf \left\{ C > 0 \; ; \; \left\| \sum_{0\preceq k\preceq n} Z_k \right\|_p \leqslant C \left(\sum_{0\preceq k\preceq n} \|Z_k\|_p^2 \right)^{1/2} \forall n \in \mathbb{Z}^d_+ \right\} \geqslant \kappa p^{d/2}. \tag{7}$$

Remark 4. A Young function ψ is a real convex nondecreasingfunction defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to\infty} \psi(t) = \infty$ and $\psi(0) = 0$. We denote by $\mathbb{L}_\psi(\Omega, \mathcal{F}, \mu)$ the Orlicz space associated to the Young function ψ, that is the space of real random variables Z defined on $(\Omega, \mathcal{F}, \mu)$ such that $\mathbb{E}\left(\psi(|Z|/c)\right) < \infty$ for some $c > 0$. The Orlicz space $\mathbb{L}_\psi(\Omega, \mathcal{F}, \mu)$ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by $\|Z\|_\psi = \inf\{\, c > 0\,;\, E[\psi(|Z|/c)] \leqslant 1\,\}$ is a Banach space. For any $p \geqslant 1$, if φ_p is the function defined by $\varphi_p(x) = x^p$ for any nonegative real x then φ_p is a Young function and the Orlicz space $\mathbb{L}_{\varphi_p}(\Omega, \mathcal{F}, \mu)$ reduces to $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ equipped with the norm $\|.\|_p$ defined by $\|Z\|_p^p = \int_\Omega |Z(\omega)|^p d\mu(\omega)$ for any Z in $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [14]. Combining Lemma 4 in [7] and Inequality (6), we obtain Kahane-Khintchine inequalities: for any $0 < q < 2/d$, there exists a positive constant κ_q such that for any $n \succeq 0$ in $\mathbb{Z}^d$,

$$\left\| \sum_{0\preceq k\preceq n} X_k \right\|_{\psi_q} \leqslant \kappa_q \left(\sum_{0\preceq k\preceq n} \|X_k\|^2_{\psi_{\beta(q)}} \right)^{1/2} \tag{8}$$

where $\beta(q) = 2q/(2 - dq)$ and ψ_α is the Young function defined for any $x \in \mathbb{R}^+$ by

$$\psi_\alpha(x) = \exp((x + h_\alpha)^\alpha) - \exp(h_\alpha^\alpha) \quad \text{with} \quad h_\alpha = ((1-\alpha)/\alpha)^{1/\alpha} \mathbb{1}_{\{0<\alpha<1\}}$$

for any real $\alpha > 0$. Moreover, (8) still hold for $q = 2/d$ if the random variables $(X_k)_{k\in\mathbb{Z}^d}$ are assumed to be uniformly bounded. Finally, using Markov inequality and the definition

of the Luxembourg norm $\|.\|_{\psi_q}$, we derive the following large deviations inequalities: for any $0 < q < 2/d$, there exists a positive constant κ_q such that for any $n \succeq 0$ in $\mathbb{Z}^d$ and any $x > 0$,

$$\mu\left(\left|\sum_{0\preceq k\preceq n} X_k\right| > x\right) \leqslant (1+e^{h_q^q})\exp\left(-\left(\frac{x}{\kappa_q\sqrt{\sum_{0\preceq k\preceq n}\|X_k\|^2_{\psi_{\beta(q)}}}} + h_q\right)^q\right).$$

Again, the above exponential inequality still hold for $q = 2/d$ when the random variables $(X_k)_{k\in\mathbb{Z}^d}$ are uniformly bounded.

Combining Proposition 1 and Theorem 1, we obtain the following result.

Proposition 2 *Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field and $(\mathcal{F}_i)_{i\in\mathbb{Z}^d}$ be a commuting filtration such that each X_i is $\mathcal{F}_i$-measurable. If there exists $p > 1$ such that X_0 belongs to $\mathbb{L}^p(\Omega,\mathcal{F}_0,\mu) \ominus \mathbb{L}^p(\Omega,\cap_{k\geqslant 1}\mathcal{F}_{k,s},\mu)$ and*

$$\sum_{k\geqslant 1} k^{d-1}\left\|\mathbb{E}\left(X_0 \mid \mathcal{F}_{k,s}\right)\right\|_p < \infty \tag{9}$$

for any s in $\langle d\rangle$ where $\mathcal{F}_{k,s} = \vee_{\substack{i=(i_1,..,i_d)\in\mathbb{Z}^d\\ i_s\leqslant -k}}\mathcal{F}_i$ then (6) still holds.

Now, we are able to investigate the WIP for random fields. Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field defined on a probability space $(\Omega,\mathcal{F},\mu)$. Let also $\mathcal{A}$ be a collection of Borel subsets of $[0,1]^d$ and consider the process $\{S_n(A)\,;\, A\in\mathcal{A}\}$ defined by

$$S_n(A) = \sum_{i\in\langle n\rangle^d} \lambda(nA\cap R_i)X_i \tag{10}$$

where $R_i =]i_1-1,i_1]\times ...\times]i_d-1,i_d]$ is the unit cube with upper corner at $i=(i_1,..,i_d)$ in $\langle n\rangle^d$ and λ is the Lebesgue measure on $\mathbb{R}^d$. The collection $\mathcal{A}$ is equipped with the pseudo-metric ρ defined by $\rho(A,B) = \sqrt{\lambda(A\Delta B)}$ for any A and B in $\mathcal{A}$. Let $\varepsilon > 0$ and let $H(\mathcal{A},\rho,\varepsilon)$ be the logarithm of the smallest number $N(\mathcal{A},\rho,\varepsilon)$ of open balls of radius ε with respect to ρ which form a covering of $\mathcal{A}$. The function $H(\mathcal{A},\rho,.)$ is called the metric entropy of the class $\mathcal{A}$ and allows us to control the size of the collection $\mathcal{A}$. Let $(\mathcal{C}(\mathcal{A}),\|.\|_{\mathcal{A}})$ be the Banach space of continuous real functions on $\mathcal{A}$ equipped with the uniform norm $\|.\|_{\mathcal{A}}$ defined by $\|f\|_{\mathcal{A}} = \sup_{A\in\mathcal{A}}|f(A)|$. A standard Brownian motion indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in $\mathcal{C}(\mathcal{A})$ and $\mathrm{Cov}(W(A),W(B)) = \lambda(A\cap B)$. From Dudley [6] we know that such a process is well defined if $\int_0^1\sqrt{H(\mathcal{A},\rho,\varepsilon)}\,d\varepsilon < \infty$. Following [18], we recall the definition of Vapnik-Chervonenkis classes (VC-classes) of sets: let $\mathcal{C}$ be a collection of subsets of a set $\mathcal{X}$. An arbitrary set of n points $F_n := \{x_1,...,x_n\}$ possesses 2^n subsets. Say that $\mathcal{C}$ *picks out* a certain subset from F_n if this can be formed as a set of the form $C\cap F_n$ for a C in $\mathcal{C}$. The collection $\mathcal{C}$ is said to *shatter* F_n if each of its 2^n subsets can be picked out in this manner. The *VC-index* $V(\mathcal{C})$ of the class $\mathcal{C}$ is the smallest n for which no set of size n is shattered by $\mathcal{C}$. Clearly, the more refined $\mathcal{C}$ is, the larger is its index. Formally, we have

$$V(\mathcal{C}) = \inf\left\{n\,;\,\max_{x_1,...,x_n}\Delta_n(\mathcal{C},x_1,...,x_n) < 2^n\right\}$$

where $\Delta_n(\mathcal{C}, x_1, ..., x_n) = \#\{C \cap \{x_1, ..., x_n\}\,;\, C \in \mathcal{C}\}$. Two classical examples of VC-classes are the collection $\mathcal{Q}_d = \left\{[0,t]\,;\, t \in [0,1]^d\right\}$ and $\mathcal{Q}_d^{'} = \left\{[s,t]\,;\, s,t \in [0,1]^d,\, s \preceq t\right\}$ with index $d+1$ and $2d+1$ respectively.

In the sequel, since the CLT does not hold for general OMD random fields (see [21], example 1, page 12), we restrict ourselves to the case of a commuting filtration generated by iid random variables.

Theorem 2 *Let $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ be an iid real random field defined on a probability space $(\Omega, \mathcal{F}, \mu)$. Denote by $(\mathcal{F}_i)_{i\in\mathbb{Z}^d}$ the commuting filtration where $\mathcal{F}_i$ is the σ-algebra generated by ε_j for $j \preceq i$ and i in $\mathbb{Z}^d$ and consider the σ-algebra $\mathcal{F}_{k,s}$ defined in Proposition 2 for any positive integer k and any s in $\langle d\rangle$. Let $(X_i)_{i\in\mathbb{Z}^d}$ be a stationary real random field and $\mathcal{A}$ be a VC-class of regular Borel subsets of $[0,1]^d$ with index V. Assume that there exists $p > 2(V-1)$ such that X_0 belongs to $\mathbb{L}^p(\Omega, \mathcal{F}_0, \mu) \ominus \mathbb{L}^p(\Omega, \cap_{k\geqslant 1}\mathcal{F}_{k,s}, \mu)$ for any s in $\langle d\rangle$ and (9) holds. Then the sequence of processes $\{n^{-d/2}S_n(A)\,;\, A \in \mathcal{A}\}$ converges in distribution in $\mathcal{C}(\mathcal{A})$ to $\sqrt{\mathbb{E}\,(X_0^2)}W$ where W is a standard Brownian motion indexed by $\mathcal{A}$.*

If we consider the particular case $\mathcal{A} = \mathcal{Q}_d$ where $\mathcal{Q}_d$ is the class of all quadrants $[0,t]$ for t in $[0,1]^d$ then Theorem 2 ensures that the WIP holds for p-integrable OMD random fields with $p > 2d$. In fact, our next result shows that the WIP still holds for $p = 2$ and can be viewed as an extension of the Donsker's invariance principle for iid random variables (see [5]).

