# Stability in shape optimization with second variation

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**Abstract**

We are interested in the question of stability in the field of shape optimization. We focus on the strategy using second order shape derivative: more precisely, we identify structural hypotheses on the hessian of the considered shape functions, so that critical stable domains (i.e. such that the first order derivative vanishes and the second order one is positive) are local minima for smooth perturbations. These conditions are quite general and are satisfied by a lot of classical functionals, involving the perimeter, the Dirichlet energy or the first Laplace-Dirichlet eigenvalue. We also explain how we can easily deal with a volume constraint and/or translation invariance of the functionals. As an application, we retrieve or improve previous results from the existing literature, and provide new local stability results. We finally test the sharpness of our hypotheses by giving counterexamples of critical stable domains that are not local minima.

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## Contents



## 1 Introduction

In this paper, we are interested in the question of stability in the field of shape optimization. More precisely, given $J : \mathcal{A} \to \mathbb{R}$ defined on $\mathcal{A} \subset \{\Omega \text{ smooth enough open sets in } \mathbb{R}^d\}$, we consider the optimization problem

$$\min \{J(\Omega), \ \ \Omega \in \mathcal{A}\}, \tag{1}$$

and we ask the following question:

*if $\Omega^* \in \mathcal{A}$ is a critical domain satisfying a stability condition (that is to say a strict second order optimality condition), can we conclude that $\Omega^*$ is a strict local minimum for* (1) *in the sense that*

$$J(\Omega) - J(\Omega^*) \geq c d_1(\Omega, \Omega^*)^2, \qquad \text{for every } \Omega \in \mathcal{V}(\Omega^*) \tag{2}$$

*where $c \in (0, \infty)$, $d_1$ is a distance among sets, and $\mathcal{V}(\Omega^*) = \{\Omega \in \mathcal{A}, d_2(\Omega, \Omega^*) < \eta\}$ is a neighborhood of $\Omega^*$, relying on a (possibly different) distance $d_2$?*
(Note that the word distance is used here and in the rest of the paper as an intuitive notion here, asserting that $\Omega$ is far or close from the fixed shape $\Omega^*$, and do not refer in general to the formal mathematical notion of distance).

**Origin of the question:**

For example in [42], the following terminology is used: given a function $f : X \to \mathbb{R}$ where $X$ is a Banach space, the property that a critical point $x \in X$ has a positive second order derivative is called *linear*-stability, and implies that $t \mapsto f(x+ty)$ has a local minimum at $t = 0$ for every $y \in X$, while *nonlinear*-stability requires that $f(x)$ is less that $f(z)$ for any $z$ close to $x$. It is classical that, when dealing with infinitely dimensional parameters, these two notions do not coincide in general.

In the framework of shapes, this question has been raised in different settings, and its answer has sometimes been mistakenly considered as easily valid: for example, in the context of stable constant mean curvature surfaces, literature has focused for a while on giving sufficient conditions so that *linear*-stability would occur, without proving that it actually implied local minimality. This point was raised by Finn in [20], and some answers followed quickly, see [28, 42, 43], though in the particular case of the ball and the isoperimetric problem, the difficulty was already handled by Fuglede in [21]. In the context of shape functionals involving PDE, the issue was raised by Descloux in [17] and a first solution was given in [14, 12].

**Quantitative isoperimetric inequalities: different strategies**

During the last decade, starting with [23], this type of question gained interest in the community of isoperimetric inequalities and shape optimization, in particular three main methods were developed in a quite extensive literature, in order to get a stability result of the form (2) for the most classical problems (1):

- Symmetrization technique,
- Mass transportation approach,
- Second order shape derivative approach.

As an example, we quote the $\mathrm{L}^1$-stability result for the perimeter: for every $V_0 \in (0, \infty)$, there exists $c \in (0, \infty)$ such that

$$P(\Omega) - P(B) \geq c d_F(\Omega, B)^2, \qquad \text{for every (measurable set) } \Omega \text{ such that } |\Omega| = V_0, \tag{3}$$

where $P$ denotes the perimeter (in the sense of geometric measure theory), $|\cdot|$ is the volume, $B$ is any ball of volume $V_0$, and

$$d_F(\Omega, B) = \inf_{\tau \in \mathbb{R}^d} \frac{|(\Omega - \tau) \Delta B|}{|B|}$$

is known as the Fraenkel asymmetry (which can be seen as the $\mathrm{L}^1$-distance to the ball, up to translations). For this specific example, all of these three strategies have been successfully applied, see [23, 19, 11].

Note in particular that the result is global: in that case a local result (in an $\mathrm{L}^1$-neighborhood) implies a non-local one as it is shown in [23, Lemma 5.1 and Lemma 2.3].

In this paper, **we focus on the third strategy**, which recently receveid even more attention as in some examples, the other techniques could not be applied, or provided non-optimal results: as an example we quote the $\mathrm{L}^1$-stability for the Faber-Krahn inequality, which was solved with symmetrization technique in [24], but provided a higher (and less strong) exponent in (2), and has been improved to an optimal exponent recently in [8] using the third strategy (see also [25]).

One specific difficulty for this strategy is to define a framework of differential calculus within shapes. This can be done for example with the notion of shape derivatives, but one main drawback is that this is available only for reasonably smooth deformations of the initial shape, or in other words, for a rather strong distance $d_2$ (otherwise it is clear that classical functionals are not differentiable for non-smooth perturbations). However, as it is shown for example in [2], the strategy can also provide results for very weak distances (as the Fraenkel asymmetry), and can be decomposed in two main steps:

- first, with the help of the differential setting and the fact that $\Omega^*$ satisfies a strict second order optimality condition, prove a stability result for small and smooth perturbations of $\Omega^*$; in other words, prove that (2) is valid where $d_2$ is a strong distance (and $d_1$ is limited by the properties of $J$, and is in general different from $d_2$, see below),
- second, deduce from this first step that (2) is valid where $d_1 = d_2$ is a weak distance (for example the Fraenkel asymmetry).

For the perimeter functional, the first step goes back to [21], and the second step is inspired by results in [44, 33], though the complete result was achieved in [11]. These two steps rely on very different arguments: in particular, the second step usually requires to adapt the regularity theory related to the optimization problem (1), namely the notion of quasi-minimizer of the perimeter when the functional $J$ contains a perimeter term, or the regularity of free boundaries when $J$ involves an energy related to a PDE functional (see [2, 8] respectively), so it strongly relies on specific properties of the functional $J$ under study. However, as we aim to show in this paper, the first step has a very large range of applications, and is valid under rather weak assumptions on the functionals.

*The aim of this paper is to describe a general framework so that the first step of the above strategy applies: while this has been done in a few places in the literature, every time specifically for the functional that was under study, we aim at giving some general statements, and then show that these statements both apply to the examples already handled in the literature, and also to new examples. Despite getting a wider degree of generality, we also simplify many proofs and strategies found in the previous literature, as we describe below.*

**Neighborhood of shapes**

In order to describe the details of the strategy, we briefly introduce two classical ways to parametrize shapes in a neighborhood of a fixed one:

- **Diffeomorphism and shape derivatives:** we consider a shape to be a neighbor of $\Omega$ if it is a deformation of $\Omega$ by a diffeomorphism which is close to the identity. More precisely, $\Theta$ being a Banach space such that $\mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d) \subset \Theta \subset W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, we consider shapes of the form $\Omega_\theta := (Id + \theta)(\Omega)$ where $\|\theta\|_\Theta$ is small.

  In this framework, we can consider the distance introduced by Micheletti :

  $$d_\Theta(\Omega_1, \Omega_2) := \inf \left\{ \|\theta\|_\Theta + \|(Id+\theta)^{-1} - I\|_\Theta, \ \ \theta \in \Theta \text{ diffeomorphism such that } (Id+\theta)(\Omega_1) = \Omega_2 \right\}.$$

  Morover, this leads to the notion of shape derivatives, first introduced by Hadamard, then developed by Murat-Simon and Delfour-Zolesio. One defines the function $\mathcal{J}_\Omega$ on a neighborhood of 0 in $\Theta$ by

  $$\forall \theta \in \Theta, \quad \mathcal{J}_\Omega(\theta) = J[(Id+\theta)(\Omega)].$$

  One then uses (in the whole paper) the usual notion of Fréchet-differentiability: shape derivatives of $J$ at $\Omega$ are the successive derivatives of $\mathcal{J}_\Omega$ at 0, when they exist. In particular, the first shape derivative is $J'(\Omega) := \mathcal{J}'_\Omega(0)$, a continuous linear form on $\Theta$ (the shape gradient), and the second order shape derivative is $J''(\Omega) := \mathcal{J}''_\Omega(0)$, a continuous symmetric bilinear form on $\Theta$ (the shape hessian).

- **Normal graphs:**
  On the other hand, assuming that $\Omega$ is $\mathrm{C}^1$ (and $\boldsymbol{n} = \boldsymbol{n}_{\partial\Omega}$ is its outer unit normal vector) we can consider

"normal graph" on $\partial\Omega$, that is $\Omega_h$ (there should be no confusion between $\Omega_\theta$, $\Omega_h$ (and later $\Omega_t$ with $t \in \mathbb{R}$) as $\theta$ and $h$ live in different spaces) such that

$$\partial\Omega_h = \{x + h(x)\boldsymbol{n}(x), x \in \partial\Omega\}, \tag{4}$$

where $h \in X$, and $X$ is a Banach space of scalar functions such that $\mathrm{C}^\infty(\partial\Omega) \subset X \subset W^{1,\infty}(\partial\Omega)$.

Then one can see

$$\widetilde{d}_X(\Omega_h, \Omega) := \|h\|_X \tag{5}$$

as a measure of the distance between $\Omega_h$ and $\Omega$ (though it is not formally a distance), and one can also define derivatives in this framework, as $\mathrm{j}_\Omega(h) := J(\Omega_h)$ is defined on a Banach space. Notice that given $\Omega_1$ and $\Omega_2$, there is at most one function $h$ defined on $\partial\Omega_1$ such that $\partial\Omega_2 = (\partial\Omega_1)_h$ as defined in (4), so we can denote this function $h = h_{\Omega_1,\Omega_2}$ if it exists, and therefore definition (5) is valid.

Let us emphasize that even if the second method seems more restrictive, the two methods are equivalent in a neighborhood of $\Omega$ (if $\Omega$ is smooth enough) in the sense that one describes as many shapes with each methods (for suitable $\Theta$ and $X$): first, a normal graph $\Omega_h$ is a deformation of $\Omega$ for any $\boldsymbol{\xi}_h$ which is an extension to $\mathbb{R}^d$ of $h\boldsymbol{n}$ (and then $\mathrm{j}_\Omega(h) = \mathcal{J}_\Omega(\xi_h)$), see the introduction of Section 4. Second, if we consider diffeomorphims that are close to the identity, the boundaries of the perturbed domains are graphs over the boundary of the initial domain: in other words, for any domain $\Omega_\theta = (Id + \theta)(\Omega)$ with $\theta \in \Theta$ close to 0, there is a unique real-valued function $h = h_{\Omega,\Omega_\theta} \in X$ defined on $\partial\Omega$ such that (4), see for example Lemma 3.1 in [36].

However, it is not clear a priori that computing derivatives for normal graphs (derivatives of $\mathrm{j}_\Omega : h \mapsto J(\Omega_h)$) is enough to describe shape derivatives (derivatives of $\mathcal{J}_\Omega : \theta \mapsto J(\Omega_\theta)$): this issue is handled in the first point below. Also, it may be of interest to focus on paths $t \in \mathbb{R} \mapsto \Omega_t$ (as we will need below), and in that case $\Omega_{t+s}$ is not in general a normal graph over $\Omega_t$; it is therefore important to understand the framework of diffeomorphisms.

We may also be interested in having distances taking into account some invariance with translation. For example, we define

$$\bar{d}_X(\Omega_2, \Omega_1) = \inf_{\tau \in \mathbb{R}^d} \|h_{\Omega_1,\Omega_2+\tau}\|_X \tag{6}$$

where the infimum is taken over $\tau$ such that $\partial\Omega_2 + \tau$ is a normal graph on $\partial\Omega_1$.

## Main contributions of the paper:

*We are now in position to describe the main steps of the strategy to obtain a stability inequality of the form* (2) *for a strong distance $d_2$: we describe here these steps, and insist on the contributions of the present paper in each step.*

- **Structure of derivatives:** The way of differentiating shape functionals with diffeomorphism described just above is very convenient as most classical shape functionals are easily proven to be smooth in this setting (usually not using any regularity on the initial shape $\Omega^*$, see more details in [29] for example), and as noticed before, it is clear that computing derivatives in the sense of normal graphs is just a particular case (while the opposite seems not clear). Nevertheless, one main drawback for our purpose is that as we are dealing with shapes, there is a lot of invariance for $\mathcal{J}_\Omega$: any non-trivial diffeomorphisms that leaves $\Omega$ invariant (at first, second order) must lead to vanishing derivatives. It is therefore unreasonable to expect that the stability condition for optimal shapes writes $J''(\Omega^*).(\xi,\xi) > 0$ for $\xi \in \Theta \setminus \{0\}$. For first order shape derivatives, this difficulty is well-known since Hadamard, who observed (in particular examples) that the shape gradient is a distribution supported on the boundary of the domain, acting only on the normal component of the deformation. In other words, for any smooth domain $\Omega$, there is a linear form $\ell_1 = \ell_1[J](\Omega)$ acting on scalar functions defined on $\partial\Omega$, such that

$$\forall \xi \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d), \quad J'(\Omega) \cdot \xi = \ell_1(\xi_{|\partial\Omega} \cdot \boldsymbol{n}), \tag{7}$$

see classical monographs, or [32] in a non-smooth setting. A similar observation can be made about second order shape derivatives, though the situation is more involved. Let us first describe a simple case,

which happens to be particularly relevant for our purpose here: if $\Omega^*$ is a critical domain for $J$, that is a domain such that the shape gradient of $J$ vanishes, then the shape hessian reduces to a symmetric bilinear form $\ell_2 = \ell_2[J](\Omega^*)$, also acting only on normal components of diffeomorphisms. When we do not assume that $\Omega^*$ is a critical domain, contrary to the first order one, the second order shape derivative may involve the tangential component of the deformation. However, as it has been proven in [36], there is a general structure, involving a quadratic form acting only on normal components (as in the critical case) and another term involving the first order derivative: more precisely, for any smooth domain $\Omega$, there exists $\ell_2 = \ell_2[J](\Omega)$ acting on normal components such that

$$\forall \xi \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d), \quad J''(\Omega).(\xi, \xi) = \ell_2(\xi \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) + \ell_1(Z_\xi) \tag{8}$$

where

$$Z_\xi = \boldsymbol{B}(\xi_\tau, \xi_\tau) - 2\nabla_\tau(\xi \cdot \boldsymbol{n}) \cdot \xi_\tau,$$

$\xi_\tau$ is the tangential component of $\xi$, and $\boldsymbol{B} = D_\tau \boldsymbol{n}$ is the tangential differential of $\boldsymbol{n}$, also known as the second fondamental form of $\partial\Omega$. This structure is often observed in the literature on specific examples and after lengthy computations, while it can be used a priori to simplify the computations: indeed, once we know the shape functional is smooth (which can be shown without computations), this result implies that the computation of shape derivatives for purely normal deformations is sufficient to describe the second order derivative for any deformation (see also Remark 2.6). In particular, using the framework of normal graphs, we get $\mathrm{j}''_\Omega(0)(h,h) = \ell_2[J](\Omega)(h,h)$.

*The first contribution of this paper is to give a new proof of the structure* (8)*, see* **Theorem 2.1**. *Though the strategy in [36] is quite natural as it shows that any small deformation of a shape can be seen (in a smooth way) as a normal deformation defined on the boundary, up to a change of parametrization of the boundary, we believe this new proof is less technical, and also quite natural as it only relies on the invariance properties mentioned before, and is therefore closer to the usual proof for the structure of first order shape derivatives.*

Even if it may seem that for our purpose this result in only helpful in the particular case where $\Omega^*$ is a critical shape, as we will notice in the following items, we actually need to deal with second order shape derivatives at non-critical shapes as well. See also [4] for recent use of these structure results and application to numerical methods.

- **Coercivity assumption:** As noticed earlier, we are dealing with infinite dimensional differential calculus, and it is well-known that usual sufficient conditions for getting optimality is that the second order derivative is coercive rather than just assuming that it is positive. In particular, we need to wonder for which norm the coercivity might be valid, with view of applications. We will see that in most examples we are dealing with, the quadratic form $\ell_2[J](\Omega)$ (normal part of the hessian of $J$ at $\Omega$) satisfies a structural property emphasizing a particular norm, that will restrict the choice of $d_1$ for applications: for $s_2 \in (0,1]$, we say that the bilinear form $\ell$ acting on $\mathrm{C}^\infty(\partial\Omega)$ satisfies condition $(\mathbf{C}_{\mathrm{H}^{s_2}})$ if (and by extension we say that $J$ satisfies the condition at $\Omega^*$ if $\ell_2[J](\Omega^*)$ does):

  $(\mathbf{C}_{\mathrm{H}^{s_2}})$ there exists $s_1 \in [0, s_2)$ and $c_1 > 0$ such that $\ell = \ell_m + \ell_r$ with

  $$\begin{cases} \ell_m \text{ is lower semi-continuous in } \mathrm{H}^{s_2}(\partial\Omega) \\ \ell_m(\varphi,\varphi) \geq c_1 |\varphi|^2_{\mathrm{H}^{s_2}(\partial\Omega)}, \ \forall \varphi \in \mathrm{C}^\infty(\partial\Omega), \\ \ell_r \text{ continuous in } \mathrm{H}^{s_1}(\partial\Omega). \end{cases}$$

  where $|\cdot|_{\mathrm{H}^{s_2}(\partial\Omega)}$ denote the $\mathrm{H}^{s_2}(\partial\Omega)$ semi-norm. In that case, $\ell$ is naturally extended (by a density argument) to the space $\mathrm{H}^{s_2}(\partial\Omega)$, and we prove that under this assumption

  $$\ell > 0 \text{ on } \mathrm{H}^{s_2}(\partial\Omega) \setminus \{0\} \Leftrightarrow \exists \lambda > 0, \quad \forall \varphi \in \mathrm{H}^{s_2}(\partial\Omega), \quad \ell(\varphi,\varphi) \geq \lambda \|\varphi\|^2_{\mathrm{H}^{s_2}(\partial\Omega)}, \tag{9}$$

  (here $\ell$ is a quadratic form, so $\ell > 0$ on $X \setminus \{0\}$ means $\ell(\varphi,\varphi) > 0$ for any $\varphi \in X \setminus \{0\}$). Notice also that this statement holds for general bilinear forms on Sobolev spaces and is not restricted to shape hessians.

