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Stability in shape optimization with second variation

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Abstract

We are interested in the question of stability in the field of shape optimization, with focus on the strategy using second order shape derivative. More precisely, we identify structural hypotheses on the hessian of the considered shape function, so that critical stable domains (i.e. such that the first order derivative vanishes and the second order one is positive) are local minima for smooth perturbations; as we are in an infinite dimensional framework, and that in most applications there is a norm-discrepancy phenomenon, this type of result require a lot of work. We show that these hypotheses are satisfied by classical functionals, involving the perimeter, the Dirichlet energy or the first Laplace-Dirichlet eigenvalue. We also explain how we can easily deal with constraints and/or invariance of the functionals. As an application, we retrieve or improve previous results from the existing literature, and provide new local stability results. We finally test the sharpness of our results by showing that the local minimality is in general not valid for non-smooth perturbations.

2000MSC : 49K20, 49Q10.

Keywords : isoperimetric inequalities, shape optimization, second order sensitivity, stability in shape optimization.

1 Introduction

1.1 Motivation and literature

In this paper, we are interested in the question of stability in the field of shape optimization. More precisely, given $J : \mathcal{A} \to \mathbb{R}$ defined on $\mathcal{A} \subset \{\Omega \text{ smooth enough open sets in } \mathbb{R}^d\}$, we consider the optimization problem

$$\min \{J(\Omega), \quad \Omega \in \mathcal{A}\}, \tag{1}$$

and we ask the following question:

if $\Omega^* \in \mathcal{A}$ is a critical domain satisfying a stability condition (that is to say a strict second order optimality condition), can we conclude that Ω^* is a strict local minimum for (1) *in the sense that*

$$J(\Omega) - J(\Omega^*) \geq c d_1(\Omega, \Omega^*)^2, \qquad \text{for every } \Omega \in \mathcal{V}(\Omega^*) \tag{2}$$

where $c \in (0, \infty)$, d_1 is a distance among sets, and $\mathcal{V}(\Omega^*) = \{\Omega \in \mathcal{A}, d_2(\Omega, \Omega^*) < \eta\}$ is a neighborhood of Ω^*, relying on a (possibly different) distance d_2?

Note first that the word distance is used here and in the rest of the paper as an intuitive notion, asserting that Ω is far or close from the fixed shape Ω^*, and does not refer in general to the formal mathematical notion of distance. Note also that in the case where solving (1) is still an open problem, obtaining (2) with $c = 0$ is already an interesting achievment, nevertheless the method used in this paper will always provide a result with $c > 0$.

During the last decade, starting with [20], this type of question gained interest in the community of isoperimetric inequalities and shape optimization, in particular three main methods were developed in a quite extensive literature, in order to get a stability result of the form (2) for the most classical problems (1): symmetrization technique, mass transportation approach, second order shape derivative approach. In this paper, we focus on the third strategy, which recently received even more attention as in some examples, the other techniques could not be applied, see for example [7].

One specific difficulty for this strategy is that differential calculus within shapes is available only for rather smooth deformations of the initial shape . Nevertheless, as it is shown for example in [2], the strategy can also provide results for very weak distances (as the Fraenkel asymmetry which can be seen as the L^1-'distance' to the ball, up to translations, see [2] for example), when it is combined with a regularization procedure. For the

perimeter functional, the stability result for smooth perturbations goes back to [19], and the regularization step is inspired by results in [40, 31], though the complete result was achieved in [11]. These two steps rely on very different arguments.

The aim of this paper is to describe a general framework so that the first step of the above strategy applies: while this has been done in a few places in the literature, every time specifically for the functional that was under study, we aim at giving some general statements, and then show that these statements both apply to the examples already handled in the literature, and also to new examples. Despite getting a wider degree of generality, we also simplify many proofs and strategies found in the previous literature, as we describe in the rest of this introduction. We also show that the second step of the above strategy does not work with a similar degree of generality.

In order to tackle this question, we have to face two main difficulties:

- The first difficulty for this strategy is to define a suitable framework of differential calculus within shapes. This can be done for example with the notion of shape derivatives, but one can not expect positivity of the second order derivative (see Section 2 for more details). We can restrict to normal deformations, but this leads to difficulties when dealing with the next point, see the proof of Theorem 1.4.
- The second difficulty is the classical two discrepancy phenomenon in optimization in infinite dimension. The studied function J is twice differentiable with respect to one norm but the coercivity inequality $J''(u)(h,h) \geq C\|h\|_w^2$ holds only for a strictly weaker norm. Hence the general theorem on sufficient conditions for the local optimality does not apply. This has been studied for example in [27], and mainly applied in the framework of optimal control, see for example [30, 8]. *We present an adaptation of these results to the context of shape optimization.*

  In the context of shapes, this norm-discrepancy issue was noticed for the perimeter functional by Fuglede in [19] (studying the case of the euclidian ball) and [24, Proof of Theorem 6], [6, Equation 3.23], [40, Equation (1)] in the more general framework of constant mean curvature surfaces.Various geometric examples have been handled in the literature since these first examples, see [15, 17, 5, 32]. In the specific context of shape optimization involving PDE, the issue was first overcome in [14, 12]. More recently a very similar approach can be found in [2], see also Section 5.1.

1.2 Main results

1.2.1 Stability results

Our first results provide an answer to the main goal of the paper: it gives the suitable assumptions on the functional so that linear stability implies non-linear stability. In other words if $\Omega^* \in \mathcal{A}$ is a critical domain satisfying a strict second order optimality condition, then Ω^* is a strict local minimum for (1). Before giving the main statement, we precise the notations and the assumptions.

Definition of derivatives: for the sake of simplicity, we define everything here using normal deformations, which is enough to give the main statements. We refer to Section 2 for more details. Assuming that Ω is C^1 and $\boldsymbol{n} = \boldsymbol{n}_{\partial\Omega}$ is its outer unit normal vector we can consider "normal graphs" on $\partial\Omega$, that is Ω_h such that

$$\partial\Omega_h = \{x + h(x)\boldsymbol{n}(x), x \in \partial\Omega\},$$

where $h \in X$, and X is a Banach space of scalar functions on $\partial\Omega$. If J is a shape functional, then we can define $\mathrm{j}_\Omega(h) = J(\Omega_h)$ for $h \in X$ close to 0, and if j_Ω is twice Fréchet differentiable around 0 in X, we can define

$$J'(\Omega).h = \mathrm{j}'_\Omega(0).h \quad \text{and} \quad J''(\Omega).(h,h) = \mathrm{j}''_\Omega(0).(h,h).$$

The first assumption will help to deal with the coercivity of the second derivative, and also to identify which distance d_1 may be expected in (2). To that end, we formulate

Assumption ($\mathbf{C}_{\mathrm{H}^{s_2}}$): for $s_2 \in (0,1]$, we say that the bilinear form ℓ acting on $\mathrm{C}^\infty(\partial\Omega)$ satisfies condition ($\mathbf{C}_{\mathrm{H}^{s_2}}$) (and by extension we say that J satisfies the condition at Ω if $\mathrm{j}''_\Omega(0)$ does) if:

($\mathbf{C}_{\mathrm{H}^{s_2}}$) there exists $s_1 \in [0, s_2)$ and $c_1 > 0$ such that $\ell = \ell_m + \ell_r$ with

$$\begin{cases} \ell_m \text{ is lower semi-continuous in } \mathrm{H}^{s_2}(\partial\Omega) \text{ and } \ell_m(\varphi,\varphi) \geq c_1|\varphi|^2_{\mathrm{H}^{s_2}(\partial\Omega)}, \ \forall\varphi \in \mathrm{C}^\infty(\partial\Omega), \\ \ell_r \text{ continuous in } \mathrm{H}^{s_1}(\partial\Omega). \end{cases}$$

where $|\cdot|_{\mathrm{H}^{s_2}(\partial\Omega)}$ denote the $\mathrm{H}^{s_2}(\partial\Omega)$ semi-norm. In that case, ℓ is extended by density to the space $\mathrm{H}^{s_2}(\partial\Omega)$.

Second, as we face the two norm discrepancy problem, we need to precise the estimation of the Taylor reminder. This is the goal of the following improved taylor condition:

Assumption ($\mathbf{IT}_{\mathrm{H}^s,X}$): given Ω, $s \in [0,1]$ and $X \subset W^{1,\infty}(\partial\Omega)$ a Banach space, and assuming that j_Ω is twice Fréchet differentiable at 0 in X, we say that J satisfies condition $(\mathbf{IT}_{\mathrm{H}^s,X})$ at Ω if:

$(\mathbf{IT}_{\mathrm{H}^s,X})$ there exist $\eta > 0$ and a modulus of continuity ω such that for every domain Ω_h with $\|h\|_X \leq \eta$,

$$\left| J(\Omega_h) - J(\Omega) - J'(\Omega).h - \frac{1}{2}J''(\Omega).(h,h) \right| \leq \omega(\|h\|_X)\|h\|^2_{\mathrm{H}^s(\partial\Omega)}.$$

We are now in position to give our first stability result in the framework of shape optimization:

Theorem 1.1 *Let Ω^* be a domain of class C^1, and J a shape functional such that j_Ω is twice Fréchet differentiable at 0 in a Banach space X such that $\mathrm{C}^\infty(\partial\Omega^*) \subset X \subset \mathrm{W}^{1,\infty}(\partial\Omega^*)$. We assume that J satisfies $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ at Ω^* for some $s_2 \in (0,1]$. Then if Ω^* is a critical and strictly stable shape for J, that is to say[1]:*

$$J'(\Omega) = 0, \qquad \text{and } J''(\Omega) > 0 \text{ on } \mathrm{H}^{s_2}(\partial\Omega^*) \setminus \{0\}, \tag{3}$$

then there exist $\eta > 0$ and $c = c(\eta) > 0$ such that

$$\forall\, \Omega = \Omega^*_h \text{ with } \|h\|_X \leq \eta, \quad J(\Omega) \geq J(\Omega^*) + c\|h\|^2_{\mathrm{H}^{s_2}(\partial\Omega^*)}.$$

The proof of the result is given in Section 3.2.

Remark 1.2 Usually, when dealing with sufficient condition for local optimality in an infinite dimensional setting, (3) is replaced by a coercivity assumption. In fact we will prove in Section 3.1 that both conditions are equivalent under condition $(\mathbf{C}_{\mathrm{H}^{s_2}})$. In practice, this improvment is rather secondary as in most applications, we directly prove coercivity, see Section 5. Our main interest in this new formulation is that it is an easy way to identify the value of s_2, which does not require to diagonalize the second order derivative.

