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Abstract

With any max-stable random process η on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$, we associate a random tessellation of the parameter space $\mathcal{X}$. The construction relies on the Poisson point process representation of the max-stable process η which is seen as the pointwise maximum of a random collection of functions $\Phi = \{\phi_i, i \geq 1\}$. The tessellation is constructed as follows: two points $x, y \in \mathcal{X}$ are in the same cell if and only if there exists a function $\phi \in \Phi$ that realizes the maximum η at both points x and y, i.e. $\phi(x) = \eta(x)$ and $\phi(y) = \eta(y)$. We characterize the distribution of cells in terms of coverage and inclusion probabilities. Most interesting is the stationary case where the asymptotic properties of the cells are strongly related to the ergodic and mixing properties of the max-stable process η and to its conservative/dissipative and positive/null decompositions.

Key words: max-stable random field, random tessellation, non-singular flow representation, ergodic properties.
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1 Introduction

Max-stable random fields provide popular and meaningful models for spatial extremes, see, e.g., de Haan and Ferreira [2]. The reason is that

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they appear as the only possible non-degenerate limits for normalized pointwise maxima of independent and identically distributed random fields. The one-dimensional marginal distributions of max-stable fields belong to the parametric class of Generalized Extreme Value distributions. Being interested mostly in the dependence structure, we will restrict our attention to max-stable fields $\eta = (\eta(x))_{x\in\mathcal{X}}$ on $\mathcal{X} \subset \mathbb{R}^d$ with standard unit Fréchet margins, i.e. satisfying

$$\mathbb{P}[\eta(x) \leq z] = \exp(-1/z) \quad \text{for all } x \in \mathcal{X} \text{ and } z > 0. \tag{1}$$

The max-stability property has then the simple form

$$n^{-1} \bigvee_{i=1}^{n} \eta_i \stackrel{d}{=} \eta \quad \text{for all } n \geq 1,$$

where $(\eta_i)_{1\leq i\leq n}$ are i.i.d. copies of η, $\bigvee$ is the pointwise maximum, and $\stackrel{d}{=}$ denotes the equality of finite-dimensional distributions.

A fundamental tool in the study of max-stable processes is their spectral representation (see, e.g., de Haan [1], Giné *et al.* [7]): any stochastically continuous max-stable process η can be written as

$$\eta(x) = \bigvee_{i\geq 1} U_i Y_i(x), \quad x \in \mathcal{X}, \tag{2}$$

where

- $(U_i)_{i\geq 1}$ is the decreasing enumeration of the points of a Poisson point process on $(0, +\infty)$ with intensity measure $u^{-2}\mathrm{d}u$,
- $(Y_i)_{i\geq 1}$ are i.i.d. copies of a non-negative stochastic process Y on $\mathcal{X}$ such that $\mathbb{E}[Y(x)] = 1$ for all $x \in \mathcal{X}$,
- the sequences $(U_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ are independent.

In this paper, we focus on max-stable random fields defined on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$. In the case $\mathcal{X} = \mathbb{R}^d$ we always assume that η has continuous sample paths. Equivalently, the spectral process Y has continuous sample paths and

$$\mathbb{E}\Big[\sup_{x\in K} Y(x)\Big] < \infty \text{ for every compact set } K \subset \mathbb{R}^d. \tag{3}$$

Note that the equivalence follows for instance from de Haan and Ferreira [2, Corollary 9.4.5].

Representation (2) has a nice interpretation pointed out by Smith [22] and Schlather [20]. In the context of a rainfall model, we can interpret each index $i \geq 1$ as a *storm event*, where U_i stands for the

intensity of the storm and Y_i stands for its shape; then $U_iY_i(x)$ represents the amount of precipitation due to the storm event i at point $x \in \mathcal{X}$, and $\eta(x)$ is the maximal precipitation over all storm events at this point. This interpretation raises a natural question: what is the shape of the region $C_i \subset \mathcal{X}$ where the storm i is extremal? More formally, we define the *cell* associated with the storm event $i \geq 1$ by $C_i = \{x \in \mathcal{X};\ U_iY_i(x) = \eta(x)\}$. It is a (possibly empty) *random closed subset* of $\mathcal{X}$ and each point $x \in \mathcal{X}$ belongs almost surely to a unique cell (the point process $\{U_iY_i(x)\}_{i\geq 1}$ is a Poisson point process with intensity $u^{-2}\mathrm{d}u$ so that the maximum $\eta(x)$ is almost surely attained for unique i).

A drawback of this approach is that the distribution of the cell C_i depends on the specific representation (2). For instance, with the convention that the sequence $(U_i)_{i\geq 1}$ is decreasing, the cell C_1 is stochastically larger than the other cells. To avoid this, we introduce a canonical way to define the tessellation.

Definition 1. *For $x \in \mathcal{X}$, the cell of x is the random closed subset*

$$C(x) = \{y \in \mathcal{X};\ \exists i \geq 1,\ U_iY_i(x) = \eta(x)\ \textit{and}\ U_iY_i(y) = \eta(y)\}. \quad (4)$$

The cell $C(x)$ is non-empty since it contains x. In the case $\mathcal{X} = \mathbb{Z}^d$, for any two points $x_1, x_2 \in \mathbb{Z}^d$, the cells $C(x_1)$ and $C(x_2)$ are almost surely either equal or disjoint. In the case $\mathcal{X} = \mathbb{R}^d$, for any two points $x_1, x_2 \in \mathbb{R}^d$, the cells $C(x_1)$ and $C(x_2)$ are almost surely either equal or have disjoint interiors.

The purpose of this paper is to study some properties of the *random tessellation* $(C(x))_{x\in\mathcal{X}}$. Let us stress that in this paper the terms *cell* and *tessellation* are meant in a broader sense than in stochastic geometry where they originated. Here, a cell is a general (not necessarily convex or connected) random closed set and a tessellation is a random covering of $\mathcal{X}$ by closed sets with pairwise disjoint interiors. The following lemma provides a first simple but important observation.

Lemma 2. *The distribution of the tessellation $(C(x))_{x\in\mathcal{X}}$ depends on the distribution of the max-stable process η only and not on the specific representation* (2).

To prove the lemma, introduce the functional point process (which will play a key role in the sequel)

$$\Phi = \{\phi_i,\ i \geq 1\} \quad \text{where } \phi_i = U_iY_i,\ i \geq 1. \quad (5)$$

Note that ϕ_i are elements of $\mathcal{F}_0 = \mathcal{F}(\mathcal{X}, [0, +\infty)) \setminus \{0\}$, the set of non-negative and continuous functions on $\mathcal{X}$ excluding the zero function. (We may assume without loss of generality that Y does not vanish

identically). The set $\mathcal{F}_0$ is endowed with the σ-algebra generated by the coordinate mappings. It follows from the transformation theorem that Φ is a Poisson point process on $\mathcal{F}_0$ with intensity measure μ given by

$$\mu(A) = \int_0^\infty \mathbb{P}[uY \in A] u^{-2} \mathrm{d}u, \quad A \subset \mathcal{F}_0 \text{ Borel.} \tag{6}$$

The measure μ is called the exponent measure or max-Lévy measure and is related to the multivariate cumulative distribution functions of η by

$$\begin{aligned} &\mathbb{P}[\eta(x_j) \leq z_j, j = 1, \ldots, n] \\ &\quad = \quad \exp\left(-\mu(\{f \in \mathcal{F}_0;\ f(x_j) > z_j \text{ for some } j = 1, \ldots, n\})\right) \end{aligned}$$

for all $n \geq 1$, $x_1, \ldots, x_n \in \mathcal{X}$ and $z_1, \ldots, z_n > 0$. In particular, this shows that μ depends on the distribution of η only and does not depend on the specific representation (2). Now, Lemma 2 follows easily since the tessellation $(C(x))_{x \in \mathcal{X}}$ is a functional of the Poisson point process Φ with intensity μ.