*The proof of this fact is rather simple, and similar arguments can be found in [28, 2]. Our contribution principally lies in the fact that we explicitly formulate the underlying assumption so that positivity implies coercivity, see* **Lemma 3.3** *in Section 3.1.*

Note in particular that the value of $s_2$ is determined by the shape functional $J$ (in practice $s_2$ usually does not depend on $\Omega$): the choice of the distance $d_1$ in (2) is therefore limited by this coercivity property. In other words, we can not expect (2) to be valid for any distance $d_1$ stronger than $\widetilde{d}_{\mathrm{H}^{s_2}}$ (see also [21] where an upper bound of the isoperimetric deficit is given, in a smooth neighborhood). As we will notice through computations, when $J$ contains a perimeter term, $s_2 = 1$, while for PDE functionals we are dealing with here (see below when we describe examples), $s_2 = 1/2$. For an interesting result about the choice of $d_1$ in a non-smooth setting, see [22] where the authors obtain an improved version of (3) with $d_1$ being a stronger distance than the Fraenkel asymmetry (see also [34] for the anisotropic case).

- **Norm discrepancy:** With the help of the previous items, we are able to properly state the stability question: if $\Omega^*$ is such that

$$\ell_1 = 0, \quad \text{and } \ell_2 > 0 \text{ on } \mathrm{H}^{s_2}(\partial\Omega^*) \setminus \{0\}$$

where $\ell_1, \ell_2$ are associated to $J'(\Omega^*)$ and $J''(\Omega^*)$ respectively, through the structure of shape derivatives (7), (8), can we conclude to a *nonlinear*-stability inequality of the form (2), for every small and smooth deformation $\Omega = (Id + \theta)(\Omega^*)$ of the set $\Omega^*$?

From the Taylor formula, we can write:

$$J((Id + \theta)(\Omega^*)) - J(\Omega^*) = \frac{1}{2}\, \ell_2(\theta \cdot \boldsymbol{n}, \theta \cdot \boldsymbol{n}) + o(\|\theta\|^2_\Theta).$$

which leads to two issues:

  - first, the remainder depends a priori on the full norm of $\theta$, while the second order term is only controlled with the norm of $\theta \cdot \boldsymbol{n}$,
  - the norm of differentiability $\Theta$ is in most cases stronger than the norm of coercivity given in the previous item, namely $\mathrm{H}^{s_2}$,

so it is a priori not possible to control the sign of the term $\frac{1}{2}\, \ell_2(\theta \cdot \boldsymbol{n}, \theta \cdot \boldsymbol{n}) + o(\|\theta\|^2_\Theta)$. To solve the first issue, one could only work with normal deformation (which is enough to describe every small and smooth deformation of $\Omega^*$), but this is in some cases a restriction (for example when trying to make volume preserving deformations, see [12, 2], though we solve this issue in a different (and easier) way, see below). The second issue is more serious. In the literature, this phenomenon was first observed when minimizing the perimeter as it is naturally differentiable in $\mathrm{W}^{1,\infty}$ while the coercivity may only happen for the $\mathrm{H}^1$-norm. This issue has been observed and solved in two places in the literature:

  - Fuglede in [21] proved a local stability result for the classical isoperimetric problem whose solution is the ball (see the next step to explain how we handle the translation invariance of the functional and the volume constraint). To that end, he considers *nearly spherical domains* $B_u = \{(r, \omega) \in [0, \infty) \times \mathbb{S}^{d-1},\ r < 1 + u(\omega)\}$ (which is equivalent to a normal graph in this case) with $u$ preserving the barycenter and the volume (at the second order), and proved that

$$P(B_u) - P(B) \geq \alpha \|u\|^2_{\mathrm{H}^1(\mathbb{S}^{d-1})} \quad \text{if } \|u\|_{\mathrm{W}^{1,\infty}} \leq \eta.$$

    where $\alpha, \eta > 0$ are explicit;
  - for stable constant mean curvature surfaces, a specific differentiability statement for the surface area is quoted in [28, Proof of Theorem 6], [7, Equation 3.23], [44, Equation (1)], and this result is not restricted to the ball. See also [42, 43] for similar observations with a different parametrization.

This difficulty is well-known in the literature on second order optimality conditions in infinite dimension. We may think that we could change the space of differentiability for the functional (choosing for example the space for which there is coercivity), unfortunately in most examples, the shape functional is not

differentiable in the space for which there is coercivity, which is weaker than $\mathrm{W}^{1,\infty}$. Various geometric examples have been handled in the literature since these first examples, see [15, 18, 6, 34].

With the previous remarks, it is natural to define the following property: given $\Omega^*$, $s \in [0,1]$ and $X \subset W^{1,\infty}(\partial\Omega^*)$ a Banach space, and assuming that $\mathrm{j}_{\Omega^*}$ is $\mathrm{C}^2$ in a neighborhood of 0 in $X$, we say that $J$ satisfies condition $\mathbf{(IT}_{\mathrm{H}^s,X}\mathbf{)}$ (for "improved Taylor" expansion) at $\Omega^*$ if:

$\mathbf{(IT}_{\mathrm{H}^s,X}\mathbf{)}$ there exist $\eta > 0$ and a modulus of continuity $\omega$ such that for every domain $\Omega_h^*$ with $\|h\|_X \leq \eta$,

$$\left| J(\Omega_h^*) - J(\Omega^*) - \ell_1(h) - \frac{1}{2}\ell_2(h,h) \right| \leq \omega(\|h\|_X)\|h\|_{\mathrm{H}^s}^2,$$

where $\ell_1 = \ell_1[J](\Omega^*) = \mathrm{j}'_{\Omega^*}(0),\ \ell_2 = \ell_2[J](\Omega^*) = \mathrm{j}''_{\Omega^*}(0)$.

*Again, the main contribution in this step is to identify explicitely the proper assumption needed to obtain non-linear stability. We will provide many examples.*

- **PDE functionals:** In the specific context of shape optimization involving PDE, the stability question was raised in the work of Descloux [17], and was overcome in [14, 12]. More recently a very similar approach can be found in [2], see also Section 5.1. The situation is much more involved than for geometric functionals, as it is much harder to write the remainder term in order to show condition $\mathbf{(IT}_{\mathrm{H}^s,X}\mathbf{)}$ for suitable spaces.

  In this case, we define a slightly different condition: given $\Omega^*$, $s \in [0,1]$ and $X \subset W^{1,\infty}(\partial\Omega^*)$ a Banach space, assuming that $\mathrm{j}_{\Omega^*}$ is $\mathrm{C}^2$ in a neighborhood of 0 in $X$, we say that $J$ satisfies condition $\mathbf{(IC}_{\mathrm{H}^s,X}\mathbf{)}$ (for "improved continuity" in $h$) at $\Omega^*$ if:

  $\mathbf{(IC}_{\mathrm{H}^s,X}\mathbf{)}$ there exist $\eta > 0$ and a modulus of continuity $\omega$ such that for every domain $\Omega_h^*$ with $\|h\|_X \leq \eta$, and all $t \in [0,1]$:

  $$\left| j''(t) - j''(0) \right| \leq \omega(\|h\|_X)\|h\|_{\mathrm{H}^s}^2,$$

  where $j : t \in [0,1] \mapsto J(\Omega_t)$ for the path $(\Omega_t)_{t\in[0,1]}$ connecting $\Omega^*$ to $\Omega_h^*$, and defined through its boundary

  $$\partial\Omega_t = \{x + \ th(x)\boldsymbol{n}(x),\ x \in \partial\Omega^*\}. \tag{10}$$

  Using the Taylor formula with integral remainder:

  $$J(\Omega_h^*) - J(\Omega^*) = \ell_1(h) + \frac{1}{2}\ell_2(h,h) + \int_0^1 [j''(t) - j''(0)](1-t)dt,$$

  it is easy to see that condition $\mathbf{(IC}_{\mathrm{H}^s,X}\mathbf{)}$ implies $\mathbf{(IT}_{\mathrm{H}^s,X}\mathbf{)}$.

  *Our main contribution about this step is to insist on the list of examples of functionals satisfying this condition, and also on possible choices of spaces $X$, depending on the functionals.*

  In [14, 12] the authors chose Hölder-spaces of the form $\mathrm{C}^{2,\alpha}$, while in [2] the authors prefer Sobolev spaces, namely $\mathrm{W}^{2,p}$ for $p$ large enough, which is a better result as it leads to a larger neighborhood in the stability result (2).

  *We prove that for the PDE functionals from [14, 12], which seems less smooth than the functional from [2] (see also Section 5.1), condition* $\mathbf{(IC}_{\mathrm{H}^s,X}\mathbf{)}$ *is also valid for $X = \mathrm{W}^{2,p}$, for $p$ large enough, therefore improving the stability properties proven in [14, 12]. Finally we also provide new examples, as the first eigenvalue for the Dirichlet Laplacian,* **see Proposition 4.7**. *Note that thanks to the next item, it is enough to focus on the path* (10) *for which computations are simplier.*

  Note that this improvement about spaces is not just a technical issue, as in [2] the choice of $\mathrm{W}^{2,p}$ rather than $\mathrm{C}^{2,\alpha}$ is relevant for the second step of the strategy (described page 3) when proving stability in an $\mathrm{L}^1$-neighborhood ([2, Section 4]): indeed their regularization procedure needs to allow discontinuities of the mean curvature, see equation (4.9) in the proof of [2, Theorem 4.3]. This difficulty is handled in another way in [8] for the quantitative Faber-Krahn inequality as a stability in $\mathrm{C}^{2,\alpha}$ is enough to make the regularization procedure work.

- **Stability result:** We are now in position to state the main stability result in the framework of shape optimization. For a similar statement for problems with volume constraint and translation invariant functionals, which is very useful in practice, see the next item and **Theorem 3.2**:

  **Theorem 1.1** *Let $\Omega^*$ be a domain of class $\mathrm{C}^3$, and $J$ a shape functional, twice Fréchet differentiable on a neighborhood of $\Omega^*$ for $d_{\mathrm{W}^{1,\infty}}$. We denote $\ell_1 = \ell_1[J](\Omega^*)$ and $\ell_2 = \ell_2[J](\Omega^*)$ (given by Theorem 2.1), and assume that $J$ satisfies $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ at $\Omega^*$ for some $s_2 \in (0,1]$ and $X$ a Banach space such that $\mathrm{C}^\infty(\mathbb{R}^d) \subset X \subset \mathrm{W}^{1,\infty}(\mathbb{R}^d)$.*
  *Then if $\Omega^*$ is a critical and strictly stable shape for $J$, that is to say*

$$\ell_1 = 0, \qquad \text{and } \ell_2 > 0 \text{ on } \mathrm{H}^{s_2}(\partial\Omega, \mathbb{R}) \setminus \{0\}, \tag{11}$$

  *then $\Omega^*$ is an $\mathrm{H}^{s_2}$-**stable local minimum of $J$ in a $X$-neighborhood**, that is to say there exists $\eta > 0$ and $c = c(\eta) > 0$ such that*

$$\forall\, \Omega \text{ such that } \widetilde{d}_X(\Omega, \Omega^*) \leq \eta, \quad J(\Omega) \geq J(\Omega^*) + c\widetilde{d}_{\mathrm{H}^{s_2}}(\Omega, \Omega^*)^2$$

  *where $\widetilde{d}_{\mathrm{H}^{s_2}}$ is defined in* (5).

  *With the help of the previous steps, the proof of this result is rather easy, and its main interest lies in the fact that its hypotheses are valid in practice for many examples, as we show in Section 5, see also the end of this introduction.*

- **Constraints and invariance:** as in the isoperimetric problem, whose quantitative version is recalled in (3), we have to handle two extra difficulties: first, the functional is translation invariant, and second, there is a volume constraint in the optimization problem. Therefore one cannot expect (11) to be satisfied, and it should be replaced by

$$\ell_1[J](\Omega^*) = \mu\ell_1[\mathrm{Vol}](\Omega^*), \qquad \text{and } \ell_2[J](\Omega^*) - \mu\ell_2[\mathrm{Vol}](\Omega^*) > 0 \text{ on } T(\partial\Omega^*) \setminus \{0\}$$
$$\text{where } T(\partial\Omega^*) := \left\{ \varphi \in \mathrm{H}^{s_2}(\partial\Omega^*), \ \int_{\partial\Omega^*} \varphi = 0 \text{ and } \int_{\partial\Omega^*} \varphi \overrightarrow{x} = \overrightarrow{0} \right\}. \tag{12}$$

  Here $\mu \in \mathbb{R}$ is the Lagrange multiplier, handling the notion of criticality when there is a volume constraint, and $T(\partial\Omega^*)$ can be seen as the tangent space to the constraint ($\int_{\partial\Omega^*} \varphi$ being the first order derivative of the volume) and to the invariance ($\int_{\partial\Omega^*} \varphi \overrightarrow{x}$ being the first order derivative of the barycenter functional). In [14, 12], the authors carefully handle the volume constraint by building a path preserving the volume and being almost normal, and prove that an estimate like $(\mathbf{IC}_{\mathrm{H}^{s_2},X})$ is valid for this more involved path. In [2], a very similar approach (adapted to $\mathrm{W}^{2,p}$-spaces) is given, and they also handle the translation-invariance (which is not there in the example of [14, 12]) which implies a lot a technicalities.

  *Inspired by the strategy of [28] who deals with the volume constraint for area minimizing surfaces, we drastically simplify the presentation of [14, 12, 2] by using an exact penalization method. More precisely (see* **Theorem 3.2***), we prove that under the assumptions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ for $\ell_2[J](\Omega^*)$, the constrained optimality conditions* (12) *implies the unconstrained conditions* (11) *when $J$ is replaced by*

$$J_{\mu,C} = J - \mu\mathrm{Vol} + C\left(\mathrm{Vol} - V_0\right)^2 + C\left\|\mathrm{Bar} - \mathrm{Bar}(\Omega^*)\right\|^2,$$

  *where $\mu$ is the Lagrange multiplier, $C \in (0,\infty)$ is large enough, and* Bar *is the barycenter functional. We can therefore apply Theorem 1.1 to $J_{\mu,C}$, see the proof of Theorem 3.2, and this clearly implies the constrained local minimality. It is clear, looking at the proofs of our results, that the situation we aim to describe in this paper is quite general and can be applied to many other constraints or invariance; in particular if there is a volume constraint but no translation invariance (or the opposite), the same strategy applies.*

**Old and new applications:**

In order to justify the interest of our general statements, we provide several examples of functionals for which Theorems 1.1 or 3.2 apply. We give here a short list of them, see Section 5 for more details.

- First, we want to show that our Theorem allows to retrieve classical statements already existing in the literature. Mainly relying on the computation of the first and second derivatives of the functionals (see Section 2), and the fact that they satisfy conditions $\mathbf{(C_{H^{s_2}})}$ and $\mathbf{(IT_{H^{s_2},X})}$ (for suitable $s_2$ and $X$), we believe that despite the degree of generality of our approach, the proofs are sometimes less technical and more straightforward than the existing literature. This includes the examples of [14, 12, 2, 8], see Section 5.1 for a more detailed description.

- Second, we noticed ealier that if we were only interested in linear stability, namely that for every smooth path $t \mapsto \Omega_t$, there exists $t_0$ small enough and $c$ such that $J(\Omega_t) - J(\Omega) \geq ct^2$ for every $t \in ]-t_0, t_0[$, then most of the work relies only on the computation of derivatives, and proving the positivity of the second order derivative $\frac{d^2}{dt^2} J(\Omega_t)_{|t=0}$. In other words, most of the previous difficulties we described earlier do not need to be handled. But it is in general much more satisfying (for example in view of the second step of the global strategy described page 3) to obtain uniformity in $(t_0, c)$ as we require in (2). *On a couple of examples, the contribution of this paper is to get uniform stability results, while only directional results have been obtained in the literature. This includes the result in [35] (see Proposition 5.1).*

- We also provide new examples, which comes with minor cost thanks to our results. One example we have in mind is the following generic example: if $\Omega^*$ is a ball of volume $V_0 \in (0, \infty)$, $P(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$ denotes the perimeter of $\Omega$ (for reasonably smooth sets), and $E$ is the Dirichlet energy:

$$E(\Omega) = \min \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega u, \;\; u \in \mathrm{H}^1_0(\Omega) \right\}, \tag{13}$$

  then the conditions of Theorem 3.2 (version of Theorem 1.1 taking into account the translation invariance and the volume constraint) are fulfilled for the functional $J = P + \gamma E$ when $\gamma \geq \gamma_0$ and $\gamma_0 \in (-\infty, 0)$ (whose optimal value can be explicitly computed), and we can conclude from our strategy that the ball is a local minimizer (in a smooth neighborhood, say $\mathrm{C}^{2,\alpha}$ or even $\mathrm{W}^{2,p}$ for $p$ large enough) of the following optimization problem

$$\min \{P(\Omega) + \gamma E(\Omega), \;\; |\Omega| = V_0\}. \tag{14}$$

  In other words, (2) is valid for $J = P + \gamma E$, $d_1$ is the $H^1$-distance, and $d_2 = \bar{d}_{\mathrm{W}^{2,p}}$ (see (6)) with $p$ large enough.

  *For $\gamma \geq 0$ this result is not surprising, since the ball minimizes both the perimeter and the Dirichlet energy, so a stability version is a direct consequence of [21]. But this result is new and surprising when $\gamma$ is nonpositive (even considering $c = 0$ only, which means that $B$ is a local minimum for $J$ in a smooth neighborhood): indeed there is a competition between minimizing the perimeter and maximizing the Dirichlet energy. Another way to state the result is to say that*

$$\frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} \geq |\gamma_0|, \qquad \forall \Omega \in \mathcal{V}(B), \tag{15}$$

  *where $\mathcal{V}(B) = \{\Omega, |\Omega| = |B| \text{ and } \bar{d}_{\mathrm{W}^{2,p}}(\Omega, B) < \eta\}$, for some $\eta > 0$.*

  For a problem related to (14) when $\gamma < 0$, see also [27]. It is also interesting to notice that local optimality of the ball is no longer valid when one consider a neighborhood of $\Omega^*$ for a weak norm, for example the $\mathrm{L}^1$-norm. A counterexample is given in Section 6. Especially it means that the second step of the strategy described page 3 does not apply to (14) if $\gamma < 0$, despite the fact that sets are minimizing the perimeter. This shows in what way these two steps have different degree of generality.