Theorem 3 *Theorem 2 still holds with $\mathcal{A} = \mathcal{Q}_d$ and $p = 2$.*

Remark 5. El Machkouri et al. [7] and Wang and Woodroofe [21] obtained also a WIP for random fields $(X_k)_{k\in\mathbb{Z}^d}$ which can be expressed as a functional of iid real random variables but under the more restrictive condition that X_0 belongs to $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ with $p > 2$. In a recent work, Wang and Volný [20] obtained the WIP for $p = 2$ under a multidimensional version of the so-called Hannan's condition for time series. Their condition is less restrictive than (9) but condition (9) gives also an orthomartingale approximation for the considered random field which is of independent interest.

Proposition 3 *Let $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ be an iid real random field defined on a probability space $(\Omega, \mathcal{F}, \mu)$ such that ε_0 belongs to $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ for some $p \geqslant 2$. Consider the linear random field $(X_k)_{k\in\mathbb{Z}^d}$ defined for any k in $\mathbb{Z}^d$ by $X_k = \sum_{j\succeq 0} a_j\varepsilon_{k-j}$ where $(a_j)_{j\in\mathbb{Z}^d}$ is a family of real numbers satisfying $\sum_{j\succeq 0} a_j^2 < \infty$. Then the condition*

$$\sum_{k\geqslant 1} k^{d-1}\sqrt{\sum_{i\in\Lambda_{k,s}} a_i^2} \tag{11}$$

where $\Lambda_{k,s} = \{i = (i_1, .., i_d) \in \mathbb{Z}^d\,;\, i_s \geqslant k\}$ is more restrictive than (9).

In particular, Proposition 3 ensures that the conclusions of Theorem 2 and Theorem 3 still hold for linear random fields with iid innovations under assumption (11). In the other part, Wang and Woodroofe ([21], Corollary 1) obtained a WIP for stationary linear random fields

with iid innovations $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ under a weaker condition than (11) but again with the additional assumption that ε_0 belongs to $\mathbb{L}^p(\Omega,\mathcal{F},\mu)$ with $p>2$.

We now provide an application of Theorem 1 to the WIP in Hölder spaces. We consider for $0<\beta\leqslant 1$ the space $\mathbb{H}_\beta([0,1]^d)$ as the space of all continuous functions g for which there exists a constant K such that for each $s,t\in[0,1]^d$,

$$|g(s)-g(t)|\leqslant K\|s-t\|^\beta,$$

where $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^d$. We endow this function space with the norm $\|g\|:=|g(0)|+\sup_{s,t\in[0,1]^d,s\neq t}|g(t)-g(s)|/\|t-s\|^\beta$. If $(X_k)_{k\in\mathbb{Z}^d}$ is a stationary real random field, we define the partial sum process $\{n^{-d/2}S_n(t);t\in[0,1]^d\}_{n\geqslant 1}$ by

$$S_n(t)=\sum_{i\in\langle n\rangle^d}\lambda([0,nt]\cap R_i)X_i \tag{12}$$

for any positive integer n and any t in $[0,1]^d$ and we recall that λ is the Lebesgue measure on $\mathbb{R}^d$ and $R_i=]i_1-1,i_1]\times\ldots\times]i_d-1,i_d]$ is the unit cube with upper corner $i=(i_1,\ldots,i_d)$ in $\langle n\rangle^d$. We consider $(S_n(t))_{t\in[0,1]^d}$ as an element of $\mathbb{H}_\beta([0,1]^d)$. Our next result provides a sufficient condition for the weak convergence of $(n^{-d/2}S_n(\cdot))_{n\geqslant 1}$ in this function space.

Theorem 4 *If the assumptions of Theorem 2 hold with $p>4\times(\log_2(4d/(4d-3)))^{-1}$ and $\mathcal{A}=\mathcal{Q}_d$ then the WIP holds in $\mathbb{H}_\gamma([0,1]^d)$ for each $\gamma<1/2-d/p$.*

Remark 6. In [17], a necessary and sufficient condition was obtained for iid random fields to satisfy the WIP in Hölder spaces. Our result provides a sufficient condition for stationary real random fields which can be expressed as a functional of iid real random variables.

3 Proofs

In this section, the letter κ will denote a universal positive constant which the value may change from line to line. The following two lemmas will be useful in the sequel.

Lemma 1 *Let $(\Omega,\mathcal{F},\mu))$ be a probability space. For mutually independent sub-σ-algebras $\mathcal{A}_1,\mathcal{A}_2$ and $\mathcal{A}_3$ of $\mathcal{F}$ and X a random variable in $\mathbb{L}^1(\Omega,\mathcal{F},\mu)$, we have*

$$\mathbb{E}\left(\mathbb{E}\left(X\mid\mathcal{A}_1\vee\mathcal{A}_2\right)\mid\mathcal{A}_2\vee\mathcal{A}_3\right)=\mathbb{E}\left(X\mid\mathcal{A}_2\right), \tag{13}$$

where $\mathcal{A}_1\vee\mathcal{A}_2$ is the σ-algebra generated by $\mathcal{A}_1$ and $\mathcal{A}_2$.

For a proof of Lemma 1, one can refer to Proposition 8.1 in [21].

Lemma 2 *Let $(\Omega,\mathcal{F},\mu)$ be a probability space and let $T^l:\Omega\to\Omega$ be a measure-preserving operator for any l in $\mathbb{Z}^d$ such that $T^i\circ T^j=T^{i+j}$ for any i and j in $\mathbb{Z}^d$ ($d\geqslant 2$). Let $p\geqslant 1$ be*

fixed and let $\mathcal{M} \subset \mathcal{F}$ be a σ-algebra such that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ is a commuting filtration. Assume that F belongs to $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)$ and for any s in $\langle d\rangle$,

$$\sum_{k\geqslant 1} k^{d-1} \left\|\mathbb{E}\left(F \mid T_s^k\mathcal{M}\right)\right\|_p < \infty. \tag{14}$$

Then for any s in $\langle d\rangle$, there exist M_s in $\mathbb{L}^p(\Omega, \mathcal{M}, \mu)\ominus\mathbb{L}^p(\Omega, T_s\mathcal{M}, \mu)$ and G_s in $\mathbb{L}^p(\Omega, T_s\mathcal{M}, \mu)$ such that $F = M_s + (I - U_s)G_s$. Moreover, for any s and l in $\langle d\rangle$, we have

$$\sum_{k\geqslant 1} k^{d-2} \left\|\mathbb{E}\left(M_s \mid T_l^k\mathcal{M}\right)\right\|_p < \infty \quad \textit{and} \quad \sum_{k\geqslant 1} k^{d-2} \left\|\mathbb{E}\left(G_s \mid T_l^k\mathcal{M}\right)\right\|_p < \infty. \tag{15}$$

Proof of Lemma 2. The first part of the proposition is well known (see [19], Theorem 2). In fact, (14) is a sufficient condition for F to be equal to $M_s + (I - U_s)G_s$ with M_s in $\mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_s\mathcal{M}, \mu)$ and $G_s \in \mathbb{L}^p(\Omega, T_s\mathcal{M}, \mu)$ for any s in $\langle d\rangle$. Moreover, M_s and G_s are given by

$$M_s = \sum_{k\geqslant 0} \mathbb{E}\left(U_s^k F \mid \mathcal{M}\right) - \mathbb{E}\left(U_s^k F \mid T_s\mathcal{M}\right) \quad \text{and} \quad G_s = \sum_{k\geqslant 0} \mathbb{E}\left(U_s^k F \mid T_s\mathcal{M}\right).$$

Let r be a positive integer and let s and l be fixed in $\langle d\rangle$. We have

$$\left\|\mathbb{E}\left(M_s \mid T_l^r\mathcal{M}\right)\right\|_p \leqslant 2\sum_{k\geqslant 0} \left\|\mathbb{E}\left(F \mid T_s^k T_l^r\mathcal{M}\right)\right\|_p \leqslant 2r \left\|\mathbb{E}\left(F \mid T_l^r\mathcal{M}\right)\right\|_p + 2\sum_{k\geqslant r} \left\|\mathbb{E}\left(F \mid T_s^k\mathcal{M}\right)\right\|_p$$

and consequently

$$\begin{aligned}\sum_{r\geqslant 1} r^{d-2} \left\|\mathbb{E}\left(M_s \mid T_l^r\mathcal{M}\right)\right\|_p &\leqslant 2\sum_{r\geqslant 1} r^{d-1} \left\|\mathbb{E}\left(F \mid T_l^r\mathcal{M}\right)\right\|_p + 2\sum_{k\geqslant 2}\sum_{r=1}^{k} r^{d-2} \left\|\mathbb{E}\left(F \mid T_s^k\mathcal{M}\right)\right\|_p \\ &\leqslant 2\sum_{r\geqslant 1} r^{d-1} \left\|\mathbb{E}\left(F \mid T_l^r\mathcal{M}\right)\right\|_p + 2\sum_{k\geqslant 2} k^{d-1} \left\|\mathbb{E}\left(F \mid T_s^r\mathcal{M}\right)\right\|_p < \infty.\end{aligned}$$

Similarly, we have also $\sum_{r\geqslant 1} r^{d-2} \left\|\mathbb{E}\left(G_s \mid T_l^r\mathcal{M}\right)\right\|_p < \infty$. The proof of Lemma 2 is complete.