Constraints and invariance: In many shape optimization problems, we have to handle two extra difficulties: the functional is translation invariant, and there is a volume constraint. Therefore one cannot expect (3) to be satisfied. Here is the statement adapted to this framework:

Theorem 1.3 *Let Ω^* of class C^1, and J a shape functional, translation invariant and such that j_Ω is twice Fréchet differentiable at 0 in a Banach space X such that $\mathrm{C}^\infty(\partial\Omega^*) \subset X \subset \mathrm{W}^{1,\infty}(\partial\Omega^*)$. We assume:*

- **Structural hypotheses:** *there exists $s_2 \in (0,1]$ and X a Banach space with $\mathrm{C}^\infty(\mathbb{R}^d) \subset X \subset \mathrm{W}^{1,\infty}(\mathbb{R}^d)$ such that J satisfies $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ at Ω^*,*
- **Necessary optimality conditions:**
  - *Ω^* is a critical shape under volume constraint for J, that is to say there exists $\mu \in \mathbb{R}$ a Lagrange multiplier such that*
  $$J'(\Omega^*) - \mu \mathrm{Vol}'(\Omega^*) = 0,$$
  - *Ω^* is a strictly stable shape for J under volume constraint and up to translations, that is to say*
  $$\forall h \in T(\partial\Omega^*) \setminus \{0\},\ ((J - \mu\mathrm{Vol})'(\Omega^*).(h,h) > 0 \tag{4}$$
  where
  $$T(\partial\Omega^*) := \left\{ h \in \mathrm{H}^s(\partial\Omega^*),\ \int_{\partial\Omega^*} h = 0,\ \int_{\partial\Omega^*} h\overrightarrow{x} = \overrightarrow{0} \right\}, \tag{5}$$

Then there exists $\eta > 0$ and $c = c(\eta) > 0$ such that:

$$\forall\, \Omega = \Omega^*_h \text{ such that } \|h\|_X \leq \eta \text{ and } |\Omega| = |\Omega^*|, \quad J(\Omega) \geq J(\Omega^*) + c d_X(\Omega, \Omega^*)^2,$$

where

$$d_X(\Omega, \Omega^*) = \inf\{\|g\|_{\mathrm{H}^{s_2}(\partial\Omega^*)}, g \text{ such that } \exists \tau \in \mathbb{R}^d,\ \Omega + \tau = \Omega^*_g\} \tag{6}$$

[1] here $J''(\Omega)$ is a quadratic form, so $J''(\Omega^*) > 0$ on $X \setminus \{0\}$ means $J''(\Omega^*)(h,h) > 0$ for any $h \in X \setminus \{0\}$

The proof is given in Section 3.3. In some particular cases for the functional J, similar results in were already obtained: in [14, 12] the authors carefully handle the volume constraint by building a path preserving the volume and being almost normal, and prove that an estimate like $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ is valid for this more involved path. In [2] a very similar approach is given, and they also handle the translation-invariance (which is not there in the example of [14, 12]) which implies a lot a technicalities.

We drastically simplify the presentation of [14, 12, 2] by using an exact penalization method. More precisely we prove that under the assumptions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ *,* (4) *implies the unconstrained condition* (3) *when* J *is replaced by*

$$J_{\mu,C} = J - \mu \mathrm{Vol} + C\left(\mathrm{Vol} - V_0\right)^2 + C\left\|\mathrm{Bar} - \mathrm{Bar}(\Omega^*)\right\|^2,$$

where $C \in (0,\infty)$ *is large enough, and* Bar *is the barycenter functional. We then apply Theorem 1.1 to* $J_{\mu,C}$ *which implies the constrained local minimality. It is clear, looking at the proofs of our results, that the situation we describe in this paper is general and can be applied to other constraints or invariance.*

Theorem 1.1 has value only if one provides explainations on how to show assumptions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^s,X})$ on concrete examples: Theorem 1.4 below is dedicated to this issue, see also Section 2.2 and Section 4 for several other examples.

1.2.2 Condition $(\mathbf{IT}_{\mathrm{H}^s,X})$ for Dirichlet energy and first eigenvalue in Sobolev spaces

The situation of a PDE functional is much more involved than for geometric functionals, as it is much harder to write the remainder term in order to show condition $(\mathbf{IT}_{\mathrm{H}^s,X})$ for suitable spaces.

In this case, it is more convenient to define a slightly different condition: given Ω, $s \in [0,1]$ and $X \subset W^{1,\infty}(\partial\Omega)$ a Banach space, assuming that j_Ω is C^2 in a neighborhood of 0 in X, we say that J satisfies condition $(\mathbf{IC}_{\mathrm{H}^s,X})$ (for "improved continuity" in h) at Ω if:

$(\mathbf{IC}_{\mathrm{H}^s,X})$ there exist $\eta > 0$ and a modulus of continuity ω such that for every domain Ω_h with $\|h\|_X \leq \eta$, and all $t \in [0,1]$:

$$|j''(t) - j''(0)| \leq \omega(\|h\|_X)\|h\|^2_{\mathrm{H}^s},$$

where $j : t \in [0,1] \mapsto J(\Omega_t)$ for the path $(\Omega_t)_{t\in[0,1]}$ connecting Ω to Ω_h, and defined through its boundary

$$\partial\Omega_t = \{x + \ th(x)\boldsymbol{n}(x),\ x \in \partial\Omega\}. \tag{7}$$

Using the Taylor formula with integral remainder:

$$J(\Omega_h) - J(\Omega) = J'(\Omega).h + \frac{1}{2}J''(\Omega).(h,h) + \int_0^1 [j''(t) - j''(0)](1-t)dt,$$

it is easy to see that condition $(\mathbf{IC}_{\mathrm{H}^s,X})$ implies $(\mathbf{IT}_{\mathrm{H}^s,X})$.

We now recall the definition of two classical PDE functionals, E the Dirichlet energy and λ_1 the first Dirichlet-eigenvalue:

$$E(\Omega) = \min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega u,\ \ u \in \mathrm{H}^1_0(\Omega)\right\},\quad \lambda_1(\Omega) = \min\left\{\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2},\ \ u \in \mathrm{H}^1_0(\Omega)\right\} \tag{8}$$

In this paper, we prove the new following result:

Theorem 1.4 *Let* Ω *be a bounded* C^3 *domain. Then* E *and* λ_1 *satisfy* $(\mathbf{IC}_{\mathrm{H}^{1/2},\mathrm{W}^{2,p}})$ *for* $p > d$.

This result is an improvement of the previous litterature in several ways: first in [14, 12] the authors prove $(\mathbf{IC}_{\mathrm{H}^{1/2},\mathrm{C}^{2,\alpha}})$ for functionals similar to E, which is weaker. In [2] the authors obtain a condition similar to $(\mathbf{IC}_{\mathrm{H}^{1/2},\mathrm{W}^{2,p}})$ for $p > d$, but for a PDE functional which provides more regularity (see (30)). Note that this improvement about spaces is not just a technical issue, as in [2] the choice of $\mathrm{W}^{2,p}$ rather than $\mathrm{C}^{2,\alpha}$ is relevant for the second step of the strategy when proving stability in an L^1-neighborhood ([2, Section 4]): indeed their regularization procedure needs to allow discontinuities of the mean curvature, see equation (4.9) in the proof of [2, Theorem 4.3]. Finally, as far as we know, the case of λ_1 was not known in the literature.

1.3 Old and new applications

In order to justify the interest of our general statements, we provide several examples of functionals for which Theorems 1.1 or 1.3 apply. We give here a short list, see Section 5 for more details.

- First, we retrieve with our results classical statements already existing in the literature: this relies on the computation of the first and second derivatives of the functionals, and the fact that they satisfy conditions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ (for suitable s_2 and X). This includes the examples of [14, 12, 2, 7]. We believe that despite the degree of generality of our approach, the proofs are less technical and more straightforward.
- Second, we apply our result to cases where only linear stability was studied: this includes the result in [33] (see Proposition 5.1).
- We finally provide new examples, which come with minor cost thanks to our results. One generic example we have in mind is the following: if $\Omega^* = B$ is a ball of volume $V_0 \in (0, \infty)$, then the conditions of Theorem 1.3 are fulfilled for the functional $J = P + \gamma E$ (P is the perimeter and E is the Dirichlet energy) when $\gamma \geq \gamma_0$ and $\gamma_0 \in (-\infty, 0)$ (whose optimal value is given in Proposition 5.5), and we can conclude from our strategy that the ball is a local minimizer (in a $\mathrm{W}^{2,p}$ neighborhood for $p > d$) of the following optimization problem
$$\min \{P(\Omega) + \gamma E(\Omega), \quad |\Omega| = V_0\}. \tag{9}$$
For $\gamma \geq 0$ this result is not surprising, since the ball minimizes both the perimeter and the Dirichlet energy. But this result is new and surprising when γ is nonpositive: there is a competition between minimizing the perimeter and maximizing the Dirichlet energy. Another way to state the result is to say that
$$\frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} \geq |\gamma_0|, \qquad \forall \Omega \in \mathcal{V}(B),$$
where $\mathcal{V}(B) = \{\Omega = B_h, |\Omega| = |B| \text{ and } \|h\|_{\mathrm{W}^{2,p}} < \eta\}$, for some $\eta > 0$.

  For a problem related to (9) when $\gamma < 0$, see also [23]. We also notice that local optimality of the ball is no longer valid when one consider a neighborhood of Ω^* for a weak distance, for example the Frankel asymmetry, see Section 6. Especially it means that the second step of the strategy described page 1 does not apply to (9) if $\gamma < 0$, despite the fact that sets are minimizing the perimeter, and shows in what way the two steps of this strategy have different degree of generality.

  In addition to this example, we obtain several new local isoperimetric inequalities, see **Proposition 5.1** *in Section 5.2.*

In Section 2, we show a new proof of the Structure Theorem for second order shape derivatives. We also recall the classical examples of second order shape derivatives, noticing in particular in which norms they are continuous (which leads to the value of s_2 from assumption $(\mathbf{C}_{\mathrm{H}^{s_2}})$), and focus on the case of the ball for which we diagonalize the shape hessians (which leads to the classical stability properties of the ball for these functionals). In Section 3 we state the version of Theorem 1.1 adapted to the constrained/invariant case, we discuss the coercivity assumptions proving Lemma 3.1, and we prove Theorems 1.1 and 1.3. In Section 4 we discuss assumption $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$, in particular we recall and improve existing results, and show Theorem 1.4. In Section 5, we explain how our general results allow to retrieve known results, and then provide some new local isoperimetric inequalities, see Proposition 5.1. All these applications are simple corollaries of our main results, combined with the computations reminded in Section 2. In the last Section, we show that similar results in non-smooth neighborhoods cannot be achieved with the same degree of generality.

2 On Second order shape derivatives.

As for all the examples of this paper, we assume J is a shape functional such that $\theta \in \Theta \mapsto J((Id + \theta)(\Omega))$ is of class C^2 in a neighborhood of 0 in $\Theta = \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. This simplifies the presentation, though similar proofs can be adapted to other functional spaces, see Remark 2.4.

2.1 Structure Theorem

It is well-known since Hadamard's work that the shape gradient is a distribution supported on the moving boundary and acting on the normal component of the deformation field. The second order shape derivative also has a specific structure as stated by A. Novruzi and M. Pierre in [34]. We quote their result, and provide a new proof that we think more natural. The proof can be skipped in a first lecture.