  *In addition to this example, we obtain several new local isoperimetric inequalities, see* **Proposition 5.1** *in Section 5.2.*

Going deeper in the computations, it is possible, as it is done in [35] who obtains a directional version of (15) when replacing $E$ with $\lambda_1$ (the first eigenvalue of the Dirichlet-Laplacian), to compute the optimal value of the constant $\gamma_0$ such that the ball is a smooth local minimizer for (14) (or equivalently compute the optimal $\gamma_0$ in (15) when $\eta$ goes to 0), see **Proposition 5.5**.

In Section 2, we start with a new proof of the Structure Theorem for second order shape derivatives. We also recall the classical examples of second order shape derivatives, noticing in particular in which norms they are continuous (which leads to the value of $s_2$ from assumption $\mathbf{(C_{H^{s_2}})}$), and focus on the case of the ball for which we diagonalize the shape hessians (which leads to the classical stability properties of the ball for these functionals). In Section 3 we state the version of Theorem 1.1 adapted to the constrained/invariant case, we discuss the coercivity assumptions proving (9) (Lemma 3.3), and we prove Theorems 1.1 and 3.2. In Section 4 we discuss assumption $\mathbf{(IT_{H^{s_2},X})}$, in particular we recall and improve existing results, and provide a new one, namely the first eigenvalue of the Dirichlet Laplacian for which it seems this condition was not proven in the existing literature. In Section 5, we explain how our general results allow to retrieve known results, and then prove some local isoperimetric inequalities, some of which are new, see Proposition 5.1. All these applications are simple corollaries of our main results, combined with the computations reminded in Section 2. In the last Section, we show that similar results in non-smooth neighborhoods cannot be achieved with the same degree of generality.

# 2 On Second order shape derivatives.

In this section, we recall classical facts on second order shape derivatives, and give a new proof of their structure. As for all the examples of this paper, we assume $J$ is a shape functional such that $\theta \in \Theta \mapsto J((Id+\theta)(\Omega))$ is of class $\mathrm{C}^2$ in a neighborhood of 0 in $\Theta = \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. This simplifies the presentation, though similar proofs can be adapted to other functional spaces, see Remark 2.4.

## 2.1 Structure theorem

It is well-known since Hadamard's work that the shape gradient is a distribution supported on the moving boundary and acting on the normal component of the deformation field. The second order shape derivative also has a specific structure as stated by A. Novruzi and M. Pierre in [36]. We quote their result, and provide a new proof:

**Theorem 2.1 (Structure Theorem of first and second shape derivatives)** *Let $\Theta = \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\Omega$ an open bounded domain of $\mathbb{R}^d$ and $J$ a real-valued shape function defined on $\mathcal{V}(\Omega) = \{(Id+\theta)(\Omega), \|\theta\|_\Theta < 1\}$. Let us define the function $\mathcal{J}_\Omega$ on $\{\theta \in \Theta, \|\theta\|_\Theta < 1\}$ by*

$$\mathcal{J}_\Omega(\theta) = J[(Id+\theta)(\Omega)].$$

**(i)** *If $\mathcal{J}_\Omega$ is differentiable at 0 and $\Omega$ is $\mathrm{C}^2$, then there exists a continuous linear form $\ell_1$ on $\mathrm{C}^1(\partial\Omega)$ such that $\mathcal{J}'_\Omega(0)\xi = \ell_1(\xi_{|\partial\Omega} \cdot \boldsymbol{n})$ for all $\xi \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, where $\boldsymbol{n}$ denotes the unit exterior normal vector on $\partial\Omega$.*

**(ii)** *If moreover $\mathcal{J}_\Omega$ is twice differentiable at 0 and $\Omega$ is $\mathrm{C}^3$, then there exists a continuous symmetric bilinear form $\ell_2$ on $\mathrm{C}^2(\partial\Omega) \times \mathrm{C}^2(\partial\Omega)$ such that for all $(\xi, \zeta) \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)^2$*

$$\mathcal{J}''_\Omega(0)(\xi,\zeta) = \ell_2(\xi\cdot\boldsymbol{n}, \zeta\cdot\boldsymbol{n}) + \ell_1(\boldsymbol{B}(\zeta_\tau,\xi_\tau) - \nabla_\tau(\zeta\cdot\boldsymbol{n})\cdot\xi_\tau - \nabla_\tau(\xi\cdot\boldsymbol{n})\cdot\zeta_\tau),$$

*where $\nabla_\tau$ is the tangential gradient, $\xi_\tau$ and $\zeta_\tau$ stands for the tangential components of $\xi$ and $\zeta$, and $\boldsymbol{B}$ is the second fondamental form of $\partial\Omega$.*

With respect to this work, it is important to notice that at a critical domain for $J$, the shape hessian is reduced to $\ell_2$ and hence does not see the tangential components of the deformations fields.

**Remark 2.2** The requirement that $\Omega$ is bounded is made only to simplify the presentation: the result remains valid replacing $\mathrm{C}^1(\partial\Omega)$ with $\mathrm{C}^1_c(\partial\Omega)$ and localizing the test functions.

**Remark 2.3** As noticed in [29, p. 225], with this degree of generality, the regularity assumption on $\Omega$ are sharp. We could indeed wonder if $\ell_1$ can be extended as a continuous linear form on $\mathrm{C}^0(\partial\Omega)$; this is not true in general if $\Omega$ is only assumed to be $\mathrm{C}^1$, as the example of the perimeter shows (it would mean that the mean curvature is a Radon measure, which is not true for a $\mathrm{C}^1$ domain). Moreover, our strategy provides $\ell_2$ being continuous for the $\mathrm{C}^2(\partial\Omega)$-norm, while [36] gives a better result with $\ell_2$ being continuous for the $\mathrm{C}^1(\partial\Omega)$-norm. However, if we assume that $\ell_1$ can be extended as a continuous linear form on $\mathrm{C}^0(\partial\Omega)$, then point (**ii**) is valid assuming $\Omega$ of class $\mathrm{C}^2$ only, and $\ell_2$ is then continuous for the $\mathrm{C}^1$-norm; it is easy to see how the proof adapts to this case, and we retrieve then an optimal result, see also [36, Remark 2.8, Corollary 2.9], .

**Remark 2.4** Compare to the result in [36], we restricted ourself to the space $\Theta = \mathrm{W}^{1,\infty}$ (or similarly $\mathrm{C}^{1,\infty} := \mathrm{W}^{1,\infty} \cap \mathrm{C}^1$, see the proof below), as all the functional of this paper are differentiable in this space. Of course, the same proof can be adapted to spaces like $\mathrm{W}^{k,\infty}$ for $k \geq 2$, which is important to handle higher order geometric or PDE functional, but we do not handle such examples in this paper.

**Remark 2.5** When $\xi = \zeta$, we get

$$\mathcal{J}''_\Omega(0).(\xi,\xi) = \ell_2(\xi\cdot\boldsymbol{n}, \xi\cdot\boldsymbol{n}) + \ell_1(Z_\xi), \text{ where } Z_\xi = \boldsymbol{B}(\xi_\tau,\xi_\tau) - 2\nabla_\tau(\xi\cdot\boldsymbol{n})\cdot\xi_\tau.$$

As noticed in [2, Equation (7.5)], the term $Z_\xi$ can have be written in a different way:

$$Z_\xi = (\xi\cdot\boldsymbol{n})\mathrm{div}(\xi) - \mathrm{div}_\tau\left(\xi_\tau(\xi\cdot\boldsymbol{n})\right) - H(\xi.\boldsymbol{n})^2.$$

The advantage of $Z_\xi$ is usually that it clearly vanishes when $\xi_\tau = 0$, but this second formulation can also have advantages, especially when $\xi$ has a vanishing divergence (as it is the case in [2]) or when there are simplifications as it is the case for the volume (see Lemma 2.7 for the first equality):

$$\mathrm{Vol}''(\Omega).(\xi,\xi) = \int_{\partial\Omega} H(\xi\cdot\boldsymbol{n})^2 + \int_{\partial\Omega} Z_\xi = \int_{\partial\Omega}(\xi\cdot\boldsymbol{n})\mathrm{div}(\xi). \tag{16}$$

In that case, the last formula is indeed more common.

**Remark 2.6** It is sometimes considered that first and second order derivatives described in the previous theorem cannot handle the differentiation of $t \mapsto J(T_t(\Omega))$ where $T \in \mathrm{C}^2([0,\alpha[,\Theta)$ is not of the form $T_t = Id + t\xi$. This is not true, as the chain rule formula easily gives (and is allowed when we have proven the Fréchet-differentiability of the functionals, which is valid for all the functionals of this paper):

$$\frac{d^2}{dt^2}J(T_t(\Omega)) = \mathcal{J}''_\Omega(T_t - Id).\left(\frac{d}{dt}T_t, \frac{d}{dt}T_t\right) + \mathcal{J}'_\Omega(T_t - Id).\left(\frac{d^2}{dt^2}T_t\right)$$

and the structure result can then be applied. For example, if $T_t$ is the flow of the vector field $\xi$ as it is usually done in the speed method, we obtain:

$$\frac{d^2}{dt^2}J(T_t(\Omega))_{|t=0} = \mathcal{J}''_\Omega(0).(\xi,\xi) + \mathcal{J}'_\Omega(0).\left((D\xi)\cdot\xi\right).$$

Another interesting case is that if $\Omega$ is a critical shape for $J$, namely $\mathcal{J}'_\Omega(0) \equiv 0$, and if $T_t = Id + t\xi + \frac{t^2}{2}\eta + o(t^2)$ where $o(t^2)$ has to be understood with the norm $\|\cdot\|_\Theta$, then we always have

$$\frac{d^2}{dt^2}J(T_t(\Omega))_{|t=0} = \ell_2(\xi\cdot\boldsymbol{n}, \xi\cdot\boldsymbol{n}).$$

This fact is often observed through computations, but it is always true, when functionals and shapes are smooth enough.

**Proof.** We only focus on the second order derivative, as the first order one is classical (see for example [29, 16, 32]). For $k \in \mathbb{N}$, we define $\mathrm{C}^{k,\infty} := \mathrm{C}^k \cap \mathrm{W}^{k,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ equipped with the same norm as $\mathrm{W}^{k,\infty}$, which

is also a Banach space and is more adapted to approximation by smooth functions. Let $\xi, \zeta \in \mathrm{C}^\infty$ compactly supported , and denote $\gamma, \delta$ their respective flow, namely

$$\left\{ \begin{array}{rcl} \frac{d}{dt}\gamma_t(x) & = & \xi(\gamma_t(x)) \\ \gamma_0(x) & = & x \end{array} \right. \qquad \left\{ \begin{array}{rcl} \frac{d}{dt}\delta_t(x) & = & \zeta(\delta_t(x)) \\ \delta_0(x) & = & x \end{array} \right.$$

Thanks to our assumption on $\xi$, we easily check that the function $T \in \Theta \mapsto \xi \circ (T + Id) \in \Theta$ is locally Lipschitz and $\mathrm{C}^2$, and therefore these ODE admits solutions defined on $(-t_0, t_0)$ and such that $[t \mapsto \gamma_t - Id, t \mapsto \delta_t - Id]$ are in $\mathrm{C}^2((-t_0, t_0), \Theta)$. As a consequence, $(t, s) \mapsto \gamma_s \circ \delta_t - Id \in \Theta$ is well-defined in a neighborhood of $(0, 0)$ and $\mathrm{C}^2$.

Let now assume that $\zeta \cdot \boldsymbol{n} = 0$. Then from classical criterion of invariance of sets with the flow, we have $\delta_t(\Omega) = \Omega$ for every $t$ small enough, so $J(\gamma_s \circ \delta_t(\Omega)) = \mathcal{J}_\Omega(\gamma_s \circ \delta_t - Id)$ is independent of $t$. Differentiating successively with respect to $t$ and $s$ at $(0, 0)$, we obtain:

$$\mathcal{J}''_\Omega(0).(\xi, \zeta) + \mathcal{J}'_\Omega(0).(D\xi \cdot \zeta) = 0, \ \ \forall \xi \in \mathrm{C}^\infty_c, \ \ \forall \zeta \in K \cap \mathrm{C}^\infty_c,$$

where $K = \mathrm{Ker}(\Phi)$ and $\Phi : \xi \in \Theta \mapsto \xi_{|\partial\Omega} \cdot \boldsymbol{n}$.

We define $b : (\xi, \zeta) \in \mathrm{C}^{2,\infty} \times \mathrm{C}^{1,\infty} \mapsto \mathcal{J}''_\Omega(0).(\xi, \zeta) + \mathcal{J}'_\Omega(0).(D\xi \cdot \zeta)$ which is a continuous bilinear functional that vanishes for $\zeta \in K$, for any fixed $\xi$. Therefore we can write, using quotient properties, $b(\xi, \zeta) = \widetilde{b}(\xi, \zeta_{|\partial\Omega \cdot \boldsymbol{n}})$ where $\widetilde{b} : \mathrm{C}^{2,\infty} \times \mathrm{C}^1(\partial\Omega) \to \mathbb{R}$ is continuous (a priori we only get that $\widetilde{b}$ is separately continuous but with Banach-Steinhaus Theorem, it implies continuity), as $\Phi$ induces an isomorphism between $\Theta/K$ and $\Phi(\Theta) = \mathrm{C}^1(\partial\Omega)$ equipped with the $\mathrm{C}^1$ norm (using that $\Omega$ is of class $\mathrm{C}^2$). Moreover by construction we have:

$$\mathcal{J}''_\Omega(0).(\xi, \zeta) + \mathcal{J}'_\Omega(0).(D\xi \cdot \zeta) = \tilde{b}(\xi, \zeta_{|\partial\Omega} \cdot \boldsymbol{n}), \ \ \forall\, \xi, \zeta \in \mathrm{C}^{2,\infty} \times \mathrm{C}^{1,\infty}.$$

Using the symmetry of $\mathcal{J}''_\Omega(0)$, we can write, for every $(\xi, \zeta) \in \mathrm{C}^{2,\infty}$:

$$\tilde{b}(\zeta, \xi_{|\partial\Omega} \cdot \boldsymbol{n}) - \tilde{b}(\xi, \zeta_{|\partial\Omega} \cdot \boldsymbol{n}) = \mathcal{J}'_\Omega(0).(D\zeta \cdot \xi - D\xi \cdot \zeta) \tag{17}$$

Our goal is now to apply this formula to $\zeta_{\boldsymbol{n}}$ the normal component of $\zeta$, which needs to be extended as a vector field on $\mathbb{R}^d$. To that end, we introduce $P_{\partial\Omega}$ the projection on $\partial\Omega$, which is well-defined and $\mathrm{C}^2$ in a neighborhood of $\partial\Omega$, as $\Omega$ is assumed to be $\mathrm{C}^3$ (see for example [16]). Then if $\varphi$ is defined on $\partial\Omega$, we set $\widetilde{\varphi}(x) = \varphi(P_{\partial\Omega}x)\chi(x)$ where $\chi$ is a smooth function with $\chi = 1$ in a neighborhood of $\partial\Omega$, and $\chi = 0$ outside a compact set (in other words, $\varphi$ is extended so that it is constant in the normal direction). This operator $\varphi \mapsto \widetilde{\varphi}$ is continuous from $\mathrm{C}^2(\partial\Omega)$ to $\mathrm{C}^{2,\infty}$. Let us define then $\zeta_{\boldsymbol{n}} := \widetilde{(\zeta \cdot \boldsymbol{n})\boldsymbol{n}}$ the extension of the normal component of $\zeta$. Defining the bilinear form $\ell_0(\varphi_1, \varphi_2) = \tilde{b}(\widetilde{\varphi_1 \boldsymbol{n}}, \varphi_2)$, continuous on $\mathrm{C}^2(\partial\Omega) \times \mathrm{C}^1(\partial\Omega)$ (and a priori non symmetric), we obtain

$$\begin{array}{rcll} \mathcal{J}''_\Omega(0).(\xi, \zeta) & = & \tilde{b}(\xi, \zeta \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\xi \cdot \zeta) & \\ & = & \tilde{b}(\zeta_{\boldsymbol{n}}, \xi \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\zeta_{\boldsymbol{n}} \cdot \xi - D\xi \cdot \zeta_{\boldsymbol{n}}) - \mathcal{J}'_\Omega(0).(D\xi \cdot \zeta) & \text{(using (17))} \\ & = & \ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\zeta_{\boldsymbol{n}} \cdot \xi - D\xi \cdot \zeta_{\boldsymbol{n}} + D\xi \cdot \zeta) & \\ & = & \ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\zeta_{\boldsymbol{n}} \cdot \xi + D\xi \cdot \zeta_\tau) & \end{array}$$

where $\zeta_\tau = \zeta - \zeta_{\boldsymbol{n}}$. We now use $D\zeta_{\boldsymbol{n}} = D_\tau\zeta_{\boldsymbol{n}}$, because thanks to our choice of extension operator, $\zeta_{\boldsymbol{n}}$ is constant in the direction $\boldsymbol{n}$ (by definition, $D_\tau a = Da - (Da \cdot \boldsymbol{n})\boldsymbol{n}$), and therefore $D\zeta_{\boldsymbol{n}} \cdot \xi = D_\tau\zeta_{\boldsymbol{n}} \cdot \xi_\tau$. Moreover, $D\xi \cdot \zeta_\tau = D_\tau\xi \cdot \zeta_\tau$.

Using a symmetrization of the previous formula, we obtain

$$\begin{array}{rcl} \mathcal{J}''_\Omega(0).(\xi, \zeta) & = & \frac{1}{2}\Big[\ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) + \ell_0(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D_\tau\zeta_{\boldsymbol{n}} \cdot \xi_\tau + D_\tau\xi \cdot \zeta_\tau + D_\tau\xi_{\boldsymbol{n}} \cdot \zeta_\tau + D_\tau\zeta \cdot \xi_\tau)\Big] \\ & = & \ell_2(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) - \frac{1}{2}\mathcal{J}'_\Omega(0) \cdot \Big(2D_\tau\zeta \cdot \xi_\tau + 2D_\tau\xi \cdot \zeta_\tau - D_\tau\xi_\tau \cdot \zeta_\tau - D_\tau\zeta_\tau \cdot \xi_\tau\Big) \end{array}$$

where we defined $\ell_2(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}) = \frac{1}{2}(\ell_0(\zeta \cdot \boldsymbol{n}, \xi \cdot \boldsymbol{n}) + \ell_0(\xi \cdot \boldsymbol{n}, \zeta \cdot \boldsymbol{n}))$, which is a continuous bilinear form on $\mathrm{C}^2(\partial\Omega)^2$.