Proof of Theorem 1. For simplicity, we consider only the case $d = 2$ and the case $d = 3$. We start with $d = 2$. Since f is $\mathcal{M}$-measurable and $\sum_{k=1}^{\infty} k \left\|\mathbb{E}\left(f \mid T_2^k\mathcal{M}\right)\right\|_p < \infty$, there exist two functions m_2 and g_2 (see [19], Theorem 2) such that

$$f = m_2 + (I - U_2)g_2, \tag{16}$$

where $m_2 \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu)\ominus\mathbb{L}^p(\Omega, T_2\mathcal{M}, \mu)$ and $g_2 \in \mathbb{L}^p(\Omega, T_2\mathcal{M}, \mu)$. We lay emphasis on that a careful reading of the proof of Theorem 2 in [19] ensure that g_2 is $T_2\mathcal{M}$-measurable when f is $\mathcal{M}$-measurable. So, by Lemma 2, we have $\sum_{r\geqslant 1} \left\|\mathbb{E}\left(m_2 \mid T_1^r\mathcal{M}\right)\right\|_p < \infty$. Applying again Theorem 2 in [19], we obtain

$$m_2 = m_1 + (I - U_1)g_1, \tag{17}$$

where $m_1 \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_1\mathcal{M}, \mu)$ and $g_1 \in \mathbb{L}^p(\Omega, T_1\mathcal{M}, \mu)$. Consequently,

$$f = m_1 + (I - U_1)g_1 + (I - U_2)g_2. \tag{18}$$

Put $m := m_1 - \mathbb{E}(m_1 \mid T_2\mathcal{M})$ so that $\mathbb{E}(m \mid T_2\mathcal{M}) = 0$. Keeping in mind that $\mathbb{E}(m_1 \mid T_1\mathcal{M}) = 0$ and that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ is a commuting filtration, we derive $\mathbb{E}(\mathbb{E}(m_1 \mid T_2\mathcal{M}) \mid T_1\mathcal{M}) = \mathbb{E}(m_1 \mid T_1T_2\mathcal{M}) = 0$ and consequently $\mathbb{E}(m \mid T_1\mathcal{M}) = 0$. That is, $(U^i m)_{i\in\mathbb{Z}^2}$ is an OMD random field. Using (17) and $\mathbb{E}(m_2 \mid T_2\mathcal{M}) = 0$, we derive

$$\mathbb{E}(m_1 \mid T_2\mathcal{M}) = -\mathbb{E}(g_1 \mid T_2\mathcal{M}) + U_1\mathbb{E}(g_1 \mid T_1T_2\mathcal{M}).$$

Since g_1 is $T_1\mathcal{M}$-measurable and $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ is a commuting filtration, we have $\mathbb{E}(g_1 \mid T_1T_2\mathcal{M}) = \mathbb{E}(g_1 \mid T_2\mathcal{M})$ and consequently we obtain

$$\mathbb{E}(m_1 \mid T_2\mathcal{M}) = -(I - U_1)\mathbb{E}(g_1 \mid T_2\mathcal{M}). \tag{19}$$

Combining (18) and (19), we get

$$f = m + (I - U_1)[g_1 - \mathbb{E}(g_1 \mid T_2\mathcal{M})] + (I - U_2)g_2 \tag{20}$$

Now, it suffices to find a decomposition for g_2 with respect to T_1. In fact, by Lemma 2, we have $\sum_{r\geqslant 1} \|\mathbb{E}(g_2 \mid T_1^r\mathcal{M})\|_p < \infty$. So, we can write (see [19], Theorem 2),

$$g_2 = \overline{m}_1 + (I - U_1)\overline{g}_1, \tag{21}$$

where $\overline{m}_1 \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_1\mathcal{M}, \mu)$ and $\overline{g}_1 \in \mathbb{L}^p(\Omega, T_1\mathcal{M}, \mu)$. That is, $(U_1^k\overline{m}_1)_{k\in\mathbb{Z}}$ is a MD sequence. Combining (20) and (21), we derive

$$f = m + (I - U_1)[g_1 - \mathbb{E}(g_1 \mid T_2\mathcal{M})] + (I - U_2)\overline{m}_1 + (I - U_2)(I - U_1)\overline{g}_1.$$

The proof of Theorem 1 is complete for $d = 2$.

In order to convince the reader, we consider now the case $d = 3$. Applying again Theorem 2 in [19], we decompose f in the following way:

$$f = m_3 + (I - U_3)g_3 \tag{22}$$

where $m_3 \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_3\mathcal{M}, \mu)$ and $g_3 \in \mathbb{L}^p(\Omega, T_3\mathcal{M}, \mu)$. By Lemma 2, we have $\sum_{k\geqslant 1} k \left\|\mathbb{E}\left(m_3 \mid T_2^k\mathcal{M}\right)\right\|_p < \infty$. Since m_3 is $\mathcal{M}$-measurable, we obtain

$$m_3 = m_2 + (I - U_2)g_2 \tag{23}$$

where $m_2 \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_2\mathcal{M}, \mu)$ and $g_2 \in \mathbb{L}^p(\Omega, T_2\mathcal{M}, \mu)$. By Lemma 2, we have also $\sum_{k\geqslant 1} \left\|\mathbb{E}\left(m_2 \mid T_1^k\mathcal{M}\right)\right\|_p < \infty$. Since m_2 is $\mathcal{M}$-measurable, we have also

$$m_2 = m_1 + (I - U_1)g_1 \tag{24}$$

where $m_1 \in \mathbb{L}^p(\Omega, \mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_1\mathcal{M}, \mu)$ and $g_1 \in \mathbb{L}^p(\Omega, T_1\mathcal{M}, \mu)$. Combining (22) ,(23) and (24), we obtain

$$f = m_1 + (I - U_1)g_1 + (I - U_2)g_2 + (I - U_3)g_3. \tag{25}$$

Now, we define

$$m := m_1 - \mathbb{E}\left(m_1 \mid T_2\mathcal{M}\right) - \mathbb{E}\left(m_1 \mid T_3\mathcal{M}\right) + \mathbb{E}\left(m_1 \mid T_2T_3\mathcal{M}\right). \tag{26}$$

Since $\mathbb{E}\left(m_1 \mid T_1\mathcal{M}\right) = 0$ and $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^3}$ is a commuting filtration, m satisfies

$$\mathbb{E}\left(m \mid T_1\mathcal{M}\right) = \mathbb{E}\left(m \mid T_2\mathcal{M}\right) = \mathbb{E}\left(m \mid T_3\mathcal{M}\right) = 0.$$

So, $(U^i m)_{i\in\mathbb{Z}^3}$ is an OMD random field. Moreover, using (24),

$$\mathbb{E}\left(m_1 \mid T_2\mathcal{M}\right) = \mathbb{E}\left(m_2 - (I - U_1)g_1 \mid T_2\mathcal{M}\right) = -(I - U_1)\mathbb{E}\left(g_1 \mid T_1T_2\mathcal{M}\right).$$

Since again g_1 is $T_1\mathcal{M}$-measurable and $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^3}$ is a commuting filtration, we have $\mathbb{E}\left(g_1 \mid T_1T_2\mathcal{M}\right) = \mathbb{E}\left(g_1 \mid T_2\mathcal{M}\right)$ and consequently,

$$\mathbb{E}\left(m_1 \mid T_2\mathcal{M}\right) = -(I - U_1)\mathbb{E}\left(g_1 \mid T_2\mathcal{M}\right). \tag{27}$$

Similarly,

$$\mathbb{E}\left(m_1 \mid T_3\mathcal{M}\right) = -(I - U_1)\mathbb{E}\left(g_1 \mid T_3\mathcal{M}\right) - (I - U_2)\mathbb{E}\left(g_2 \mid T_3\mathcal{M}\right) \tag{28}$$

and

$$\mathbb{E}\left(m_1 \mid T_2T_3\mathcal{M}\right) = -(I - U_1)\mathbb{E}\left(g_1 \mid T_2T_3\mathcal{M}\right). \tag{29}$$

Combining (25), (26), (27), (28) and (29), we obtain

$$\begin{aligned} f = m &+ (I - U_1)[g_1 - \mathbb{E}\left(g_1 \mid T_2\mathcal{M}\right) - \mathbb{E}\left(g_1 \mid T_3\mathcal{M}\right) + \mathbb{E}\left(g_1 \mid T_2T_3\mathcal{M}\right)] \\ &+ (I - U_2)[g_2 - \mathbb{E}\left(g_2 \mid T_3\mathcal{M}\right)] + (I - U_3)g_3. \end{aligned}$$

By (22) and Lemma 2, we have $\sum_{k\geqslant 1} k \left\|\mathbb{E}\left(g_3 \mid T_1^k\mathcal{M}\right)\right\|_p < \infty$. Since g_3 is $T_3\mathcal{M}$-measurable, we derive from Theorem 2 in [19] that

$$g_3 = \overline{m}_1 + (I - U_1)\overline{g}_1 \tag{30}$$

where $\overline{m}_1 \in \mathbb{L}^p(\Omega, T_3\mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_1T_3\mathcal{M}, \mu)$ and $\overline{g}_1 \in \mathbb{L}^p(\Omega, T_1T_3\mathcal{M}, \mu)$. By Lemma 2, we have $\sum_{k\geqslant 1} \left\|\mathbb{E}\left(\overline{m}_1 \mid T_2^k\mathcal{M}\right)\right\|_p < \infty$ and consequently

$$\overline{m}_1 = \overline{m}_2 + (I - U_2)\overline{g}_2 \tag{31}$$

where $\overline{m}_2 \in \mathbb{L}^p(\Omega, T_3\mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_2T_3\mathcal{M}, \mu)$ and $\overline{g}_2 \in \mathbb{L}^p(\Omega, T_2T_3\mathcal{M}, \mu)$. Denoting

$$\overline{m} := \overline{m}_2 - \mathbb{E}\left(\overline{m}_2 \mid T_1T_3\mathcal{M}\right), \tag{32}$$

and keeping in mind that $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ is a commuting filtration, we have $\mathbb{E}\left(\overline{m} \mid T_1\mathcal{M}\right) = \mathbb{E}\left(\overline{m} \mid T_2\mathcal{M}\right) = 0$. That is $(U^i_{\{1,2\}}\overline{m})_{i\in\mathbb{Z}^2}$ is an OMD random field. Moreover, using (31), we have

$$\mathbb{E}\left(\overline{m}_2 \mid T_1T_3\mathcal{M}\right) = \mathbb{E}\left(\overline{m}_1 - (I-U_2)\overline{g}_2 \mid T_1T_3\mathcal{M}\right) = -(I-U_2)\mathbb{E}\left(\overline{g}_2 \mid T_1T_2T_3\mathcal{M}\right).$$