The aim of this paper is to study some properties of the tessellation $(C(x))_{x \in \mathcal{X}}$ and to relate them to the properties of the max-stable random field $(\eta(x))_{x \in \mathcal{X}}$. It is worth noting that some well-known tessellations like the Laguerre and some Johnson–Mehl tessellations (see, e.g., Møller [14]) are particular cases of this setting (see Examples 8 and 9 below). Furthermore, thanks to the Poisson point process representation by Giné *et al.* [7], the results from the present paper could presumably be extended to the more general framework of upper semi-continuous max-infinitely divisible processes. The connection with stochastic geometry would even be stronger via the notion of hypograph: the hypograph of an upper semi-continuous max-infinitely divisible process can be represented as the union of random closed sets from a Poisson point process. However, for the sake of simplicity, we consider only the case of continuous max-stable processes for which more results are available from the literature.

The paper is structured as follows. In Section 2, we study the law of the cell $C(x)$ and provide some formulas for the inclusion and coverage probabilities as well as some examples. In Section 3, we focus on the stationary case and establish strong connections between asymptotic properties of $C(0)$ and properties of the max-stable random field η such as ergodicity, mixing and decompositions of the non-singular flow associated with η. Theorem 19 relates the boundedness of the cell to the conservative/dissipative decomposition. Theorem 21 links the asymptotic density of the cell with the positive/null decomposition. Proofs are collected in Sections 4 and 5.

2 Basic properties and examples

2.1 Basic properties

Our first result is a simple characterization of the distribution of the cells of the tessellation.

Theorem 3. *Consider a sample continuous max-stable random field η given by representation (2). For every $x \in \mathcal{X}$ and every measurable set $K \subset \mathcal{X}$,*

$$\mathbb{P}[K \subset C(x)] = \mathbb{E}\left[\inf_{y \in K \cup \{x\}} \frac{Y(y)}{\eta(y)}\right] \tag{7}$$

and

$$\mathbb{P}[C(x) \subset K] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{y \in K^c} \frac{Y(y)}{\eta(y)}\right)^+\right], \tag{8}$$

where Y is independent of η, $K^c = \mathcal{X} \setminus K$ is the complement of the set K, and $(z)^+ = \max(z, 0)$ is the positive part of z.

It is well-known that the distribution of a random closed set $C \subset \mathcal{X}$ is completely determined by its capacity functional

$$\mathcal{X}_C(K) = \mathbb{P}[C \cap K \neq \emptyset], \quad K \subset \mathcal{X} \text{ compact},$$

see, e.g., Molchanov [13, Chapter 1]. Clearly, Theorem 3 implies that for all $x \in \mathcal{X}$ the capacity functional of the cell $C(x)$ is given by

$$\mathcal{X}_{C(x)}(K) \;=\; 1 - \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{y \in K} \frac{Y(y)}{\eta(y)}\right)^+\right].$$

Remark 4. It is worth noting that Weintraub [27] introduced (with a different terminology) the probability that two points x and y are in the same cell as a measure of dependence between $\eta(x)$ and $\eta(y)$. More precisely, he considered

$$\pi(x, y) = \mathbb{P}[y \in C(x)] = \mathbb{E}\left[\frac{Y(x)}{\eta(x)} \wedge \frac{Y(y)}{\eta(y)}\right], \quad x, y \in \mathcal{X}.$$

Clearly, $\pi(x, y) \in [0, 1]$. One can prove easily that $\pi(x, y) = 0$ holds if and only if $\eta(x)$ and $\eta(y)$ are independent, while $\pi(x, y) = 1$ if and only if $\eta(x) = \eta(y)$ almost surely. Moreover, $\pi(x, y)$ can be compared to the extremal coefficient $\theta(x, y)$ which is another well-known measure of dependence for max-stable processes defined by

$$\theta(x, y) = -\log \mathbb{P}[\eta(x) \vee \eta(y) \leq 1] \in [1, 2]. \tag{9}$$

According to Stoev [23, Proposition 5.1], we have

$$\frac{1}{2}(2-\theta(x,y)) \leq \pi(x,y) \leq 2(2-\theta(x,y)). \tag{10}$$

In the case of stationary max-stable random fields, we use the notation $\theta(h) = \theta(0,h)$ and $\pi(h) = \pi(0,h)$.

As a by-product of Theorem 3, we can provide an explicit expression for the mean volume of the cells. Denote by λ the discrete counting measure when $\mathcal{X} = \mathbb{Z}^d$ or the Lebesgue measure when $\mathcal{X} = \mathbb{R}^d$. The volume of $C(x)$ is defined by $\mathrm{Vol}(C(x)) = \lambda(C(x))$. In the discrete case, $\mathrm{Vol}(C(x))$ is the cardinality of $C(x)$.

Corollary 5. *Let $x \in \mathcal{X}$. The cell $C(x)$ has expected volume*

$$\mathbb{E}[\mathrm{Vol}(C(x))] = \int_{\mathcal{X}} \mathbb{E}\left[\frac{Y(x)}{\eta(x)} \wedge \frac{Y(y)}{\eta(y)}\right] \lambda(\mathrm{d}y).$$

This together with (10) implies that the cell $C(x)$ has finite expected volume if and only if $\int_{\mathcal{X}}(2-\theta(x,y))\lambda(\mathrm{d}y) < +\infty$. Another consequence of Theorem 3 is an expression for the probability that the cell $C(x)$ is bounded.

Corollary 6. *Let $x \in \mathcal{X}$. The cell $C(x)$ is bounded with probability*

$$\mathbb{P}[C(x) \text{ is bounded}] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{y\to\infty} \frac{Y(y)}{\eta(y)}\right)^+\right].$$

Furthermore, the following statements are equivalent:

i) the cell $C(x)$ is bounded a.s.;

ii) $\lim_{y\to\infty} \frac{Y(y)}{\eta(y)} = 0$ a.e. on the event $\{Y(x) \neq 0\}$.

Remark 7. In the case when the max-stable process η is stationary, we will see in Section 3.3 below that condition ii) can be replaced by the following one: $Y(y) \to 0$ a.s. as $y \to \infty$.

2.2 Examples

As an illustration and to get some intuition, we provide several examples. Simulations of the max-stable processes together with the associated tessellations are available on the personal webpage of the first author.

Example 8. The isotropic Smith process [22] is defined by

$$\eta(x) = \bigvee_{i \geq 1} U_i h(x - X_i), \quad x \in \mathbb{R}^d,$$

where $\{(U_i, X_i), i \geq 1\}$ is a Poisson point process on $(0, \infty) \times \mathbb{R}^d$ with intensity $u^{-2}\mathrm{d}u\mathrm{d}x$ and $h(x) = (2\pi)^{-d/2} \exp(-\|x\|^2/2)$ is the standard Gaussian d-variate density function. The Smith process is a stationary max-stable process that belongs to the class of *moving maximum processes* and is hence mixing. Surprisingly, the associated tessellation is exactly the so-called *Laguerre tessellation* studied in great detail by Lautensack and Zuyev [12]. Indeed, the cell C_i is given by

$$C_i = \{x \in \mathbb{R}^d;\ \|x - X_i\|^2 - 2\ln(U_i) \leq \|x - X_j\|^2 - 2\ln(U_j),\ \ j \neq i\}.$$

In this very specific example, the cells are convex bounded polygons.

Example 9. Consider a moving maximum process of the same form as in the previous example, but with $h(x) = \exp\{-\|x\|/v\}$, $x \in \mathbb{R}^d$, where $v > 0$ is a parameter. Then, the cell C_i is given by

$$C_i = \{x \in \mathbb{R}^d;\ \|x - X_i\|/v - \ln(U_i) \leq \|x - X_j\|/v - \ln(U_j),\ \ j \neq i\},$$

and we recover a special case of the *Johnson–Mehl tessellation*; see Møller [14].

Example 10. The stationary extremal Gaussian process originally introduced by Schlather [20] corresponds to the case when the spectral process Y in representation (2) is given by

$$Y(x) = \sqrt{\frac{\pi}{2}} \max(W(x), 0), \quad x \in \mathbb{R}^d,$$

where W is a stationary Gaussian process on $\mathbb{R}^d$ with zero mean, unit variance and correlation function $\rho(h) = \mathbb{E}[W(0)W(h)]$, $h \in \mathbb{R}^d$. The extremal coefficient function is given by

$$\theta(h) = 2T_2\left[\sqrt{\frac{2}{1-\rho(h)^2}} - \sqrt{\frac{1-\rho(h)^2}{2}}\rho(h)\right], \quad h \in \mathbb{R}^d,$$

where T_2 is the cumulative distribution function of a Student distribution with 2 degrees of freedom. Typically, $\rho(h) \to 0$ as $h \to \infty$, so that $\theta(h) \to 2T_2(\sqrt{2}) < 2$ and η is neither mixing nor ergodic (see Stoev [23] or Kabluchko and Schlather [9]). The inequalities in (10) entail that $\liminf \mathbb{P}[h \in C(0)] > 0$ as $h \to \infty$. This suggests that the cells are not bounded which is consistent with the simulations available on the

first author personal webpage. Note that the cells are neither convex nor connected and have a pretty regular shape due to the particular choice of the correlation function $\rho(h) = \exp(-\|h\|^2/2)$ that yields smooth Gaussian sample paths.