From the structure of the first order derivative, and using the formula

$${}^tD_\tau\xi_\tau \cdot \boldsymbol{n} + {}^tD_\tau\boldsymbol{n} \cdot \xi_\tau = 0$$

(obtained by tangentially differentiating $\xi_\tau \cdot \boldsymbol{n} = 0$), we finally obtain (using the $C^3$ regularity of $\partial\Omega$ so that $D_\tau \boldsymbol{n}$ belongs to the space of definition of $\ell_1$)

$$\begin{aligned}\mathcal{J}''_\Omega(0).(\xi,\zeta) &= \ell_2(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n}) - \tfrac{1}{2}\ell_1\Big((2D_\tau\zeta\cdot\xi_\tau + 2D_\tau\xi\cdot\zeta_\tau)\cdot\boldsymbol{n} - \zeta_\tau\cdot({}^tD_\tau\xi_\tau\cdot\boldsymbol{n}) - \xi_\tau\cdot({}^tD_\tau\zeta_\tau\cdot\boldsymbol{n})\Big)\\ &= \ell_2(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n}) + \ell_1\Big((D_\tau\boldsymbol{n}\cdot\zeta_\tau)\cdot\xi_\tau - \nabla_\tau(\zeta\cdot\boldsymbol{n})\cdot\xi_\tau - \nabla_\tau(\xi\cdot\boldsymbol{n})\cdot\zeta_\tau\Big)\end{aligned}$$

(where we used that $D_\tau\boldsymbol{n}$ is symmetric), which concludes the proof (a priori, $\ell_0$ depends on the extension operator that has been chosen, but as in the final formula the extension only appears in $\ell_2$ which does not depend of the extension operator). □

## 2.2 Examples of shapes derivatives on general domains.

For an open bounded (smooth enough) set $\Omega \subset \mathbb{R}^d$, we consider in this section (and in the rest of the paper) its volume $|\Omega|$, its perimeter $P(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$, its Dirichlet energy $E(\Omega)$ defined in (13)) and $\lambda_1$ the first eigenvalue of the Dirichlet Laplace operator. The existence and computations of the shape derivatives of these functionals are well known, see for example [29, Chapter 5]; we just recall here the results. Recall that the mean curvature (understood as the sum of the principal curvatures of $\partial\Omega$) is denoted by $H$, $\boldsymbol{B} = D_\tau\boldsymbol{n}$ is the second fundamental form of $\partial\Omega$, and $\|\boldsymbol{B}\|^2$ is the sum of the squares of the principal curvatures of $\partial\Omega$.

**Lemma 2.7 (Expression of shape derivatives)** *If $\Omega$ is $\mathrm{C}^2$, one has, for any $\varphi \in \mathrm{C}^\infty(\partial\Omega)$,*

- $\ell_1[\mathrm{Vol}](\Omega).\varphi = \displaystyle\int_{\partial\Omega}\varphi, \qquad \ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} H\varphi^2.$
- $\ell_1[P](\Omega).\varphi = \displaystyle\int_{\partial\Omega} H\varphi, \qquad \ell_2[P](\Omega).(\varphi,\varphi) = \int_{\partial\Omega}|\nabla_\tau\varphi|^2 + \int_{\partial\Omega}\left[H^2 - \|\boldsymbol{B}\|^2\right]\varphi^2$
- $\ell_1[E](\Omega).\varphi = -\dfrac{1}{2}\displaystyle\int_{\partial\Omega}(\partial_n u)^2\varphi, \qquad \ell_2[E](\Omega).(\varphi,\varphi) = \langle\partial_n u\ \varphi, \Lambda(\partial_n u\ \varphi)\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} + \displaystyle\int_{\partial\Omega}\left[\partial_n u + \frac{1}{2}H(\partial_n u)^2\right]\varphi^2$

  *where $u \in \mathrm{H}^1_0(\Omega)$ is the unique solution to $-\Delta u = 1$, $\Lambda :\ \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ is the Dirichlet-to-Neumann map defined as $\Lambda(\psi) = \partial_n\mathsf{H}(\psi)$ where $\mathsf{H}$ is the harmonic extension operator from $\mathrm{H}^{1/2}(\partial\Omega)$ into $\mathrm{H}^1(\Omega)$:*

$$-\Delta\mathsf{H}(\psi) = 0 \text{ in } \Omega, \quad \mathsf{H}(\psi) = \psi \text{ on } \partial\Omega,$$

- $\ell_1[\lambda_1](\Omega).\varphi = -\displaystyle\int_{\partial\Omega}(\partial_n v)^2\varphi, \qquad \ell_2[\lambda_1](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} 2w(\varphi)\ \partial_n w(\varphi) + H(\partial_n v)^2\varphi^2$

  *where $v$ is the normalized eigenfunction (solution in $\mathrm{H}^1_0(\Omega)$ of $-\Delta v = \lambda_1 v$ with $v \geq 0$ in $\Omega$ and $\|v\|_{\mathrm{L}^2(\Omega)} = 1$) and $w(\varphi)$ is the solution of*

$$\left\{\begin{array}{rcl} -\Delta w(\varphi) &=& \lambda_1 w(\varphi) - v\displaystyle\int_{\partial\Omega}(\partial_n v)^2\varphi \text{ in } \Omega,\\ w(\varphi) &=& -\varphi\partial_n v \text{ on } \partial\Omega,\\ \displaystyle\int_\Omega v\ w(\varphi) &=& 0.\end{array}\right. \tag{18}$$

A fundamental fact for this work appears here in the expression of the shape hessians. Even if they are derived for regular perturbations, they are naturally defined and continuous on different Sobolev spaces on $\partial\Omega$:

**Lemma 2.8 (Continuity of shape Hessians)** *If $\Omega$ is $\mathrm{C}^2$, there is a constant $C > 0$ such that*

$$|\ell_2[P](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^1(\partial\Omega)}, \quad |\ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{L}^2(\partial\Omega)},$$
$$|\ell_2[E](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega),} \quad |\ell_2[\lambda_1](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

Therefore, from this Lemma, it is natural to consider the extension of these bilinear forms to their space of continuity. Note that the $\mathrm{C}^2$ assumption on $\Omega$ is sufficient here, see also Remark 2.3.

### 2.3 The case of balls

When proving the second order optimality condition, one needs to explicit the shape derivatives of these functionals on the ball $B_1$ of radius 1. For the Dirichlet energy $E$, we remark that $u(x) = (1-|x|^2)/2d$ solves $-\Delta u = 1$ in $\mathrm{H}^1_0(B_1)$ and satisfies $\partial_n u = -\frac{1}{d}$ on $\partial B_1$. For $\lambda_1$, we recall that the eigenvalue and eigenfunction are

$$\lambda_1(B_1) = j^2_{d/2-1} \text{ associated to } v(x) \;=\; \alpha_d\, |x|^{1-d/2}\, J_{d/2-1}\left(j_{d/2-1}\;\; |x|\right),$$

where $j_{d/2-1}$ is the first zero of Bessel's function $J_{d/2-1}$ and the normalization constant is defined as

$$\alpha_d = \left[ |\partial B_1| \int_0^1 r J^2_{d/2-1}\left(\frac{j_{d/2-1}}{R}\, r\right)\, dr \right]^{-1/2}.$$

Moreove, from [30, p. 35], the eigenfunction satisfies

$$\partial_n v = \sqrt{\frac{2}{P(B_1)}}\, j_{d/2-1} := \gamma_d, \text{ so that } \gamma_d^2 = \frac{2\lambda_1(B_1)}{P(B_1)}. \tag{19}$$

We obtain the **shape gradients**:

$$\ell_1[\mathrm{Vol}](B_1).\varphi = \int_{\partial B_1} \varphi, \qquad \ell_1[P](B_1).\varphi = (d-1)\int_{\partial B_1} \varphi,$$
$$\ell_1[E](B_1).\varphi = -\frac{1}{2d^2}\int_{\partial B_1} \varphi, \qquad \ell_1[\lambda_1](B_1).\varphi = \;-\gamma_d^2 \int_{\partial B_1} \varphi.$$

Let us notice that these four shape gradients at balls are colinear. As a consequence, the balls are critical domains for the perimeter, the Dirichlet energy and $\lambda_1$ (or any sum of these functionals) under a volume constraint, and these formula easily provide the value of the Lagrange-multiplier.

Let us turn our attention to the **hessians**. The value of $\ell_2[\lambda_1]$ is a bit more involved, so we deal with it in the next lemma. For the other functionals, it is known from Lemma 2.7 that:

$$\ell_2[\mathrm{Vol}](B_1).(\varphi,\varphi) = (d-1)\int_{\partial B_1} \varphi^2,$$

$$\ell_2[P](B_1).(\varphi,\varphi) = \int_{\partial B_1} |\nabla_\tau \varphi|^2 + (d-1)(d-2)\int_{\partial B_1} \varphi^2,$$

$$\ell_2[E](B_1).(\varphi,\varphi) = \frac{1}{d^2}\langle \varphi, \Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} - \frac{d+1}{2d^2}\int_{\partial B_1} \varphi^2.$$

In order to see that the quadratic forms associated to the Lagrangian are coercive on their natural spaces, it is useful to study the diagonalized form of these Hessians. To that end, we use spherical harmonics defined as the restriction to the unit sphere of harmonic polynomials. We recall here facts from [41, pages 139-141]. We let $\mathcal{H}_k$ denote the space of spherical harmonics of degree $k$ (that is, the restriction to $\partial B_1$ of homogeneous polynomials in $\mathbb{R}^d$, of degree $k$). It is also the eigenspace of the Laplace-Beltrami operator on the unit sphere associated with the eigenvalue $-k(k+d-2)$. Let $(Y^{k,l})_{1\le l\le d_k}$ be an orthonormal basis of $\mathcal{H}_k$ with respect to the $\mathrm{L}^2(\partial B_1)$ scalar product. The family $(Y^{k,l})_{k\in\mathbb{N},1\le l\le d_k}$ is a Hilbert basis of $\mathrm{L}^2(\partial B_1)$. Hence, any function $\varphi$ in $\mathrm{L}^2(\partial B_1)$ can be decomposed:

$$\varphi(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} \alpha_{k,l}(\varphi) Y^{k,l}(x), \quad \text{for } |x| = 1. \tag{21}$$

Then, by construction, the function $h$ defined by

$$h(x) = \sum_{k=0}^{\infty} |x|^k \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi) Y^{k,l}\left(\frac{x}{|x|}\right), \quad \text{for } |x| \le 1, \tag{22}$$

is harmonic in $B_1$ and satisfies $h = \varphi$ on $\partial B_1$. Moreover, the sequence of coefficients $\alpha_{k,l}$ characterizes the Sobolev regularity of $\varphi$: indeed $\varphi \in \mathrm{H}^s(\partial B_1)$ if and only if the sum $\sum_k (1+k^2)^s \sum_l |\alpha_{k,l}|^2$ converges. We can now state the following lemma expressing the previous shape hessians are diagonal on this basis.

**Lemma 2.9** *Using the decomposition* (21)*, we have (*$\gamma_d$ *is the constant defined in* (19)*)*

$$\ell_2[\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(d-1)\ \alpha_{k,l}(\varphi)^2,$$

$$\ell_2[E](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{1}{d^2}\ k\ -\frac{d+1}{2d^2}\right]\alpha_{k,l}(\varphi)^2,$$

$$\ell_2[P](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[k^2+(d-2)k+(d-1)(d-2)\right]\ \alpha_{k,l}(\varphi)^2,$$

$$\ell_2[\lambda_1](B_1)(\varphi,\varphi) = \gamma_d^2\left(3\alpha_{0,1}^2(\varphi)+\sum_{k=1}^{\infty}\sum_{l=1}^{d_k}2\left[k+\frac{d-1}{2}-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)\right).$$

This result is very useful to look for the sign of $\ell_2[J-\mu\mathrm{Vol}](B_1)$ where $J$ is the functional we minimize and $\mu$ is a Lagrange multiplier, see Section 5.

**Proof.** First we check that

$$\int_{\partial B_1}\varphi^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2.$$

$$\int_{\partial B_1}|\nabla_\tau\varphi|^2 = -\int_{\partial B_1}\varphi\ \Delta_\tau\varphi = \sum_{k=0}^{\infty}k(k+d-2)\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2.$$

Then, we precise the term involving the Dirichlet-to-Neumann map that appears in the shape hessian of the Dirichlet energy. Using $h$ defined in (22) and Green formula, we have:

$$\begin{aligned}
\langle\varphi,\Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} &= \int_{\partial B_1}\varphi\partial_n h = \int_{B_1}|\nabla h|^2\\
&= \int_0^1\left(\int_{\partial B_r}\left((\partial_n h)^2+|\nabla_\tau h|^2\right)d\sigma\right)dr = \int_0^1\left(\int_{\partial B_r}\left((\partial_n h)^2-h\Delta_\tau h\right)d\sigma\right)dr\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\int_0^1 r^{d-1}\left[k^2r^{2(k-1)}+\frac{k(k+d-2)}{r^2}\ r^{2k}\right]dr\ \alpha_{k,l}(\varphi)^2\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{k^2}{2k+d-2}+\frac{k(k+d-2)}{2k+d-2}\right]\alpha_{k,l}(\varphi)^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}k\ \alpha_{k,l}(\varphi)^2.
\end{aligned}$$

We obtain $\ell_2[\mathrm{Vol}]$, $\ell_2[P]$ and $\ell_2[E]$ by gathering these elementary terms.

Let us now consider the case of the first eigenvalue. We apply [30, p. 35] (see also [38] and [40]): for a second order volume preserving path, that is $t\mapsto T_t$ such that $|T_t(\Omega)| = |\Omega|+o(t^2)$ for small $t$, we have

$$\left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} = \sum_{k=1}^{\infty}\sum_{l=1}^{d_k}2\gamma_d^2\ \left[k+d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)$$

where $\varphi = (\frac{d}{dt}T_t)_{|t=0}\cdot\boldsymbol{n}$ and we have used the recursive formula for Bessel function $J_\nu'(z) = (\nu/z)J_\nu(z)-J_{\nu+1}(z)$ to adapt his expression to our notations ([1, 9.1.27, p 361]). To deduce $\ell_2[\lambda_1]$ from this computation, we introduce $\theta$ a smooth vector field which is normal on $\partial B_1$ and denote $\varphi=\theta\cdot\boldsymbol{n}$. We assume that $\int_{\partial B_1}\varphi = \alpha_{0,1}(\varphi)=0$. It is then clear that there exists $\xi$ such that $T_t := Id+t\theta+\frac{t^2}{2}\xi$ is volume preserving at the second order, that is to say

$$\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi)+\ell_1[\mathrm{Vol}](B_1)(\psi)=0,$$

where $\psi=\xi\cdot\boldsymbol{n}$. Then we observe that for a smooth shape functional $J$ and for such $t\mapsto T_t$,

$$\left(\frac{d^2}{dt^2}J(T_t(B_1))\right)_{|t=0} = \ell_2[J](B_1)(\varphi,\varphi)+\ell_1[J](B_1)(\psi),$$

and therefore, denoting $\mu$ the Lagrange multiplier such that $\ell_1[\lambda_1 - \mu \mathrm{Vol}](B_1) = 0$, we obtain

$$\begin{aligned}\left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} &= \ell_2[\lambda_1](B_1)(\varphi,\varphi) + \ell_1[\lambda_1](B_1)(\psi) = \ell_2[\lambda_1](B_1)(\varphi,\varphi) + \mu\ell_1[\mathrm{Vol}](B_1)(\psi)\\ &= \ell_2[\lambda_1](B_1)(\varphi,\varphi) - \mu\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi)\end{aligned}$$

Then, we get, as here $\mu = -\gamma_d^2$:

$$\begin{aligned}\ell_2[\lambda_1](\varphi,\varphi) &= \left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} + \mu\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi),\\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k} 2\gamma_d^2 \left[k + d - 1 - j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi) - \gamma_d^2\sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(d-1)a_{k,l}^2(\varphi),\\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k} 2\gamma_d^2 \left[k + \frac{d-1}{2} - j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi).\end{aligned}$$

It remains to compute the coefficient associated to the mode $k = 0$. It suffices to consider the deformations as $T_t(x) = x + tx$ mapping the ball $B_1$ onto the ball $B_{1+t}$. Here $\varphi = 1$ and $\alpha_{0,1}(\varphi) = P(B_1)^{1/2}$. Since $\lambda_1$ is homogeneous of degree $-2$, we get $\lambda(t) := \lambda_1(T_t(B_1)) = (1+t)^{-2}\lambda_1(B_1)$ so that $\lambda''(0) = 6\lambda_1(B_1) = 6\frac{\lambda_1(B_1)}{P(B_1)}\alpha_{0,1}(\varphi)^2$. □

# 3 Main Theorem

In the introduction, we gave the unconstrained version of the main result of this paper, see Theorem 1.1. As in most applications we need to deal with a volume constraint and a translation invariance of the functional, we describe here the corresponding statement. In the rest of the section, we discuss condition $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and prove (9) about coercivity and we prove Theorems 1.1 and 3.2.