Theorem 2.1 (Structure Theorem of first and second shape derivatives) *Let $\Theta = \mathrm{W}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, Ω an open bounded domain of $\mathbb{R}^d$ and J a real-valued shape function defined on $\mathcal{V}(\Omega) = \{(Id+\theta)(\Omega), \|\theta\|_\Theta < 1\}$. Let us define the function $\mathcal{J}_\Omega$ on $\{\theta \in \Theta, \|\theta\|_\Theta < 1\}$ by*

$$\mathcal{J}_\Omega(\theta) = J[(Id+\theta)(\Omega)].$$

(i) *If $\mathcal{J}_\Omega$ is differentiable at 0 and Ω is C^2, then there exists a continuous linear form ℓ_1 on $\mathrm{C}^1(\partial\Omega)$ such that $\mathcal{J}'_\Omega(0)\xi = \ell_1(\xi_{|\partial\Omega} \cdot \boldsymbol{n})$ for all $\xi \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, where $\boldsymbol{n}$ denotes the unit exterior normal vector on $\partial\Omega$.*

(ii) *If moreover $\mathcal{J}_\Omega$ is twice differentiable at 0 and Ω is C^3, then there exists a continuous symmetric bilinear form ℓ_2 on $\mathrm{C}^2(\partial\Omega) \times \mathrm{C}^2(\partial\Omega)$ such that for all $(\xi, \zeta) \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)^2$*

$$\mathcal{J}''_\Omega(0)(\xi,\zeta) = \ell_2(\xi\cdot\boldsymbol{n}, \zeta\cdot\boldsymbol{n}) + \ell_1(\boldsymbol{B}(\zeta_\tau,\xi_\tau) - \nabla_\tau(\zeta\cdot\boldsymbol{n})\cdot\xi_\tau - \nabla_\tau(\xi\cdot\boldsymbol{n})\cdot\zeta_\tau),$$

where ∇_τ is the tangential gradient, ξ_τ and ζ_τ stands for the tangential components of ξ and ζ, and $\boldsymbol{B} = D_\tau \boldsymbol{n}$ is the second fondamental form of $\partial\Omega$.

With respect to this work, it is important to notice that at a critical domain for J, the shape hessian is reduced to ℓ_2 and hence does not see the tangential components of the deformation fields.

Remark 2.2 The requirement that Ω is bounded is made only to simplify the presentation: the result remains valid replacing $\mathrm{C}^1(\partial\Omega)$ with $\mathrm{C}^1_c(\partial\Omega)$ and localizing the test functions.

Remark 2.3 As noticed in [25, p. 225], with this degree of generality, the regularity assumption on Ω are sharp. We could indeed wonder if ℓ_1 can be extended as a continuous linear form on $\mathrm{C}^0(\partial\Omega)$; this is not true in general if Ω is only assumed to be C^1, as the example of the perimeter shows (it would mean that the mean curvature is a Radon measure, which is not true for a C^1 domain). Moreover, our strategy provides ℓ_2 being continuous for the $\mathrm{C}^2(\partial\Omega)$-norm, while [34] gives a better result with ℓ_2 being continuous for the $\mathrm{C}^1(\partial\Omega)$-norm. However, if we assume that ℓ_1 can be extended as a continuous linear form on $\mathrm{C}^0(\partial\Omega)$, then point (**ii**) is valid assuming Ω of class C^2 only, and ℓ_2 is then continuous for the C^1-norm; it is easy to see how the proof adapts to this case, and we retrieve then an optimal result, see also [34, Remark 2.8, Corollary 2.9], .

Remark 2.4 Compare to the result in [34], we restricted ourself to the space $\Theta = \mathrm{W}^{1,\infty}$ (or similarly $\mathrm{C}^{1,\infty} := \mathrm{W}^{1,\infty} \cap \mathrm{C}^1$, see the proof below), as all the functionals of this paper are differentiable in this space. Of course, the same proof can be adapted to spaces like $\mathrm{W}^{k,\infty}$ for $k \geq 2$, which is important to handle higher order geometric or PDE functional.

Remark 2.5 When $\xi = \zeta$, we get

$$\mathcal{J}''_\Omega(0).(\xi,\xi) = \ell_2(\xi\cdot\boldsymbol{n}, \xi\cdot\boldsymbol{n}) + \ell_1(Z_\xi), \text{ where } Z_\xi = \boldsymbol{B}(\xi_\tau,\xi_\tau) - 2\nabla_\tau(\xi\cdot\boldsymbol{n})\cdot\xi_\tau.$$

As noticed in [2, Equation (7.5)], the term Z_ξ can have be written in a different way:

$$Z_\xi = (\xi\cdot\boldsymbol{n})\mathrm{div}(\xi) - \mathrm{div}_\tau(\xi_\tau(\xi\cdot\boldsymbol{n})) - H(\xi.\boldsymbol{n})^2.$$

The advantage of Z_ξ is that it clearly vanishes when $\xi_\tau = 0$, but this second formulation can also have advantages, especially when ξ has a vanishing divergence (as it is the case in [2]) or when there are simplifications as it is the case for the volume (see Lemma 2.7 for the first equality):

$$\mathrm{Vol}''(\Omega).(\xi,\xi) = \int_{\partial\Omega} H(\xi\cdot\boldsymbol{n})^2 + \int_{\partial\Omega} Z_\xi = \int_{\partial\Omega} (\xi\cdot\boldsymbol{n})\mathrm{div}(\xi). \tag{10}$$

Remark 2.6 It is sometimes considered that first and second order derivatives described in the previous theorem cannot handle the differentiation of $t \mapsto J(T_t(\Omega))$ where $T \in \mathrm{C}^2([0,\alpha[,\Theta)$ is not of the form $T_t = Id + t\xi$. This is not true, as the chain rule formula easily gives (and is allowed when we have proven the Fréchet-differentiability of the functionals, which is valid for all the functionals of this paper):

$$\frac{d^2}{dt^2}J(T_t(\Omega)) = \mathcal{J}''_\Omega(T_t - Id).\left(\frac{d}{dt}T_t, \frac{d}{dt}T_t\right) + \mathcal{J}'_\Omega(T_t - Id).\left(\frac{d^2}{dt^2}T_t\right)$$

and the structure result can then be applied. For example, if T_t is the flow of the vector field ξ as it is usually done in the speed method, we obtain:

$$\frac{d^2}{dt^2}J(T_t(\Omega))_{|t=0} = \mathcal{J}''_\Omega(0).(\xi,\xi) + \mathcal{J}'_\Omega(0).((D\xi)\cdot\xi).$$

Another interesting case is that if Ω is a critical shape for J, namely $\mathcal{J}'_\Omega(0) \equiv 0$, and if $T_t = Id + t\xi + \frac{t^2}{2}\eta + o(t^2)$ where $o(t^2)$ has to be understood with the norm $\|\cdot\|_\Theta$, then we always have

$$\frac{d^2}{dt^2}J(T_t(\Omega))_{|t=0} = \ell_2(\xi\cdot\boldsymbol{n}, \xi\cdot\boldsymbol{n}).$$

Proof of Theorem 2.1 We only focus on the second order derivative, as the first order one is classical (see for example [25, 16, 29]). For $k \in \mathbb{N}$, we define $\mathrm{C}^{k,\infty} := \mathrm{C}^k \cap \mathrm{W}^{k,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ equipped with the same norm as $\mathrm{W}^{k,\infty}$, which is also a Banach space and is more adapted to approximation by smooth functions. Let $\xi,\zeta \in \mathrm{C}^\infty$ compactly supported, and denote γ,δ their respective flow, namely

$$\left\{\begin{array}{rcl} \frac{d}{dt}\gamma_t(x) &=& \xi(\gamma_t(x)) \\ \gamma_0(x) &=& x \end{array}\right. \qquad \left\{\begin{array}{rcl} \frac{d}{dt}\delta_t(x) &=& \zeta(\delta_t(x)) \\ \delta_0(x) &=& x \end{array}\right.$$

Thanks to our assumption on ξ, we easily check that the function $T \in \Theta \mapsto \xi\circ(T+Id) \in \Theta$ is locally Lipschitz and C^2, and therefore these ODE admits solutions defined on $(-t_0,t_0)$ and such that $[t \mapsto \gamma_t - Id, t \mapsto \delta_t - Id]$ are in $\mathrm{C}^2((-t_0,t_0),\Theta)$. As a consequence, $(t,s) \mapsto \gamma_s\circ\delta_t - Id \in \Theta$ is well-defined in a neighborhood of $(0,0)$ and C^2.

Let now assume that $\zeta\cdot\boldsymbol{n} = 0$. Then from classical criterion of invariance of sets with the flow, we have $\delta_t(\Omega) = \Omega$ for every t small enough, so $J(\gamma_s\circ\delta_t(\Omega)) = \mathcal{J}_\Omega(\gamma_s\circ\delta_t - Id)$ is independent of t. Differentiating successively with respect to t and s at $(0,0)$, we obtain:

$$\mathcal{J}''_\Omega(0).(\xi,\zeta) + \mathcal{J}'_\Omega(0).(D\xi\cdot\zeta) = 0, \ \ \forall\xi \in \mathrm{C}^\infty_c, \ \ \forall\zeta \in K\cap\mathrm{C}^\infty_c,$$

where $K = \mathrm{Ker}(\Phi)$ and $\Phi : \xi \in \Theta \mapsto \xi_{|\partial\Omega}\cdot\boldsymbol{n}$.

We define $b : (\xi,\zeta) \in \mathrm{C}^{2,\infty}\times\mathrm{C}^{1,\infty} \mapsto \mathcal{J}''_\Omega(0).(\xi,\zeta) + \mathcal{J}'_\Omega(0).(D\xi\cdot\zeta)$ which is a continuous bilinear functional that vanishes for $\zeta \in K$, for any fixed ξ. Therefore we can write, using quotient properties, $b(\xi,\zeta) = \widetilde{b}(\xi,\zeta_{|\partial\Omega}\cdot\boldsymbol{n})$ where $\widetilde{b} : \mathrm{C}^{2,\infty}\times\mathrm{C}^1(\partial\Omega) \to \mathbb{R}$ is continuous (a priori we only get that $\widetilde{b}$ is separately continuous but with Banach-Steinhaus Theorem, it implies continuity), as Φ induces an isomorphism between Θ/K and $\Phi(\Theta) = \mathrm{C}^1(\partial\Omega)$ equipped with the C^1 norm (using that Ω is of class C^2). Moreover by construction we have:

$$\mathcal{J}''_\Omega(0).(\xi,\zeta) + \mathcal{J}'_\Omega(0).(D\xi\cdot\zeta) = \tilde{b}(\xi,\zeta_{|\partial\Omega}\cdot\boldsymbol{n}), \ \ \forall\,\xi,\zeta \in \mathrm{C}^{2,\infty}\times\mathrm{C}^{1,\infty}.$$

Using the symmetry of $\mathcal{J}''_\Omega(0)$, we can write, for every $(\xi,\zeta) \in \mathrm{C}^{2,\infty}$:

$$\tilde{b}(\zeta,\xi_{|\partial\Omega}\cdot\boldsymbol{n}) - \tilde{b}(\xi,\zeta_{|\partial\Omega}\cdot\boldsymbol{n}) = \mathcal{J}'_\Omega(0).(D\zeta\cdot\xi - D\xi\cdot\zeta) \tag{11}$$