Example 11. Brown–Resnick processes [10] form a flexible class of max-stable processes. They are given by (2) with the spectral process of the form

$$Y(x) = \exp\left(W(x) - \frac{1}{2}\sigma^2(x)\right), \quad x \in \mathbb{R}^d,$$

where W is a stationary increment centered Gaussian process on $\mathbb{R}^d$, and $\sigma^2(x) = \operatorname{Var} W(x)$. Surprisingly, the process η is stationary [10]. Its distribution is completely characterized by the variogram

$$\gamma(h) = \operatorname{Var}(W(x+h) - W(x)), \quad h \in \mathbb{R}^d.$$

The extremal coefficient function is given by ([10, p. 2063])

$$\theta(h) = 2G\left(\frac{1}{2}\sqrt{\gamma(h)}\right), \quad h \in \mathbb{R}^d,$$

where G is the cumulative distribution function of the standard normal distribution. Typically, $\gamma(h) \to \infty$ as $h \to \infty$, so that $\theta(h) \to 2$ and η is mixing [23, 9]. The inequalities in (10) entail that $\lim \mathbb{P}[h \in C(0)] = 0$ as $h \to \infty$ suggesting that the cells become asymptotically independent at large distances. Since $1 - G(u) \sim 1/(\sqrt{2\pi}u)\,\mathrm{e}^{-u^2/2}$, $u \to +\infty$, Corollary 5 implies that the cell $C(0)$ has finite expected volume (and hence, is a.s. bounded) provided that the following condition is satisfied:

$$\liminf_{h\to\infty} \frac{\gamma(h)}{\log \|h\|} > 8d.$$

Simulations with the variogram $\gamma(h) = 2\|h\|$ are available on the first author's personal webpage. They show that the cells may have a very rough shape, due to the particular choice of the variogram that yields rough Gaussian paths.

3 The stationary case: asymptotic properties of cells

In the sequel, we focus on the case when η is a *stationary* sample continuous max-stable random field on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$. We show strong connections between the ergodic and mixing properties of the random field η, its conservative/dissipative and positive/null decompositions and the geometry of the cells.

3.1 Ergodic properties and geometry of the cells

Ergodic and mixing properties of max-stable random fields have been studied intensively by Stoev [23, 24] and Kabluchko and Schlather [9]. A simple characterization using the extremal coefficient is known (see, e.g., [9, Theorems 1.1 and 1.2] where the more general case of max-infinitely divisible processes is considered).

Theorem 12 (Stoev (2008), Kabluchko and Schlather (2010))**.**
Let η be a stationary max-stable random field on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$.

- *η is ergodic if and only if $\theta(h) \to 2$ in Cesàro mean as $h \to \infty$;*
- *η is mixing if and only if $\theta(h) \to 2$ as $h \to \infty$.*

Interestingly, these results can be reinterpreted in terms of the geometric properties of the tessellation. For $r > 0$, we write $B_r = [-r, r]^d \cap \mathcal{X}$. We equip $\mathcal{X}$ with a measure λ which is either the counting or the Lebesgue measure, when $\mathcal{X} = \mathbb{Z}^d$ or $\mathcal{X} = \mathbb{R}^d$, respectively.

Proposition 13. *Let η be a stationary, sample continuous max-stable random field on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$.*

1. *The following statements are equivalent:*
   - *(1.a) η is ergodic,*
   - *(1.b) $\lim_{r\to+\infty} \mathbb{E}\Big[\frac{\lambda(C(0)\cap B_r)}{\lambda(B_r)}\Big] = 0$.*
2. *The following statements are equivalent:*
   - *(2.a) η is mixing,*
   - *(2.b) $\lim_{x\to\infty} \mathbb{P}[x \in C(0)] = 0$.*

Next we focus on strong mixing properties of max-stable processes, see Dombry and Eyi-Minko [4]. The β-mixing coefficients of the random process η are defined as follows: for disjoint closed subsets $S_1, S_2 \subset \mathcal{X}$, we define

$$\beta(S_1, S_2) = \sup\Big\{|\mathcal{P}_{S_1\cup S_2}(C) - (\mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2})(C)|;\ C \in \mathcal{B}_{S_1\cup S_2}\Big\}, \quad (11)$$

where $\mathcal{P}_S$ is the distribution (on the space $\mathbb{R}_+^S$) of the restriction of η to the set S, and $\mathcal{B}_S$ is the product σ-algebra on the space $\mathbb{R}_+^S$. Given a closed subset $S \subset \mathcal{X}$ and $r > 0$, we define

$$\beta_r(S) = \beta(S, S_r^c) \text{ with } S_r^c = \{x \in S; d(x, S) \geq r\}$$

where $d(x, S)$ denotes the distance between the point x and the set S. We say that η is strongly β-mixing if for all compact sets $S \subset \mathcal{X}$,

$$\lim_{r\to+\infty} \beta_r(S) = 0.$$

Proposition 14. *If η is a stationary max-stable random field such that $C(0)$ is almost surely bounded, then η is strongly β-mixing.*

We conjecture that the converse implication is also true:

Conjecture 15. *If η is a strongly β-mixing stationary max-stable random field, then $C(0)$ is almost surely bounded.*

We were not able to prove the conjecture, mainly because we lack a lower bound for the β-mixing coefficient $\beta(S_1, S_2)$ (only an upper bound is given in [4]). The intuition is relatively clear though: if the cell remains unbounded with positive probability, then the value $\eta(0)$ of the random field at the origin may have an impact at infinity via the unbounded cell $C(0)$. In view of Corollary 20, see below, the conjecture can also be stated as follows: a stationary max-stable field η is strongly β-mixing if and only if η is purely dissipative.

3.2 Preliminaries on stationary max-stable processes

The structure of stationary max-stable processes was first investigated by de Haan and Pickands [3]. Recently, further results were obtained by exploiting the analogy between the theory of max-stable and sum-stable processes. Inspired by the works of Rosiński [15, 16], Rosiński and Samorodnitsky [17] and Samorodnitsky [18, 19], the representation theory of stationary max-stable random fields via non-singular flows was developed independently by Kabluchko [8], Wang and Stoev [26] and Wang *et al.* [25]. See also Kabluchko and Stoev [11] for an extension to sum- and max-infinitely divisible processes. In these works, the conservative/dissipative and positive/null decompositions of the non-singular flow play a major role.

To avoid technical details of non-singular ergodic theory, we use a naive approach based on cone decompositions of max-stable processes (see, e.g., Wang and Stoev [26, Theorem 5.2]). The links between this approach and the non-singular ergodic theory are explored in Dombry and Kabluchko [6].

The following simple lemma about cone decompositions of max-stable processes will be useful. Recall that $\mathcal{F}_0 = \mathcal{F}(\mathcal{X}, [0, +\infty)) \setminus \{0\}$ denotes the set of continuous, non-negative functions on $\mathcal{X}$ excluding the zero function. A measurable subset $\mathcal{C} \subset \mathcal{F}_0$ is called a *cone* if for all $f \in \mathcal{C}$ and $u > 0$, $uf \in \mathcal{C}$. The cone $\mathcal{C}$ is said to be *shift-invariant* if for all $f \in \mathcal{C}$ and $x \in \mathcal{X}$, we have $f(\cdot + x) \in \mathcal{C}$.