We start with the suitable definitions of critical and stable domains for problems with volume constraint and translation invariant functionals:

**Definition 3.1** *Let $\Omega^*$ be a shape (smooth enough so that Theorem 2.1 applies) and $J$ a shape functional defined and twice shape differentiable in a neighborhood of $\Omega^*$ for $d_{\mathrm{W}^{1,\infty}}$.*

- *We say that $\Omega^*$ is a critical domain for $J$ under volume constraint if*

$$\forall \varphi \in \mathrm{C}^\infty(\partial\Omega^*) \text{ such that } \ell_1[\mathrm{Vol}](\Omega^*).\varphi = \int_{\partial\Omega^*}\varphi = 0, \qquad \ell_1[J](\Omega^*).(\varphi) = 0.$$

  *It is equivalent to the existence of $\mu \in \mathbb{R}$ such that $(\ell_1[J] - \mu\ell_1[\mathrm{Vol}])(\Omega^*) = 0$, $\mu$ is called a Lagrange multiplier.*

- *When $\Omega^*$ is a critical domain for $J$ under volume constraint, we say that $\Omega^*$ is a strictly stable shape for $J$ under volume constraint and up to translations if*

$$\forall \varphi \in T(\partial\Omega)\setminus\{0\},\ (\ell_2[J] - \mu\ell_2[\mathrm{Vol}])(\Omega).(\varphi,\varphi) > 0 \tag{23}$$

  *where*

$$T(\partial\Omega) := \left\{\varphi \in \mathrm{H}^s(\partial\Omega),\ \int_{\partial\Omega}\varphi = 0,\ \int_{\partial\Omega}\varphi\overrightarrow{x} = \overrightarrow{0}\right\}, \tag{24}$$

  *$\mu$ is the Lagrange multiplier associated and $s \geq 0$ is the lowest index so that $\ell_2[J](\Omega)$ is continuous on $\mathrm{H}^s(\partial\Omega)$ (see Lemma 2.8).*

Here is the main result of this paper:

**Theorem 3.2** *Let $\Omega^*$ of class $\mathrm{C}^3$, and $J$ a shape functional, translation invariant and twice Fréchet differentiable on a neighborhood of $\Omega^*$ for $d_{\mathrm{W}^{1,\infty}}$. We assume:*

- **Structural hypotheses:** *there exists* $s_2 \in (0,1]$ *and* $X$ *a Banach space with* $\mathrm{C}^\infty(\mathbb{R}^d) \subset X \subset \mathrm{W}^{1,\infty}(\mathbb{R}^d)$ *such that* $J$ *satisfies* $(\mathbf{C}_{\mathrm{H}^{s_2}})$ *and* $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ *at* $\Omega^*$,
- **Necessary optimality conditions:**
  - $\Omega^*$ *is a critical shape under volume constraint for* $J$,
  - $\Omega^*$ *is a strictly stable shape for* $J$ *under volume constraint and up to translations.*

*Then* $\Omega^*$ *is an* $\mathrm{H}^{s_2}$**-stable local minimum of** $J$ **in a** $X$**-neighborhood under volume constraint**, *that is to say there exists* $\eta > 0$ *and* $c = c(\eta) > 0$ *such that:*

$$\forall\, \Omega \text{ such that } \bar{d}_X(\Omega,\Omega^*) \le \eta \text{ and } |\Omega| = |\Omega^*|, \quad J(\Omega) \ge J(\Omega^*) + c\bar{d}_{\mathrm{H}^{s_2}}(\Omega,\Omega^*)^2.$$

*where* $\bar{d}_X, \bar{d}_{\mathrm{H}^{s_2}}$ *are defined in* (6).

### 3.1 About coercivity and condition $(\mathbf{C}_{\mathrm{H}^s})$

Usually the coercivity property for the second order derivative (of the functional or of the Lagrangian) has to be proven by hand on each specific example by studying the lower bound of the spectrum of the bilinear form $\ell_2$ defined in Theorem 2.1, typically thanks to Lemma 2.9. Nevertheless, when $\ell_2$ enjoys some structural property, coercivity can be more easily checked as a consequence of the following general lemma.

**Lemma 3.3** *Let* $M$ *be a Lipschitz manifold,* $s_2 \in [0,1]$*, and* $V$ *a vectorial subspace of* $\mathrm{H}^{s_2}(M)$*, closed for the weak convergence in* $\mathrm{H}^{s_2}(M)$*. If* $\ell$*, a quadratic form defined on* $\mathrm{H}^{s_2}(M)$ *satisfies condition* $(\mathbf{C}_{\mathrm{H}^{s_2}})$*, namely*

$(\mathbf{C}_{\mathrm{H}^{s_2}})$ *there exists* $0 \le s_1 < s_2$ *and* $c_1 > 0$ *such that* $\ell = \ell_m + \ell_r$ *with*

$$\begin{cases} \ell_m \text{ is lower semi-continuous in } \mathrm{H}^{s_2}(M) \text{ and } \ell_m(\varphi,\varphi) \ge c_1|\varphi|^2_{\mathrm{H}^{s_2}},\ \forall \varphi \in \mathrm{C}^\infty(M), \\ \ell_r \text{ continuous in } \mathrm{H}^{s_1}(M). \end{cases}$$

*then the following propositions are equivalent:*

**(i)** $\ell(\varphi,\varphi) > 0$ *for any* $\varphi \in V \setminus \{0\}$.

**(ii)** $\exists \lambda > 0, \quad \ell(\varphi,\varphi) \ge \lambda \|\varphi\|^2_{\mathrm{H}^{s_1}}$ *for any* $\varphi \in V$.

**(iii)** $\exists \lambda > 0, \quad \ell(\varphi,\varphi) \ge \lambda \|\varphi\|^2_{\mathrm{H}^{s_2}}$ *for any* $\varphi \in V$.

**Remark 3.4** In practice, we apply this lemma to the spaces $\mathrm{H}^s(\partial\Omega)$ where $\partial\Omega$ is smooth enough, and $V$ is either $\mathrm{H}^{s_2}(\partial\Omega)$ or $T(\partial\Omega)$ defined in (24).

**Proof.** Since the implications $(iii) \implies (ii)$ and $(ii) \implies (i)$ are trivial, it suffices to prove $(i) \implies (iii)$. To that end, let $(\varphi_k)_k$ a minimizing sequence for the problem

$$\inf\left\{\ell(\varphi,\varphi),\ \varphi \in V, \|\varphi\|_{\mathrm{H}^{s_2}} = 1\right\}.$$

Up to a subsequence, $\varphi_k$ weakly converges in $\mathrm{H}^{s_2}(M)$ to some $\varphi_\infty \in V$. By the compactness of the embedding of $\mathrm{H}^{s_2}(M)$ into $\mathrm{H}^{s_1}(M)$, $\varphi_k \to \varphi_\infty$ in $\mathrm{H}^{s_1}(M)$ so that $\ell_r(\varphi_k,\varphi_k) \to \ell_r(\varphi_\infty,\varphi_\infty)$. We distinguish two cases: if $\varphi_\infty \neq 0$, $\liminf_k \ell_m(\varphi_k,\varphi_k) \ge \ell_m(\varphi_\infty,\varphi_\infty)$ by the lower semi continuity of $\ell_m$, so that $\liminf_k \ell(\varphi_k,\varphi_k) \ge \ell(\varphi_\infty,\varphi_\infty) > 0$ by assumption **(i)**. Now, if $\varphi_\infty = 0$, then as the norm $\|\cdot\|_{\mathrm{H}^{s_2}}$ is equivalent to the norm $\|\cdot\|_{\mathrm{H}^{s_1}} + |\cdot|_{\mathrm{H}^{s_2}}$, we know that $|\varphi_k|_{\mathrm{H}^{s_2}}$ is bounded from below by a positive constant, and using $(\mathbf{C}_{\mathrm{H}^{s_2}})$,

$$\liminf_k \ell(\varphi_k,\varphi_k) = \liminf_k \ell_m(\varphi_k,\varphi_k) \ge c_1 \liminf_k |\varphi_k|^2_{\mathrm{H}^{s_2}} > 0.$$

□

**Remark 3.5** The equivalence between coercivity in $\mathrm{L}^2$ and $\mathrm{H}^1$ was already known in the context of stable minimal surface it appears in the work [28] of Grosse-Brauckmann. Also in [2], the previous lemma is proven in the particular case of the functional under study (see also Section 5.1).

**Remark 3.6** When one applies this lemma to a shape hessian, assumption (i) may seem unnatural. Indeed, shape derivatives are usually defined for regular perturbations that are dense subsets of $\mathrm{H}^s(\partial\Omega)$ and one could expect to assume only $\ell(\varphi,\varphi) > 0$ for $\varphi \in V \setminus \{0\}$ smooth enough. But, in that case, the proof (nor the result) is not valid since $\varphi_\infty$ may not be smooth and therefore not admissible to test the positivity property. Therefore, the shape hessian $\ell$ has to be extended by continuity to the whole $\mathrm{H}^s(\partial\Omega)$ (see assumption (11) in Theorem 1.1 and (23) for Theorem 3.2). Notice that this extension is for free once the expression of the shape derivative has been computed as illustrated by Lemma 2.7.

We conclude this section noticing that the shape hessians of the model functionals from Section 2 satisfies $(\mathbf{C}_{\mathrm{H}^{s_2}})$:

- The perimeter satisfies $(\mathbf{C}_{\mathrm{H}^1})$ with

$$\ell_m[P](\varphi,\varphi) = \int_{\partial\Omega} |\nabla_\tau \varphi|^2 \quad \text{and} \quad \ell_r[P](\varphi,\varphi) = \int_{\partial\Omega} \left[H^2 - \|\boldsymbol{B}\|^2\right] \varphi^2 \qquad \text{(here we can choose } s_1 = 0\text{)}.$$

- The Dirichlet energy and $\lambda_1$ satisfy $(\mathbf{C}_{\mathrm{H}^{1/2}})$ (again $s_1 = 0$):

$$\ell_m[E](\varphi,\varphi) = \langle \partial_n u\varphi, \Lambda(\partial_n u\varphi)\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} \quad \text{and} \quad \ell_r[E](\varphi,\varphi) = \int_{\partial\Omega} \left[\partial_n u + \frac{1}{2}H(\partial_n u)^2\right] \varphi^2,$$
$$\ell_m[\lambda_1](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} 2w(\varphi)\ \partial_n w(\varphi) \quad \text{and} \quad \ell_r[\lambda_1](\varphi,\varphi) = \int_{\partial\Omega} H(\partial_n v)^2\varphi^2.$$

**Remark 3.7** Let us emphasize that condition $(\mathbf{C}_{\mathrm{H}^s})$ may not be valid in some interesting examples. Shape functionals used for domain reconstruction from boundary measurements provide in general non-coercive Hessians. With the examples treated in [3], [5] one can find critical shape whose hessian is positive but is not coercive (for any $\mathrm{H}^s$-norm).

More precisely, for a reconstruction function $J$ related to this kind of inverse problem (for example the least square fitting to data), the Riesz operator corresponding to the shape Hessian $\ell_2[J]$ at a critical domain is compact. This means, that one cannot expect an estimate of the kind $J(\Omega_t) - J(\Omega_0) \geq ct^2$ with a constant $c$ uniform in the deformation direction. This explains also why regularization is required in the numerical treatment of this type of problem. This fact is well-known in the inverse problem community.

There are also situations where the objective is flat up to fourth order (see [13]).

### 3.2 Proof of Theorem 1.1

Let $\Omega^*$ be a domain satisfying the assumption of Theorem 1.1. Let $\eta > 0$ and let $\Omega$ be such that $\widetilde{d}_X(\Omega,\Omega^*) < \eta$. Then, there exists $h \in X$ such that the boundary of $\Omega$ is the set $\{x+h(x)\boldsymbol{n}(x),\ x \in \partial\Omega^*\}$. Then from $(\mathbf{IT}_{\mathrm{H}^s,X})$ we have

$$J(\Omega) - J(\Omega^*) = \underbrace{\ell_1[J](\Omega^*)(h)}_{=0} + \frac{1}{2}\ell_2[J](\Omega^*)(h,h) + \omega(\|h\|_X)\|h\|^2_{\mathrm{H}^s}$$

Using $(\mathbf{C}_{\mathrm{H}^{s_2}})$, we can apply Lemma 3.3 and there is a constant $\lambda > 0$ such that

$$\ell_2[J](\Omega^*).(h,h) \geq \lambda\|h\|^2_{\mathrm{H}^{s_2}}.$$

Therefore there exists $\eta$ small enough such that if $\|h\|_X \leq \eta$, then $\omega(\|h\|_X) \leq \frac{\lambda}{4}$ and then

$$J(\Omega) - J(\Omega^*) \geq \frac{\lambda}{4}\|h\|^2_{\mathrm{H}^{s_2}}.$$

□

### 3.3 Proof of Theorem 3.2

We denote $\mu$ the Lagrange multiplier associated to $J$. Therefore we consider $J_\mu = J - \mu\mathrm{Vol}$ and $\Omega^*$ satisfies $J'_\mu(\Omega^*) = 0$.

**Step 1: Stability under volume and barycenter constraint:** Under the structural hypotheses on $\ell_2[J](\Omega^*) = \ell_m + \ell_r$ and the fact that $\ell_2[\mathrm{Vol}](\Omega^*)$ is continuous in the $\mathrm{L}^2$-norm, we can applied Lemma 3.3 to $\ell_2[J_\mu](\Omega^*)$, so there are constants $c_1, c_2, c_3$ and $c_4 > 0$ such that

$$\forall \varphi \in \mathrm{H}^{s_2}(\partial\Omega^*), \quad |\ell_m(\varphi,\varphi)| \geq c_1 |\varphi|^2_{\mathrm{H}^{s_1}} \qquad |\ell_r(\varphi,\varphi)| \leq c_2 \|\varphi\|^2_{\mathrm{H}^{s_1}}, \qquad |\ell_2[\mathrm{Vol}](\Omega^*).(\varphi,\varphi)| \leq c_3 \|\varphi\|^2_{\mathrm{L}^2}, \tag{25}$$

$$\forall \varphi \in T(\partial\Omega^*), \quad \ell_2[J - \mu\mathrm{Vol}](\Omega^*).(\varphi,\varphi) \geq c_4 \|\varphi\|^2_{\mathrm{H}^{s_2}}. \tag{26}$$

**Step 2: Stability without constraint:** In order to deal with the volume constraint and the invariance with respect to translations, we use an idea of [44, 28] by considering

$$J_{\mu,C} = J - \mu\mathrm{Vol} + C\,(\mathrm{Vol} - V_0)^2 + C\,\|\mathrm{Bar} - \mathrm{Bar}(\Omega^*)\|^2,$$

where $\mathrm{Bar}(\Omega) := \int_\Omega x$ and $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^d$. The shape $\Omega^*$ still satisfies $J'_{\mu,C}(\Omega^*) = 0$. We claim that $\Omega^*$ is a strictly stable shape for $J_{\mu,C}$ on the entire space $\mathrm{H}^{s_2}(\partial\Omega^*)$ when $C$ is big enough, that is to say for all $\varphi$ in $\mathrm{H}^{s_2} \setminus \{0\}$,

$$\ell_2[J_{\mu,C}](\Omega^*).(\varphi,\varphi) > 0. \tag{27}$$

Indeed, if it was not the case, we would have the existence of $\varphi_n \in \mathrm{H}^{s_2}(\partial\Omega^*) \setminus \{0\}$ such that

$$\ell_2[J_{\mu,n}](\Omega^*).(\varphi_n,\varphi_n) \leq 0.$$

Using (25), this leads to

$$c_1|\varphi_n|^2_{\mathrm{H}^{s_2}} - c_2\|\varphi_n\|^2_{\mathrm{H}^{s_1}} - |\mu|c_3\|\varphi_n\|^2_{\mathrm{L}^2} + 2n\left(\int_{\partial\Omega^*}\varphi_n\right)^2 + 2n\left\|\int_{\partial\Omega^*}\varphi_n x\right\|^2 \leq 0. \tag{28}$$

Assuming by homogeneity that $\|\varphi_n\|_{\mathrm{H}^{s_1}} = 1$ for every $n$, (28) implies that $(\varphi_n)_n$ is bounded in $\mathrm{H}^{s_2}$ and using the compactness of $\mathrm{H}^{s_2}(\partial\Omega^*)$ in $\mathrm{H}^{s_1}(\partial\Omega^*)$, we have, up to a subsequence, that $\varphi_n$ converges to $\varphi$, weakly in $\mathrm{H}^{s_2}$ and strongly in $\mathrm{H}^{s_1}$ and $\mathrm{L}^2$. Therefore, (28) implies first that $2n[\mathrm{Vol}'(\varphi_n)^2 + \mathrm{Bar}'(\varphi_n)^2]$ is bounded, then that $\varphi \in T(\partial\Omega^*)$, that is to say

$$\int_{\partial\Omega^*}\varphi = 0 \text{ and } \int_{\partial\Omega^*}\varphi x = 0,$$

and then the semi-lower continuity assumption in $\mathbf{(C_{H^{s_2}})}$ implies

$$\ell_2[J_\mu](\Omega^*).(\varphi,\varphi) \leq 0, \quad \text{with } \|\varphi\|_{\mathrm{H}^{s_1}} = 1$$

which contradicts (26), since $\varphi \neq 0$.

**Step 3: Stability:** It is now easy to see that $J_{\mu,C}$ satisfies both $\mathbf{(C_{H^{s_2}})}$ and $\mathbf{(IT_{H^{s_2},X})}$ at $\Omega^*$ (using that Vol and Bar satisfy $\mathbf{(IT_{H^0,W^{1,\infty}})}$, see Section 4.1), and for $C$ large enough we have (27), so applying Theorem 1.1, there exists $c > 0$ and $\eta > 0$ such that for every $\Omega = \Omega_h$ with $\widetilde{d}_X(\Omega,\Omega^*) < \eta$,

$$J_{\mu,C}(\Omega) - J_{\mu,C}(\Omega^*) \geq c\|h\|^2_{\mathrm{H}^{s_2}},$$

Writing this inequality in particular for shapes $\Omega$ having the same volume and barycenter as $\Omega^*$,

$$J(\Omega) - J(\Omega^*) \geq c\|h\|^2_{\mathrm{H}^{s_2}}.$$

We conclude using the invariance of $J$ with translations that this estimate is valid for any $\Omega$ having the same volume as $\Omega^*$ and such that $\bar{d}_X(\Omega,\Omega^*) < \eta$. □

# 4 About Condition $(\mathbf{IT}_{\mathrm{H}^s,X})$

In this section, we show that our main examples satisfy condition $(\mathbf{IT}_{\mathrm{H}^s,X})$ where $s$ is given in Section 3.1, and $X$ is hoped to be as large as possible.

Given $\Omega$ an open set and $h : \partial\Omega \to \mathbb{R}$, we recall that $\Omega_h$ is defined so that

$$\partial\Omega_h = \{x + h(x)\boldsymbol{n}(x), x \in \partial\Omega\}.$$

It may be useful to see $\Omega_h$ as a deformation with a vector field. To that end, we assume $\Omega$ of class $\mathrm{C}^2$ so that the projection $\pi_{\partial\Omega}$ on $\partial\Omega$ is well-defined and $\mathrm{C}^1$ in a neighborhood of $\partial\Omega$, and we define

$$h(x) = h(\pi_{\partial\Omega}(x)) \qquad \text{and} \qquad \boldsymbol{n}(x) = \boldsymbol{n}(\pi_{\partial\Omega}(x)),$$

in order to extend $h$ and $\boldsymbol{n}$ in a neighborhood of $\partial\Omega$ and so that it is constant in the direction given by $\boldsymbol{n}$.

Then we define $\boldsymbol{\xi}_h(x) = h(x)\boldsymbol{n}(x)$ in this neighborhood and we have $\mathrm{div}\boldsymbol{\xi}_h = \mathrm{div}(\boldsymbol{n})h$. We then extend it smoothly to $\mathbb{R}^d$, so that $\boldsymbol{\xi}_h \in \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Denoting $T_h = Id + \boldsymbol{\xi}_h$, we have $\Omega_h = T_h(\Omega)$, and $\mathrm{j}(h) = \mathcal{J}_\Omega(\boldsymbol{\xi}_h) = J(\Omega_h)$ (where $J$ is the shape functional under study). In this section, the notation $\widehat{w_h}$ stands for $w_h \circ T_h$ where $w_h$ is defined on $\Omega_h$ or $\partial\Omega_h$.