Our goal is now to apply this formula to $\zeta_{\boldsymbol{n}}$ the normal component of ζ, which needs to be extended as a vector field on $\mathbb{R}^d$. To that end, we introduce $P_{\partial\Omega}$ the projection on $\partial\Omega$, which is well-defined and C^2 in a neighborhood of $\partial\Omega$, as Ω is assumed to be C^3 (see for example [16]). Then if φ is defined on $\partial\Omega$, we set $\widetilde{\varphi}(x) = \varphi(P_{\partial\Omega}x)\chi(x)$ where χ is a smooth function with $\chi = 1$ in a neighborhood of $\partial\Omega$, and $\chi = 0$ outside a compact set (in other words, φ is extended so that it is constant in the normal direction). This operator $\varphi \mapsto \widetilde{\varphi}$ is continuous from $\mathrm{C}^2(\partial\Omega)$ to $\mathrm{C}^{2,\infty}$. Let us define then $\zeta_{\boldsymbol{n}} := \widetilde{(\zeta\cdot\boldsymbol{n})}\boldsymbol{n}$ the extension of the normal component of ζ. Defining the bilinear form $\ell_0(\varphi_1,\varphi_2) = \tilde{b}(\widetilde{\varphi_1\boldsymbol{n}},\varphi_2)$, continuous on $\mathrm{C}^2(\partial\Omega)\times\mathrm{C}^1(\partial\Omega)$ (and a priori non symmetric), we obtain

$$\begin{array}{rcll} \mathcal{J}''_\Omega(0).(\xi,\zeta) &=& \tilde{b}(\xi,\zeta\cdot\boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\xi\cdot\zeta) & \\ &=& \tilde{b}(\zeta_{\boldsymbol{n}},\xi\cdot\boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\zeta_{\boldsymbol{n}}\cdot\xi - D\xi\cdot\zeta_{\boldsymbol{n}}) - \mathcal{J}'_\Omega(0).(D\xi\cdot\zeta) & \text{(using (11))} \\ &=& \ell_0(\zeta\cdot\boldsymbol{n},\xi\cdot\boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\zeta_{\boldsymbol{n}}\cdot\xi - D\xi\cdot\zeta_{\boldsymbol{n}} + D\xi\cdot\zeta) & \\ &=& \ell_0(\zeta\cdot\boldsymbol{n},\xi\cdot\boldsymbol{n}) - \mathcal{J}'_\Omega(0).(D\zeta_{\boldsymbol{n}}\cdot\xi + D\xi\cdot\zeta_\tau) & \end{array}$$

where $\zeta_\tau = \zeta - \zeta_{\boldsymbol{n}}$. We now use $D\zeta_{\boldsymbol{n}} = D_\tau\zeta_{\boldsymbol{n}}$, because thanks to our choice of extension operator, $\zeta_{\boldsymbol{n}}$ is constant in the direction $\boldsymbol{n}$ (by definition, $D_\tau a = Da - (Da\cdot\boldsymbol{n})\boldsymbol{n}$), and therefore $D\zeta_{\boldsymbol{n}}\cdot\xi = D_\tau\zeta_{\boldsymbol{n}}\cdot\xi_\tau$. Moreover, $D\xi\cdot\zeta_\tau = D_\tau\xi\cdot\zeta_\tau$.

Using a symmetrization of the previous formula, we obtain

$$\begin{aligned}\mathcal{J}''_\Omega(0).(\xi,\zeta) &= \tfrac{1}{2}\Big[\ell_0(\zeta\cdot\boldsymbol{n},\xi\cdot\boldsymbol{n})+\ell_0(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n})-\mathcal{J}'_\Omega(0).(D_\tau\zeta_{\boldsymbol{n}}\cdot\xi_\tau+D_\tau\xi\cdot\zeta_\tau+D_\tau\xi_{\boldsymbol{n}}\cdot\zeta_\tau+D_\tau\zeta\cdot\xi_\tau)\Big]\\ &= \ell_2(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n})-\tfrac{1}{2}\mathcal{J}'_\Omega(0)\cdot\Big(2D_\tau\zeta\cdot\xi_\tau+2D_\tau\xi\cdot\zeta_\tau-D_\tau\xi_\tau\cdot\zeta_\tau-D_\tau\zeta_\tau\cdot\xi_\tau\Big)\end{aligned}$$

where we defined $\ell_2(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n})=\frac{1}{2}(\ell_0(\zeta\cdot\boldsymbol{n},\xi\cdot\boldsymbol{n})+\ell_0(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n}))$, which is a continuous bilinear form on $\mathrm{C}^2(\partial\Omega)^2$.

From the structure of the first order derivative, and using the formula

$${}^tD_\tau\xi_\tau\cdot\boldsymbol{n}+{}^tD_\tau\boldsymbol{n}\cdot\xi_\tau=0$$

(obtained by tangentially differentiating $\xi_\tau\cdot\boldsymbol{n}=0$), we finally obtain (using the C^3 regularity of $\partial\Omega$ so that $D_\tau\boldsymbol{n}$ belongs to the space of definition of ℓ_1)

$$\begin{aligned}\mathcal{J}''_\Omega(0).(\xi,\zeta) &= \ell_2(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n})-\tfrac{1}{2}\ell_1\Big((2D_\tau\zeta\cdot\xi_\tau+2D_\tau\xi\cdot\zeta_\tau)\cdot\boldsymbol{n}-\zeta_\tau\cdot({}^tD_\tau\xi_\tau\cdot\boldsymbol{n})-\xi_\tau\cdot({}^tD_\tau\zeta_\tau\cdot\boldsymbol{n})\Big)\\ &= \ell_2(\xi\cdot\boldsymbol{n},\zeta\cdot\boldsymbol{n})+\ell_1\Big((D_\tau\boldsymbol{n}\cdot\zeta_\tau)\cdot\xi_\tau-\nabla_\tau(\zeta\cdot\boldsymbol{n})\cdot\xi_\tau-\nabla_\tau(\xi\cdot\boldsymbol{n})\cdot\zeta_\tau\Big)\end{aligned}$$

(where we used that $D_\tau\boldsymbol{n}$ is symmetric), which concludes the proof (a priori, ℓ_0 depends on the extension operator that has been chosen, but as in the final formula the extension only appears in ℓ_2 which does not depend of the extension operator). □

2.2 Examples of shapes derivatives

For an open bounded (smooth enough) set $\Omega\subset\mathbb{R}^d$, we consider in this section (and in the rest of the paper) its volume $|\Omega|$, its perimeter $P(\Omega)=\mathcal{H}^{d-1}(\partial\Omega)$, its Dirichlet energy $E(\Omega)$ and its first eigenvalue of the Dirichlet Laplace operator $\lambda_1(\Omega)$ (see (8)). The existence and computations of the shape derivatives of these functionals are well known, see for example [25, Chapter 5]. We denote the mean curvature (understood as the sum of the principal curvatures of $\partial\Omega$) by H, $\boldsymbol{B}=D_\tau\boldsymbol{n}$ is the second fundamental form of $\partial\Omega$, and $\|\boldsymbol{B}\|^2$ is the sum of the squares of the principal curvatures of $\partial\Omega$.

Lemma 2.7 (Expression of shape derivatives) *If Ω is C^2, one has, for any $\varphi\in\mathrm{C}^\infty(\partial\Omega)$,*

- $\ell_1[\mathrm{Vol}](\Omega).\varphi=\displaystyle\int_{\partial\Omega}\varphi,\qquad \ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi)=\int_{\partial\Omega}H\varphi^2.$
- $\ell_1[P](\Omega).\varphi=\displaystyle\int_{\partial\Omega}H\varphi,\qquad \ell_2[P](\Omega).(\varphi,\varphi)=\int_{\partial\Omega}|\nabla_\tau\varphi|^2+\int_{\partial\Omega}\left[H^2-\|\boldsymbol{B}\|^2\right]\varphi^2$
- $\ell_1[E](\Omega).\varphi=-\dfrac{1}{2}\displaystyle\int_{\partial\Omega}(\partial_n u)^2\varphi,\qquad \ell_2[E](\Omega).(\varphi,\varphi)=\langle\partial_n u\ \varphi,\Lambda(\partial_n u\ \varphi)\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}}+\displaystyle\int_{\partial\Omega}\left[\partial_n u+\frac{1}{2}H(\partial_n u)^2\right]\varphi^2$

  where $u\in\mathrm{H}^1_0(\Omega)$ is the unique solution to $-\Delta u=1$, $\Lambda:\ \mathrm{H}^{1/2}(\partial\Omega)\to\mathrm{H}^{-1/2}(\partial\Omega)$ is the Dirichlet-to-Neumann map defined as $\Lambda(\psi)=\partial_n\mathsf{H}(\psi)$ where H is the harmonic extension operator from $\mathrm{H}^{1/2}(\partial\Omega)$ into $\mathrm{H}^1(\Omega)$:

$$-\Delta\mathsf{H}(\psi)=0\ \text{in}\ \Omega,\quad \mathsf{H}(\psi)=\psi\ \text{on}\ \partial\Omega,$$

- $\ell_1[\lambda_1](\Omega).\varphi=-\displaystyle\int_{\partial\Omega}(\partial_n v)^2\varphi,\qquad \ell_2[\lambda_1](\Omega).(\varphi,\varphi)=\int_{\partial\Omega}2w(\varphi)\ \partial_n w(\varphi)+H(\partial_n v)^2\varphi^2$

  where v is the normalized eigenfunction (solution in $\mathrm{H}^1_0(\Omega)$ of $-\Delta v=\lambda_1 v$ with $v\geq 0$ in Ω and $\|v\|_{\mathrm{L}^2(\Omega)}=1$) and $w(\varphi)$ is the solution of

$$\left\{\begin{aligned}-\Delta w(\varphi) &= \lambda_1 w(\varphi)-v\int_{\partial\Omega}(\partial_n v)^2\varphi\ \text{in}\ \Omega,\\ w(\varphi) &= -\varphi\partial_n v\ \text{on}\ \partial\Omega,\\ \int_\Omega v\ w(\varphi) &= 0.\end{aligned}\right.\tag{12}$$

A fundamental fact for this work appears here in the expression of the shape hessians. Even if they are derived for regular perturbations, they are naturally defined and continuous on different Sobolev spaces on $\partial\Omega$:

Lemma 2.8 (Continuity of shape Hessians) *If Ω is C^2, there is a constant $C > 0$ such that*

$$|\ell_2[P](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^1(\partial\Omega)}, \qquad |\ell_2[\mathrm{Vol}](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{L}^2(\partial\Omega)},$$
$$|\ell_2[E](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}, \qquad |\ell_2[\lambda_1](\Omega).(\varphi,\varphi)| \leq C\|\varphi\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}.$$

Therefore, from this Lemma, it is natural to consider the extension of these bilinear forms to their space of continuity.

2.3 The case of balls

In this section, we describe the shape derivatives of the previous functionals when the set Ω is a ball. This will be very efficient when studying if one can apply Theorems 1.1 and 1.3 to the ball, see Section 5.

Let us focus on the ball B_1 of radius 1. For the Dirichlet energy E, we remark that $u(x) = (1-|x|^2)/2d$ solves $-\Delta u = 1$ in $\mathrm{H}^1_0(B_1)$ and satisfies $\partial_n u = -\frac{1}{d}$ on ∂B_1. For λ_1, we recall that the eigenvalue and eigenfunction are

$$\lambda_1(B_1) = j^2_{d/2-1} \text{ associated to } v(x) \;=\; \alpha_d\, |x|^{1-d/2}\, J_{d/2-1}\left(j_{d/2-1}\,|x|\right),$$

where $j_{d/2-1}$ is the first zero of Bessel's function $J_{d/2-1}$ and α_d a normalization constant. Moreover, from [26, p. 35], the eigenfunction satisfies

$$\partial_n v = \sqrt{\frac{2}{P(B_1)}}\, j_{d/2-1} := \beta_d, \text{ so that } \beta_d^2 = \frac{2\lambda_1(B_1)}{P(B_1)}. \tag{13}$$

We obtain the **shape gradients**:

$$\ell_1[\mathrm{Vol}](B_1).\varphi = \int_{\partial B_1} \varphi, \qquad \ell_1[P](B_1).\varphi = (d-1)\int_{\partial B_1} \varphi,$$
$$\ell_1[E](B_1).\varphi = -\frac{1}{2d^2}\int_{\partial B_1} \varphi, \qquad \ell_1[\lambda_1](B_1).\varphi = \;-\beta_d^2 \int_{\partial B_1} \varphi.$$

Let us notice that these four shape gradients at balls are colinear. As a consequence, the balls are critical domains for the perimeter, the Dirichlet energy and λ_1 (or any sum of these functionals) under a volume constraint, and these formula easily provide the value of the Lagrange-multiplier.