Lemma 16. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two measurable, shift-invariant cones such that $\mathcal{F}_0 = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Let η be a stationary max-stable process given by representation* (2)*. Consider the decomposition $\eta = \eta_1 \vee \eta_2$ with*

$$\eta_1(x) = \bigvee_{i\geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{C}_1\}} \quad \text{and} \quad \eta_2(x) = \bigvee_{i\geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{C}_2\}}.$$

Then, η_1 and η_2 are stationary and independent max-stable processes[1] *whose distribution depends only on the distribution of η and not on the specific representation* (2)*.*

The notion of Brown–Resnick stationarity introduced in Kabluchko *et al.* [10] will be useful.

Definition 17. *We say that the process $Y = (Y(x))_{x\in\mathcal{X}}$ is Brown–Resnick stationary if the associated max-stable process η defined by* (2) *is stationary.*

For future reference, we gather in the next lemma several properties of Brown-Resnick stationary processes. A shift-invariant cone $\mathcal{F}_L$ is said to be *localizable* if there exist mappings $L_1 : \mathcal{F}_L \to \mathcal{X}$ and $L_2 : \mathcal{F}_L \to (0, +\infty)$ such that for all $f \in \mathcal{F}_L$, $x \in \mathcal{X}$ and $u > 0$,

- $L_1(f(\cdot + x)) = L_1(f) - x$ and $L_1(uf) = L_1(f)$,
- $L_2(f(\cdot + x)) = L_2(f)$ and $L_2(uf) = uL_2(f)$.

A typical example of localizable cone is the cone $\{f \in \mathcal{F}_0; \lim_\infty f = 0\}$ with $L_1(f) = \arg\max f$ and $L_2(f) = \max f$ (if the maximum is attained at several points, we define the arg max as the smallest such point with respect to the lexicographic order).

Lemma 18. *Let Y and Y' be independent Brown–Resnick stationary processes. In the case $\mathcal{X} = \mathbb{R}^d$, we assume for statements iii) and iv) that the associated max-stable process has continuous sample paths.*

i) The product YY' is also Brown–Resnick stationary.

ii) Let C be a shift-invariant cone, then $Y 1_{\{Y\in C\}}$ is Brown–Resnick stationary.

iii) Let $K \subset \mathcal{X}$ be compact. In the case $\mathcal{X} = \mathbb{R}^d$, we suppose that the interior of K is non-empty. Then, modulo null sets,

$$\left\{\lim_{x\to\infty} Y(x) = 0\right\} = \left\{\int_{\mathcal{X}} \sup_{y\in K} Y(x+y)\lambda(\mathrm{d}x) < \infty\right\}.$$

[1] with margins differing from the standardized form (1) by a multiplicative constant

iv) The cone $\mathcal{F}_L = \{f \in \mathcal{F}_0; \sup_{x\in\mathcal{X}} f(x) > \limsup_{x\to\infty} f(x)\}$ *is localizable and, modulo null sets,*

$$\{Y \in \mathcal{F}_L\} \subset \left\{ \lim_{x\to\infty} Y(x) = 0 \right\}.$$

In fact, the latter inclusion holds for any localizable cone.

Statement i) is due to Kabluchko *et al.* [10, Corollary 8], statement ii) is a by-product of Lemma 16 and its proof. Statements iii) and iv) are closely related to Proposition 10 and its proof in Kabluchko and Dombry [6]. In the proof of [6, Proposition 10], we show that $\mathcal{F}_D = \mathcal{F}'_D = \tilde{\mathcal{F}}_D$ wich implies iii); statement iv) is proven with the same arguments as for the proof of $\tilde{\mathcal{F}}_D \subset \mathcal{F}_D$ in [6], see the appendix for more details.

3.3 Boundedness of cells

We prove that the boundedness of the cell $C(x)$, $x \in \mathcal{X}$, is strongly connected with the conservative/dissipative decomposition of the max-stable process η. Introduce the following shift-invariant cones of functions:

$$\mathcal{F}_C = \left\{ f \in \mathcal{F}_0;\ \limsup_{x\to\infty} f(x) > 0 \right\}, \tag{12}$$

$$\mathcal{F}_D = \left\{ f \in \mathcal{F}_0;\ \lim_{x\to\infty} f(x) = 0 \right\}. \tag{13}$$

The conservative/dissipative decomposition of η is given by

$$\eta_C(x) = \bigvee_{i\geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{F}_C\}}, \tag{14}$$

$$\eta_D(x) = \bigvee_{i\geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{F}_D\}}. \tag{15}$$

According to Lemma 16, the processes η_C and η_D are independent stationary max-stable processes such that $\eta = \eta_C \vee \eta_D$. In dimension $d = 1$, this cone decomposition is known to be related to the the conservative/dissipative decomposition of the non-singular flow generating η (see, e.g., Wang and Stoev [26, Theorem 5.2]). The following theorem relates this conservative/dissipative decomposition to the boundedness of the cell $C(x)$.

Theorem 19. *Let* $x \in \mathcal{X}$*. The following events are equal modulo null sets:*

$$\{C(x) \text{ is unbounded}\} = \{\eta_C(x) > \eta_D(x)\}, \tag{16}$$

$$\{C(x) \text{ is bounded}\} = \{\eta_D(x) > \eta_C(x)\}. \tag{17}$$

We denote by α_C and α_D the scale parameters of the 1-Fréchet random variables $\eta_C(x)$ and $\eta_D(x)$ respectively, i.e. for all $z > 0$,

$$\mathbb{P}[\eta_C(x) \leq z] = \exp(-\alpha_C/z), \quad \mathbb{P}[\eta_D(x) \leq z] = \exp(-\alpha_D/z). \quad (18)$$

Note that $\alpha_D + \alpha_C = 1$ and that α_C and α_D do not depend on $x \in \mathcal{X}$. We say that η is purely conservative (resp. purely dissipative) if $\alpha_C = 1$ (resp. $\alpha_D = 1$).

Corollary 20. *Let $x \in \mathcal{X}$. We have:*

i) $\mathbb{P}[C(x)$ *is unbounded*$] = \alpha_C$,

ii) $\mathbb{P}[C(x)$ *is bounded*$] = \alpha_D$,

iii) $C(x)$ *is unbounded a.s. if and only if η is purely conservative,*

iv) $C(x)$ *is bounded a.s. if and only if η is purely dissipative.*

3.4 Asymptotic density of cells

Next we consider the decomposition of η into positive and null components and relate it to the asymptotic density of the cell $C(x)$. For this purpose, we introduce a new construction of the positive/null decomposition of max-stable processes which simplifies and extends to the dimension $d \geq 1$ the construction from Samorodnitsky [19] and Wang and Stoev [26, Example 5.4].

Recall that we write $B_r = [-r, r]^d \cap \mathcal{X}$ for $r > 0$ and that λ is either the counting or the Lebesgue measure on $\mathcal{X}$, when $\mathcal{X} = \mathbb{Z}^d$ or $\mathcal{X} = \mathbb{R}^d$, respectively. Consider the shift-invariant cones of functions

$$\mathcal{F}_P = \left\{ f \in \mathcal{F}_0;\ \lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f(x)\lambda(\mathrm{d}x) > 0 \right\}, \quad (19)$$

$$\mathcal{F}_N = \left\{ f \in \mathcal{F}_0;\ \liminf_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f(x)\lambda(\mathrm{d}x) = 0 \right\}. \quad (20)$$

In the definition of $\mathcal{F}_P$, we assume that the limit exists. The stationarity of η implies that $Y \in \mathcal{F}_P \cup \mathcal{F}_N$ a.s.; see Dombry and Kabluchko [6]. According to Lemma 16, the corresponding decomposition is

$$\eta_P(x) = \bigvee_{i\geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{F}_P\}}, \quad (21)$$

$$\eta_N(x) = \bigvee_{i\geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{F}_N\}}, \quad (22)$$

where the processes η_N and η_P are independent, stationary, max-stable and $\eta = \eta_P \vee \eta_N$. This decomposition based on cones is equal to the positive/null decomposition based on the underlying non-singular flow

(see e.g., Wang and Stoev [26, Theorem 5.3] in dimension $d = 1$, Wang *et al.* [25] in dimension $d \geq 1$, Dombry and Kabluchko [6]).