Since $\overline{g}_2$ is $T_2T_3\mathcal{M}$-measurable and $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^d}$ is a commuting filtration, we have also $\mathbb{E}\left(\overline{g}_2 \mid T_1T_2T_3\mathcal{M}\right) = \mathbb{E}\left(\overline{g}_2 \mid T_1\mathcal{M}\right)$ and consequently

$$\mathbb{E}\left(\overline{m}_2 \mid T_1T_3\mathcal{M}\right) = -(I-U_2)\mathbb{E}\left(\overline{g}_2 \mid T_1\mathcal{M}\right). \tag{33}$$

Combining (30), (31), (32) and (33), we obtain

$$g_3 = \overline{m} + (I-U_1)\overline{g}_1 + (I-U_2)[\overline{g}_2 - \mathbb{E}\left(\overline{g}_2 \mid T_1\mathcal{M}\right)].$$

As before, since $\overline{g}_1$ is $T_1T_3\mathcal{M}$-measurable and $\sum_{k\geqslant 1}\left\|\mathbb{E}\left(\overline{g}_1 \mid T_2^k\mathcal{M}\right)\right\|_p < \infty$ (by Lemma 2), we have

$$\overline{g}_1 = \overline{\overline{m}}_2 + (I-U_2)\overline{\overline{g}}_2$$

where $\overline{\overline{m}}_2 \in \mathbb{L}^p(\Omega, T_1T_3\mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_1T_2T_3\mathcal{M}, \mu)$ and $\overline{\overline{g}}_2 \in \mathbb{L}^p(\Omega, T_1T_2T_3\mathcal{M}, \mu)$. In particular, $(U_2^k\overline{\overline{m}}_2)_{k\in\mathbb{Z}}$ is a MD sequence. Consequently,

$$\begin{aligned} f = m &+ (I-U_1)[g_1 - \mathbb{E}\left(g_1 \mid T_2\mathcal{M}\right) - \mathbb{E}\left(g_1 \mid T_3\mathcal{M}\right) + \mathbb{E}\left(g_1 \mid T_2T_3\mathcal{M}\right)] \\ &+ (I-U_2)[g_2 - \mathbb{E}\left(g_2 \mid T_3\mathcal{M}\right)] \\ &+ (I-U_3)\overline{m} + (I-U_3)(I-U_2)[\overline{g}_2 - \mathbb{E}\left(\overline{g}_2 \mid T_1\mathcal{M}\right)] \\ &+ (I-U_3)(I-U_1)\overline{\overline{m}}_2 + (I-U_3)(I-U_1)(I-U_2)\overline{\overline{g}}_2. \end{aligned}$$

Since g_2 is $T_2\mathcal{M}$-measurable and $\sum_{k\geqslant 1}\left\|\mathbb{E}\left(g_2 \mid T_1^k\mathcal{M}\right)\right\|_p < \infty$ (by Lemma 2), we have

$$g_2 = \overline{\overline{m}}_1 + (I-U_1)\overline{\overline{g}}_1 \tag{34}$$

where $\overline{\overline{m}}_1 \in \mathbb{L}^p(\Omega, T_2\mathcal{M}, \mu) \ominus \mathbb{L}^p(\Omega, T_1T_2\mathcal{M}, \mu)$ and $\overline{\overline{g}}_1 \in \mathbb{L}^p(\Omega, T_1T_2\mathcal{M}, \mu)$. Denoting

$$\overline{\overline{m}} := \overline{\overline{m}}_1 - \mathbb{E}\left(\overline{\overline{m}}_1 \mid T_3\mathcal{M}\right), \tag{35}$$

and applying Lemma 2, we have $\mathbb{E}\left(\overline{\overline{m}} \mid T_1\mathcal{M}\right) = \mathbb{E}\left(\overline{\overline{m}} \mid T_3\mathcal{M}\right) = 0$. So, $(U^i_{\{1,3\}}\overline{\overline{m}})_{i\in\mathbb{Z}^2}$ is an OMD random field. Moreover, using (34), we derive

$$\mathbb{E}\left(\overline{\overline{m}}_1 \mid T_3\mathcal{M}\right) = \mathbb{E}\left(g_2 \mid T_3\mathcal{M}\right) - (I-U_1)\mathbb{E}\left(\overline{\overline{g}}_1 \mid T_1T_3\mathcal{M}\right). \tag{36}$$

Since $\overline{\overline{g}}_1$ is $T_1\mathcal{M}$-measurable and $(T^{-i}\mathcal{M})_{i\in\mathbb{Z}^3}$ is a commuting filtration, we know that $\mathbb{E}\left(\overline{\overline{g}}_1 \mid T_1T_3\mathcal{M}\right) = \mathbb{E}\left(\overline{\overline{g}}_1 \mid T_3\mathcal{M}\right)$. Combining (34), (35) and (36), we obtain

$$g_2 - \mathbb{E}\left(g_2 \mid T_3\mathcal{M}\right) = \overline{\overline{m}} + (I-U_1)[\overline{\overline{g}}_1 - \mathbb{E}\left(\overline{\overline{g}}_1 \mid T_3\mathcal{M}\right)].$$

Finally,

$$\begin{aligned} f = m &+ (I-U_1)[g_1 - \mathbb{E}\,(g_1 \mid T_2\mathcal{M}) - \mathbb{E}\,(g_1 \mid T_3\mathcal{M}) + \mathbb{E}\,(g_1 \mid T_2T_3\mathcal{M})] \\ &+ (I-U_2)\overline{\overline{m}} + (I-U_2)(I-U_1)[\overline{\overline{g}}_1 - \mathbb{E}\,(\overline{\overline{g}}_1 \mid T_3\mathcal{M})] \\ &+ (I-U_3)\overline{m} + (I-U_3)(I-U_2)[\overline{g}_2 - \mathbb{E}\,(\overline{g}_2 \mid T_1\mathcal{M})] \\ &+ (I-U_3)(I-U_1)\overline{\overline{m}}_2 + (I-U_3)(I-U_1)(I-U_2)\overline{\overline{g}}_2. \end{aligned}$$

The proof of Theorem 1 is complete for $d = 3$. The proof of Theorem 1 for $d \geqslant 4$ can be done in the same way. It is left to the reader.

Proof of Proposition 1. Let $(X_i)_{i\in\mathbb{Z}^d}$ be an OMD random field with respect to a commuting filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}^d}$. Again, for simplicity, we consider only the case $d = 2$. Let $n = (n_1, n_2) \succeq 0$ be fixed in $\mathbb{Z}^2$ and consider $(Y_i)_{i\in\mathbb{Z}}$ defined for any i in $\mathbb{Z}$ by $Y_i = \sum_{j=0}^{n_2} X_{i,j}$. One can notice that $(Y_i)_{i\in\mathbb{Z}}$ is a MD sequence with respect to the filtration $(\vee_{j\in\mathbb{Z}}\mathcal{F}_{(i,j)})_{i\in\mathbb{Z}}$. Consequently, by Burkholder's inequality (see [10], Theorem 2.10), we have

$$\left\|\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} X_{i,j}\right\|_p \leqslant \kappa\sqrt{p}\left(\sum_{i=0}^{n_1}\|Y_i\|_p^2\right)^{1/2}.$$

Moreover, since for any i in $\mathbb{Z}$, $(X_{i,j})_{j\in\mathbb{Z}}$ is a MD sequence with respect to the filtration $(\vee_{i\in\mathbb{Z}}\mathcal{F}_{(i,j)})_{j\in\mathbb{Z}}$, we have also

$$\|Y_i\|_p = \left\|\sum_{j=0}^{n_2} X_{i,j}\right\|_p \leqslant \kappa\sqrt{p}\left(\sum_{j=0}^{n_2}\|X_{i,j}\|_p^2\right)^{1/2}.$$

Consequently, we obtain

$$\left\|\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} X_{i,j}\right\|_p \leqslant \kappa p\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\|X_{i,j}\|_p^2\right)^{1/2}. \tag{37}$$

In order to prove the optimality of the constant p in (37), arguing as in Wang and Woodroofe [21] (Example 1, page 12), we consider a sequence $(\eta_i)_{i\in\mathbb{Z}}$ of iid real random variables satisfying $\mu(\eta_0 = 1) = \mu(\eta_0 = -1) = 1/2$. Let also $(\eta_i^{'})_{i\in\mathbb{Z}}$ be an independent copy of $(\eta_i)_{i\in\mathbb{Z}}$ and consider the filtrations $(\mathcal{G}_k)_{k\in\mathbb{Z}}$ and $(\mathcal{H}_k)_{k\in\mathbb{Z}}$ defined for any k in $\mathbb{Z}$ by $\mathcal{G}_k = \sigma(\eta_s; s \leqslant k)$ and $\mathcal{H}_k = \sigma(\eta_s^{'}; s \leqslant k)$. For any (i, j) in $\mathbb{Z}^2$, we denote $Z_{i,j} = \eta_i\eta_j^{'}$. Then $(Z_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is an OMD random field with respect to the commuting filtration $(\mathcal{F}_{i,j})_{(i,j)\in\mathbb{Z}^2}$ defined by $\mathcal{F}_{i,j} = \mathcal{G}_i \vee \mathcal{H}_j$ for any (i, j) in $\mathbb{Z}^2$. Let C be a positive constant such that for any $n = (n_1, n_2) \succeq 0$,

$$\left\|\sum_{i=0}^{n_1}\eta_i\right\|_p \times \left\|\sum_{j=0}^{n_2}\eta_j^{'}\right\|_p = \left\|\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} Z_{i,j}\right\|_p \leqslant C\left(\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\|Z_{i,j}\|_p^2\right)^{1/2} \leqslant C\sqrt{n_1n_2}.$$

Applying the CLT for iid real random variables, we derive $C \geqslant \|N\|_p^2$ where N is a standard normal random variable. Since there exists $\kappa > 0$ such that $\|N\|_p^2 \geqslant \kappa p$, we derive (7). The proof of Proposition 1 is complete.