When studying condition $(\mathbf{IC}_{\mathrm{H}^s,X})$ (which implies $(\mathbf{IT}_{\mathrm{H}^s,X})$), we focus on the path $\Omega_t$ defined in (10), and we have $\Omega_t = (Id + t\boldsymbol{\xi}_h)(\Omega)$ and $j''(t) = \mathcal{J}''_\Omega(t\boldsymbol{\xi}_h).(\boldsymbol{\xi}_h, \boldsymbol{\xi}_h)$ for all $t \in [0, 1]$, where $j(t) = J(\Omega_t)$. Note that in this case we will notify the dependence of quantities with respect to $t$, but there is also a dependence in $h$ that we will not recall in order to simplify the notations : for example $\boldsymbol{n}_h$ will denote the exterior normal vector to $\Omega_h$ and $\boldsymbol{n}_t$ the normal vector to $\Omega_t$ while we should use $\boldsymbol{n}_{th}$. Also, as we chose a vector field that is constant along the normal vector in a neighborhood of $\partial\Omega$, we have

$$j''(t) = J''(\Omega_t) \cdot (\boldsymbol{\xi}_h, \boldsymbol{\xi}_h) = \mathcal{J}''_{\Omega_t}(0)(\boldsymbol{\xi}_h, \boldsymbol{\xi}_h) \tag{29}$$

if $\|h\|_\infty$ is small enough, which allows to use the results in Section 2.

## 4.1 Geometric quantities

**• The volume:**

**Proposition 4.1** *If $\Omega$ is $\mathrm{C}^2$, then* Vol *satisfies* $(\mathbf{IC}_{\mathrm{L}^2,\mathrm{W}^{1,\infty}})$ *condition at $\Omega$.*

**Remark 4.2** More generally (with a similar proof), we have that $\Omega \mapsto \int_\Omega f$ also satisfies $(\mathbf{IC}_{\mathrm{L}^2,\mathrm{W}^{1,\infty}})$ if $f \in C^1(\mathbb{R}^d)$. This is true in particular for the barycenter functional.

Before proving this result, we give a geometric Lemma, inspired by the results in [12]. We recall that $J_{\partial\Omega}(h) := \det DT_h|({}^tDT_h^{-1})\boldsymbol{n}|$ is the surface jacobian, appearing when changing variables between $\partial\Omega_h$ and $\partial\Omega$.

**Lemma 4.3** *We have the following Taylor expensions, where $\mathcal{O}$ denote a domination uniform in $x \in \partial\Omega$,*

- $J_{\partial\Omega}(h)(x) = 1 + \ell_1^J(h(x), \nabla h(x)) + \frac{1}{2}\ell_2^J(h(x), \nabla h(x)) + \mathcal{O}\left(\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}\left(|h(x)|^2 + |\nabla h(x)|^2\right)\right)$,
- $\widehat{\boldsymbol{n}}_h(x) = \boldsymbol{n}(x) + \ell_1^{\boldsymbol{n}}(h(x), \nabla h(x)) + \frac{1}{2}\ell_2^{\boldsymbol{n}}(h(x), \nabla h(x)) + \mathcal{O}\left(\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}\left(|h(x)|^2 + |\nabla h(x)|^2\right)\right)$.

*where $(\ell_1^J, (\ell_1^{\boldsymbol{n}})_{i\in[\![1,d]\!]})$, $(\ell_2^J, (\ell_2^{\boldsymbol{n}})_{i\in[\![1,d]\!]})$ are respectively linear and quadratic forms on $\mathbb{R}^{d+1}$.*

**Proof of Lemma 4.3:** The first part follows simply from the fact that $A \in M_d(\mathbb{R}) \mapsto \det(A)\left|({}^tA^{-1})\boldsymbol{n}\right|$ is smooth in a neighborhood of $Id$, and the fact that $D\boldsymbol{\xi}_h = h(D\boldsymbol{n}) + \nabla h \otimes \boldsymbol{n}$.
For the second part, we use a level-set parametrization: there exists $\phi$ of class $\mathrm{C}^2$ such that $\Omega = \{\phi < 0\}$ and $\nabla\phi$ does not vanish in a neighborhood of $\partial\Omega$, and then $\Omega = \{\phi \circ T_h^{-1} < 0\}$. Therefore

$$\widehat{\boldsymbol{n}}_h - \boldsymbol{n} = \frac{\nabla(\phi \circ T_h^{-1})}{|\nabla(\phi \circ T_h^{-1})|} \circ T_h - \frac{\nabla\phi}{|\nabla\phi|} = \frac{{}^tDT_h^{-1}.\nabla\phi}{|{}^tDT_h^{-1}.\nabla\phi|} - \frac{\nabla\phi}{|\nabla\phi|},$$

and we conclude using the smoothness of $A \mapsto {}^tA^{-1}$ and $w \in \mathbb{R}^d \mapsto \frac{w}{|w|}$ in the neighborhood of $Id$ and $\nabla\phi$ respectively. □

**Proof of Proposition 4.1:** We denote $H = \mathrm{div}(\boldsymbol{n})$ the mean curvature of $\partial\Omega$, extended to $\mathbb{R}^d$ and constant in the normal direction of $\partial\Omega$. We use (16), and the fact that $\mathrm{div}(\boldsymbol{\xi}_h) = h\mathrm{div}(\boldsymbol{n})$ (as $h$ is constant in the direction of $\boldsymbol{n}$). Therefore if $v(t) = \mathrm{Vol}(\Omega_t)$, we have (see (16) and (29)):

$$v''(t) = \int_{\partial\Omega_t} \boldsymbol{\xi}_h \cdot \boldsymbol{n}_t \mathrm{div}(\boldsymbol{\xi}_h) = \int_{\partial\Omega_t} \mathrm{div}(\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{n}_t)h^2 = \int_{\partial\Omega} H(\boldsymbol{n}\cdot\widehat{\boldsymbol{n}}_t)h^2 J_{\partial\Omega}(t).$$

With Lemma 4.3, we easily obtain

$$|v''(t) - v''(0)| \leq Ct\|h\|_{\mathrm{W}^{1,\infty}}\|h\|^2_{\mathrm{L}^2} \leq C\|h\|_{\mathrm{W}^{1,\infty}}\|h\|^2_{\mathrm{L}^2}.$$

□

**Remark 4.4** We could try a direct proof estimating

$$|\Omega_h| - |\Omega| = \int_\Omega \left(\det(Id + D\boldsymbol{\xi}_h) - 1\right),$$

but a priori this only leads to the fact that the volume satisfies $\mathbf{(IT_{H^1,W^{1,\infty}})}$. In the spirit of [34, Lemma 4.1], we could also try:

$$|\Omega| = \frac{1}{d}\int_{\partial\Omega} x\cdot\boldsymbol{n}_h = \frac{1}{d}\int_{\partial\Omega}(x + h(x)\boldsymbol{n}(x))\cdot\widehat{\boldsymbol{n}}_h J_{\partial\Omega}(h)$$

but this leads to the same issue (see also Remark 4.6).

• **The perimeter:**

**Proposition 4.5** *If $\Omega$ is $\mathrm{C}^2$, then $P$ satisfies $\mathbf{(IT_{H^1,W^{1,\infty}})}$ condition at $\Omega$.*

**Proof.** We follow exactly the second proof suggested in Remark 4.4:

$$P(\Omega_h) = \int_{\partial\Omega_h} 1 = \int_{\partial\Omega} J_{\partial\Omega}(h) = P(\Omega) + \ell_1[P](\Omega)(h) + \frac{1}{2}\ell_2[P](\Omega)(h,h) + \mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}}\|h\|^2_{\mathrm{H}^1}),$$

where we used Lemma 4.3. □

**Remark 4.6** It is interesting to compare the two strategies used for the volume and for the perimeter: indeed, for the volume we prefered to use condition **(IC)**, while a similar strategy for the perimeter, as it is done in [12] or in [2, Proof of Theorem 3.9] (but for a different path of shapes) lead to weaker results, namely $\mathbf{(IC_{H^1,C^{2,\alpha}})}$ and $\mathbf{(IC_{H^1,W^{2,p}})}$ respectively).

## 4.2 PDE energy

For PDE energies, a condition of the type $\mathbf{(IC_{H^s,X})}$ was studied first in [14] where it is proven that in dimension two the Dirichlet energy satisfy $\mathbf{(IC_{H^{1/2},C^{2,\alpha}})}$ (for a volume preserving path instead of a normal path), then a similar result is proven for general PDE functionals in any dimension in [12], either for the path (10) or a volume preserving path. More recently in [2], it was proven that the functional described in (**??**) involving the sum of the perimeter and a PDE functional (of a different kind than in [12]) satisfies $\mathbf{(IC_{H^1,W^{2,p}})}$ for $p$ large enough, also for a volume preserving path, see also Section 5.1. Finally, condition $\mathbf{(IC_{H^{1/2},C^{2,\alpha}})}$ is also established for the drag in a Stokes flow in [9]. Thanks to our method to handle the volume constraint (see Section 3.2), we only need to deal with the normal path (10).

In this section, we prove the following result, which includes the case of $\lambda_1$ (which seemed not to be handled in the literature), and we improve the space $X$ compare to [12]:

**Proposition 4.7** *Let $\Omega$ be a bounded $\mathrm{C}^3$ domain. Then $E$ and $\lambda_1$ satisfy $\mathbf{(IC_{H^{1/2},W^{2,p}})}$ for $p > d$.*

We give 4 preliminary steps to prove this result. We only give the details for $\lambda_1$; the case of $E$ is easier and the reader can follow [12] or [8, Appendix] and use the ideas below where we explain how to get $X$ to be $\mathrm{W}^{2,p}$ instead of $\mathrm{C}^{2,\alpha}$.

• Step 1: **Computing the second derivative along the path.**

Denoting $v_t$ the first normalized eigenfunction on $\Omega_t$ and applying the structure Theorem to $\lambda_1$ (Lemma 2.7) and (29), we get

$$\begin{aligned}\lambda_1''(\Omega_t).(\boldsymbol{\xi}_h,\boldsymbol{\xi}_h) &= 2\int_{\partial\Omega_t} v_t'\partial_{\boldsymbol{n}_t}v_t' + \int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\left[H_t(\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t)^2 - \boldsymbol{B}_t((\boldsymbol{\xi}_h)_{\tau_t},(\boldsymbol{\xi}_h)_{\tau_t}) + 2\nabla_{\tau_t}(\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t)(\boldsymbol{\xi}_h)_{\tau_t}\right]\\ &= \underbrace{2\int_{\partial\Omega_t} v_t'\partial_{\boldsymbol{n}_t}v_t'}_{\mathcal{T}_1(t)} + \underbrace{\int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\left[H_t\alpha_t^2 - \boldsymbol{B}_t(\beta_t,\beta_t) - 2\nabla_{\tau_t}(\alpha_t)\cdot\beta_t\right]h^2}_{\mathcal{T}_2(t)}\\ &\qquad \underbrace{-2\int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\alpha_t\left(\beta_t\cdot\nabla_{\tau_t}h\right)h}_{\mathcal{T}_3(t)}\end{aligned}$$

where

$$\alpha_t = \boldsymbol{n}_t\cdot\boldsymbol{n}, \qquad \beta_t = \alpha_t\boldsymbol{n}_t - \boldsymbol{n}.$$

• Step 2: **Geometric estimates:**

Similarly to Section 4.1, we denote $\widehat{w}_h = w_h\circ(Id+\boldsymbol{\xi}_h)$ where $w_h$ is defined on $\Omega_h$ of $\partial\Omega_h$. The following Lemma follows easily from Lemma 4.3 (see [12] for more details).

**Lemma 4.8** *There is a constant $C>0$ depending on $\Omega$ such that for all $h$ in a neighborhood of 0 in $\mathrm{W}^{2,p}(\partial\Omega)$,*

- $\|\widehat{J_{\partial\Omega}(h)}-1\|_{\mathrm{L}^\infty(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}, \quad \|\widehat{J_{\partial\Omega}(h)}-1\|_{\mathrm{W}^{1,p}(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$
- $\|\widehat{H_h}-H\|_{\mathrm{L}^p(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}, \quad \|\widehat{\boldsymbol{B}_h}-\boldsymbol{B}\|_{\mathrm{L}^p(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$
- $\|\widehat{\alpha_h}-1\|_{\mathrm{L}^\infty(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}, \quad \|\widehat{\nabla_{\tau_h}\alpha_h}\|_{\mathrm{L}^p(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$
- $\|\widehat{\beta_h}\|_{\mathrm{L}^\infty(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}, \quad \|\widehat{\beta_h}\|_{\mathrm{W}^{1,p}(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}.$

• Step 3: **Estimate of $\|\widehat{v_\theta}-v\|_{\mathrm{W}^{2,p}}$** The statement in this step is not specific to our chosen deformations $\boldsymbol{\xi}_h$ hence we present it for general deformations $\theta\in\mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, that is $\Omega_\theta=(Id+\theta)(\Omega)$, $v_\theta$ is the first Dirichlet eigenfunction on $\Omega_\theta$. Precisely, we prove

**Lemma 4.9** *Let $\Omega$ be a bounded $\mathrm{C}^3$ domain in $\mathbb{R}^d$. If $p>d$, the map $\theta\mapsto\widehat{v_\theta}$ from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ with values in $\mathrm{W}^{2,p}(\Omega)$ is analytic around 0. As a consequence, there is a neighborhood of 0 in $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ and $C$ depending on $\Omega$ only so that*

$$\|\widehat{v_\theta}-v_0\|_{\mathrm{W}^{2,p}(\Omega)} \le C\|\theta-Id\|_{\mathrm{W}^{2,p}}.$$

**Proof.** We use the same strategy as in [29, Proof of Theorem 5.7.4] and [30] but with different functional spaces: precisely, we will apply the implicit function theorem to $\mathcal{F}:\ X\times Y\times\mathbb{R}\to Z\times\mathbb{R}$ defined by

$$\mathcal{F}(\theta,v,\lambda)=\left(-\mathrm{div}A(\theta)\nabla v-\lambda J(\theta)v, \int_\Omega v^2J(\theta)-1\right) \quad \text{where} \quad \begin{cases} J(\theta)=\det(Id+D\theta),\\ A(\theta)=J(\theta)(Id+D\theta)^{-1}(Id+{}^tD\theta)^{-1},\end{cases}$$

for suitable spaces $X,Y,Z$.

By Neumann series expansion and using that $\mathrm{W}^{1,p}$ is an algebra for $p>d$, the maps $J$ and $A$ are analytic around 0 from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ into $\mathrm{W}^{1,p}(\mathbb{R}^d,\mathbb{R}^d)$. As a consequence, by Sobolev's injection, the map $\mathcal{F}$ is analytic around $(0,v_0,\lambda_0)$ from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)\times\mathrm{W}^{2,p}(\Omega)\cap H_0^1(\Omega)\times\mathbb{R}$ into $\mathrm{L}^p(\Omega)$ (here we denoted $\lambda_0:=\lambda_1(\Omega)$). One checks that $\mathcal{F}(0,v_0,\lambda_0)=(0,0)$ and that the differential

$$\partial_{w,\lambda}\mathcal{F}(0,v_0,\lambda_0).[w,\lambda]=\left((-\Delta-\lambda_0)w-\lambda v_0, 2\int_\Omega v_0w\right)$$

is an isomorphism from $\mathrm{W}^{2,p}(\Omega)\cap H_0^1(\Omega)\times\mathbb{R}$ into $\mathrm{L}^p(\Omega)\times\mathbb{R}$ (see [29, Lemma 5.7.3] for details) and the conclusion follows from the implicit function theorem. □

• **Step 4: estimation of the variation of the shape derivative of the eigenfunction:**
The objective of this step is to prove the following estimate:

**Lemma 4.10** *There is $C,\eta$ depending only on $\Omega$ such that, if $\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\leq\eta$, then*

$$\|\widehat{v_t'}-v_0'\|_{\mathrm{H}^1(\Omega)}\leq C\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\ \|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}. \tag{31}$$

This step is the most involved one when dealing with $\lambda_1$ instead of the Dirichlet Energy: the latter reduces indeed to the second step in the following proof.