Given a measurable subset $C \subset \mathcal{X}$, we define its lower and upper asymptotic densities by

$$\delta^-(C) = \liminf_{r\to+\infty} \frac{\lambda(C \cap B_r)}{\lambda(B_r)}, \quad \delta^+(C) = \limsup_{r\to+\infty} \frac{\lambda(C \cap B_r)}{\lambda(B_r)}.$$

If $\delta^-(C) = \delta^+(C)$, the common value is called the asymptotic density of C and denoted by $\delta(C)$. The following theorem relates the positive/null decomposition of η to the asymptotic density of the cell $C(x)$.

Theorem 21. *Let $x \in \mathcal{X}$. The following events are equal modulo null sets:*

$$\{\delta(C(x)) > 0\} = \{\eta_P(x) > \eta_N(x)\}, \tag{23}$$

$$\{\delta^-(C(x)) = 0\} = \{\eta_N(x) > \eta_P(x)\}, \tag{24}$$

where the notation $\delta(C(x)) > 0$ means that the asymptotic density $\delta(C(x))$ exists and is positive.

We denote by α_P and α_N the scale parameters of the 1-Fréchet random variables $\eta_P(x)$ and $\eta_N(x)$ respectively, i.e. for all $z > 0$,

$$\mathbb{P}[\eta_P(x) \leq z] = \exp(-\alpha_P/z) \text{ and } \mathbb{P}[\eta_N(x) \leq z] = \exp(-\alpha_N/z).$$

Note that $\alpha_P + \alpha_N = 1$ and that α_P and α_N do not depend on x. We say that the max-stable process η is generated by a positive (resp. null) flow if $\alpha_P = 1$ (resp. $\alpha_N = 1$).

Corollary 22. *Let $x \in \mathcal{X}$. We have:*

i) $\mathbb{P}[\delta(C(x)) > 0] = \alpha_P$,

ii) $\mathbb{P}[\delta^-(C(x)) = 0] = \alpha_N$,

iii) $\delta(C(x)) > 0$ *a.s. if and only if η is generated by a positive flow,*

iv) $\delta^-(C(x)) = 0$ *a.s. if and only if η is generated by a null flow.*

4 Proofs related to Section 2

4.1 Proof of Theorem 3

Proof of Theorem 3. We first prove (7). For $f, g : \mathcal{X} \to \mathbb{R}$ and $K \subset \mathcal{X}$, we use the notation

$$f >_K g \quad \text{if and only if} \quad f(x) > g(x) \text{ for all } x \in K.$$

For $i \geq 1$, we write $m_i = \bigvee_{j \neq i} \phi_j$ where $\phi_i = U_i Y_i$ is defined by (5). Fix some $x \in \mathcal{X}$. Note that $x \in C_i$ if and only if $\phi_i(x) \geq m_i(x)$, whence (modulo null sets)

$$\begin{aligned} \{K \subset C(x)\} &= \{\exists i \geq 1,\ \phi_i(x) > m_i(x) \text{ and } \forall y \in K,\ \phi_i(y) > m_i(y)\} \\ &= \{\exists i \geq 1,\ \phi_i >_{K \cup \{x\}} m_i\}. \end{aligned}$$

The events $\{\phi_i >_{K \cup \{x\}} m_i\}$, $i \geq 1$, are pairwise disjoint so that

$$1_{\{K \subset C(x)\}} = \sum_{i \geq 1} 1_{\{\phi_i >_{K \cup \{x\}} m_i\}} \quad \text{a.s.}$$

Hence, we obtain

$$\mathbb{P}[K \subset C(x)] = \mathbb{E}\Big[\sum_{i \geq 1} 1_{\{\phi_i >_{K \cup \{x\}} m_i\}} \Big].$$

This expectation can be computed thanks to the Slivnyak–Mecke formula (see, e.g., Schneider and Weil [21, page 68]). Recall from (5) and (6) that $\Phi = \{\phi_i, i \geq 1\}$ is a Poisson point process with intensity μ and that m_i is a functional of $\Phi \setminus \{\phi_i\}$. The Slivnyak–Mecke formula implies that

$$\mathbb{P}[K \subset C(x)] = \int_{\mathcal{F}_0} \mathbb{E}\left[1_{\{f >_{K \cup \{x\}} \eta\}} \right] \mu(\mathrm{d}f).$$

Using (6), we compute

$$\begin{aligned} \int_{\mathcal{F}_0} \mathbb{E}\left[1_{\{f >_{K \cup \{x\}} \eta\}} \right] \mu(\mathrm{d}f) &= \int_0^\infty \mathbb{E}\left[1_{\{uY >_{K \cup \{x\}} \eta\}} \right] u^{-2} \mathrm{d}u \\ &= \mathbb{E}\left[\int_0^\infty 1_{\{u > \sup_{K \cup \{x\}} \eta/Y\}} u^{-2} \mathrm{d}u \right] \\ &= \mathbb{E}\left[\inf_{K \cup \{x\}} Y/\eta \right]. \end{aligned}$$

This proves (7).

In the same spirit as in the proof of (7), we have

$$\{C(x) \subset K\} = \{\exists i \geq 1,\ \phi_i(x) > m_i(x) \text{ and } \phi_i <_{K^c} m_i\}$$

whence we deduce

$$\mathbb{P}[C(x) \subset K] = \mathbb{E}\Big[\sum_{i \geq 1} 1_{\{\phi_i(x) > m_i(x)\}} 1_{\{\phi_i <_{K^c} m_i\}} \Big].$$

We obtain (8) thanks to the Slivnyak–Mecke and straightforward computations. □

4.2 Proof of Corollaries 5 and 6

Proof of Corollary 5. By Fubini's Theorem, the expected volume of the cell $C(x)$ is equal to

$$\mathbb{E}[\mathrm{Vol}(C(x))] = \mathbb{E}\left[\int_{\mathcal{X}} 1_{\{y\in C(x)\}}\lambda(\mathrm{d}y)\right] = \int_{\mathcal{X}} \mathbb{P}[y \in C(x)]\lambda(\mathrm{d}y)$$

and, according to Theorem 3,

$$\mathbb{P}[y \in C(x)] = \mathbb{E}\left[\frac{Y(x)}{\eta(x)} \wedge \frac{Y(y)}{\eta(y)}\right].$$

□

Proof of Corollary 6. For $n \geq 1$, we recall that $B_n = [-n, n]^d \cap \mathcal{X}$. The sequence of events $\{C(x) \subset B_n\}$, $n \geq 1$, is non-decreasing and we have

$$\{C(x) \text{ bounded}\} = \bigcup_{n\geq 1} \{C(x) \subset B_n\},$$

whence

$$\mathbb{P}[C(x) \text{ bounded}] = \lim_{n\to\infty} \mathbb{P}[C(x) \subset B_n].$$

Using (8), we get

$$\mathbb{P}[C(x) \subset B_n] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{B_n^c} \frac{Y}{\eta}\right)^+\right].$$

As $n \to +\infty$, the sequence $\sup_{B_n^c} Y/\eta$ decreases to $\limsup_{y\to\infty} Y(y)/\eta(y)$. The monotone convergence theorem entails that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{B_n^c} \frac{Y}{\eta}\right)^+\right] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{y\to\infty} \frac{Y(y)}{\eta(y)}\right)^+\right],$$

whence we deduce

$$\mathbb{P}[C(x) \text{ bounded}] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{y\to\infty} \frac{Y(y)}{\eta(y)}\right)^+\right].$$

In order to prove the equivalence of the statements (i) and (ii), we note that

$$0 \leq \left(\frac{Y(x)}{\eta(x)} - \limsup_{y\to\infty} \frac{Y(y)}{\eta(y)}\right)^+ \leq \frac{Y(x)}{\eta(x)}.$$

Note also that $\mathbb{E}[Y(x)/\eta(x)] = 1$ since $Y(x)$ is independent of $1/\eta(x) \sim \mathrm{Exp}(1)$. Using the fact that $(a-b)^+ = a$ (for $a, b \geq 0$) if and only if $a = 0$ or $b = 0$, we can deduce that the equality

$$\mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{y\to\infty} \frac{Y(y)}{\eta(y)}\right)^+\right] = 1$$

occurs if and only if $\limsup_{y\to\infty} Y(y)/\eta(y) = 0$ a.e. on the event $\{Y(x) \neq 0\}$. This proves the equivalence of (i) and (ii). □