Let us turn our attention to the **hessians**. The value of $\ell_2[\lambda_1]$ is a bit more involved, so we deal with it in the next lemma. For the other functionals, it is known from Lemma 2.7 that:

$$\ell_2[\mathrm{Vol}](B_1).(\varphi,\varphi) = (d-1)\int_{\partial B_1} \varphi^2,$$

$$\ell_2[P](B_1).(\varphi,\varphi) = \int_{\partial B_1} |\nabla_\tau \varphi|^2 + (d-1)(d-2)\int_{\partial B_1} \varphi^2,$$

$$\ell_2[E](B_1).(\varphi,\varphi) = \frac{1}{d^2}\langle \varphi, \Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} - \frac{d+1}{2d^2}\int_{\partial B_1} \varphi^2.$$

In order to see that the quadratic forms associated to the Lagrangian are coercive on their natural spaces, it is useful to study the diagonalized form of these Hessians. To that end, we use spherical harmonics defined as the restriction to the unit sphere of harmonic polynomials. We recall here facts from [39, pages 139-141]. We let $\mathcal{H}_k$ denote the space of spherical harmonics of degree k (that is, the restriction to ∂B_1 of homogeneous polynomials in $\mathbb{R}^d$, of degree k). It is also the eigenspace of the Laplace-Beltrami operator on the unit sphere associated with the eigenvalue $-k(k+d-2)$. Let $(Y^{k,l})_{1\leq l\leq d_k}$ be an orthonormal basis of $\mathcal{H}_k$ with respect to the $\mathrm{L}^2(\partial B_1)$ scalar product. The family $(Y^{k,l})_{k\in\mathbb{N},1\leq l\leq d_k}$ is a Hilbert basis of $\mathrm{L}^2(\partial B_1)$. Hence, any function φ in $\mathrm{L}^2(\partial B_1)$ can be decomposed:

$$\varphi(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} \alpha_{k,l}(\varphi) Y^{k,l}(x), \quad \text{for } |x| = 1. \tag{15}$$

Then, by construction, the function h defined by

$$h(x) = \sum_{k=0}^{\infty} |x|^k \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi) Y^{k,l}\left(\frac{x}{|x|}\right), \quad \text{for } |x| \leq 1, \tag{16}$$

is harmonic in B_1 and satisfies $h = \varphi$ on ∂B_1. Moreover, the sequence of coefficients $\alpha_{k,l}$ characterizes the Sobolev

regularity of φ: indeed $\varphi \in \mathrm{H}^s(\partial B_1)$ if and only if the sum $\sum_k (1+k^2)^s \sum_l |\alpha_{k,l}|^2$ converges. We can now state the following lemma expressing the previous shape hessians are diagonal on this basis.

Lemma 2.9 *Using the decomposition* (15)*, we have (β_d is the constant defined in* (13)*)*

$$\ell_2[\mathrm{Vol}](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(d-1)\ \alpha_{k,l}(\varphi)^2,$$

$$\ell_2[E](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{1}{d^2}\ k\ -\frac{d+1}{2d^2}\right]\alpha_{k,l}(\varphi)^2,$$

$$\ell_2[P](B_1).(\varphi,\varphi) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[k^2+(d-2)k+(d-1)(d-2)\right]\ \alpha_{k,l}(\varphi)^2,$$

$$\ell_2[\lambda_1](B_1)(\varphi,\varphi) = \beta_d^2\left(3\alpha_{0,1}^2(\varphi)+\sum_{k=1}^{\infty}\sum_{l=1}^{d_k}2\left[k+\frac{d-1}{2}-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)\right).$$

Proof. First we check that

$$\int_{\partial B_1}\varphi^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2, \qquad \int_{\partial B_1}|\nabla_\tau\varphi|^2 = -\int_{\partial B_1}\varphi\ \Delta_\tau\varphi = \sum_{k=0}^{\infty}k(k+d-2)\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)^2.$$

Then, we precise the term involving the Dirichlet-to-Neumann map that appears in the shape hessian of the Dirichlet energy. Using h defined in (16) and Green formula, we have:

$$\begin{aligned}
\langle\varphi,\Lambda\varphi\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} &= \int_{\partial B_1}\varphi\partial_n h = \int_{B_1}|\nabla h|^2\\
&= \int_0^1\left(\int_{\partial B_r}\left((\partial_n h)^2+|\nabla_\tau h|^2\right)d\sigma\right)dr = \int_0^1\left(\int_{\partial B_r}\left((\partial_n h)^2-h\Delta_\tau h\right)d\sigma\right)dr\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\int_0^1 r^{d-1}\left[k^2r^{2(k-1)}+\frac{k(k+d-2)}{r^2}\ r^{2k}\right]dr\ \alpha_{k,l}(\varphi)^2\\
&= \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\left[\frac{k^2}{2k+d-2}+\frac{k(k+d-2)}{2k+d-2}\right]\alpha_{k,l}(\varphi)^2 = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}k\ \alpha_{k,l}(\varphi)^2.
\end{aligned}$$

We obtain $\ell_2[\mathrm{Vol}]$, $\ell_2[P]$ and $\ell_2[E]$ by gathering these elementary terms.

Let us now consider the case of the first eigenvalue. We apply [26, p 35] (see also [36] and [38]): for a second order volume preserving path, that is $t\mapsto T_t$ such that $|T_t(\Omega)| = |\Omega| + o(t^2)$ for small t, we have

$$\left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} = \sum_{k=1}^{\infty}\sum_{l=1}^{d_k}2\beta_d^2\ \left[k+d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi)$$

where $\varphi = (\frac{d}{dt}T_t)_{|t=0}\cdot\boldsymbol{n}$ and we have used the recursive formula for Bessel function $J_\nu'(z) = (\nu/z)J_\nu(z) - J_{\nu+1}(z)$ to adapt his expression to our notations ([1, section 9.1.27, p 361]).

To deduce $\ell_2[\lambda_1]$ from this computation, we introduce θ a smooth vector field which is normal on ∂B_1 and denote $\varphi = \theta\cdot\boldsymbol{n}$. We assume that $\int_{\partial B_1}\varphi = \alpha_{0,1}(\varphi) = 0$. It is then clear that there exists ξ such that $T_t := Id + t\theta + \frac{t^2}{2}\xi$ is volume preserving at the second order, that is to say

$$\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi) + \ell_1[\mathrm{Vol}](B_1)(\psi) = 0,$$

where $\psi = \xi\cdot\boldsymbol{n}$. Then we observe that for a smooth shape functional J and for such $t\mapsto T_t$,

$$\left(\frac{d^2}{dt^2}J(T_t(B_1))\right)_{|t=0} = \ell_2[J](B_1)(\varphi,\varphi) + \ell_1[J](B_1)(\psi),$$

and therefore, denoting μ the Lagrange multiplier such that $\ell_1[\lambda_1 - \mu\mathrm{Vol}](B_1) = 0$, we obtain

$$\begin{aligned}
\left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} &= \ell_2[\lambda_1](B_1)(\varphi,\varphi) + \ell_1[\lambda_1](B_1)(\psi) = \ell_2[\lambda_1](B_1)(\varphi,\varphi) + \mu\ell_1[\mathrm{Vol}](B_1)(\psi)\\
&= \ell_2[\lambda_1](B_1)(\varphi,\varphi) - \mu\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi)
\end{aligned}$$

Then, we get, as here $\mu = -\beta_d^2$:

$$\begin{aligned}
\ell_2[\lambda_1](\varphi,\varphi) &= \left(\frac{d^2}{dt^2}\lambda_1(T_t(B_1))\right)_{|t=0} + \mu\ell_2[\mathrm{Vol}](B_1)(\varphi,\varphi),\\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k} 2\beta_d^2 \left[k+d-1-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi) - \beta_d^2\sum_{k=0}^{\infty}\sum_{l=1}^{d_k}(d-1)a_{k,l}^2(\varphi),\\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{d_k} 2\beta_d^2 \left[k+\frac{d-1}{2}-j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]\alpha_{k,l}^2(\varphi).
\end{aligned}$$

It remains to compute the coefficient associated to the mode $k=0$. It suffices to consider the deformations as $T_t(x) = x + tx$ mapping the ball B_1 onto the ball B_{1+t}. Here $\varphi = 1$ and $\alpha_{0,1}(\varphi) = P(B_1)^{1/2}$. Since λ_1 is homogeneous of degree -2, we get $\lambda(t) := \lambda_1(T_t(B_1)) = (1+t)^{-2}\lambda_1(B_1)$ so that

$$\lambda''(0) = 6\lambda_1(B_1) = 6\frac{\lambda_1(B_1)}{P(B_1)}\alpha_{0,1}(\varphi)^2.$$

□

3 Stability Theorems

3.1 About coercivity and condition $(\mathbf{C}_{\mathrm{H}^s})$

Usually the coercivity property for the second order derivative (of the functional or of the Lagrangian) has to be proven by hand on each specific example by studying the lower bound of the spectrum of the bilinear form ℓ_2 defined in Theorem 2.1, typically thanks to Lemma 2.9. Nevertheless, when ℓ_2 enjoys some structural property, coercivity can be more easily checked as a consequence of the following general lemma.

Lemma 3.1 *Let M be the boundary of a Lipschitz-domain in $\mathbb{R}^d$, $s_2 \in [0,1]$, and V a linear subspace of $\mathrm{H}^{s_2}(M)$, closed for the weak convergence in $\mathrm{H}^{s_2}(M)$. If ℓ, a quadratic form defined on $\mathrm{H}^{s_2}(M)$ satisfies condition $(\mathbf{C}_{\mathrm{H}^{s_2}})$ (see page 2), then the following propositions are equivalent:*

- **(i)** $\ell(\varphi,\varphi) > 0$ *for any* $\varphi \in V \setminus \{0\}$.
- **(ii)** $\exists\gamma > 0, \quad \ell(\varphi,\varphi) \geq \gamma\|\varphi\|^2_{\mathrm{H}^{s_2}(M)}$ *for any* $\varphi \in V$.

Remark 3.2 In practice, we apply this lemma when V is either $\mathrm{H}^{s_2}(\partial\Omega)$ or $T(\partial\Omega)$ defined in (5).

