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We study the dynamics of Bogoliubov excitations of a Bose-Einstein condensate in the quasiperiodic kicked rotor. In the weakly interacting regime, the condensate is stable and both the condensate and the excitations undergo a phase transition from a quasilocalized to a diffusive regime. The corresponding critical exponents are identical for the condensate and the excitations, and compare very well with the value $\nu \approx 1.6$ for non-interacting particles.

Ultracold atoms are clean, controllable, and flexible systems whose dynamics can be modeled from first principles. Interacting ultracold bosons are often well described in the frame of a mean-field approximation leading to the Gross-Pitaevskii equation (GPE) [1], which is useful in a wealth of situations of experimental interest: Superfluidity and vortex formation [2], chaotic behavior [3], soliton propagation [4], etc. Ultracold atom systems are thus increasingly used to realize simple models that are inaccessible experimentally in other areas of physics [5].

Ultracold gases in a disordered optical potential have been used as an emulator for the Anderson model [6], allowing the direct observation of the Anderson localization [7]. The quantum kicked rotor (QKR), obtained by placing cold atoms in a pulsed standing wave, is also a (less obvious) emulator for the Anderson physics [8]: It displays *dynamical localization*, a suppression of chaotic diffusion in the momentum space, recognized to be equivalent to the Anderson localization. Recent studies suggest that interactions (treated in the frame of the GPE) lead to a progressive destruction of the dynamical localization, which is replaced by a subdiffusive regime [9, 10] in analogy with what is numerically observed for the 1D Anderson model itself [11].

Applying standing-wave pulses (kicks) to a Bose-Einstein condensate (BEC) may lead to a dynamical instability which transfers atoms from the condensed to the non-condensed fraction, a phenomenon which is not described by the GPE. The most common "higher order" approximation in this context is the Bogoliubov-de Gennes (BdG) approach [12]. The BdG theory considers "excitations" – described as independent bosonic quasiparticles – of the Bose gas, and thus indicates how (and how much) it differs from a perfectly condensed gas. It has been applied both to describe the dynamical instability of the periodic kicked rotor [13] and to study a one-dimensional weakly interacting BEC [14] in a disordered potential; it was found in the latter case that the quasiparticles may also display Anderson localization. Interestingly, a modified version of the QKR, the *quasiperiodic* kicked rotor (QPKR), in the absence of interactions, emulates the dynamics of a *3D* Anderson-like

model, and displays the Anderson metal-insulator transition [15]. With this system a rather complete theoretical and experimental study of this transition has been performed [16–18]. In the present work we apply the BdG approach to the QPKR, both to study the stability of the condensate and the dynamics of its Bogoliubov excitations. We show that for weak enough interactions, the condensate remains stable for experimentally relevant times, and that the Bogoliubov quasiparticles also display the Anderson phase transition.

A kicked rotor is realized by submitting ultracold atoms to short kicks of a standing wave at times separated by a constant interval T. If such kicks have a constant amplitude, one realizes the standard (periodic) kicked rotor; which exhibits dynamical localization [8, 19]. If the amplitude of the kicks is modulated with a quasiperiodic function $f(t) = 1 + \epsilon \cos(\omega_2 t + \varphi_2) \cos(\omega_3 t + \varphi_3)$, where $\omega_2 T$, $\omega_3 T$ and $\bar{k} \equiv 4\hbar k_L^2 T/M$ (the reduced Planck constant) are incommensurable (k_L is the wave-vector of the standing wave and M is the mass of the atoms), the QPKR is obtained [15]. In the absence of particle-particle interactions, the Hamiltonian of the QPKR, in conventional normalized units [19, 20], is:

$$H = \frac{p^2}{2} + K \cos x \; f(t) \sum_{n \in \mathbb{N}} \delta(t - n). \quad (1)$$

where K is proportional to the average standing wave intensity. In such units the time interval between kicks is $T = 1$, lengths are measured in units of $(2k_L)^{-1}$. Throughout this work we take $\omega_2 = 2\pi\sqrt{5}$, $\omega_3 = 2\pi\sqrt{13}$ and $\bar{k} = 2.89$ corresponding to typical experimental values [16–18]. In the absence of interactions the QPKR displays, for low values of K and ϵ, dynamical localization at long times (i.e. $\langle p^2 \rangle \sim$ constant); for $K \gg 1$, $\epsilon \approx 1$ one observes a diffusive regime $\langle p^2 \rangle \sim t$, and in between there is a critical region which displays a subdiffusive behavior $\langle p^2 \rangle \sim t^{2/3}$ [20].