Proof of Theorem 2. The convergence of the finite-dimensional laws of the partial sums process is a direct consequence of the CLT by Wang and Woodroofe ([21], Theorem 5). So, it suffices to establish the tightness property of the partial sums process. Assume that $\mathcal{A}$ is a VC-class with index V and there exists $p > 2(V-1)$ such that X_0 belongs to $\mathbb{L}^p(\Omega, \mathcal{F}, \mu)$ and (9) holds. Then there exists a positive constant K such that for any $0 < \varepsilon < 1$, we have (see Van der Vaart and Wellner [18], Theorem 2.6.4)

$$N(\mathcal{A}, \rho, \varepsilon) \leqslant KV(4e)^V \left(\frac{1}{\varepsilon}\right)^{2(V-1)}$$

where $N(\mathcal{A}, \rho, \varepsilon)$ is the smallest number of open balls of radius ϵ with respect to ρ which form a covering of $\mathcal{A}$. Since $p > 2(V-1)$, we have

$$\int_0^1 (N(\mathcal{A}, \rho, \varepsilon))^{\frac{1}{p}} \, d\varepsilon < \infty. \tag{38}$$

Moreover, using Proposition 2, we derive

$$\left\| n^{-d/2} S_n(A) - n^{-d/2} S_n(B) \right\|_p \leqslant \kappa p^{d/2} \rho(A, B) \, \|X_0\|_p \tag{39}$$

for any positive integer n and any A and B in $\mathcal{A}$. Combining (38) and (39) and applying Theorem 11.6 in Ledoux and Talagrand [15], we obtain that for each positive ϵ there exists a positive real δ, depending on ϵ and on the value of the entropy integral (38) but not on n, such that

$$\mathbb{E}\left(\sup_{\substack{A,B \in \mathcal{A} \\ \rho(A,B) < \delta}} |n^{-d/2} S_n(A) - n^{-d/2} S_n(B)| \right) < \epsilon. \tag{40}$$

In particular, for any $x > 0$, we have

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mu \left(\sup_{\substack{A,B \in \mathcal{A} \\ \rho(A,B) < \delta}} \left| n^{-d/2} S_n(A) - n^{-d/2} S_n(B) \right| > x \right) = 0.$$

Consequently, the partial sum process $\{n^{-d/2} S_n(A); A \in \mathcal{A}\}_{n \geqslant 1}$ is tight in the space $\mathcal{C}(\mathcal{A})$ and the WIP holds. The proof of Theorem 2 is complete.

Proof of Theorem 3. Let $(\Omega, \mathcal{F}, \mu, \{T^k\}_{k \in \mathbb{Z}^d})$ be a dynamical system (i.e. $(\Omega, \mathcal{F}, \mu)$ is a probability space and $T^k : \Omega \to \Omega$ is a measure-preserving transformation for any k in $\mathbb{Z}^d$ satisfying $T^i \circ T^j = T^{i+j}$ for any i and any j in $\mathbb{Z}^d$) and let $(\varepsilon_i)_{i \in \mathbb{Z}^d}$ be a field of iid real random variables defined on $(\Omega, \mathcal{F}, \mu)$. Let $\mathcal{M} \subset \mathcal{F}$ be the σ-algebra generated by the random

variables ε_i for $i \preceq 0$ and let $f : \Omega \to \mathbb{R}$ be $\mathcal{M}$-measurable. We consider the stationary real random field $(f \circ T^i)_{i\in\mathbb{Z}^d}$ and the partial sum process $\left\{S_n(f,t)\,;\, t \in [0,1]^d\right\}_{n\geqslant 1}$ defined for any integer $n \geqslant 1$ and any t in $[0,1]^d$ by

$$S_n(f,t) = \sum_{i\in\langle n\rangle^d} \lambda([0,nt]\cap R_i) f \circ T^i \tag{41}$$

where λ is the Lebesgue measure on $\mathbb{R}^d$ and $R_i =]i_1-1,i_1]\times ...\times]i_d-1,i_d]$ is the unit cube with upper corner $i=(i_1,...,i_d)$ in $\langle n\rangle^d$. Again, the convergence of the finite-dimensional laws of the process $\left\{n^{-d/2}S_n(f,t)\,;\, t\in[0,1]^d\right\}_{n\geqslant 1}$ is a direct consequence of the CLT established by Wang and Woodroofe ([21], Theorem 3.2). In order to obtain the tightness property of the partial sum process, it suffices to establish for any $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \mu\left(\sup_{\substack{s,t\in[0,1]^d\\ |s-t|<\delta}} n^{-d/2}|S_n(f,s)-S_n(f,t)| > \varepsilon\right) = 0$$

where $|x| = \max_{k\in\langle d\rangle}|x_k|$ for any $x=(x_1,...,x_d)$ in $[0,1]^d$. For simplicity, we are going to consider only the case $d=2$. By Theorem 1, we have

$$f = m + (I-U_1)m_1 + (I-U_2)m_2 + (I-U_1)(I-U_2)g, \tag{42}$$

where m, m_1, m_2 and g are square-integrable functions defined on Ω such that $(U^i m)_{i\in\mathbb{Z}^2}$ is an OMD random field and $(U_2^k m_1)_{k\in\mathbb{Z}}$ and $(U_1^k m_2)_{k\in\mathbb{Z}}$ are MD sequences. In the sequel, for any real x, we denote by $[x]$ the integer part of x. Let $n\geqslant 1$ and $t=(t_1,t_2)$ in $[0,1]^2$. For any $1\leqslant i\leqslant [nt_1]+1$ and any $1\leqslant j\leqslant [nt_2]+1$, we denote $\lambda_{i,j}(t) = \lambda\left([0,nt]\cap R_{(i,j)}\right)$. We have

$$S_n((I-U_1)m_1,t) = \sum_{i=1}^{[nt_1]+1}\sum_{j=1}^{[nt_2]+1} \lambda_{i,j}(t)U^{(i,j)}(I-U_1)m_1 = \sum_{j=1}^{[nt_2]+1} U_2^j \sum_{i=1}^{[nt_1]+1} \lambda_{i,j}(t)\left(U_1^i m_1 - U_1^{i+1}m_1\right).$$

Using Abel's transformation and noting that $\lambda_{i+1,j}(t) = \lambda_{i,j}(t)$ for any $1\leqslant i\leqslant [nt_1]-1$ and any $1\leqslant j\leqslant [nt_2]+1$, we obtain that $S_n((I-U_1)m_1,t)$ equals

$$\sum_{j=1}^{[nt_2]+1} U_2^j\left\{\lambda_{[nt_1]+1,j}(t)\left(U_1m_1 - U_1^{[nt_1]+2}m_1\right) - \sum_{i=1}^{[nt_1]}\left(U_1m_1-U_1^{i+1}m_1\right)\left(\lambda_{i+1,j}(t)-\lambda_{i,j}(t)\right)\right\}$$

$$= \sum_{j=1}^{[nt_2]+1} U_2^j\left\{\lambda_{[nt_1]+1,j}(t)\left(U_1m_1 - U_1^{[nt_1]+2}m_1\right) - \left(U_1m_1-U_1^{[nt_1]+1}m_1\right)\left(\lambda_{[nt_1]+1,j}(t)-\lambda_{[nt_1],j}(t)\right)\right\}$$

$$= U_1(I-U_1^{[nt_1]+1})\sum_{j=1}^{[nt_2]+1}\lambda_{[nt_1]+1,j}(t)\,U_2^j m_1 - U_1(I-U_1^{[nt_1]})\sum_{j=1}^{[nt_2]+1}\left(\lambda_{[nt_1]+1,j}(t)-\lambda_{[nt_1],j}(t)\right)U_2^j m_1.$$

Moreover, since $\lambda_{i,j}(t) = \lambda_{i,1}(t)$ for any $1 \leqslant i \leqslant [nt_1]+1$ and any $1 \leqslant j \leqslant [nt_2]$, we derive

$$\begin{aligned} S_n((I-U_1)m_1,t) &= U_1(I-U_1^{[nt_1]+1})\lambda_{[nt_1]+1,1}(t)\sum_{j=1}^{[nt_2]} U_2^j m_1 \\ &\quad + U_1(I-U_1^{[nt_1]+1})\lambda_{[nt_1]+1,[nt_2]+1}(t)\, U_2^{[nt_2]+1} m_1 \\ &\quad - U_1(I-U_1^{[nt_1]})\left(\lambda_{[nt_1]+1,1}(t)-\lambda_{[nt_1],1}(t)\right)\sum_{j=1}^{[nt_2]} U_2^j m_1 \\ &\quad - U_1(I-U_1^{[nt_1]})\left(\lambda_{[nt_1]+1,[nt_2]+1}(t)-\lambda_{[nt_1],[nt_2]+1}(t)\right) U_2^{[nt_2]+1} m_1. \end{aligned}$$

So, we obtain

$$\sup_{t\in[0,1]^2} |S_n((I-U_1)m_1,t)| \leqslant 4 \max_{1\leqslant l,k\leqslant n+2} U_1^l U_2^k |m_1| + 4 \max_{1\leqslant l,k\leqslant n+2} U_1^l \left|\sum_{j=1}^{k} U_2^j m_1\right|. \tag{43}$$

Let $x>0$ be fixed. Since $m_1 \in \mathbb{L}^2(\Omega,\mathcal{F},\mu)$, we have

$$\mu\left(\max_{1\leqslant l,k\leqslant n+2} U_1^l U_2^k |m_1| > nx\right) \leqslant \kappa n^2 \mu\left(m_1^2 > n^2x^2\right) \xrightarrow[n\to\infty]{} 0. \tag{44}$$

In the other part,

$$\mu\left(\max_{1\leqslant l,k\leqslant n+2} U_1^l \left|\sum_{j=1}^{k} U_2^j m_1\right| > xn\right) = \mu\left(\max_{1\leqslant l\leqslant n+2} U_1^l \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1\right)^2 > nx^2\right). \tag{45}$$