Proof. The implication **(ii)** $\Longrightarrow$ **(i)** is trivial. Assume **(i)** and let $(\varphi_k)_k$ a minimizing sequence for the problem

$$\inf\left\{\ell(\varphi,\varphi),\ \varphi \in V, \|\varphi\|_{\mathrm{H}^{s_2}} = 1\right\}.$$

Up to a subsequence, φ_k weakly converges in $\mathrm{H}^{s_2}(M)$ to some $\varphi_\infty \in V$. By the compactness of the embedding of $\mathrm{H}^{s_2}(M)$ into $\mathrm{H}^{s_1}(M)$, $\varphi_k \to \varphi_\infty$ in $\mathrm{H}^{s_1}(M)$ so that $\ell_r(\varphi_k,\varphi_k) \to \ell_r(\varphi_\infty,\varphi_\infty)$. We distinguish two cases: if $\varphi_\infty \neq 0$, $\liminf_k \ell_m(\varphi_k,\varphi_k) \geq \ell_m(\varphi_\infty,\varphi_\infty)$ by the lower semi continuity of ℓ_m, so that $\liminf_k \ell(\varphi_k,\varphi_k) \geq \ell(\varphi_\infty,\varphi_\infty) > 0$ by assumption **(i)**. Now, if $\varphi_\infty = 0$, then as the norm $\|\cdot\|_{\mathrm{H}^{s_2}}$ is equivalent to the norm $\|\cdot\|_{\mathrm{H}^{s_1}} + |\cdot|_{\mathrm{H}^{s_2}}$, we know that $|\varphi_k|_{\mathrm{H}^{s_2}}$ is bounded from below by a positive constant, and using $(\mathbf{C}_{\mathrm{H}^{s_2}})$,

$$\liminf_k \ell(\varphi_k,\varphi_k) = \liminf_k \ell_m(\varphi_k,\varphi_k) \geq c_1 \liminf_k |\varphi_k|^2_{\mathrm{H}^{s_2}} > 0.$$

□

Remark 3.3 The equivalence between coercivity in L^2 and H^1 was already known in the context of stable minimal surface it appears in the work [24] of Grosse-Brauckmann. In [2], the previous lemma is proven in the particular case of the functional under study (see also Section 5.1).

Remark 3.4 When one applies this lemma to a shape hessian, assumption (i) may seem unnatural. Indeed, shape derivatives are usually defined for regular perturbations that are dense subsets of $\mathrm{H}^s(\partial\Omega)$ and one could expect to assume only $\ell(\varphi,\varphi) > 0$ for $\varphi \in V \setminus \{0\}$ smooth enough. But this assumption may not be sufficient: indeed the

function φ_∞ in the proof above may not be smooth and therefore not admissible to test the positivity property. Therefore, the shape hessian ℓ has to be first extended by continuity to the whole $\mathrm{H}^s(\partial\Omega)$ (see assumption (3) in Theorem 1.1 and (4) for Theorem 1.3), see Lemma 2.7 for such an extension in classical examples. However in some cases, we may expect regularity for φ_∞, see for example [15, Remark 1].

We conclude this section noticing that the shape hessians of the model functionals from Section 2 satisfies $(\mathbf{C}_{\mathrm{H}^{s_2}})$:

- The perimeter satisfies $(\mathbf{C}_{\mathrm{H}^1})$ with
$$\ell_m[P](\Omega)(\varphi,\varphi) = \int_{\partial\Omega} |\nabla_\tau \varphi|^2 \quad \text{and} \quad \ell_r[P](\Omega)(\varphi,\varphi) = \int_{\partial\Omega} \left[H^2 - \|\boldsymbol{B}\|^2\right] \varphi^2 \qquad \text{(here we can choose } s_1 = 0\text{)}.$$
- The Dirichlet energy and λ_1 satisfy $(\mathbf{C}_{\mathrm{H}^{1/2}})$ (again $s_1 = 0$):
$$\ell_m[E](\Omega)(\varphi,\varphi) = \langle \partial_n u\varphi, \Lambda(\partial_n u\varphi)\rangle_{\mathrm{H}^{1/2}\times\mathrm{H}^{-1/2}} \quad \text{and} \quad \ell_r[E](\Omega)(\varphi,\varphi) = \int_{\partial\Omega} \left[\partial_n u + \frac{1}{2}H(\partial_n u)^2\right]\varphi^2,$$
$$\ell_m[\lambda_1](\Omega).(\varphi,\varphi) = \int_{\partial\Omega} 2w(\varphi)\ \partial_n w(\varphi) \quad \text{and} \quad \ell_r[\lambda_1](\Omega)(\varphi,\varphi) = \int_{\partial\Omega} H(\partial_n v)^2\varphi^2.$$

Remark 3.5 Let us emphasize that condition $(\mathbf{C}_{\mathrm{H}^s})$ may not be valid in some interesting examples. Shape functionals used for domain reconstruction from boundary measurements provide in general non-coercive Hessians. With the examples treated in [3], [4] one can find critical shape whose hessian is positive but is not coercive (for any H^s-norm).

More precisely, for a reconstruction function J related to this kind of inverse problem (for example the least square fitting to data), the Riesz operator corresponding to the shape Hessian $\ell_2[J]$ at a critical domain is compact. This means, that one cannot expect an estimate of the kind $J(\Omega_t) - J(\Omega_0) \geq ct^2$ with a constant c uniform in the deformation direction. This explains also why regularization is required in the numerical treatment of this type of problem. This fact is well-known in the inverse problem community.

There are also situations where the objective is flat up to fourth order (see [13]).

3.2 Proof of Theorem 1.1

Let Ω^* be a domain satisfying the assumption of Theorem 1.1. Let $\eta > 0$ and let $\Omega = \Omega_h^*$ with $\|h\|_X < \eta$. Then from $(\mathbf{IT}_{\mathrm{H}^s,X})$ we have
$$J(\Omega) - J(\Omega^*) = \underbrace{\ell_1[J](\Omega^*)(h)}_{=0} + \frac{1}{2}\ell_2[J](\Omega^*)(h,h) + \omega(\|h\|_X)\|h\|^2_{\mathrm{H}^s}$$
Using $(\mathbf{C}_{\mathrm{H}^{s_2}})$, we can apply Lemma 3.1 and there is a constant $\gamma > 0$ such that
$$\ell_2[J](\Omega^*).(h,h) \geq \gamma\|h\|^2_{\mathrm{H}^{s_2}}.$$
Therefore there exists η small enough such that if $\|h\|_X \leq \eta$, then $\omega(\|h\|_X) \leq \frac{\gamma}{4}$ and then
$$J(\Omega) - J(\Omega^*) \geq \frac{\gamma}{4}\|h\|^2_{\mathrm{H}^{s_2}}.$$
□

3.3 Proof of Theorem 1.3

We denote μ the Lagrange multiplier associated to J. Therefore we consider $J_\mu = J - \mu\mathrm{Vol}$ and Ω^* satisfies $J'_\mu(\Omega^*) = 0$.

Step 1: Stability under volume and barycenter constraint: Under the structural hypotheses on $\ell_2[J](\Omega^*) = \ell_m + \ell_r$ and the fact that $\ell_2[\mathrm{Vol}](\Omega^*)$ is continuous in the L^2-norm, we can applied Lemma 3.1 to $\ell_2[J_\mu](\Omega^*)$, so there are constants c_1, c_2, c_3 and $c_4 > 0$ such that
$$\forall\varphi \in \mathrm{H}^{s_2}(\partial\Omega^*), \quad |\ell_m(\varphi,\varphi)| \geq c_1|\varphi|^2_{\mathrm{H}^{s_1}} \qquad |\ell_r(\varphi,\varphi)| \leq c_2\|\varphi\|^2_{\mathrm{H}^{s_1}}, \qquad |\ell_2[\mathrm{Vol}](\Omega^*).(\varphi,\varphi)| \leq c_3\|\varphi\|^2_{\mathrm{L}^2}, \tag{17}$$
$$\forall\varphi \in T(\partial\Omega^*), \quad \ell_2[J - \mu\mathrm{Vol}](\Omega^*).(\varphi,\varphi) \geq c_4\|\varphi\|^2_{\mathrm{H}^{s_2}}. \tag{18}$$

Step 2: Stability without constraint: We consider

$$J_{\mu,C} = J - \mu \mathrm{Vol} + C\left(\mathrm{Vol} - V_0\right)^2 + C\left\|\mathrm{Bar} - \mathrm{Bar}(\Omega^*)\right\|^2,$$

where $\mathrm{Bar}(\Omega) := \int_\Omega x$ and $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^d$. The shape Ω^* still satisfies $J'_{\mu,C}(\Omega^*) = 0$. We claim that Ω^* is a strictly stable shape for $J_{\mu,C}$ on the entire space $\mathrm{H}^{s_2}(\partial\Omega^*)$ when C is big enough, that is to say for all φ in $\mathrm{H}^{s_2}(\partial\Omega^*) \setminus \{0\}$,

$$\ell_2[J_{\mu,C}](\Omega^*).(\varphi,\varphi) > 0. \tag{19}$$

Indeed, if it was not the case, we would have the existence of $\varphi_n \in \mathrm{H}^{s_2}(\partial\Omega^*) \setminus \{0\}$ such that

$$\ell_2[J_{\mu,n}](\Omega^*).(\varphi_n,\varphi_n) \leq 0.$$

Using (17), this leads to

$$c_1|\varphi_n|^2_{\mathrm{H}^{s_2}} - c_2\|\varphi_n\|^2_{\mathrm{H}^{s_1}} - |\mu|c_3\|\varphi_n\|^2_{\mathrm{L}^2} + 2n\left(\int_{\partial\Omega^*}\varphi_n\right)^2 + 2n\left\|\int_{\partial\Omega^*}\varphi_n x\right\|^2 \leq 0. \tag{20}$$

Assuming by homogeneity that $\|\varphi_n\|_{\mathrm{H}^{s_1}} = 1$ for every n, (20) implies that $(\varphi_n)_n$ is bounded in H^{s_2} and using the compactness of $\mathrm{H}^{s_2}(\partial\Omega^*)$ in $\mathrm{H}^{s_1}(\partial\Omega^*)$, we have up to a subsequence that φ_n converges to φ weakly in H^{s_2} and strongly in H^{s_1}. Therefore (20) implies first that $2n[\mathrm{Vol}'(\varphi_n)^2 + \mathrm{Bar}'(\varphi_n)^2]$ is bounded, then that $\varphi \in T(\partial\Omega^*)$ and then the semi-lower continuity assumption in $(\mathbf{C}_{\mathrm{H}^{s_2}})$ implies

$$\ell_2[J_\mu](\Omega^*).(\varphi,\varphi) \leq 0, \quad \text{with } \|\varphi\|_{\mathrm{H}^{s_1}} = 1$$

which contradicts (18).

Step 3: Stability: It is now easy to see that $J_{\mu,C}$ satisfies both $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ at Ω^* (using that Vol and Bar satisfy $(\mathbf{IT}_{\mathrm{H}^0,\mathrm{W}^{1,\infty}})$, see Section 4.1), and for C large enough we have (19), so applying Theorem 1.1, there exists $c > 0$ and $\eta > 0$ such that for every $\Omega = \Omega_h$ with $\|h\|_X < \eta$,

$$J_{\mu,C}(\Omega) - J_{\mu,C}(\Omega^*) \geq c\|h\|^2_{\mathrm{H}^{s_2}},$$

We then write this inequality in particular for shapes Ω having the same volume and barycenter as Ω^*, and conclude the proof using the invariance of J with translations. □

4 About Condition $(\mathbf{IT}_{\mathrm{H}^s,X})$

In this section, we show that our main examples satisfy condition $(\mathbf{IT}_{\mathrm{H}^s,X})$ where s is given in Section 3.1, and X is hoped to be as large as possible. Let us start with the notations we will use in this section.