5 Proofs related to Section 3

5.1 Proofs of Propositions 13 and 14

Proof of Proposition 13. According to Theorem 12, η is ergodic if and only if

$$\lim_{r\to+\infty} \frac{1}{\lambda(B_r)} \int_{B_r} (2 - \theta(h))\lambda(\mathrm{d}h) = 0, \tag{25}$$

and η is mixing if and only if

$$\lim_{h\to\infty} (2 - \theta(h)) = 0. \tag{26}$$

Clearly, in view of the inequalities (10), (25) is equivalent to

$$\lim_{r\to+\infty} \frac{1}{\lambda(B_r)} \int_{B_r} \mathbb{P}[h \in C(0)]\lambda(\mathrm{d}h) = \lim_{r\to+\infty} \mathbb{E}\Big[\frac{\lambda(C(0) \cap B_r)}{\lambda(B_r)}\Big] = 0$$

and (26) is equivalent to $\lim_{h\to\infty} \mathbb{P}[h \in C(0)] = 0$. □

Proof of Proposition 14. We use here an upper bound for the β-mixing coefficient provided by Dombry and Eyi-Minko [4, Theorem 3.1]: the β-mixing coefficient $\beta(S_1, S_2)$ defined by (11) satisfies

$$\beta(S_1, S_2) \leq 2\mathbb{P}[A(S_1, S_2)],$$

where $A(S_1, S_2)$ denotes the event

$$\{\exists i \geq 1,\ \exists (s_1, s_2) \in S_1 \times S_2,\ U_i Y_i(s_1) = \eta(s_1) \text{ and } U_i Y_i(s_2) = \eta(s_2)\}.$$

Introducing the cells $C(s_1)$ with $s_1 \in S_1$, we have

$$\begin{aligned} A(S_1, S_2) &= \{\exists (s_1, s_2) \in S_1 \times S_2,\ s_2 \in C(s_1)\} \\ &= \{\cup_{s_1\in S_1} C(s_1) \cap S_2 \neq \emptyset\} \end{aligned}$$

and

$$\beta(S_1, S_2) \leq 2\mathbb{P}[\cup_{s_1\in S_1} C(s_1) \cap S_2 \neq \emptyset].$$

We deduce that, for all compact set $K \subset \mathcal{X}$ and for all $r > 0$,

$$\beta_r(K) \leq 2\mathbb{P}[\exists x \in \mathcal{X},\ d(x, K) \geq r \text{ and } x \in \cup_{s\in K} C(s)].$$

We prove below that if $C(0)$ is bounded a.s., then so is $\cup_{s\in K} C(s)$, whence the right-hand side in the above inequality converges to 0 (by

the dominated convergence theorem), and $\lim_{r\to\infty}\beta_r(K)=0$.
Suppose now that $C(0)$ is bounded a.s. In the discrete case $\mathcal{X}=\mathbb{Z}^d$, the compact set K is finite and $\cup_{s\in K}C(s)$ is a.s. bounded as a finite union of bounded sets. In the continuous case $\mathcal{X}=\mathbb{R}^d$, K may be infinite but it is known that there are a.s. only finitely many indices $i\geq 1$ such that $U_iY_i(s)=\eta(s)$ for some $s\in K$ (see Dombry and Eyi-Minko [5, Proposition 2.3]). Hence, we can extract a finite covering $\cup_{s\in K}C(s)=\cup_{j=1}^k C(s_j)$ and $\cup_{s\in K}C(s)$ is a.s. bounded as a finite union of bounded sets. □

5.2 Proof of Lemma 16

Proof of Lemma 16. By the uniqueness of the max-Lévy measure, the max-stable process η is stationary if and only if its max-Lévy measure μ is stationary. By the properties of Poisson point processes, $\Phi\cap\mathcal{C}_i$, $i=1,2$, are independent Poisson point processes with intensity measures $\mathrm{d}\mu_i=1_{\mathcal{C}_i}\mathrm{d}\mu$. The max-stable processes η_1 and η_2 are hence independent with exponent measures μ_1 and μ_2, respectively. Since the cone $\mathcal{C}_i$ is shift-invariant, so is the measure μ_i. Hence, the process η_i is stationary. Finally, the distribution of η_i is characterized by the max-Lévy measure $\mathrm{d}\mu_i=1_{\mathcal{C}_i}\mathrm{d}\mu$ and does not depend on the representation (2). □

5.3 Proofs of Theorem 19 and Corollary 20

In the next lemma, we gather some preliminary computations needed for the proof of Theorem 19.

Lemma 23. *Let $x\in\mathcal{X}$. We have:*

i) $\alpha_C=\mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_C\}}]$ *and* $\alpha_D=\mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_D\}}]$,

ii) $\mathbb{P}[\eta_C(x)>\eta_D(x)]=\alpha_C$ *and* $\mathbb{P}[\eta_D(x)>\eta_C(x)]=\alpha_D$,

iii) $\mathbb{P}[C(x)$ *is bounded,* $\eta_C(x)>\eta_D(x)]=\mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)}-\limsup_\infty\frac{Y}{\eta}\right)^+1_{\{Y\in\mathcal{F}_C\}}\right]$,

iv) $\mathbb{P}[C(x)$ *is bounded,* $\eta_D(x)>\eta_C(x)]=\mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)}-\limsup_\infty\frac{Y}{\eta}\right)^+1_{\{Y\in\mathcal{F}_D\}}\right]$.

Proof of Lemma 23. i) From (14) we get

$$\begin{aligned}\mathbb{P}[\eta_C(x)\leq y] &= \mathbb{P}[\vee_{i\geq 1}U_iY_i(x)1_{\{Y_i\in\mathcal{F}_C\}}\leq y]\\ &= \exp\Big(-\int_0^\infty\mathbb{P}[uY(x)1_{\{Y\in\mathcal{F}_C\}}>y]u^{-2}\mathrm{d}u\Big)\\ &= \exp(-\mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_C\}}]/y),\end{aligned}$$

whence we deduce that $\alpha_C = \mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_C\}}]$. The formula for α_D is obtained in the same way.

ii) The random variables $\eta_C(x)$ and $\eta_D(x)$ are independent and have Fréchet distribution with parameters α_C and α_D, respectively. Hence,

$$\begin{aligned}\mathbb{P}[\eta_C(x) > \eta_D(x)] &= \mathbb{E}[\exp(-\alpha_D/\eta_C(x))] \\ &= \int_0^{+\infty} \exp(-\alpha_D/u)\mathrm{d}(\mathrm{e}^{-\alpha_C/u}) \\ &= \alpha_C.\end{aligned}$$

For the last equality, we use $\alpha_C + \alpha_D = 1$. Similarly, $\mathbb{P}[\eta_D(x) > \eta_C(x)] = \alpha_D$.

iii) This statement is a variation of Corollary 6 and we give only the main lines of its proof. We first prove the following version of (8): for all compact sets $K \subset \mathcal{X}$,

$$\begin{aligned}&\mathbb{P}[C(x) \subset K,\ \eta_C(x) > \eta_D(x)] \\ = \ &\mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{y\in K^c}\frac{Y(y)}{\eta(y)}\right)^+ 1_{\{Y\in\mathcal{F}_C\}}\right]. \qquad (27)\end{aligned}$$

Indeed, with the same notation as in the proof of (8), we have

$$\begin{aligned}&\{C(x) \subset K,\ \eta_C(x) > \eta_D(x)\} \\ = \ &\{\exists i \geq 1,\ \phi_i(x) > m_i(x), \quad \phi_i <_{K^c} m_i \quad \text{and } \phi_i \in \mathcal{F}_C\}\end{aligned}$$

and the Slivnyak–Mecke formula entails that

$$\begin{aligned}&\mathbb{P}[C(x) \subset K,\ \eta_C(x) > \eta_D(x)] \\ = \ &\mathbb{E}\left[\sum_{i\geq 1} 1_{\{\phi_i(x)>m_i(x)\}}1_{\{\phi_i<_{K^c}m_i\}}1_{\{\phi_i\in\mathcal{F}_C\}}\right] \\ = \ &\int_{\mathcal{F}_0} \mathbb{E}\left[1_{\{f(x)>\eta(x)\}}1_{\{f<_{K^c}\eta\}}1_{\{f\in\mathcal{F}_C\}}\right]\mu(\mathrm{d}f).\end{aligned}$$