Lemma 3 *Let $(Z_n)_{n\geqslant 1}$ be a sequence of uniformly integrable real random variables. For any s in $\langle d\rangle$,*

$$\limsup_{n\to\infty} \mu\left(\max_{1\leqslant i_1,..,i_s\leqslant n} U_1^{i_1}\ldots U_s^{i_s}\,|Z_n| > n^s\right) = 0.$$

Proof of Lemma 3. Let n be a positive integer. For any s in $\langle d\rangle$, we denote

$$p_n(s) := \mu\left(\max_{1\leqslant i_1,\ldots,i_s\leqslant n} U_1^{i_1}\ldots U_s^{i_s}\,|Z_n| > n^s\right).$$

Let R be a positive real number. We have

$$p_n(s) \leqslant \frac{2R}{n^s} + n^s\,\mu\left(|Z_n|\ \mathbb{1}_{\{|Z_n|>R\}} > \frac{n^s}{2}\right) \leqslant \frac{2R}{n^s} + 2\sup_{k\geqslant 1}\mathbb{E}\left(|Z_k|\ \mathbb{1}_{\{|Z_k|>R\}}\right).$$

Consequently, $\limsup_{n\to\infty} p_n(s) \leqslant 2\sup_{k\geqslant 1}\mathbb{E}\left(|Z_k|\ \mathbb{1}_{\{|Z_k|>R\}}\right) \xrightarrow[R\to\infty]{} 0$. The proof of Lemma 3 is complete.

Lemma 4 *The sequence* $\left\{ \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1 \right)^2 ;\, n \geqslant 1 \right\}$ *is uniformly integrable.*

Proof of Lemma 4. Since $(U_2^k m_1)_{k\in\mathbb{Z}}$ is a MD sequence, using Doob's inequality, we derive

$$\left\| \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1 \right\|_2 \leqslant 2 \left\| \sum_{j=1}^{n+2} U_2^j m_1 \right\|_2 \leqslant \kappa\sqrt{n}\, \|m_1\|_2 .$$

So, $\left\{ \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1 \right)^2 ;\, n \geqslant 1 \right\}$ is bounded in $\mathbb{L}^1(\Omega, \mathcal{F}, \mu)$. Let M be a fixed positive constant. We have $m_1 = m_1^{'} + m_1^{''}$ where

$$\begin{aligned} m_1^{'} &= m_1\, \mathbb{1}_{|m_1|\leqslant M} - \mathbb{E}\left(m_1\, \mathbb{1}_{|m_1|\leqslant M} \mid T_2\mathcal{M} \right) \\ m_1^{''} &= m_1\, \mathbb{1}_{|m_1|> M} - \mathbb{E}\left(m_1\, \mathbb{1}_{|m_1|> M} \mid T_2\mathcal{M} \right). \end{aligned}$$

Moreover, if A belongs to $\mathcal{F}$ then

$$\begin{aligned} \int_A \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1 \right)^2 d\mu \leqslant\ & 2 \int_A \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1^{'} \right)^2 d\mu \\ & + 2 \int_A \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1^{''} \right)^2 d\mu. \end{aligned}$$

Since $(U_2^k m_1^{'})_{k\in\mathbb{Z}}$ and $(U_2^k m_1^{''})_{k\in\mathbb{Z}}$ are MD sequences, using Schwarz's inequality, we obtain

$$\begin{aligned} \int_A \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1 \right)^2 d\mu \leqslant\ & 2 \left\| \frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1^{'} \right\|_4^2 \sqrt{\mu(A)} \\ & + 2 \left\| \frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1^{''} \right\|_2^2 . \end{aligned}$$

Keeping in mind that $m_1^{'}$ is bounded by M and using again Doob's inequality, there exists a positive constant κ_0 such that

$$\int_A \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1 \right)^2 d\mu \leqslant \kappa_0 \left(M^2 \sqrt{\mathbb{P}(A)} + \mathbb{E}\left(m_1^2\, \mathbb{1}_{|m_1|>M} \right) \right).$$

Let $\varepsilon > 0$ be fixed and let $M > 0$ such that $\kappa_0 \mathbb{E}\left(m_1^2\, \mathbb{1}_{|m_1|>M} \right) \leqslant \frac{\varepsilon}{2}$. One can choose the measurable set A in $\mathcal{F}$ such that $\kappa_0 M^2 \sqrt{\mu(A)} \leqslant \frac{\varepsilon}{2}$ and consequently

$$\sup_{n\geqslant 1} \int_A \left(\frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n+2} \sum_{j=1}^{k} U_2^j m_1 \right)^2 d\mu \leqslant \varepsilon.$$

The proof of Lemma 4 is complete.

Combining (43), (44), (45), Lemma 3 and Lemma 4, we obtain

$$\limsup_{n\to\infty} \mu\left(\sup_{t\in[0,1]^2} |S_n((I-U_1)m_1,t)| > xn \right) = 0. \tag{46}$$

In a similar way, we derive also

$$\limsup_{n\to\infty} \mu\left(\sup_{t\in[0,1]^2} |S_n((I-U_2)m_1,t)| > xn \right) = 0. \tag{47}$$

Now, noting that $\lambda_{i,j}(t) = \lambda_{i,1}(t)$ for any $1 \leqslant i \leqslant [nt_1]+1$ and any $1 \leqslant j \leqslant [nt_2]$, we have $S_n((I-U_1)(I-U_2)m_1,t)$ equals

$$\begin{aligned}
&\sum_{i=1}^{[nt_1]+1} \sum_{j=1}^{[nt_2]+1} \lambda_{i,j}(t)\, U^{(i,j)}(I-U_1)(I-U_2)m_1 \\
&= \sum_{i=1}^{[nt_1]+1} U_1^i(I-U_1)\left(\lambda_{i,1}(t) \sum_{j=1}^{[nt_2]} (U_2^j - U_2^{j+1})m_1 + \lambda_{i,[nt_2]+1}(t) U_2^{[nt_2]+1}(I-U_2)m_1 \right) \\
&= U_2(I-U_2^{[nt_2]}) \sum_{i=1}^{[nt_1]+1} \lambda_{i,1}(t)(U_1^i - U_1^{i+1})m_1 + U_2^{[nt_2]+1}(I-U_2) \sum_{i=1}^{[nt_1]+1} \lambda_{i,[nt_2]+1}(t)(U_1^i m_1 - U_1^{i+1}m_1).
\end{aligned}$$

Since $\lambda_{i,j}(t) = \lambda_{1,j}(t)$ for any $1 \leqslant i \leqslant [nt_1]$ and any $1 \leqslant j \leqslant [nt_2]+1$, we derive

$$\begin{aligned}
S_n((I-U_1)(I-U_2)m_1,t) = {} & \lambda_{1,1}(t) U_2(I-U_2^{[nt_2]}) U_1(I-U_1^{[nt_1]})m_1 \\
& + \lambda_{[nt_1]+1,1}(t) U_2(I-U_2^{[nt_2]}) U_1^{[nt_1]+1}(I-U_1)\, m_1 \\
& + \lambda_{1,[nt_2]+1}(t) U_2^{[nt_2]+1}(I-U_2) U_1(I-U_1^{[nt_1]+1})m_1 \\
& + \lambda_{[nt_1]+1,[nt_2]+1}(t) U_2^{[nt_2]+1}(I-U_2) U_1^{[nt_1]+1}(I-U_1)m_1.
\end{aligned}$$

Thus

$$\sup_{t\in[0,1]^2} |S_n((I-U_1)(I-U_2)m_1,t)| \leqslant \kappa \max_{1\leqslant k,l\leqslant n+2} U_1^k U_2^l |m_1|$$

and for any positive x,

$$\mu\left(\sup_{t\in[0,1]^2} |S_n((I-U_1)(I-U_2)m_1,t)| > xn \right) \leqslant \kappa n^2 \mu\left(m_1^2 > n^2x^2\right) \xrightarrow[n\to\infty]{} 0. \tag{48}$$

Now, it suffices to prove the tightness of the process $\{\frac{1}{n}S_n(m,t); t\in[0,1]^2\}_{n\geqslant 1}$. That is, for any positive x,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mu\left(\sup_{\substack{s,t\in[0,1]^2 \\ |s-t|<\delta}} |S_n(m,s) - S_n(m,t)| > xn \right) = 0. \tag{49}$$

Let n be a positive integer and let $s=(s_1,s_2)$ and $t=(t_1,t_2)$ be fixed in $[0,1]^2$. We denote $\Delta_n(s,t)=S_n(m,s)-S_n(m,t)$ and for any i and j in $\langle n\rangle$,

$$\beta_{i,j}=\lambda_{i,j}(s)-\lambda_{i,j}(t)=\lambda\left([0,ns]\cap R_{(i,j)}\right)-\lambda\left([0,nt]\cap R_{(i,j)}\right).$$

Noting that $\beta_{i,j}=0$ for any $1\leqslant i\leqslant[n(s_1\wedge t_1)]$ and any $1\leqslant j\leqslant[n(s_1\wedge t_1)]$, we have $\Delta_n(s,t)=\Delta_n^{'}(s,t)+\Delta_n^{''}(s,t)$ where

$$\Delta_n^{'}(s,t)=\sum_{i=[n(s_1\wedge t_1)]+1}^{[n(s_1\vee t_1)]+1}\sum_{j=1}^{[n(s_2\wedge t_2)]+1}\beta_{i,j}\,U^{(i,j)}m\quad\text{and}\quad\Delta_n^{''}(s,t)=\sum_{i=1}^{[n(s_1\wedge t_1)]+1}\sum_{j=[n(s_2\vee t_2)]+1}^{[n(s_2\vee t_2)]+1}\beta_{i,j}\,U^{(i,j)}m.$$