We use in the present work a model slightly different of the experimentally realized QPKR; we consider a sinusoidal potential "folded" over one spatial period of the standing wave; in such case p becomes a discrete variable $p = \bar{k}l$, with $l \in \mathbb{Z}$. Mathematically, this is equivalent to

use one-period (2π in normalized units) spatial periodic boundary conditions. In the presence of weak interactions that are modeled by a mean-field nonlinear potential, the critical and the diffusive regime are not affected whereas the localized regime is replaced by a subdiffusive one $\langle p^2 \rangle \sim t^\alpha$, with $\alpha \sim 0.4$ [10, 21]. In the following, we consider the weakly interacting regime and short time scales so that this effect is negligible; we shall thus use the term "quasilocalized" to characterize this phase.

We emphasize that in our model there is no spatial dilution of the boson gas, and the average nonlinear potential, which is proportional to the density, does not vary with time. This is not the case in the usual experimental realization of the QPKR, where the atom cloud diffuses with time, causing a significant diminution of the spatial density; once the system is diluted, the nonlinearity does not play anymore an important role. Our model is thus expected to catch more clearly the physics in presence of the nonlinearity. One can use a torus-shaped confining optical potential in order to realize experimentally such a geometry [22].

In this quasi-1D geometry, we take interactions into account via the particle-number-conserving Bogoliubov formalism [23], at zero temperature. The gas of interacting bosons is separated into two parts: (i) The condensed part (or condensate) and (ii) the non-condensed part (or "excitations"). The condensate is governed by the Gross-Pitaevskii equation

$$i\hbar \frac{\partial \phi(x)}{\partial t} = H\phi(x) + g|\phi(x)|^2 \phi(x) \qquad (2)$$

where the condensate wave function ϕ is normalized to unity: $\int_0^L |\phi(x)|^2 dx = 1$ ($L = 2\pi$ is the system length) and the rescaled 1D interaction strength $g = 2\hbar\omega_\perp aN$ is proportional to the S-wave scattering length a, the number of atoms N and the transverse trapping frequency $\omega_\perp$. The non-condensed part is described in the Bogoliubov formalism as a set of independent bosonic quasiparticles, whose two-component state vector (u_k, v_k), satisfying the normalization condition $\int_0^L \left[|u_k|^2(x) - |v_k|^2(x)\right] dx = 1$, evolves according to the equation:

$$i\hbar \partial_t \begin{bmatrix} u_k \\ v_k \end{bmatrix} = \mathcal{L} \begin{bmatrix} u_k \\ v_k \end{bmatrix}. \qquad (3)$$

The operator $\mathcal{L}$ is a 2×2 matrix:

$$\mathcal{L} = \begin{bmatrix} Q \\ & Q^\dagger \end{bmatrix} \mathcal{L}_{\mathrm{GP}} \begin{bmatrix} Q \\ & Q^\dagger \end{bmatrix}$$
$$\mathcal{L}_{\mathrm{GP}} = \begin{bmatrix} H + 2g|\phi|^2 - \mu(t) & g\phi^2 \\ -g\phi^{*2} & -H - 2g|\phi|^2 + \mu(t) \end{bmatrix}, \quad (4)$$

where $\mathcal{L}_{\mathrm{GP}}$ is the usual Bogoliubov operator and $\mu(t) = \int dx \, (\phi^* H\phi + g|\phi|^4)$ is the time-dependent chemical potential. The presence of the projection operator $Q = 1 -$

$|\phi\rangle\langle\phi|$ ensures the total number-conservation of the particles [23]. The condensate wave-function and the Bogoliubov mode amplitudes can be written as a Fourier series: $f(x) = L^{-1} \sum_{l \in \mathbb{Z}} e^{ilx} \tilde{f}(l)$ and $\tilde{f}(l) = \int_0^L e^{-ilx} f(x) dx$, where $f = \phi, u_k, v_k$. For an initial uniform distribution $\phi(x, t = 0) = L^{-1/2}$, the corresponding initial values of (u_k, v_k), obtained from the diagonalization of the operator $\mathcal{L}(t = 0)$ [24], are plane waves of momentum k (in units of $\hbar$)

$$\begin{bmatrix} \tilde{u}_k(l, t = 0) \\ \tilde{v}_k(l, t = 0) \end{bmatrix} = \frac{\sqrt{L}}{2} \begin{bmatrix} \xi + 1/\xi \\ \xi - 1/\xi \end{bmatrix} \delta_{k,l} \qquad (5)$$

with $k \in \mathbb{Z}^*$, and ξ given by:

$$\xi = \left[\frac{k^2}{k^2 + 2g/\pi\hbar^2} \right]^{1/4}. \qquad (6)$$

We emphasize that the Bogoliubov modes (u_k, v_k) are momentum eigenstates only at time $t = 0$; once the kicks are applied, different components of the momentum distribution are mixed.