With similar computations as in the proof of (8), (27) is easily deduced. Then statement iii) follows from (27) exactly in the same way as Corollary 6 follows from (8).

iv) The proof is similar and is omitted. □

Proof of Theorem 19. Since $\{\eta_D(x) = \eta_C(x)\}$ is a null set, it suffices to prove the following two inclusions (modulo null sets):

$$\{\eta_D(x) > \eta_C(x)\} \subset \{C(x) \text{ is bounded}\}, \qquad (28)$$

$$\{\eta_C(x) > \eta_D(x)\} \subset \{C(x) \text{ is unbounded}\}. \qquad (29)$$

Proof of (28). We first reduce the proof of (28) to the proof of

$$\lim_{y\to\infty} \frac{Y(y)}{\eta(y)} 1_{\{Y\in\mathcal{F}_D\}} = 0 \quad \text{a.s.} \tag{30}$$

Indeed, (30) and statements i), ii) and iv) of Lemma 23 entail that

$$\begin{aligned}
& \mathbb{P}[C(x) \text{ is bounded, } \eta_D(x) > \eta_C(x)] \\
= \; & \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{\infty} \frac{Y}{\eta}\right)^+ 1_{\{Y\in\mathcal{F}_D\}}\right] \\
= \; & \mathbb{E}\left[\frac{Y(x)}{\eta(x)} 1_{\{Y\in\mathcal{F}_D\}}\right] \\
= \; & \alpha_D \\
= \; & \mathbb{P}[\eta_D(x) > \eta_C(x)],
\end{aligned}$$

and we deduce (28).

It remains to prove (30). Statements i) and iii) of Lemma 18 imply that $Y 1_{\{Y\in\mathcal{F}_D\}}$ is Brown–Resnick stationary and such that

$$\int_{\mathcal{X}} \sup_{y\in K} Y(x+y) 1_{\{Y\in\mathcal{F}_D\}} \lambda(\mathrm{d}x) < \infty \quad \text{a.s.}$$

On the other hand, let us consider the process $Z = \frac{Y}{\eta} 1_{\{Y\in\mathcal{F}_D\}}$. Since Y and $1/\eta$ are Brown–Resnick stationary and since the cone $\mathcal{F}_D$ is shift invariant, statement i) and ii) of Lemma 18 imply that $Z = \frac{Y}{\eta} 1_{\{Y\in\mathcal{F}_D\}}$ is Brown–Resnick stationary. Furthermore, for any compact set $K \subset \mathcal{X}$,

$$\begin{aligned}
& \mathbb{E}\left[\int_{\mathcal{X}} \sup_{y\in K} Z(x+y)\lambda(\mathrm{d}x) \;\middle|\; Y\right] \\
\leq \; & \mathbb{E}\left[\int_{\mathcal{X}} \frac{\sup_{y\in K} Y(x+y)}{\inf_{y\in K} \eta(x+y)} 1_{\{Y\in\mathcal{F}_D\}} \lambda(\mathrm{d}x) \;\middle|\; Y\right] \\
= \; & \mathbb{E}\left[\sup_{y\in K} \eta^{-1}(y)\right] \int_{\mathcal{X}} \sup_{y\in K} Y(x+y) 1_{\{Y\in\mathcal{F}_D\}} \lambda(\mathrm{d}x) < \infty \quad \text{a.s.}
\end{aligned}$$

In the last equation, we used the independence of Y and η, the stationarity of η and the fact that $\mathbb{E}\left[\sup_{y\in K} \eta^{-1}(y)\right] < \infty$ (see Dombry and Eyi-Minko [4, Theorem 2.2]). As a consequence,

$$\int_{\mathcal{X}} \sup_{y\in K} Z(x+y)\lambda(\mathrm{d}x) < \infty \quad \text{a.s.}$$

and Lemma 18-iii) implies that $\lim_{x\to\infty} Z(x) = 0$ a.s., thus proving (30).

Proof of (29). We consider the shift-invariant cone

$$\mathcal{F}_L = \left\{ f \in \mathcal{F}_0; \sup_{\mathcal{X}} f > \limsup_{\infty} f \right\}.$$

We will prove that the process $Z = \frac{Y}{\eta} 1_{\{Y\in\mathcal{F}_C\}}$ is Brown–Resnick stationary and satisfies

$$\mathbb{P}[Z \in \mathcal{F}_L] = 0. \tag{31}$$

After this has been done, (29) can be deduced as follows: (31) implies that

$$\frac{Y(x)}{\eta(x)} 1_{\{Y\in\mathcal{F}_C\}} \le \sup_{\mathcal{X}} \left(\frac{Y}{\eta} 1_{\{Y\in\mathcal{F}_C\}} \right) \le \left(\limsup_{\infty} \frac{Y}{\eta} 1_{\{Y\in\mathcal{F}_C\}} \right) \quad \text{a.s.},$$

whence

$$\left(\frac{Y(x)}{\eta(x)} - \limsup_{\infty} \frac{Y}{\eta} \right)^+ 1_{\{Y\in\mathcal{F}_C\}} = 0 \quad \text{a.s.}$$

According to Lemma 23, statement iii), we obtain that

$$\begin{aligned} & \mathbb{P}[C(x) \text{ is bounded, } \eta_C(x) > \eta_D(x)] \\ = \; & \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{\infty} \frac{Y}{\eta} \right)^+ 1_{\{Y\in\mathcal{F}_C\}} \right] \\ = \; & 0, \end{aligned}$$

and this implies (29).

We now consider (31). Statements i) and ii) of Lemma 18 imply that the process Z is Brown–Resnick stationary. Lemma 18-iv) entails that $\mathbb{P}[Z \in \mathcal{F}_L] \le \mathbb{P}[Z \in \mathcal{F}_D]$. So, it suffices to prove that $\mathbb{P}[Z \in \mathcal{F}_D] = 0$. Suppose by contradiction that $\mathbb{P}[Z \in \mathcal{F}_D] > 0$. Recalling that $Z = \frac{Y}{\eta} 1_{\{Y\in\mathcal{F}_C\}}$, we see that

$$\{Z \in \mathcal{F}_D\} = \{Y \in \mathcal{F}_C\} \cap \{Y/\eta \in \mathcal{F}_D\}.$$

On the set $\{Y \in \mathcal{F}_C\} = \{\limsup_\infty Y > 0\}$, one can construct a $\sigma(Y)$-measurable random sequence $x_n \to \infty$ such that $Y(x_n) \ge \frac{1}{2} \limsup_\infty Y > 0$. Then, on $\{Z \in \mathcal{F}_D\} \subset \{Y/\eta \in \mathcal{F}_D\} = \{\lim_\infty Y/\eta = 0\}$, we have necessarily $\eta(x_n) \to +\infty$. But η is stationary and independent of Y, so that $\eta(x_n)$ has a unit Fréchet distribution that does not depend on n. This leads to a contradiction and we must hence have $\mathbb{P}[Z \in \mathcal{F}_D] = 0$. This concludes the proof of (31). □

Proof of Corollary 20. Theorem 19 and Lemma 23-ii) together yield

$$\mathbb{P}[C(x) \text{ is unbounded}] = \mathbb{P}[\eta_C(x) > \eta_D(x)] = \alpha_C,$$

proving statement i). Statement ii) is proved similarly. Furthermore, η is purely dissipative if $\eta_C = 0$, which is equivalent to $\alpha_C = 0$. We deduce easily that η is purely dissipative if and only if $C(x)$ is bounded a.s. and this proves iii). The proof of iv) is similar. □

5.4 Proofs of Theorem 21 and Corollary 22

Proof of Theorem 21. It suffices to prove the following two inclusions (modulo null sets):

$$\{\eta_N(x) > \eta_P(x)\} \subset \{\delta^-(C(x)) = 0\} \tag{32}$$

and

$$\{\eta_P(x) > \eta_N(x)\} \subset \{\delta(C(x)) > 0\}. \tag{33}$$

Proof of (32). Let us consider the cell of x with respect to the null component only. It is defined by

$$C_N(x) = \{y \in \mathcal{X};\, \exists i \geq 1,\, Y_i \in \mathcal{F}_N,\, U_iY_i(x) = \eta_N(x),\, U_iY_i(y) = \eta_N(y)\}.$$

Clearly, $\eta_N(x) > \eta_P(x)$ implies that $C(x) \subset C_N(x)$. We will prove that $\delta^-(C_N(x)) = 0$ on $\{\eta_N(x) > \eta_P(x)\}$ and this implies (32).