Moreover, $\Delta_n^{'}(s,t)=\Delta_{1,n}^{'}(s,t)+\Delta_{2,n}^{'}(s,t)+\Delta_{3,n}^{'}(s,t)+\Delta_{4,n}^{'}(s,t)$ where

$$\begin{aligned}
\Delta_{1,n}^{'}(s,t)&=\sum_{i=[n(s_1\wedge t_1)]+2}^{[n(s_1\vee t_1)]}\sum_{j=1}^{[n(s_2\wedge t_2)]}\beta_{i,j}\,U^{(i,j)}m\\
\Delta_{2,n}^{'}(s,t)&=\sum_{j=1}^{[n(s_2\wedge t_2)]}\beta_{[n(s_1\vee t_1)]+1,j}\,U^{([n(s_1\vee t_1)]+1,j)}m\\
\Delta_{3,n}^{'}(s,t)&=\sum_{j=1}^{[n(s_2\wedge t_2)]}\beta_{[n(s_1\wedge t_1)]+1,j}\,U^{([n(s_1\wedge t_1)]+1,j)}m\\
\Delta_{4,n}^{'}(s,t)&=\sum_{i=[n(s_1\wedge t_1)]+1}^{[n(s_1\vee t_1)]+1}\beta_{i,[n(s_2\wedge t_2)]+1}\,U^{(i,[n(s_2\wedge t_2)]+1)}m.
\end{aligned}$$

Let α in $\{-1,+1\}$ such that $\beta_{i,j}=\alpha$ if $[n(s_1\wedge t_1)]+2\leqslant i\leqslant[n(s_1\vee t_1)]$ and $1\leqslant j\leqslant[n(s_2\wedge t_2)]$ So,

$$\Delta_{1,n}^{'}(s,t)=\alpha\sum_{i=[n(s_1\wedge t_1)]+2}^{[n(s_1\vee t_1)]}\sum_{j=1}^{[n(s_2\wedge t_2)]}U^{(i,j)}m$$

and for any positive x,

$$\begin{aligned}
\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{1,n}^{'}(s,t)|>nx\right)&\leqslant\sum_{k=0}^{[\frac{1}{\delta}]}\mu\left(\max_{\substack{1\leqslant p\leqslant n\\r\in[0,\delta]}}\left|\sum_{i=[nk\delta]+2}^{[n(k\delta+r)]}\sum_{j=1}^{p}U^{(i,j)}m\right|>nx\right)\\
&=\sum_{k=0}^{[\frac{1}{\delta}]}\mu\left(\max_{\substack{1\leqslant p\leqslant n\\r\in[0,\delta]}}\left|\sum_{i=1}^{[n(k\delta+r)]-[nk\delta]-1}\sum_{j=1}^{p}U^{(i,j)}m\right|>nx\right).
\end{aligned}$$

Since $[n(k\delta+r)]-[nk\delta]-1$ is an integer smaller than $[nr]$, we obtain

$$\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{1,n}^{'}(s,t)|>nx\right)\leqslant\left(1+\frac{1}{\delta}\right)\mu\left(\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left|\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right|>nx\right)$$
$$=\left(1+\frac{1}{\delta}\right)\mu\left(\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left(\frac{1}{n\sqrt{\delta}}\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right)^2>\frac{x^2}{\delta}\right)$$
$$\leqslant\left(\frac{1+\delta}{x^2}\right)\mathbb{E}_{\frac{x^2}{\delta}}\left(\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left(\frac{1}{n\sqrt{\delta}}\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right)^2\right)$$

where we used the notation $\mathbb{E}_A(Z)=\mathbb{E}\left(Z\,\mathbb{1}_{|Z|>A}\right)$ for any $A>0$ and any Z in $\mathbb{L}^1(\Omega,\mathcal{F},\mu)$.

Lemma 5 *The family* $\left\{\max_{\substack{1\leqslant p\leqslant n\\1\leqslant q\leqslant[n\delta]}}\left(\frac{1}{n\sqrt{\delta}}\sum_{i=1}^{q}\sum_{j=1}^{p}U^{(i,j)}m\right)^2\;;\;n\geqslant 1,\delta>0\right\}$ *is uniformly integrable.*

Proof of Lemma 5. The proof follows the same lines as the proof of Lemma 4 using Cairoli's maximal inequality for orthomartingales (see [12], Theorem 2.3.1) instead of Doob's inequality for martingales. The proof of Lemma 5 is complete.

So, we obtain

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{1,n}^{'}(s,t)|>nx\right)=0. \tag{50}$$

In the other part, since $\beta_{[n(s_1\vee t_1)]+1,j}=\beta_{[n(s_1\vee t_1)]+1,1}$ for any $1\leqslant j\leqslant[n(s_2\wedge t_2)]$, we have

$$\Delta_{2,n}^{'}(s,t)=\beta_{[n(s_1\vee t_1)]+1,1}U_1^{[n(s_1\vee t_1)]+1}\sum_{j=1}^{[n(s_2\wedge t_2)]}U_2^jm$$

and consequently

$$\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{2,n}^{'}(s,t)|\leqslant\max_{\substack{1\leqslant k\leqslant n+1\\1\leqslant l\leqslant n}}U_1^k\left|\sum_{j=1}^{l}U_2^jm\right|.$$

So,

$$\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{2,n}^{'}(s,t)|>nx\right)\leqslant\mu\left(\max_{1\leqslant k\leqslant n+1}U_1^k\left(\max_{1\leqslant l\leqslant n}\frac{1}{\sqrt{n}}\left|\sum_{j=1}^{l}U_2^jm\right|\right)^2>nx^2\right). \tag{51}$$

Since $(U_2^k m)_{k\in\mathbb{Z}}$ is a MD sequence, arguing as in Lemma 4, the sequence $\left\{\left(\max_{1\leqslant l\leqslant n}\frac{1}{\sqrt{n}}\left|\sum_{j=1}^{l}U_2^j m\right|\right)^2\right\}_{n\geqslant 1}$ is uniformly integrable. Combining (51) and Lemma 3, we derive that for any $\delta>0$,

$$\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{2,n}^{'}(s,t)|>nx\right)=0. \tag{52}$$

Similarly, we have also

$$\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{3,n}^{'}(s,t)|>nx\right)=0 \tag{53}$$

for any $\delta>0$. Moreover, for any $[n(s_1\wedge t_1)]+1\leqslant i\leqslant[n(s_1\vee t_1)]$, we have $\beta_{i,[n(s_2\wedge t_2)]+1}=\beta_{[n(s_1\wedge t_1)]+1,[n(s_2\wedge t_2)]+1}$ and consequently

$$\begin{aligned}\Delta_{4,n}^{'}(s,t)=\beta_{[n(s_1\wedge t_1)]+1,[n(s_2\wedge t_2)]+1}\,U_2^{[n(s_2\wedge t_2)]+1}\sum_{i=[n(s_1\wedge t_1)]+1}^{[n(s_1\vee t_1)]}U_1^i m\\ +\beta_{[n(s_1\vee t_1)]+1,[n(s_2\wedge t_2)]+1}\,U^{([n(s_1\vee t_1)]+1,[n(s_2\wedge t_2)]+1)}m\end{aligned}$$

and

$$\begin{aligned}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{4,n}^{'}(s,t)|>nx\right)\leqslant\mu\left(\max_{1\leqslant k\leqslant n+1}U_2^k\left(\max_{1\leqslant l\leqslant[n\delta]}\frac{1}{\sqrt{n\delta}}\left|\sum_{j=1}^{l}U_1^j m\right|\right)^2>\frac{nx^2}{2\delta}\right)\\ +2n^2\mu\left(m^2>\frac{n^2x^2}{4}\right)\end{aligned}$$

Arguing as in Lemma 3, the family $\left\{\left(\max_{1\leqslant l\leqslant[n\delta]}\frac{1}{\sqrt{n\delta}}\left|\sum_{j=1}^{l}U_1^j m\right|\right)^2;\ n\geqslant 1,\delta>0\right\}$ is uniformly integrable since $(U_1^k m)_{k\in\mathbb{Z}}$ is a MD sequence. By Lemma 4, we obtain for any $\delta>0$,

$$\limsup_{n\to\infty}\mu\left(\max_{1\leqslant k\leqslant n+1}U_2^k\left(\max_{1\leqslant l\leqslant[n\delta]}\frac{1}{\sqrt{n\delta}}\left|\sum_{j=1}^{l}U_1^j m\right|\right)^2>\frac{nx^2}{2\delta}\right)=0.$$

Moreover, $n^2\mu\left(m^2>\frac{n^2x^2}{4}\right)$ goes to zero as n goes to infinity since m belongs to $\mathbb{L}^2(\Omega,\mathcal{F},\mu)$. Consequently, for any $\delta>0$,

$$\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\|s-t|<\delta}}|\Delta_{4,n}^{'}(s,t)|>nx\right)=0. \tag{54}$$

Combining (50), (52),(53) and (54), we obtain

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\ |s-t|<\delta}}|\Delta_n^{'}(s,t)|>nx\right)=0. \tag{55}$$

Similarly, one can check that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mu\left(\sup_{\substack{s,t\in[0,1]^2\\ |s-t|<\delta}}|\Delta_n^{''}(s,t)|>nx\right)=0. \tag{56}$$

Finally, keeping in mind $\Delta_n(s,t)=\Delta_n^{'}(s,t)+\Delta_n^{''}(s,t)$ and combining (55) and (56), we obtain (49). The proof of Theorem 3 is complete.