Let us now consider the stability of the condensate by estimating the number of non-condensed atoms (the quantum depletion) at zero temperature of the gas which is given by $\delta N = \sum_k N_k$, where the number of excitations N_k in the mode k is

$$N_k = \int_0^L |v_k(x)|^2 dx = \frac{1}{L} \sum_l |\tilde{v}_k(l)|^2. \qquad (7)$$

As the total number of particles is fixed, the number of condensed particles is $N - \delta N$ and the non-condensed fraction is simply $\delta N/N$. As long as δN is much smaller that the typical number of atoms $\approx 10^5$ used in a experiment, the kicked condensate is stable. Due to the inversion symmetry of the problem, we can restrict the study to $k > 0$. We will focus on the initially most populated mode $k = 1$.

For the periodic kicked rotor, several studies showed the emergence of an instability at large positive values of g (repulsive interactions) [13], which manifests itself by an exponential increase of the number of excitations. We shall now study this instability in the *quasiperiodic* kicked rotor for $g > 0$. Equations (2) and (3) can be integrated simultaneously by a split-step method. Numerical data are averaged over 500 random realizations of the phases $\varphi_2, \varphi_3 \in [0, 2\pi)$. Figure 1(a) displays the time evolution of N_1 for different values of g in the *quasilocalized* regime. For low interaction strengths $g \leq 0.1$ the system is stable in the considered time range, the average number of excitations $N_1 \leq 0.1$ being very small. For $g = 1$ the interplay between the kicks and interactions leads to a slow exponential increase of the number of quasiparticles: $N_1(t = 1000) \approx 10$, and for $g = 4$, the condensate is clearly unstable, the number of excitations exceeding the typical number of condensed atoms N after a few hundred kicks. Figure 1(b) shows the number
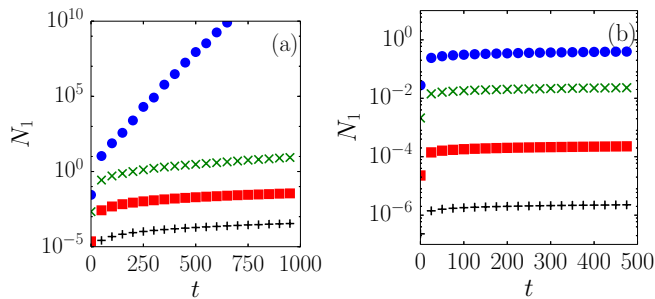
Figure 1. (Color online) Evolution of the population of the mode $k = 1$ for $g = 10^{-2}$ (black +), $g = 10^{-1}$ (red $\square$), $g = 1$ (green $\times$) and $g = 4$ (blue $\circ$) (a) in the quasilocalized regime $K = 4$, $\epsilon = 0.1$ and (b) in the diffusive regime $K = 9$, $\epsilon = 0.8$ (b).

of excitations for a larger kick amplitude $K = 9$ with $\epsilon = 0.8$ corresponding to the diffusive regime. In this case, the condensate is less affected by the presence of interactions, as the kinetic energy grows linearly with time and eventually dominates the constant interaction energy $\simeq g$. In the following, we thus focus on very low interacting strengths $g \leq 0.1$ for which $N_1 \ll N$ meaning that the condensate is stable and the Bogoliubov formalism is valid.