Given Ω an open set and $h : \partial\Omega \to \mathbb{R}$, we recall that Ω_h is defined so that

$$\partial\Omega_h = \{x + h(x)\boldsymbol{n}(x), x \in \partial\Omega\}.$$

It will be useful to see Ω_h as a deformation with a vector field. To that end, we assume Ω of class C^2 so that the projection $\pi_{\partial\Omega}$ on $\partial\Omega$ is well-defined and C^1 in a neighborhood of $\partial\Omega$, and we define

$$h(x) = h(\pi_{\partial\Omega}(x)) \qquad \text{and} \qquad \boldsymbol{n}(x) = \boldsymbol{n}(\pi_{\partial\Omega}(x)),$$

in order to extend h and $\boldsymbol{n}$ in a neighborhood of $\partial\Omega$, and then we define $\boldsymbol{\xi}_h(x) = h(x)\boldsymbol{n}(x)$ in this neighborhood. With this construction, $\boldsymbol{\xi}_h$ is constant in the normal direction, so $\mathrm{div}\boldsymbol{\xi}_h = \mathrm{div}(\boldsymbol{n})h$. We can then extend it smoothly to $\mathbb{R}^d$, so that $\boldsymbol{\xi}_h \in \mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$. Denoting $T_h = Id + \boldsymbol{\xi}_h$, we have $\Omega_h = T_h(\Omega)$, and $\mathrm{j}_\Omega(h) = \mathcal{J}_\Omega(\boldsymbol{\xi}_h) = J(\Omega_h)$ (where J is the shape functional under study). In this section, the notation $\widehat{w_h}$ stands for $w_h \circ T_h$ where w_h is defined on Ω_h or $\partial\Omega_h$.

When studying condition $(\mathbf{IC}_{\mathrm{H}^s,X})$ (which implies $(\mathbf{IT}_{\mathrm{H}^s,X})$), we focus on the path Ω_t defined in (7), and we have $\Omega_t = (Id + t\boldsymbol{\xi}_h)(\Omega)$ and $j''(t) = \mathcal{J}''_\Omega(t\boldsymbol{\xi}_h).(\boldsymbol{\xi}_h,\boldsymbol{\xi}_h)$ for all $t \in [0,1]$, where $j(t) = J(\Omega_t)$. Note that in this case we will notify the dependence of quantities with respect to t, but there is also a dependence in h that we will not recall in order to simplify the notations : for example $\boldsymbol{n}_h$ will denote the exterior normal vector to Ω_h and $\boldsymbol{n}_t$ the normal vector to Ω_t while we should use $\boldsymbol{n}_{th}$. Also, as we chose a vector field that is constant along the normal vector in a neighborhood of $\partial\Omega$, we have (if $\|h\|_\infty$ is small enough)

$$j''(t) = J''(\Omega_t)\cdot(\boldsymbol{\xi}_h,\boldsymbol{\xi}_h) = \mathcal{J}''_{\Omega_t}(0)(\boldsymbol{\xi}_h,\boldsymbol{\xi}_h). \tag{21}$$

4.1 Geometric quantities

• The volume:

Proposition 4.1 *If Ω is C^2, then* Vol *satisfies* $(\mathbf{IC}_{\mathrm{L}^2,\mathrm{W}^{1,\infty}})$ *at Ω.*

Remark 4.2 More generally (with a similar proof), we have that $\Omega \mapsto \int_\Omega f$ also satisfies $(\mathbf{IC}_{\mathrm{L}^2,\mathrm{W}^{1,\infty}})$ if $f \in C^1(\mathbb{R}^d)$. This is true in particular for the barycenter functional.

Before proving this result, we give a geometric Lemma, inspired by the results in [12]. We recall that $J_{\partial\Omega}(h) := \det DT_h|({}^tDT_h^{-1})\boldsymbol{n}|$ is the surface jacobian, appearing when changing variables between $\partial\Omega_h$ and $\partial\Omega$.

Lemma 4.3 *We have the following Taylor expensions, where $\mathcal{O}$ denote a domination uniform in $x \in \partial\Omega$,*

- $J_{\partial\Omega}(h)(x) = 1 + \ell_1^J(h(x), \nabla h(x)) + \frac{1}{2}\ell_2^J(h(x), \nabla h(x)) + \mathcal{O}\left(\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}\left(|h(x)|^2 + |\nabla h(x)|^2\right)\right)$,
- $\widehat{\boldsymbol{n}}_h(x) = \boldsymbol{n}(x) + \ell_1^{\boldsymbol{n}}(h(x), \nabla h(x)) + \frac{1}{2}\ell_2^{\boldsymbol{n}}(h(x), \nabla h(x)) + \mathcal{O}\left(\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}\left(|h(x)|^2 + |\nabla h(x)|^2\right)\right)$.

where $(\ell_1^J, (\ell_1^{\boldsymbol{n}})_{i\in[\![1,d]\!]})$, $(\ell_2^J, (\ell_2^{\boldsymbol{n}})_{i\in[\![1,d]\!]})$ are respectively linear and quadratic forms on $\mathbb{R}^{d+1}$.

Proof of Lemma 4.3: The first part follows simply from the fact that $A \in M_d(\mathbb{R}) \mapsto \det(A)\left|({}^tA^{-1})\boldsymbol{n}\right|$ is smooth in a neighborhood of Id, and the fact that $D\boldsymbol{\xi}_h = h(D\boldsymbol{n}) + \nabla h \otimes \boldsymbol{n}$.
For the second part, we use a level-set parametrization: there exists ϕ of class C^2 such that $\Omega = \{\phi < 0\}$ and $\nabla\phi$ does not vanish in a neighborhood of $\partial\Omega$, and then $\Omega = \{\phi \circ T_h^{-1} < 0\}$. Therefore

$$\widehat{\boldsymbol{n}}_h - \boldsymbol{n} = \frac{\nabla(\phi \circ T_h^{-1})}{|\nabla(\phi \circ T_h^{-1})|} \circ T_h - \frac{\nabla\phi}{|\nabla\phi|} = \frac{{}^tDT_h^{-1}.\nabla\phi}{|{}^tDT_h^{-1}.\nabla\phi|} - \frac{\nabla\phi}{|\nabla\phi|},$$

and we conclude using the smoothness of $A \mapsto {}^tA^{-1}$ and $w \in \mathbb{R}^d \mapsto \frac{w}{|w|}$ in the neighborhood of Id and $\nabla\phi$ respectively. □

Proof of Proposition 4.1: We use (10), (21), and the fact that $\mathrm{div}(\boldsymbol{\xi}_h) = h\mathrm{div}(\boldsymbol{n})$ (as h is constant in the direction of $\boldsymbol{n}$). Therefore if $v(t) = \mathrm{Vol}(\Omega_t)$, we have:

$$v''(t) = \int_{\partial\Omega_t} \boldsymbol{\xi}_h \cdot \boldsymbol{n}_t \mathrm{div}(\boldsymbol{\xi}_h) = \int_{\partial\Omega_t} \mathrm{div}(\boldsymbol{n})(\boldsymbol{n} \cdot \boldsymbol{n}_t)h^2 = \int_{\partial\Omega} H(\boldsymbol{n} \cdot \widehat{\boldsymbol{n}}_t)h^2 J_{\partial\Omega}(t).$$

With Lemma 4.3, we easily obtain

$$|v''(t) - v''(0)| \le Ct\|h\|_{\mathrm{W}^{1,\infty}}\|h\|_{\mathrm{L}^2}^2 \le C\|h\|_{\mathrm{W}^{1,\infty}}\|h\|_{\mathrm{L}^2}^2.$$

□

Remark 4.4 We could try a direct proof estimating

$$|\Omega_h| - |\Omega| = \int_\Omega \left(\det(Id + D\boldsymbol{\xi}_h) - 1\right),$$

but a priori this only leads to the fact that the volume satisfies $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$. In the spirit of [32, Lemma 4.1], we could also try:

$$|\Omega| = \frac{1}{d}\int_{\partial\Omega} x \cdot \boldsymbol{n}_h = \frac{1}{d}\int_{\partial\Omega} (x + h(x)\boldsymbol{n}(x)) \cdot \widehat{\boldsymbol{n}}_h J_{\partial\Omega}(h)$$

but this leads to the same issue (see also Remark 4.6).

• The perimeter:

Proposition 4.5 *If Ω is C^2, then P satisfies* $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ *condition at Ω.*

Proof. We follow exactly the second proof suggested in Remark 4.4 and use Lemma 4.3:

$$P(\Omega_h) = \int_{\partial\Omega_h} 1 = \int_{\partial\Omega} J_{\partial\Omega}(h) = P(\Omega) + \ell_1[P](\Omega)(h) + \frac{1}{2}\ell_2[P](\Omega)(h,h) + \mathcal{O}(\|h\|_{\mathrm{W}^{1,\infty}}\|h\|_{\mathrm{H}^1}^2).$$

□

Remark 4.6 It is interesting to compare the two strategies used for the volume and for the perimeter: indeed, for the volume we prefered to use condition **(IC)**, while a similar strategy for the perimeter, as it is done in [12] or in [2, Proof of Theorem 3.9] (but for a different path of shapes) lead to weaker results, namely $(\mathbf{IC}_{\mathrm{H}^1,\mathrm{C}^{2,\alpha}})$ and $(\mathbf{IC}_{\mathrm{H}^1,\mathrm{W}^{2,p}})$ respectively).

4.2 PDE energies

For PDE energies, a condition of the type $(\mathbf{IC}_{\mathrm{H}^s,X})$ was studied first in [14] where it is proven that in dimension two the Dirichlet energy satisfy $(\mathbf{IC}_{\mathrm{H}^{1/2},\mathrm{C}^{2,\alpha}})$ (for a volume preserving path instead of a normal path), then a similar result is proven for general PDE functionals in any dimension in [12], either for the path (7) or a volume preserving path. More recently in [2], it was proven that the functional described in (30) involving the sum of the perimeter and a PDE functional (of a different kind than in [12]) satisfies $(\mathbf{IC}_{\mathrm{H}^1,\mathrm{W}^{2,p}})$ for p large enough, also for a volume preserving path, see also Section 5.1. Finally, condition $(\mathbf{IC}_{\mathrm{H}^{1/2},\mathrm{C}^{2,\alpha}})$ is also established for the drag in a Stokes flow in [9]. Thanks to our method to handle the volume constraint (see Section 3.3), we only need to deal with the normal path (7).

In this section, we prove Theorem 1.4, which includes the case of λ_1 (which seemed not to be handled in the literature), and we improve the result from [12] by proving $(\mathbf{IC}_{\mathrm{H}^s,X})$ with a smaller space X. We give 4 preliminary steps to prove this result. We only give the details for λ_1, as the case of E is easier and the reader can follow [12] or [7, Appendix] and use the ideas below where we explain how to get X to be $\mathrm{W}^{2,p}$ instead of $\mathrm{C}^{2,\alpha}$. We assume Ω to be C^3.