Proof of Proposition 3. We shall use Rosenthal's inequality ([10], Theorem 2.12). Let $p\geqslant 2$ be fixed. There exists a constant C depending only on p such that if $(Y_j)_{j\geqslant 1}$ is a sequence of independent zero-mean random variables and n a positive integer then

$$\frac{1}{C}\left(\sum_{j=1}^n\mathbb{E}[Y_j^2]\right)^{p/2}+\frac{1}{C}\sum_{j=1}^n\mathbb{E}|Y_j|^p\leqslant\mathbb{E}\left|\sum_{j=1}^n Y_j\right|^p\leqslant C\left(\sum_{j=1}^n\mathbb{E}[Y_j^2]\right)^{p/2}+C\sum_{j=1}^n\mathbb{E}|Y_j|^p. \tag{57}$$

Keeping in mind that $\Lambda_{k,s}=\{i=(i_1,..,i_d)\in\mathbb{Z}^d\,;\,i_s\geqslant k\}$ for any $k\geqslant 1$ and any s in $\langle d\rangle$, we have $\mathbb{E}\left(X_0\mid\mathcal{F}_{k,s}\right)=\sum_{i\in\Lambda_{k,s}}a_i\varepsilon_{-i}$. Since $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ is an iid real random field with ε_0 in $\mathbb{L}^p(\Omega,\mathcal{F},\mu)$, we apply (57) and the series $\sum_{k\geqslant 1}k^{d-1}\|\mathbb{E}[X_0\mid\mathcal{F}_{k,s}]\|_p$ is convergent if and only if

$$\sum_{k\geqslant 1}k^{d-1}\left\{\left(\sum_{i\in\Lambda_{k,s}}|a_i|^2\right)^{1/2}+\left(\sum_{i\in\Lambda_{k,s}}|a_i|^p\right)^{1/p}\right\}<\infty.$$

The result follows from the fact that $\sum_{i\in\mathbb{Z}}|c_i|^p\leqslant\left(\sum_{i\in\mathbb{Z}}|c_i|^2\right)^{p/2}$ for any sequence $(c_i)_{i\in\mathbb{Z}}$ of real numbers. The proof of Proposition 3 is complete.

Proof of Theorem 4. We shall use Theorem 1 in [13] which states that if a sequence of random processes $\{Y_n(t)\,;\,t\in[0,1]^d\}_{n\geqslant 1}$ whose finite dimensional distributions are weakly convergent and for some constants α, β and K such that

$$\beta\in(0,1]\quad\text{and}\quad\alpha\beta>\frac{2}{\log_2\left(\frac{4d}{4d-3}\right)}$$

and

$$\mu\left\{|Y_n(t)-Y_n(s)|\geqslant\varepsilon\right\}\leqslant\frac{K}{\varepsilon^\alpha}\|s-t\|^{\alpha\beta} \tag{58}$$

for any s and t in $[0,1]^d$, any $\varepsilon > 0$ and any positive integer n then $(Y_n(\cdot))_{n\geqslant 1}$ converges weakly to some process in $\mathbb{H}_\gamma([0,1]^d)$ where $0 < \gamma < \beta - m/\alpha$. Since the finite-dimensional laws of the process $\{n^{-d/2}S_n(t)\,;\, t \in [0,1^d]\}_{n\geqslant 1}$ are weakly convergent (cf. Theorem 3), it suffices to convert the moment inequality given by Proposition 2 into an inequality involving $\mu\left\{|S_n(t) - S_n(s)| \geqslant n^{d/2}\varepsilon\right\}$ in order to check that condition (58) is satisfied with $\alpha = p$, $\beta = 1/2$ and $Y_n(t) = n^{-d/2}S_n(t)$. We shall do the proof for $d = 2$. Let $s = (s_1, s_2)$ and $t = (t_1, t_2)$ be fixed in $[0,1]^2$ and n be a positive integer. Without loss of generality, we assume that $s_1 > t_1$ and $s_2 < t_2$ (similar arguments can be used to threat the general case). Let $s_1' = k_1/n$ and $t_1' = (l_1+1)/n$ where (k_1, l_1) is the unique element of $\langle n\rangle^2$ such that $k_1/n \leqslant s_1 < (k_1+1)/n$ and $l_1/n \leqslant t_1 < (l_1+1)/n$. In other words, keeping in mind that [.] denotes the integer part function, we have $s_1' = [ns_1]/n$ and $t_1' = ([nt_1]+1)/n$ and similarly, we define $s_2' = ([ns_2]+1)/n$ and $t_2' = [nt_2]/n$. With these notations, we have

$$\begin{aligned}
|S_n(t) - S_n(s)| &= |S_n(t_1,t_2) - S_n(s_1,s_2)| \\
&\leqslant |S_n(t_1,t_2) - S_n(t_1,t_2')| + |S_n(t_1',t_2') - S_n(t_1,t_2')| \\
&\quad + |S_n(t_1',t_2') - S_n(s_1',s_2')| + |S_n(s_1',s_2') - S_n(s_1',s_2)| \\
&\quad + |S_n(s_1',s_2) - S_n(s_1,s_2)|.
\end{aligned}$$

Since

$$|S_n(t_1,t_2) - S_n(t_1,t_2')| = (t_2 - t_2')\left|\sum_{i=1}^{[nt_1]} X_{i,[nt_2]} + (t_1' - t_1)X_{[nt_1]+1,[nt_2]}\right|$$

and $t_2 - t_2' \leqslant 1/n$, we have

$$\mathbb{E}\,|S_n(t_1,t_2) - S_n(t_1,t_2')|^p \leqslant \kappa(t_2 - t_2')^p n^{p/2}\mathbb{E}|X_{0,0}|^p \leqslant \kappa(t_2 - t_2')^{p/2}\mathbb{E}|X_{0,0}|^p. \tag{59}$$

Similarly,

$$\mathbb{E}\,|S_n(t_1',t_2') - S_n(t_1,t_2')|^p \leqslant \kappa(t_1' - t_1)^{p/2}\mathbb{E}|X_{0,0}|^p, \tag{60}$$

$$\mathbb{E}\,|S_n(s_1',s_2') - S_n(s_1',s_2)|^p \leqslant \kappa(s_2' - s_2)^{p/2}\mathbb{E}|X_{0,0}|^p, \tag{61}$$

$$\mathbb{E}\,|S_n(s_1',s_2) - S_n(s_1,s_2)|^p \leqslant \kappa(s_1 - s_1')^{p/2}\mathbb{E}|X_{0,0}|^p. \tag{62}$$

Moreover, from Proposition 2, for any positive n and any i and j in $\langle n\rangle^2$, we have

$$\mathbb{E}\left|\frac{1}{n}S_n\left(\frac{i}{n}\right) - \frac{1}{n}S_n\left(\frac{j}{n}\right)\right|^p \leqslant \kappa\mathbb{E}|X_{0,0}|^p\left\|\frac{i}{n} - \frac{j}{n}\right\|^{p/2}. \tag{63}$$

Combining (59), (60), (61), (62) and (63) and using the elementary convexity inequality $(a_1+a_2+a_3+a_4+a_5)^p \leqslant 5^{p-1}(a_1^p + a_2^p + a_3^p + a_4^p + a_5^p)$for any non-negative a_1, a_2, a_3, a_4 and a_5, we derive that

$$\mathbb{E}|S_n(t) - S_n(s)|^p \leqslant \kappa\|s - t\|^{p/2}.$$

Finaly, using Markov's inequality, we obtain (58). The proof of Theorem 4 is complete.

Acknowledgements. The authors would like to thank D. Volný and Y. Wang for many usefull comments and discussions.

References

[1] A. K. Basu and C. C. Y. Dorea. On functional central limit theorem for stationary martingale random fields. *Acta. Math. Hung.*, 33:307–316, 1979.

[2] P. Billingsley. The Lindeberg-Lévy theorem for martingales. *Proc. Amer. Math. Soc.*, 12:788–792, 1961.

[3] P. Billingsley. *Convergence of probability measures*. Wiley, 1968.

[4] R. Cairoli. Un théorème de convergence pour martingales à indices multiples. *C. R. Acad. Sci. Paris Sér. A-B*, 269:A587–A589, 1969.

[5] M. D. Donsker. An invariance principle for certain probability limit theorems. *Mem. Amer. Math. Soc.*, 6:1–12, 1951.

[6] R. M. Dudley. Sample functions of the Gaussian process. *Ann. Probab.*, 1:66–103, 1973.

[7] M. El Machkouri, D. Volný, and W. B. Wu. A central limit theorem for stationary random fields. *Stochastic Process. Appl.*, 123(1):1–14, 2013.

[8] M. I. Gordin. The central limit theorem for stationary processes. *Soviet Math.Dokl.*, pages 1174–1176, 1969.

[9] M. I. Gordin. Martingale-co-boundary representation for a class of stationary random fields. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 364(Veroyatnost i Statistika. 14.2):88–108, 236, 2009.

[10] P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic Press, New York, 1980.

[11] I. A. Ibragimov. A central limit theorem for a class of dependent random variables. *Theory Probab. Appl.*, 8:83–89, 1963.

[12] D. Khoshnevisan. *Multiparameter processes*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2002.

[13] Jana Klicnarová. Central limit theorem for Hölder processes on $\mathbb{R}^m$-unit cube. *Comment. Math. Univ. Carolin.*, 48(1):83–91, 2007.

[14] M. A. Krasnosel'skii and Y. B. Rutickii. *Convex Functions and Orlicz Spaces*. P. Noordhoff LTD-Groningen-The Netherlands, 1961.

[15] M. Ledoux and M. Talagrand. *Probability in Banach spaces*. Springer, New York, 1991.

[16] B. Nahapetian and A. N. Petrosian. Martingale-difference Gibbs random fields and central limit theorem. *Ann. Acad. Sci. Fenn., Series A-I Math.*, 17:105–110, 1992.

[17] A. Račkauskas, C. Suquet, and V. Zemlys. A Hölderian functional central limit theorem for a multi-indexed summation process. *Stochastic Process. Appl.*, 117(8):1137–1164, 2007.

[18] A. W. Van der Vaart and J. A. Wellner. *Weak convergence and empirical processes with applications to statistics*. Springer, 1996.

[19] D. Volný. Approximating martingales and the central limit theorem for strictly stationary processes. *Stochastic Processes and Their Applications*, 44:41–74, 1993.

[20] D. Volný and Wang Y. Invariance principle for stationary random fields under Hannan's condition. arXiv:1403.4613, 2014.

[21] Y. Wang and M. Woodroofe. A new condition for the invariance principle for stationary random fields. arXiv:1101.5195, 2012.