We can suppose without loss of generality that $\eta = \eta_N$ is generated by a null flow and prove that the lower asymptotic density of $C(x) = C_N(x)$ is equal to zero. According to Wang *et al.* [25, Theorem 4.1] or Kabluchko [8, Theorem 8], max-stable random fields generated by null flows are ergodic, whence Proposition 13 implies

$$\mathbb{E}\Big[\frac{\lambda(C(0) \cap B_r)}{\lambda(B_r)}\Big] \to 0 \quad \text{as } r \to \infty.$$

This implies the convergence in probability

$$\frac{\lambda(C(0) \cap B_r)}{\lambda(B_r)} \xrightarrow{\mathbb{P}} 0, \quad \text{as } r \to +\infty$$

and hence almost sure converge to 0 along a subsequence. We deduce that $\delta^-(C(0)) = 0$ almost surely and, by stationarity, the same holds true for $C(x)$, $x \in \mathcal{X}$.

Proof of Equation (33). Possibly changing representation (2), we may suppose without loss of generality that for any $i \geq 1$, the random process $\tilde{Y}_i = Y_i 1_{\{Y_i \in P\}}$ is stationary. We consider the cells

$$\tilde{C}_i = \{y \in \mathcal{X},\, U_i\tilde{Y}_i(y) = \eta(y)\}, \quad i \geq 1.$$

We will prove below that for every $i \geq 1$ with probability one,

$$\text{either } \delta(\tilde{C}_i) > 0 \text{ or } \lambda(\tilde{C}_i) = 0. \tag{34}$$

We show that this implies (33). On the event $\{\eta_P(x) > \eta_N(x)\}$, there is a random index $i(x)$ such that $C(x) = \tilde{C}_{i(x)}$. Furthermore, since $x \in C(x)$, we have $\lambda(\tilde{C}_{i(x)}) > 0$ (this is clear in the discrete case, in the continuous case, $C(x)$ contains a neighborhood of x). According to (34), we obtain $\delta(C_{i(x)}) = \delta(C_x) > 0$, proving (33).

It remains to prove (34). Recall that the U_i's are arranged in the decreasing order. Fix $i \geq 1$ and observe that the distribution of $(U_i, \tilde{Y}_i, \eta)$ is invariant under the shift

$$T_x(u, f_1, f_2) = (u, f_1(\cdot + x), f_2(\cdot + x)), \quad u > 0,\ f_1, f_2 \in \mathcal{F}_0.$$

Then we observe that

$$\begin{aligned}
\frac{\lambda(\tilde{C}_i \cap B_r)}{\lambda(B_r)} &= \frac{1}{\lambda(B_r)} \int_{B_r} 1_{\{x \in \tilde{C}_i\}} \lambda(\mathrm{d}x) \\
&= \frac{1}{\lambda(B_r)} \int_{B_r} 1_{\{U_i \tilde{Y}_i(x) = \eta(x)\}} \lambda(\mathrm{d}x) \\
&= \frac{1}{\lambda(B_r)} \int_{B_r} 1_{\{T_x(U_i, \tilde{Y}_i, \eta) \in A\}} \lambda(\mathrm{d}x)
\end{aligned}$$

with $A = \{(u, f_1, f_2); u f_1(0) = f_2(0)\}$. We can then apply the multiparameter ergodic theorem (see, e.g., [25, Theorem 2.8]) and conclude that

$$\lim_{r \to +\infty} \frac{\lambda(\tilde{C}_i \cap B_r)}{\lambda(B_r)} = \mathbb{E}[1_A(U_i, \tilde{Y}_i, \eta) \mid \mathcal{I}] \quad \text{a.s.},$$

where $\mathcal{I}$ denotes the σ-algebra of shift-invariant sets. This shows that $\tilde{C}_i$ has an asymptotic density,

$$\delta(\tilde{C}_i) = \mathbb{E}[1_{\{0 \in \tilde{C}_i\}} \mid \mathcal{I}] \quad \text{a.s.}$$

Furthermore, we observe that shift-invariance implies that

$$\mathbb{E}[1_{\{0 \in \tilde{C}_i\}} \mid \mathcal{I}] = \mathbb{E}[1_{\{x \in \tilde{C}_i\}} \mid \mathcal{I}], \quad x \in \mathcal{X}.$$

Using the fact that $\{\delta(\tilde{C}_i) = 0\} \in \mathcal{I}$, we deduce that

$$\begin{aligned}
\mathbb{E}[\lambda(\tilde{C}_i) 1_{\{\delta(\tilde{C}_i) = 0\}} \mid \mathcal{I}] &= 1_{\{\delta(\tilde{C}_i) = 0\}} \int_{\mathcal{X}} \mathbb{E}[1_{\{x \in \tilde{C}_i\}} \mid \mathcal{I}] \, \lambda(\mathrm{d}x) \\
&= 0.
\end{aligned}$$

Taking the expectation, we obtain that

$$\mathbb{E}[\lambda(\tilde{C}_i) 1_{\{\delta(\tilde{C}_i) = 0\}}] = 0$$

and we conclude that $\lambda(\tilde{C}_i) = 0$ on the event $\{\delta(\tilde{C}_i) = 0\}$, proving (34). □

Proof of Corollary 22. For the sake of brevity, we omit the proof which is quite straightforward from Theorem 21 and very similar to the proof of Corollary 20. □

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A Proof of Lemma 18 iv)

Proof of Lemma 18 iv). To check that $\mathcal{F}_L$ is localizable, take $L_1(f) = \arg\max(f)$ and $L_2(f) = \max(f)$ in the definition of a localizable cone (note that we are working with continuous functions so that the supremum is a maximum).

For the proof of the inclusion $\{Y \in \mathcal{F}_L\} \subset \{\lim_{x\to\infty} Y(x) = 0\}$, we prove that $\eta_L = \vee_{i\geq 1} U_i Y_i 1_{\{Y_i \in \mathcal{F}_L\}}$ admits a mixed moving maximum representation. According to [6, Proposition 10], this implies that $Y_L \in \tilde{\mathcal{F}}_D$ almost surely and hence the inclusion $\{Y \in \mathcal{F}_L\} \subset \{\lim_{x\to\infty} Y(x) = 0\}$ modulo null sets. For simplicity, we omit the subscript L and assume that $Y \in \mathcal{F}_L$ almost surely. We prove that $\eta = \vee_{i\geq 1} U_i Y_i$ admits a mixed moving maximum representation. In fact, the proof works if $\mathcal{F}_L$ is replaced by any localizable cone. We follow the proof of Theorem 14 in Kabluchko et al. [10] and we sketch only the main lines. We introduce the random variables

$$X_i = \operatorname*{argmax}_{x\in\mathcal{X}} Y_i(x), \quad Z_i(\cdot) = \frac{Y_i(X_i + \cdot)}{\max_{x\in\mathcal{X}} Y_i(x)}, \quad V_i = U_i \max_{x\in\mathcal{X}} Y_i(x).$$

Note that X_i is well-defined because of the definition of $\mathcal{F}_L$. If the maximum is attained at several points, we take the lexicographically smallest one. Clearly, we have $U_i Y_i(x) = V_i Z_i(x - X_i)$ for all $x \in \mathcal{X}$ so that

$$\eta(x) = \bigvee_{i\geq 1} V_i Z_i(x - X_i).$$

It remains to check that $(V_i, X_i, Z_i)_{i\geq 1}$ is a Poisson point process with intensity measure $u^{-2}\mathrm{d}u\lambda(\mathrm{d}x)Q(\mathrm{d}f)$, where Q is a probability measure on $\mathcal{F}_0$. Clearly, $(V_i, X_i, Z_i)_{i\geq 1}$ is a Poisson point process as the image of the original point process $(U_i, Y_i)_{i\geq 1}$. Its intensity is the image of the intensity of the original point process. With a straightforward transposition of the arguments of [10, Theorem 14], one can check that it has the required form. □

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