**Proof.**
We recall (see (18)) that

$$\left\{\begin{array}{rcl} -\Delta v_t' &=& \lambda_1(t)v_t'+\lambda_1'(t)v_t \text{ in } \Omega_t,\\ v_t' &=& -(\partial_{\boldsymbol{n}_t}v_t)\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t \text{ on } \partial\Omega_t,\\ \displaystyle\int_\Omega v_t'v_t &=& 0.\end{array}\right.$$

*1. Splitting.* We introduce $\mathsf{H}_t$ the harmonic extension on $\Omega_t$ of $(\partial_{\boldsymbol{n}_t}v_t)\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t$. Noticing that

$$\lambda_1'(t)=-\int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t=\lambda_1(t)\langle v_t,\mathsf{H}_t\rangle$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2(\Omega_t)$, we decompose

$$v_t'=-\pi_t\mathsf{H}_t+w_t$$

where $\pi_t$ is the orthogonal projection on $E(t):=\{v_t\}^\perp$, and $w_t$ solves

$$\left\{\begin{array}{rcl} (-\Delta-\lambda_1(t))w_t &=& -\lambda_1(t)\pi_t\mathsf{H}_t \text{ in } \Omega_t,\\ w_t &=& 0 \text{ on } \partial\Omega_t,\\ \displaystyle\int_{\Omega_t}v_tw_t &=& 0.\end{array}\right.$$

We will now prove that each term of the splitting satisfies estimates like (31). *2. Estimate of the harmonic extension.* Let us define $\mathcal{L}_t=\mathrm{div}(A_t\nabla\cdot)$ where $A_t=J_t.(Id+tD\boldsymbol{\xi}_h)^{-1}(Id+t^tD\boldsymbol{\xi}_h)^{-1}$ and $J_t=\det(Id+tD\boldsymbol{\xi}_h)$, so that $\mathcal{L}_t\widehat{f_t}=\widehat{\Delta f_t}$ if $f_t$ is defined on $\Omega_t$. Then as $\Delta(\widehat{\mathsf{H}_t}-\mathsf{H}_0)=-\mathrm{div}((A_t-Id)\nabla\widehat{\mathsf{H}_t})$, from classical elliptic estimate (see [26, Corollary 8.7]), we obtain:

$$\begin{array}{rcl}\|\widehat{\mathsf{H}_t}-\mathsf{H}_0\|_{\mathrm{H}^1(\Omega)} &\leq& C\|(A_t-Id)\nabla\widehat{\mathsf{H}_t}\|_{\mathrm{L}^2(\Omega)}+C\|\widehat{\mathsf{H}_t}-\mathsf{H}_0\|_{\mathrm{H}^{1/2}(\partial\Omega)}\\ &\leq& C\|A_t-Id\|_{\mathrm{L}^\infty(\Omega)}\|\nabla\widehat{\mathsf{H}_t}\|_{\mathrm{L}^2(\Omega)}+C\left\|\left((\widehat{\partial_{\boldsymbol{n}_t}v_t})\widehat{\alpha_t}-\partial_{\boldsymbol{n}}v_0\right)h\right\|_{\mathrm{H}^{1/2}(\partial\Omega)}\\ &\leq& C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}\left(\|\nabla\widehat{\mathsf{H}_t}-\nabla\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)}+\|\nabla\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)}\right)\\ &&\qquad +C\|\widehat{(\partial_{\boldsymbol{n}_t}v_t)}\widehat{\alpha_t}-\partial_{\boldsymbol{n}}v_0\|_{\mathrm{W}^{1-1/p,p}(\partial\Omega)}\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\\ &\leq& C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{\mathsf{H}_t}-\mathsf{H}_0\|_{\mathrm{H}^1(\Omega)}+\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right).\end{array}$$

Here we used that $\|\nabla\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)}=\|\mathsf{H}_0\|_{\mathrm{H}^{1/2}(\partial\Omega)}\leq C\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$, Lemmas 4.8 and 4.9, and also an estimate of a product norm in $\mathrm{H}^{1/2}$. More precisely, from [39, Th 2 p 177], there exists $C$ such that for every functions $u,v:\mathbb{R}^n\to\mathbb{R}$:

$$\|uv\|_{\mathrm{H}^{1/2}(\mathbb{R}^n)}\leq C\|u\|_{\mathrm{W}^{s,p}(\mathbb{R}^n)}\|v\|_{\mathrm{H}^{1/2}(\mathbb{R}^n)} \tag{34}$$

if $s>n/p$. Using smooth enough maps between $\partial\Omega$ and $\mathbb{R}^{d-1}$ we obtain that these inequalities are valid (when $s>(d-1)/p$) for Sobolev spaces on $\partial\Omega$ if in addition $s\leq 1$. We apply it here to $s=1-1/p$ which is valid as $p>d$.

Equation (33) leads to the first estimate

$$\|\widehat{\mathsf{H}_t} - \mathsf{H}_0\|_{\mathrm{H}^1(\Omega)} \leq C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$$

as soon as $\|h\|_{\mathrm{W}^{2,p}} \leq 1/(2C)$.

*3. Estimates on the variation of $w_t$.* We look at the PDE satisfied by $\widehat{w_t} - w_0$:

$$\left\{\begin{array}{rcl} (-\Delta - \lambda_1(0))(\widehat{w_t} - w_0) & = & \Big[(-\Delta - \lambda_1(0)) - (-\mathcal{L}_t - \lambda_1(t))\Big](\widehat{w_t}) - \lambda_1(t)\widehat{\pi_t\mathsf{H}_t} + \lambda_1(0)\pi_0\mathsf{H}_0 \text{ in } \Omega, \\ \widehat{w_t} - w_0 & = & 0 \text{ on } \partial\Omega, \end{array}\right.$$

and we know that $(-\Delta - \lambda_1(0))$ is an isomorphism on $\{v_0\}^\perp$. Therefore

$$\|\widehat{w_t} - w_0 - \gamma_t v_0\|_{\mathrm{H}^1(\Omega)} \leq C\|(A_t - Id)\nabla\widehat{w_t}\|_{\mathrm{L}^2(\Omega)} + |\lambda_1(t) - \lambda_1(0)|\|\widehat{w_t}\|_{\mathrm{L}^2(\Omega)} + \|\lambda_1(t)\widehat{\pi_t\mathsf{H}_t} - \lambda_1(0)\pi_0\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)}$$

where $\gamma_t$ is chosen so that $\widehat{w_t} - w_0 - \gamma_t v_0 \in \{v_0\}^\perp$. From there and using the previous step, we obtain

$$\|\widehat{w_t} - w_0 - \gamma_t v_0\|_{\mathrm{H}^1(\Omega)} \leq C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{w_t}\|_{\mathrm{H}^1} + \|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right).$$

But we have:

$$|\gamma_t| = \left|\int_\Omega (\widehat{w_t} - w_0)v_0\right| = \left|\int_\Omega \widehat{w_t}\big[\widehat{v_t}J_t - v_0\big]\right| \leq C\|h\|_{\mathrm{W}^{2,p}}\|\widehat{w_t}\|_{\mathrm{L}^2(\Omega)},$$

leading to

$$\|\widehat{w_t} - w_0\|_{\mathrm{H}^1(\Omega)} \leq C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{w_t}\|_{\mathrm{H}^1} + \|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right) \leq C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{w_t} - w_0\|_{\mathrm{H}^1} + \|w_0\|_{\mathrm{H}^1} + \|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right).$$

Using now $\|w_0\|_{\mathrm{H}^1} \leq C\|\mathsf{H}_0\|_{\mathrm{L}^2} \leq C\|h\|_{\mathrm{H}^{1/2}}$ and again that $\|h\|_{\mathrm{W}^{2,p}}$ is small enough, this leads to

$$\|\widehat{w_t} - w_0\|_{\mathrm{H}^1(\Omega)} \leq C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$$

and concludes the proof of this lemma. □

**Proof of Proposition 4.7:** We deal separately with the terms of the decomposition (30):
*Estimate of $\mathcal{T}_1(t) - \mathcal{T}_1(0)$.* We first observe that

$$\mathcal{T}_1(t) = \int_{\Omega_t} |\nabla v_t'|^2 - \lambda_1(t)\int_{\Omega_t} v_t^2,$$

and also that $\|v_0'\|_{\mathrm{H}^1(\Omega)} \leq \|w_0\|_{\mathrm{H}^1(\Omega)} + \|\mathsf{H}_0\|_{\mathrm{H}^1(\Omega)} \leq C\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$. Therefore using Lemma 4.10, we get

$$\begin{array}{rcl} \left|\displaystyle\int_{\Omega_t} |\nabla v_t'|^2 - \int_{\Omega_0} |\nabla v_0'|^2\right| & = & \left|\displaystyle\int_\Omega (A_t - Id)|\nabla\widehat{v_t'}|^2 + \nabla(\widehat{v_t'} - v_0')\cdot\nabla(\widehat{v_t'} + v_0')\right| \\ & \leq & C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}. \end{array}$$

and

$$\begin{array}{rcl} \left|\lambda_1(t)\displaystyle\int_{\Omega_t} |v_t'|^2 - \lambda_1(0)\int_{\Omega_0} |v_0'|^2\right| & = & \left|(\lambda_1(t) - \lambda_1(0))\displaystyle\int_{\Omega_t} |v_t'|^2 + \lambda_1(0)\int_{\Omega_0} (J_t - 1)|\widehat{v_t'}|^2 + (\widehat{v_t'} - v_0')(\widehat{v_t'} + v_0')\right| \\ & \leq & C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}. \end{array}$$

*Estimate of $\mathcal{T}_2(t) - \mathcal{T}_2(0)$.* After change of variable, we have $\mathcal{T}_2(t) = \int_{\partial\Omega}\sigma_t h^2$ where

$$\sigma_t = (\widehat{\partial_{\boldsymbol{n}_t} v_t})^2\left[\widehat{H_t}\widehat{\alpha_t}^2 - \widehat{\boldsymbol{B}_t}(\widehat{\beta_t},\widehat{\beta_t}) - 2\widehat{\nabla_{\tau_t}(\alpha_t)}\cdot\widehat{\beta_t}\right]J_{\partial\Omega}(t)$$

and from Lemmas 4.8 and 4.9, we easily get

$$\|\sigma_t - \sigma_0\|_{\mathrm{L}^p(\partial\Omega)} \leq C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}.$$

Notice that the control holds only in $\mathrm{L}^p$ and not in $\mathrm{L}^\infty$ as in [12] or [8, Appendix], hence we do not obtain a control with the $\mathrm{L}^2$ norm of $h$. However, by Hölder inequality, it comes $|\mathcal{T}_2(t)-\mathcal{T}_2(0)| \le \|\sigma_t-\sigma_0\|_{\mathrm{L}^p}\|h\|^2_{\mathrm{L}^{\tilde p}}$ for any $\tilde p \ge 2p/(p-1)$. Since $\|h\|_{\mathrm{L}^{\tilde p}} \le C\|h\|_{\mathrm{H}^{1/2}}$ when $\tilde p < 2d/(d-1)$ by Sobolev's injection, such a $\tilde p$ can be chosen provided $p > d$. Then, it holds

$$|\mathcal{T}_2(t)-\mathcal{T}_2(0)| \le \|\sigma_t-\sigma_0\|_{\mathrm{L}^p}\|h\|^2_{\mathrm{H}^{1/2}} \le C\|h\|_{\mathrm{W}^{2,p}}\|h\|^2_{\mathrm{H}^{1/2}}.$$

*Estimate of* $\mathcal{T}_3(t)-\mathcal{T}_3(0)$. After change of variable, we have $\mathcal{T}_3(t) = \int_{\partial\Omega} \rho_t \cdot (\nabla_{\widehat{\tau_t}} h)h$ where

$$\rho_t = (\widehat{\partial_{\boldsymbol{n}_t} v_t})^2 \widehat{\alpha_t}\widehat{\beta_t} J_{\partial\Omega}(t), \qquad \text{and} \qquad \nabla_{\widehat{\tau_t}} h = \nabla h - (\nabla h \cdot \widehat{\boldsymbol{n}_t})\widehat{\boldsymbol{n}_t}$$

and we obtain (recall that $\nabla h \cdot \boldsymbol{n} = 0$):

$$\begin{aligned}
|\mathcal{T}_3(t)-\mathcal{T}_3(0)| &\le \left|\int_{\partial\Omega} \rho_t \cdot (\nabla_{\widehat{\tau_t}} h - \nabla_\tau h)\, h\right| + \left|\int_{\partial\Omega} (\rho_t-\rho_0)\cdot \nabla_\tau h)\, h\right| \\
&\le \|\nabla_{\widehat{\tau_t}} h - \nabla_\tau h\|_{\mathrm{H}^{-1/2}} \|\rho_t h\|_{\mathrm{H}^{1/2}} + \|(\rho_t-\rho_0)h\|_{\mathrm{H}^{1/2}} \|\nabla_\tau h\|_{\mathrm{H}^{-1/2}} \\
&\le \|\nabla h \cdot (\widehat{\boldsymbol{n}_t} - \boldsymbol{n})\|_{\mathrm{H}^{-1/2}} \|\rho_t h\|_{\mathrm{H}^{1/2}} + \|(\rho_t-\rho_0)h\|_{\mathrm{H}^{1/2}} \|h\|_{\mathrm{H}^{1/2}}
\end{aligned}$$

In addition to (34), we also have from [39, Th 2 p 173]:

$$\|uv\|_{\mathrm{H}^{-1/2}(\partial\Omega)} \le C\|u\|_{\mathrm{W}^{s,p}(\partial\Omega)} \|v\|_{\mathrm{H}^{-1/2}(\partial\Omega)} \tag{38}$$

if $\max\{1/2, (d-1)/p\} < s \le 1$. Using again Lemmas 4.3, 4.8 and 4.9, we get

$$\|\rho_t-\rho_0\|_{\mathrm{W}^{1-1/p,p}(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}, \qquad \|\widehat{\boldsymbol{n}_t} - \boldsymbol{n}_0\|_{\mathrm{W}^{1,p}(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$$

which combined with (37), (34) and (38), concludes the estimate of this term and hence the proof. □

# 5 Applications

## 5.1 Retrieving some examples from the literature

We review here some results from the literature for particular functionals, and explain how our main result Theorem 3.2 applies in these cases.

**Isoperimetric inequalities**

According to the previous sections, the perimeter satisfy conditions $(\mathbf{C}_{\mathrm{H}^1})$ and $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ at any smooth enough set, and in particular for the ball. Moreover, as shows Section 2.3, we have

$$\ell_1[P](B_1) = (d-1)\ell_1[\mathrm{Vol}](B_1), \qquad \text{and} \qquad \ell_2[P-(d-1)\mathrm{Vol}](B_1)(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} (k-1)(k+d-1)\ \alpha_{k,l}(\varphi)^2.$$

Moreover, $\varphi \in T(\partial B_1)$ if and only if $\alpha_{0,1}(\varphi) = \alpha_{1,i}(\varphi) = 0$ for $i \in \{1,\dots,d\}$. Therefore $B_1$ is a critical and strictly stable shape for $P$ under volume constraint, and up to translations: Theorem 3.2 applies, and we retrieve Fuglede's result from [21] about nearly spherical domains.

Recently in [34], different improved versions (even with a better distance than the Fraenkel asymmetry for $d_1$ in (2)) of the quantitative isoperimetric inequality has been achieved for the anisotropic perimeter

$$P_f(\Omega) = \int_{\partial\Omega} f(\boldsymbol{n}_{\partial\Omega})$$

where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a convex positively 1-homogeneous function, whose minimizer under volume constraint is an homothetic version of the Wulff shape $K = \{f_* < 1\}$ where $f_*$ is the gauge function of $f$. In particular in [34, Theorem 1.3 and Section 4], focused on the case where $K$ is assumed to be $\mathrm{C}^2$ and uniformly convex, a strategy based on the second variation is used: the author proves in [34, Lemma 4.1] that $P_f$ satisfies

conditions $(\mathbf{C}_{\mathrm{H}^1})$ and $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$. Therefore, this falls into the hypothesis of our Theorem 3.2, so if we prove that $K$ is critical and stricly stable (in the sense of (23)), then we obtain a Fuglede type result, see [34, Proposition 1.9] (we assumed the shape to be $C^3$, but here this can be reduced to $C^2$, as noticed in Remark 2.3). It is interesting to notice though that in order to show that $K$ is strictly stable in the sense that $\ell_2[P_f - \mu\mathrm{Vol}](K) > 0$ on $T(\partial K) \setminus \{0\}$, the author in [34] needs to use the quantitative Wulff isoperimetric inequality from [19] (obtained with optimal transport method). They apply then the regularization procedure mentioned in the introduction. Therefore, up to our knowledge, there is no proof "from scratch" of the quantitative anisotropic isoperimetric inequality using a result similar to Theorem 3.2.

**The Ohta-Kawasaki model**

In the paper [2], both steps of the strategy described in the introduction are achieved in order to deal with the following functional, formulated in the periodic sense, and which includes a non-local term:

$$J(\Omega) = P_{\mathbb{T}^N}(\Omega) + \gamma G(\Omega) \quad \text{where } G(\Omega) = \int_{\mathbb{T}^N} |\nabla w_\Omega|^2 dx$$

$$\text{and } w_\Omega \text{ solves } \begin{cases} -\Delta w_\Omega &= \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c} - m \ \text{ in } \mathbb{T}^N \\ \displaystyle\int_{\mathbb{T}^N} w_\Omega dx &= 0 \end{cases}$$

where $\mathbb{T}^N$ is the $N$-dimensional flat torus of unit volume, and $m = |\Omega| - |\Omega^c| \in (-1, 1)$ is fixed. Again, there is an invariance with translation and a volume constraint.

In order to handle the first step of the strategy, the authors in [2] prove a stability result for the $\mathrm{W}^{2,p}$-topology, for $p$ large enough. The strategy is very similar to [12], but in the framework of $\mathrm{W}^{2,p}$-spaces rather than $\mathrm{C}^{2,\alpha}$-spaces. Note that this difference in the choice of spaces is not just a detail as it is relevant for the second step of the strategy when proving stability in an $\mathrm{L}^1$-neighborhood as it is done in [2, Section 4]: their regularization procedure needs to allow discontinuity of the mean curvature, see equation (4.9) in the proof of [2, Theorem 4.3]. From the computations of [10], we obtain

$$\ell_1[G](\Omega)(\varphi) = 4\int_{\partial\Omega} w_\Omega \varphi,$$

$$\ell_2[G](\Omega)(\varphi,\varphi) = 8\int_{\mathbb{T}^N} |\nabla z_\varphi|^2 dx + 4\int_{\partial\Omega} (\partial_{\boldsymbol{n}} w_\Omega + H)\varphi^2, \text{ where } -\Delta z_\varphi = \varphi \mathcal{H}^{N-1} \lfloor \partial\Omega$$

therefore $G$ satisfies $(\mathbf{C}_{\mathrm{H}^{1/2}})$ and $J$ satisfies $(\mathbf{C}_{\mathrm{H}^1})$, the dominant term being contained in the perimeter term. As we have seen that the perimeter satisfies $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ condition, it just remains to handle functional $G$, which is proven to satisfy $(\mathbf{IC}_{\mathrm{H}^1,\mathrm{W}^{2,p}})$ for $p > d$ in [2]. Therefore Theorem 3.2 applies, and we retrieve [2, Theorem 3.9], as $\bar{d}_{\mathrm{H}^1}(\Omega^*, \Omega)$ easily dominates the Fraenkel asymmetry.

**The Faber-Krahn inequality**

In [8] (see also [25]) a quantitative version of the Faber-Krahn inequality is achieved, using again the two steps mentioned in the introduction: in order to achieve the first step, they use the Kohler-Jobin inequality ([31]), which implies that the Faber-Krahn deficit is controlled by the deficit of the Dirichlet energy $E$. However, as we show here, it is possible to achieve this step without this "trick". Indeed, we have seen that $\lambda_1$ satisfies $(\mathbf{C}_{\mathrm{H}^{1/2}})$ and $(\mathbf{IC}_{\mathrm{H}^{1/2},\mathrm{W}^{2,p}})$ for $p > d$, and for any $\varphi \in C^\infty(\partial B_1)$ such that $\int_{\partial B_1} \varphi = 0$, we have

$$\ell_1[\lambda_1](B_1) = -\gamma_d^2 \ell_1[\mathrm{Vol}](B_1), \qquad \text{and} \qquad \ell_2[\lambda_1 + \gamma_d^2 \mathrm{Vol}](B_1)(\varphi,\varphi) = 2\gamma_d^2 \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} Q_k \ \alpha_{k,l}(\varphi)^2.$$

where (using [1, 9.1.27, p 361])

$$Q_k = j_{d/2-1} \frac{J'_{k+d/2-1}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} + \frac{d}{2} = k + d - 1 - j_{d/2-1} \frac{J_{k+d/2}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} = j_{d/2-1} \frac{J_{k+d/2-2}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} - k + 1.$$

With the last formula, we easily notice that $Q_1 = 0$. The sign of $Q_k$ can be obtained using the argument of [37, 6.5 page 133] (done when $d = 2$, but as noticed in [30], valid for any $d$): indeed, their computations imply

$$j_{d/2-1}\frac{J'_{k+d/2-1}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} \geq k - d/2 - 1, \forall n \in \mathbb{N}^*,$$

which leads to

$$\forall k \geq 2, Q_k \geq k - 1.$$

Therefore Theorem 3.2 applies, and we retrieve a Faber-Krahn quantitative inequality for the $\bar{d}_{\mathrm{H}^{1/2}}$ distance in a $\mathrm{W}^{2,p}$-neighborhood of the ball.