We now consider the (normalized) momentum distribution of the condensate, $n_c(p = \hbar l) = |\tilde{\phi}(l)|^2/(L\hbar)$ and that of the quasiparticles, $n_b(p = \hbar l) = |\tilde{v}_1(l)|^2/(LN_1\hbar)$. For $g \leq 0.1$ and for short enough times, one expects the condensate to display quasilocalization if $K < \tilde{K}$ and diffusion if $K > \tilde{K}$, $\tilde{K}$ being the critical point. Our numerical simulations show that this is also the case for the quasiparticles. We have checked that momentum distributions remain essentially centered around the origin so that their first moments $\langle p \rangle_i = \hbar^2 \sum_l l n_i(p)$ $(i = c, b)$ are small. Figure 2(a) shows the second moment of the distribution variance of the distribution $\sigma_i^2 = \langle p^2 \rangle_i - \langle p \rangle_i^2$, with $\langle p^2 \rangle_i = \hbar^3 \sum l^2 n_i(p)$, for both the condensate ($i = c$, blue empty markers) and of the excitations ($i = b$, red full markers) in the quasilocalized regime. For the two values of the interacting strength $g = 10^{-4}$ (triangles) and $g = 10^{-1}$ (squares), the second moment of the condensate saturates to a constant value $\sigma_c^2 = 50$, showing that the wave-packet is quasilocalized. Assuming an exponential profile $n_c(p) \propto \exp(-|p|/\xi)$, the localization length [25] of the momentum distribution at $t = 10^4$ is given by $\xi = \sigma_c/\sqrt{2} \sim 5$, which evolves very slowly with time up to $t = 10^4$ [cf. Fig. 2(a)]. In other words, for very weak interactions and at the short time scales accessible to experiments, the condensate behaves as a single particle and displays a behavior very close to the Anderson localization. More interestingly, Fig. 2(a) shows that *Bogoliubov quasiparticles also tend to "localize"*, although with a larger value of the second moment $\sigma_b^2 \approx 180$. The

momentum distributions in the quasilocalized regime at $t = 10^4$ are shown in Fig. 2(b) which displays n_c and n_b [same graphical conventions as in Fig. 2(a)]. The momentum distribution of the condensate has a typical exponential profile associated with Anderson localization, whereas that of the quasiparticles presents two peaks with exponential wings. This peculiar shape is probably due to the asymmetric initial condition, the initial momentum distribution of the mode $k = 1$ [Eq. (5)] is centered at $p = \hbar$. The wings of the quasiparticle momentum distribution have approximately the same slope as the condensate distribution, showing that quasiparticles have the same localization length in this example. Because $n_b(p)$ has a much flatter top than $n_c(p)$, the associated second moment σ_b is however significantly larger than σ_c.

In the diffusive regime, the condensate and the quasiparticles have the same dynamical behavior: Figure 2(c), which is the equivalent of Fig. 2(a) for $K = 9$, $\epsilon = 0.8$, actually shows that σ_c^2 and σ_b^2 increase linearly with time and that the evolution is very similar for $g = 10^{-4}$ and $g = 10^{-1}$. The diffusion coefficients $D_c = \sigma_c^2/(2t) \sim 20$ and $D_b = \sigma_b^2/(2t) \sim 25$ are also similar. Figure 2(d) represents the corresponding momentum distributions at $t = 500$. Both have the typical Gaussian shape associated with a diffusion process.

These results show that the nature of the two phases of the Anderson metal-insulator transition in the QPKR are not changed, at least for relatively short times, by weak interactions.

We now study the critical behavior. As Bogoliubov quasiparticles behave like real (non-interacting) particles in the quasilocalized and diffusive regimes, it is reasonable to expect that they also display the Anderson transition. The universality of this second-order phase-transition has been recently demonstrated experimentally in the absence of interactions [18], by showing that the experimental value of the critical exponent $\nu = 1.63 \pm 0.05$ is independent of the microscopic parameters of the system, and consistent with the numerically predicted value 1.58 ± 0.02 [26]. In order to access the critical properties of the system, we use a finite-time scaling method [20, 26] which allows one to characterize the critical regime of the phase transition and to extract its critical exponent ν. The critical regime is shown to correspond to a subdiffusive expansion with exponent $2/3$: $\sigma^2 \propto t^{2/3}$ [20]. We cross the transition along the path $\epsilon(K) = 0.1 + 0.14(K - 4)$ used in [16]. Figure 3 shows that, for small nonlinearities, the critical exponent is the same for both components and compares very well with the (non-interacting) experimental measurement $\nu = 1.63 \pm 0.05$ [18], but their values tend to become different for higher values of g. The critical point is also found to be the same for both the condensate and the excitations. Its value, $\tilde{K} \approx 6.38 \pm 0.05$ at $g = 0$, practically does not change (inside the sta-
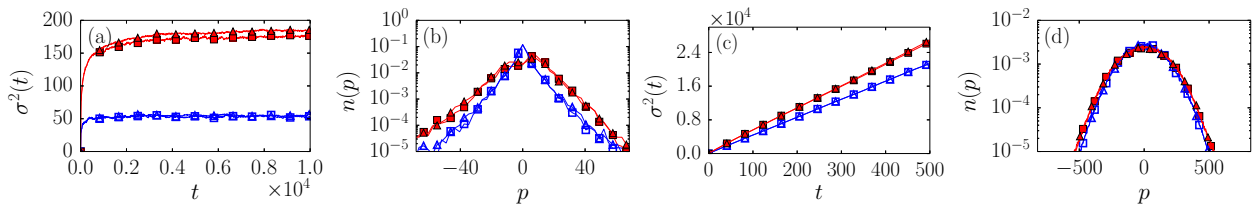
Figure 2. (Color online) Dynamics of the condensate and of the quasiparticles. Variance of the momentum of the condensate $\sigma_c^2(t)$ (empty blue markers) and of the quasiparticles $\sigma_b^2(t)$ (full red markers) for $g = 10^{-4}$ (triangles) and $g = 10^{-1}$ (squares) in the (a) quasilocalized regime ($K = 4, \epsilon = 0.1$) and (c) in the diffusive regime ($K = 9, \epsilon = 0.8$). Momentum distributions of the condensate $n_c(p)$ (empty markers) and of the quasiparticles $n_b(p)$ (full markers) at $t = 10^4$ (logarithmic scale) in the (b) quasilocalized regime and (d) diffusive regime.