### 5.2 Examples with competition

In this section, $B$ is a ball, $X = \mathrm{W}^{2,p}(\partial B)$ for $p > d$ and we denote for $\eta > 0$ :

$$\mathcal{V}_\eta = \{\Omega, \bar{d}_X(\Omega, B) \leq \eta \text{ and } |\Omega| = |B|\}.$$

Combining Theorem 3.2 to the computations from Section 2.1, we easily obtain the following result:

**Proposition 5.1** *There exists $\gamma_0 \in (0, \infty)$ such that for every $\gamma \in [-\gamma_0, \infty)$, there exists $\eta = \eta(\gamma) > 0$ and $c = c(\gamma) > 0$ such that for every $\Omega \in \mathcal{V}_\eta$,*

$$(P + \gamma E)(\Omega) \ \geq \ (P + \gamma E)(B) + c\bar{d}_{\mathrm{H}^1}(\Omega, B)^2, \quad (P + \gamma\lambda_1)(\Omega) \ \geq \ (P + \gamma\lambda_1)(B) + c\bar{d}_{\mathrm{H}^1}(\Omega, B)^2$$

$$(E + \gamma\lambda_1)(\Omega) \ \geq \ (E + \gamma\lambda_1)(B) + c\bar{d}_{\mathrm{H}^{1/2}}(\Omega, B)^2, \quad (\lambda_1 + \gamma E)(\Omega) \ \geq \ (\lambda_1 + \gamma E)(B) + c\bar{d}_{\mathrm{H}^{1/2}}(\Omega, B)^2.$$

**Proof of Proposition 5.1**: It suffices to prove that Theorem 3.2 can be applied to $\Omega^* = B$ and

$$J \in \{P + \gamma E, P + \gamma\lambda_1, E + \gamma\lambda_1, \lambda_1 + \gamma E)\}.$$

It is shown in Sections 3.1 and 4 that $(P, E, \lambda_1)$ satisfy $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ for suitable values of $s_2$, and with Lemmata 2.9 and 2.8 we easily check that the ball is a critical and strictly stable domain for $J$ under volume constraint and up to translations, either if $\gamma \geq 0$ or if $\gamma < 0$ is small enough. □

**Corollary 5.2** *With the same notations as in Proposition 5.1, we have, with $\eta_0 = \eta(\gamma_0)$:*

$$\forall \Omega \in \mathcal{V}_{\eta_0}, \quad \frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} \ \geq \gamma_0, \qquad \frac{P(\Omega) - P(B)}{\lambda_1(\Omega) - \lambda_1(B)} \ \geq \gamma_0$$

$$\gamma_0 \leq \frac{\lambda_1(\Omega) - \lambda_1(B)}{E(\Omega) - E(B)} \ \leq \gamma_0^{-1}.$$

**Remark 5.3** In [35], the second inequality in Corollary 5.2 is also investigated, but we provide here a uniform neighborhood so that this estimate applies. We also refer to [37] for some result of this kind.

**Remark 5.4** To the contrary to the last two-sided inequality, it is not possible to bound the first two ratio from above. Indeed, for every $\gamma \in (0, \infty)$, there exists $\Omega_\gamma = (Id + \theta_\gamma)(B)$ of class $\mathrm{C}^\infty$ such that

$$|\Omega_\gamma| = |B|, \ \ \|\theta_\gamma\|_{\mathrm{W}^{2,p}(\mathbb{R}^d)} \leq \gamma^{-1} \text{ and } \frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} \ > \gamma.$$

This is due to the fact that the functionals $P$ and $(E, \lambda_1)$ satisfy conditions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ for different values of $s_2$.

**Explicit constants**

We want to go further and compute explicit numbers $\gamma$ such that the inequalities of Proposition 5.1 holds. To simplify the expressions, we restrict ourselves to the case of the unit ball. In the first two cases, we find the optimal constant, see Remark 5.6 about the other cases.

**Proposition 5.5** *Using notations of Proposition 5.1 and $\gamma_d$ defined in* (19),

**(i)** *if $\gamma > -(d+1)d^2$, then $B_1$ is a local strict minimizer of $P+\gamma E$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (P+\gamma E)(\Omega) \ \geq \ (P+\gamma E)(B).$$

*Moreover, when $\gamma = -(d+1)d^2$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -(d+1)d^2$, the ball is a saddle shape for $P+\gamma E$.*

**(ii)** *if $\gamma > -\dfrac{d(d+1)}{2\gamma_d^2(d+j_{d/2-1}^2)}$, then $B_1$ is a local strict minimizer of $P+\gamma\lambda_1$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (P+\gamma \lambda_1)(\Omega) \ \geq \ (P+\gamma\lambda_1)(B);$$

*Moreover, when $\gamma = -\dfrac{d(d+1)}{2\gamma_d^2(d+j_{d/2-1}^2)}$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -\dfrac{d(d+1)}{2\gamma_d^2(d+j_{d/2-1}^2)}$, the ball is a saddle shape for $P+\gamma\lambda_1$.*

**(iii)** *if $\gamma > -\dfrac{1}{d^2(d+1)\gamma_d^2}$, then $B_1$ is a local strict minimizer of $E+\gamma\lambda_1$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (E+\gamma \lambda_1)(\Omega) \ \geq \ (E+\gamma\lambda_1)(B);$$

**(iv)** *if $\gamma > -\gamma_d^2 d^2$, then $B_1$ is a local strict minimizer of $\lambda_1 + \gamma E$: there exists $\eta = \eta(\gamma) > 0$ such that*

$$\forall \Omega \in \mathcal{V}_\eta, (\lambda_1+\gamma E)(\Omega) \ \geq \ (\lambda_1+\gamma E)(B).$$

**Remark 5.6** In the cases (iii) and (iv), the constants we compute are not optimal, in particular we do not claim the ball is a saddle point once we go beyond the computed value. Nevertheless computing the optimal value only requires to compute $\sup_{k\geq 2}\tau_k'$ and $\sup_{k\geq 2}\tau_k''$ (see the notations in the proof below) as it is done in the cases (i) and (ii). As it is seen in the second case handled by Nitsch in [35], these computations can be rather technical.

**Proof of Proposition 5.5:**
**(i)** We first compute the Lagrange multiplier $\mu(t)$ associated to the volume constraint at $B_1$: it is defined as $\ell_1[P+tE)+\mu(t)\mathrm{Vol}] = 0$ that is from the expression of the shape gradients of Vol, $P$ and $E$:

$$\mu(t) = \frac{1}{2d^2}\, t \ -(d-1).$$

Let us now turn our attention to hessian of the function $P+tE+\mu(t)\mathrm{Vol}$ on the balls $B_1$. As a consequence of Lemma 2.9, the shape hessian of the lagrangian $P+tE+\mu(t)\mathrm{Vol}$ at balls is

$$\ell_2[P+tE+\mu(t)\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty} c_k(t) \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = k^2 + \left[(d-2)+\frac{1}{d^2}\,t\right] \ k - \left[(d-1)+\frac{1}{d^2}\,t\right] = (k-1)\left[k+(d-1)+\frac{1}{d^2}\,t\right].$$

Therefore, the hessian of the Lagrangian $\ell_2[P+tE+\mu(t)\mathrm{Vol}](B_1)$ is coercive in $T(\partial B_1)$ if and onmly if $t$ solves the inequalities

$$k+(d-1)+\frac{1}{d^2}\,t > 0$$

for all $k \geq 2$. Of course, it suffices to solves that inequality in the special case $k = 2$ that provides $t > -(d+1)d^2$.

**(ii)** With the same notions as in (i) with $P + t\lambda_1 + \mu(t)\text{Vol}$, we obtain :

$$\mu(t) = \gamma_d^2\ t\ -(d-1), \qquad c_k(t) = k^2 + (d-2+t\gamma_d^2)k - (d-1) + t\gamma_d^2 \left[ d - 1 - j_{d/2-1} \frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})} \right].$$

We introduce the sequences $a_k = J_{k-1+d/2}(j_{d/2-1})$ and $b_k = a_{k+1}/a_k$ so that:

$$c_k(t) = k^2 + (d-2)k - (d-1) + 2t\gamma_d^2 \left[ k + d - 1 - j_{d/2-1} b_k \right].$$

For a given integer $k \geq 2$, $c_k(t) > 0$ holds when $t > \tau_k$ defined as

$$\tau_k = -\frac{(k-1)(k+d-1)}{2\gamma_d^2(k+d-1-j_{d/2-1}b_k)}.$$

In order to obtain to find the optimal value of $t$ so that these inequalities are satisfied for every $k \geq 2$, we need to compute the supremum of $\{\tau_k, k \geq 2\}$. It is proven by Nitsch in [35, p. 332, proof of Lemma 2.3] that for all $k \geq 2, \tau_k \leq \tau_2$, so the ball is strictly stable if and only if $t > \tau_2$. We describe here how one can obtain a more explicit version of $\tau_2$: from the recurrence formula for Bessel function ([1, 9.1.27, p 361])

$$(2\nu/z)J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z)$$

applied to $\nu = k - 1 + d/2$ and $z = j_{d/2-1}$, the sequences $a_k$ and $b_k$ satisfy the recurrence property

$$a_{k+1} = \frac{2(k-1)+d}{j_{d/2-1}}\, a_k - a_{k-1} \text{ and } b_{k+1} = \frac{2(k-1)+d}{j_{d/2-1}} \ -\ \frac{1}{b_k}$$

with the initial terms $a_0 = 0$ and $a_1 = J_{d/2}(j_{d/2-1})$ so that $b_1 = a_2/a_1 = d/j_{d/2-1}$ (which explains $c_1(t) = 0$ for any $t$, as known for the invariance by translations of all the involved functions). Therefore, we have:

$$b_2 = \frac{d}{j_{d/2-1}} - \frac{j_{d/2-1}}{d} = \frac{d^2 - j_{d/2-1}^2}{d j_{d/2-1}}$$

and as a consequence, we obtain that

$$\tau_2 = -\ \frac{d(d+1)}{2\gamma_d^2(d + j_{d/2-1}^2)}.$$

**(iii)** With the same notions as in (i) with $E + t\lambda_1 + \mu(t)\text{Vol}$, we obtain :

$$\mu(t) = (1/d^2) + t\gamma_d^2, \qquad c_k(t) = \left( \frac{1}{d^2} + t\gamma_d^2 \right) k - \frac{1}{d^2} + t\gamma_d^2 \left[ d - 1 - j_{d/2-1} b_k \right].$$

Again $c_1(t) = 0$ and $c_k(t) > 0$ if and only if

$$t > \tau_k' = -\frac{k-1}{d^2\gamma_d^2(k+d-1-j_{d/2-1}b_k)}.$$

Using that $b_1 \geq b_k > 0$, we obtain

$$\tau_k' < -\frac{1}{d^2\gamma_d^2}\ \frac{k-1}{k+d-1} = -\frac{1}{d^2\gamma_d^2}\ \left( 1 - \frac{d}{k+d-1} \right) \leq -\frac{1}{d^2(d+1)\gamma_d^2}.$$

Therefore, if $t > -\dfrac{1}{d^2(d+1)\gamma_d^2}$ then for any $k \geq 2$, $t > \tau_k'$, which leads to the result.

**(iv)** With the same notions as in (i) with $\lambda_1 + tE + \mu(t)\mathrm{Vol}$, we obtain :

$$\mu(t) = (t/d^2) + \gamma_d^2, \quad c_k(t) = \left(\frac{t}{d^2} + \gamma_d^2\right) k - \frac{t}{d^2} + \gamma_d^2 \left[d - 1 - j_{d/2-1} b_k\right].$$

We check $c_1(t) = 0$, and $c_k(t) > 0$ if and only if

$$t > \tau_k'' = -\gamma_d^2 d^2 \left(1 + \frac{d - j_{d/2-1} b_k}{k-1}\right).$$

Using that $b_1 \geq b_k > 0$, we obtain $\tau_k'' \leq -\gamma_d^2 d^2$, and therefore, if $t > -\gamma_d^2 d^2$ then for any $k \geq 2$, $t > \tau_k''$, which leads to the result. □

# 6 Counterexample for non smooth perturbations

We show in this section that even if the ball is a local minimum in a smooth neighborhood, it may not be a local minimum in a non-smooth neighborhood.

Consider $\Omega^* = B$ a ball of volume $V_0$. We have seen in Proposition 5.1 that there is a real number $\gamma_0 \in (0, \infty)$ such that for every $\gamma \in (-\gamma_0, \infty)$, $B$ is a stable local minimum for $P + \gamma E$.

For $\gamma \geq 0$ this is not surprising: since the ball minimizes $E$ among sets of given volume, it is enough to prove that the ball is a stable minimizer for the perimeter, which goes back to Fuglede [21]. Moreover, it has been proven that $B$ is an $\mathrm{L}^1$-stable minimizer of the perimeter in a $\mathrm{L}^1$-neighborhood of the ball, that is to say there exists $\eta > 0$ such that

$$\forall\, \Omega \text{ such that } |\Omega \Delta B| \leq \eta, |\Omega| = |B|, \quad P(\Omega) - P(B) \geq c|\Omega \Delta B|^2$$

where we assume the barycenter of $\Omega$ to be the same as the one of $B$ (actually this result is no longer local, see [23]). Therefore a similar inequality is valid for $P + \gamma E$ if $\gamma \geq 0$.

However, *for $\gamma < 0$, the fact that the ball is a local minimizer is no longer trivial*: there is a competition between the minimization of the perimeter and maximization the Dirichlet energy. If the coefficient in $E$ is small enough, our result shows that $B$ is still a local minimizer in a $\mathrm{W}^{2,p}$-neighborhood. Nevertheless, in that case $B$ is no longer a local minimizer in a $\mathrm{L}^1$-neighborhood :

**Proposition 6.1** *Let $B$ be a ball. For every $\gamma < 0$ and any $\varepsilon > 0$ one can find $\Omega_\varepsilon$ such that*

$$|\Omega_\varepsilon \Delta B| < \varepsilon, \quad |\Omega_\varepsilon| = |B|, \quad \textit{and} \quad (P + \gamma E)(\Omega_\varepsilon) < (P + \gamma E)(B).$$

To prove this result, we use the idea of topological derivative : it is well known that if one consider a small hole of size $\varepsilon$ in the interior of a fixed shape, the energy will change at order $\varepsilon^{d-2}$ if $d \geq 3$ and $1/\log(\varepsilon)$ if $d = 2$, which is strictly bigger than the change of perimeter which is of order $\varepsilon^{d-1}$, and therefore will strictly decrease the energy $P + \gamma E$ when $\gamma < 0$. For the sake of completeness, we provide a proof of this fact for a centered hole.

**Proof.** We can assume without loss of generality (using translation and scaling properties) that $B = B_1$ is the centered ball of radius 1, and we define $\Omega_\varepsilon = B_1 \setminus B(0, \varepsilon)$. Using that $\Delta u = \partial_{rr} u + \frac{d-1}{r} \partial_r u$ when $u$ is radial, the state function is:

$$u_{\Omega_\varepsilon}(r) = \frac{(\varepsilon^{d-2} - \varepsilon^d) r^{2-d} + \varepsilon^d - 1}{2d(\varepsilon^{d-2} - 1)} - \frac{r^2}{2d}, \quad \text{if } d \geq 3$$

$$u_{\Omega_\varepsilon}(r) = \frac{1 - \varepsilon^2}{-4\log(\varepsilon)} \log(r) + \frac{1 - r^2}{4}, \quad \text{if } d = 2$$

and therefore

$$\begin{aligned} \text{if } d \geq 3, \quad E(\Omega_\varepsilon) &= -\frac{1}{2} \int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{d(1-\varepsilon^2)^2 \varepsilon^{d-2} - 2(1-\varepsilon^d)^2}{8d^2(1-\varepsilon^{d-2})} + \frac{1 - \varepsilon^{d+2}}{4d(d+2)}\right] P(B_1) \\ &= \left[-\frac{1}{2d^2(d+2)} + \frac{d-2}{8d^2} \varepsilon^{d-2} + o(\varepsilon^{d-2})\right] P(B_1), \end{aligned}$$

$$\text{if } d=2,\quad E(\Omega_\varepsilon) = -\frac{1}{2}\int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{(1-\varepsilon^2)}{-8\log(\varepsilon)}(1-\varepsilon^2(1-2\log(\varepsilon))) - \frac{1}{16}(1-\varepsilon^2+\frac{\varepsilon^4}{2})\right] P(B_1)$$
$$= \left[-\frac{1}{16} - \frac{1}{8\log(\varepsilon)} + o\left(\frac{1}{\log(\varepsilon)}\right)\right] P(B_1).$$

We now define $\widetilde{\Omega_\varepsilon} = \mu_\varepsilon \Omega_\varepsilon$ where $\mu_\varepsilon = (1-\varepsilon^d)^{-1/d}$ so that

$$|\widetilde{\Omega_\varepsilon}| = |B_1|, \quad P(\widetilde{\Omega_\varepsilon}) - P(B_1) = \left[\mu_\varepsilon^{d-1}(1+\varepsilon^{d-1}) - 1\right] P(B_1) \sim_{\varepsilon\to 0} \varepsilon^{d-1} P(B_1)$$

$$E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon\to 0} \frac{(d-2)P(B_1)}{8d^2}\varepsilon^{d-2} > 0, \quad \text{if } d \geq 3, \quad E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon\to 0} \frac{P(B_1)}{-8\log(\varepsilon)} > 0, \quad \text{if } d = 2$$

so that in both cases, for any negative $\gamma$, $(P+\gamma E)(\Omega_\varepsilon) - (P+\gamma E)(B_1) < 0$ for small $\varepsilon$. □

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