tistical uncertainty) up to $g = 0.1$, in accordance with the self-consistent theory prediction of [10]. This is the main result of the present work: Bogoliubov quasiparticles undergo a second-order phase transition and the corresponding critical exponent has the same value as the one observed for a non-interacting system of independent particles. This suggests that the concept of universality is valid for independent particles, for interacting condensates and for Bogoliubov quasiparticles: All these conceptually different objects undergo a phase transition with the *same* critical exponent. For $g \geq 0.1$, the value of the critical exponents starts to deviate from the universal value illustrating that the system enters a new regime where the kicked condensate is changed by the presence of interactions, that is, the subdiffusive character of the quasilocalized regime becomes important even for short times [10].

The above study is restricted to the Bogoliubov mode $k = 1$. Experimentally, Bogoliubov modes can be selectively excited using two laser waves whose directions are chosen so that their wave-vector difference Δk_L corresponds to the wave-vector k of the desired mode [27]. Considering another mode $k \neq 1$ is equivalent to a change of the initial condition in the Bogoliubov equations and should not change our results. We checked numerically that theses modes, which are initially less populated than the $k = 1$ mode, display the same behavior, but are much more affected by finite-time effects, as their initial momentum distribution is more asymmetric [see Eq. (5)]. We also mention that it would be interesting to extend the present study to higher values of g and/or longer time scales where dynamical localization is supposed to be effectively destroyed, and see if quasiparticles display the same subdiffusive dynamics *below* the critical point, as predicted for the condensate by the self-consistent theory [10], but such a study is outside the scope of the present work.

In conclusion, in a quasiperiodic kicked rotor, in the very weakly interacting regime, the condensate is stable for times that are larger than the experimental time-scale (presently up to 1000 kicks). In this regime, both quasiparticles and the condensate behave like single parti-
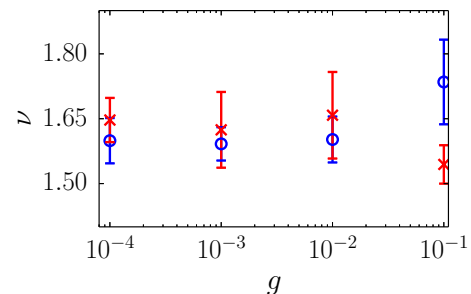

Figure 3. (Color online) Critical exponent ν *vs* interacting strength g for both the condensed fraction (blue circles) and the Bogoliubov quasiparticles (red crosses). Error bars are calculated via a standard bootstrap method [28].

cles undergoing the Anderson transition from a localized regime to a diffusive regime, and display the same critical exponent, which is compatible with the one observed for non-interacting particles; the universality of the phase transition is thus valid irrespectively of the type of particle. The above findings confirm the potentialities of the quasiperiodic kicked rotor for the experimental study of the effect of interactions on the Anderson transition. For low positive values of g the noninteracting regime can be experimentally observed, and, by increasing interactions via a Feshbach resonance one can observe the onset of nonlinear effects. The present work opens the way for such an experiment, which would represent an important advance in the physics of interacting disordered systems.

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