- Step 1: **Computing the second derivative along the path.**

Denoting v_t the first normalized eigenfunction on Ω_t and applying the structure Theorem to λ_1 (Lemma 2.7) and (21), we get

$$\begin{aligned}\lambda_1''(\Omega_t).(\boldsymbol{\xi}_h,\boldsymbol{\xi}_h) &= 2\int_{\partial\Omega_t} v_t'\partial_{\boldsymbol{n}_t}v_t' + \int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\left[H_t(\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t)^2 - \boldsymbol{B}_t((\boldsymbol{\xi}_h)_{\tau_t},(\boldsymbol{\xi}_h)_{\tau_t}) + 2\nabla_{\tau_t}(\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t)(\boldsymbol{\xi}_h)_{\tau_t}\right]\\ &= 2\underbrace{\int_{\partial\Omega_t} v_t'\partial_{\boldsymbol{n}_t}v_t'}_{\mathcal{T}_1(t)} + \underbrace{\int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\left[H_t\alpha_t^2 - \boldsymbol{B}_t(\beta_t,\beta_t) - 2\nabla_{\tau_t}(\alpha_t)\cdot\beta_t\right]h^2}_{\mathcal{T}_2(t)} \\ &\qquad - 2\underbrace{\int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\alpha_t\left(\beta_t\cdot\nabla_{\tau_t}h\right)h}_{\mathcal{T}_3(t)} \qquad \text{where } \alpha_t = \boldsymbol{n}_t\cdot\boldsymbol{n}, \qquad \beta_t = \alpha_t\boldsymbol{n}_t - \boldsymbol{n}.\end{aligned} \tag{22}$$

- Step 2: **Geometric estimates:**

Similarly to Section 4.1, we denote $\widehat{w}_h = w_h\circ(Id+\boldsymbol{\xi}_h)$ where w_h is defined on Ω_h or $\partial\Omega_h$. The following Lemma follows easily from Lemma 4.3 (see [12] for more details).

Lemma 4.7 *There is a constant C depending on Ω such that for all h in a neighborhood of 0 in $\mathrm{W}^{2,p}(\partial\Omega)$,*

- $\|\widehat{J_{\partial\Omega}(h)} - 1\|_{\mathrm{L}^\infty(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}, \quad \|\widehat{J_{\partial\Omega}(h)} - 1\|_{\mathrm{W}^{1,p}(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$
- $\|\widehat{H_h} - H\|_{\mathrm{L}^p(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}, \quad \|\widehat{\boldsymbol{B}_h} - \boldsymbol{B}\|_{\mathrm{L}^p(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$
- $\|\widehat{\alpha_h} - 1\|_{\mathrm{L}^\infty(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}, \quad \|\widehat{\nabla_{\tau_h}\alpha_h}\|_{\mathrm{L}^p(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$
- $\|\widehat{\beta_h}\|_{\mathrm{L}^\infty(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}, \quad \|\widehat{\beta_h}\|_{\mathrm{W}^{1,p}(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}.$

- Step 3: **Estimate of $\|\widehat{v_\theta} - v\|_{\mathrm{W}^{2,p}}$:** This step is not specific to our chosen deformations $\boldsymbol{\xi}_h$ hence we present it for general deformations $\theta\in\mathrm{W}^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, that is v_θ is the first Dirichlet eigenfunction on $(Id+\theta)(\Omega)$.

Lemma 4.8 *If $p>d$, the map $\theta\mapsto\widehat{v_\theta}$ from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ with values in $\mathrm{W}^{2,p}(\Omega)$ is C^∞ around 0. As a consequence, there is a neighborhood of 0 in $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ and C depending on Ω only so that*

$$\|\widehat{v_\theta} - v_0\|_{\mathrm{W}^{2,p}(\Omega)} \le C\|\theta - Id\|_{\mathrm{W}^{2,p}}.$$

Proof. We use the same strategy as in [25, Proof of Theorem 5.7.4] and [26] but with different functional spaces: precisely, we will apply the implicit function theorem to $\mathcal{F}:\ X\times Y\times\mathbb{R}\to Z\times\mathbb{R}$ defined by

$$\mathcal{F}(\theta,v,\lambda)=\left(-\mathrm{div}A(\theta)\nabla v-\lambda J(\theta)v,\int_\Omega v^2J(\theta)-1\right)\quad\text{where}\quad\begin{cases}J(\theta)=\det(Id+D\theta),\\ A(\theta)=J(\theta)(Id+D\theta)^{-1}(Id+{}^tD\theta)^{-1},\end{cases}$$

for suitable spaces X,Y,Z. Using that $\mathrm{W}^{1,p}$ is an algebra for $p>d$, we easily obtain that the maps J and A are C^∞ around 0 from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)$ into $\mathrm{W}^{1,p}(\mathbb{R}^d,\mathbb{R}^d)$. As a consequence, by Sobolev's embedding, the map $\mathcal{F}$ is C^∞ around $(0,v_0,\lambda_0:=\lambda_1(\Omega))$ from $\mathrm{W}^{2,p}(\mathbb{R}^d,\mathbb{R}^d)\times\mathrm{W}^{2,p}(\Omega)\cap H^1_0(\Omega)\times\mathbb{R}$ into $\mathrm{L}^p(\Omega)\times\mathbb{R}$. Besides $\mathcal{F}(0,v_0,\lambda_0)=(0,0)$ and the differential

$$\partial_{v,\lambda}\mathcal{F}(0,v_0,\lambda_0).[w,\lambda]=\left((-\Delta-\lambda_0)w-\lambda v_0,2\int_\Omega v_0w\right)$$

is an isomorphism from $\mathrm{W}^{2,p}(\Omega)\cap H^1_0(\Omega)\times\mathbb{R}$ into $\mathrm{L}^p(\Omega)\times\mathbb{R}$ (see [25, Lemma 5.7.3] for details) and the conclusion follows. □

- **Step 4: estimation of the variation of the shape derivative of the eigenfunction:**

The objective of this step is to prove the following estimate:

Lemma 4.9 *There is C,η depending only on Ω such that, if $\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\le\eta$, then*

$$\|\widehat{v_t'}-v_0'\|_{\mathrm{H}^1(\Omega)}\le C\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\ \|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}. \tag{23}$$

This step is the most involved one when dealing with λ_1 instead of the Dirichlet Energy: the latter reduces in fact to the second step in the following proof.

Proof.

We recall (see (12)) that

$$\begin{cases}-\Delta v_t' &=& \lambda_1(t)v_t'+\lambda_1'(t)v_t\text{ in }\Omega_t,\\ v_t' &=& -(\partial_{\boldsymbol{n}_t}v_t)\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t\text{ on }\partial\Omega_t,\\ \displaystyle\int_\Omega v_t'v_t &=& 0.\end{cases}$$

1. Splitting. We introduce H_t the harmonic extension on Ω_t of $(\partial_{\boldsymbol{n}_t}v_t)\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t$. Noticing that

$$\lambda_1'(t)=-\int_{\partial\Omega_t}(\partial_{\boldsymbol{n}_t}v_t)^2\boldsymbol{\xi}_h\cdot\boldsymbol{n}_t=\lambda_1(t)\langle v_t,\mathsf{H}_t\rangle$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2(\Omega_t)$, we decompose

$$v_t'=-\pi_t\mathsf{H}_t+w_t$$

where π_t is the orthogonal projection on $E(t):=\{v_t\}^\perp$, and w_t solves

$$\begin{cases}(-\Delta-\lambda_1(t))w_t &=& -\lambda_1(t)\pi_t\mathsf{H}_t\text{ in }\Omega_t,\\ w_t &=& 0\text{ on }\partial\Omega_t,\\ \displaystyle\int_{\Omega_t}v_tw_t &=& 0.\end{cases}$$

We will now prove that each term of the splitting satisfies estimates like (23).

2. Estimate of the harmonic extension. Let us define $\mathcal{L}_t=\mathrm{div}(A_t\nabla\cdot)$ where $A_t=J_t.(Id+tD\boldsymbol{\xi}_h)^{-1}(Id+t^tD\boldsymbol{\xi}_h)^{-1}$ and $J_t=\det(Id+tD\boldsymbol{\xi}_h)$, so that $\mathcal{L}_t\widehat{f_t}=\widehat{\Delta f_t}$ if f_t is defined on Ω_t. Then as $\Delta(\widehat{\mathsf{H}_t}-\mathsf{H}_0)=-\mathrm{div}((A_t-Id)\nabla\widehat{\mathsf{H}_t})$, from classical elliptic estimate (see [22, Corollary 8.7 p 183]), we obtain:

$$\begin{aligned}\|\widehat{\mathsf{H}_t}-\mathsf{H}_0\|_{\mathrm{H}^1(\Omega)}&\le C\|(A_t-Id)\nabla\widehat{\mathsf{H}_t}\|_{\mathrm{L}^2(\Omega)}+C\|\widehat{\mathsf{H}_t}-\mathsf{H}_0\|_{\mathrm{H}^{1/2}(\partial\Omega)}\\ &\le C\|A_t-Id\|_{\mathrm{L}^\infty(\Omega)}\|\nabla\widehat{\mathsf{H}_t}\|_{\mathrm{L}^2(\Omega)}+C\left\|\left((\widehat{\partial_{\boldsymbol{n}_t}v_t})\widehat{\alpha_t}-\partial_{\boldsymbol{n}}v_0\right)h\right\|_{\mathrm{H}^{1/2}(\partial\Omega)}\\ &\le C\|h\|_{\mathrm{W}^{1,\infty}(\partial\Omega)}\left(\|\nabla\widehat{\mathsf{H}_t}-\nabla\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)}+\|\nabla\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)}\right)\\ &\qquad+C\|(\widehat{\partial_{\boldsymbol{n}_t}v_t})\widehat{\alpha_t}-\partial_{\boldsymbol{n}}v_0\|_{\mathrm{W}^{1-1/p,p}(\partial\Omega)}\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\\ &\le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{\mathsf{H}_t}-\mathsf{H}_0\|_{\mathrm{H}^1(\Omega)}+\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right).\end{aligned} \tag{24}$$

Here we used that $\|\nabla \mathsf{H}_0\|_{\mathrm{L}^2(\Omega)} = \|\mathsf{H}_0\|_{\mathrm{H}^{1/2}(\partial\Omega)} \le C\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$, Lemmas 4.7 and 4.8, and the following estimate of a product norm in $\mathrm{H}^{1/2}$:

$$\|uv\|_{\mathrm{H}^{1/2}(\partial\Omega)} \le C\|u\|_{\mathrm{W}^{s,p}(\partial\Omega)}\|v\|_{\mathrm{H}^{1/2}(\partial\Omega)} \tag{25}$$

if $(d-1)/p < s \le 1$. One can find this inequality in [37, Theorem 2 p 177] for functions on $\mathbb{R}^{d-1}$, with the condition $(d-1)/p < s$; using smooth maps between $\partial\Omega$ and $\mathbb{R}^{d-1}$ we obtain (25) if in addition $s \le 1$. We apply it here to $s = 1 - 1/p$ which is valid as $p > d$. Equation (24) leads to the first estimate

$$\|\widehat{\mathsf{H}_t} - \mathsf{H}_0\|_{\mathrm{H}^1(\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$$

as soon as $\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)} \le 1/(2C)$.

3. Estimates on the variation of w_t. We look at the PDE satisfied by $\widehat{w_t} - w_0$:

$$\begin{cases} (-\Delta - \lambda_1(0))(\widehat{w_t} - w_0) &= \Big[(-\Delta - \lambda_1(0)) - (-\mathcal{L}_t - \lambda_1(t))\Big](\widehat{w_t}) - \lambda_1(t)\widehat{\pi_t \mathsf{H}_t} + \lambda_1(0)\pi_0\mathsf{H}_0 \text{ in } \Omega, \\ \widehat{w_t} - w_0 &= 0 \text{ on } \partial\Omega, \end{cases}$$

and we know that $(-\Delta - \lambda_1(0))$ is an isomorphism on $\{v_0\}^\perp$. Therefore

$$\|\widehat{w_t} - w_0 - \gamma_t v_0\|_{\mathrm{H}^1(\Omega)} \le C\|(A_t - Id)\nabla\widehat{w_t}\|_{\mathrm{L}^2(\Omega)} + |\lambda_1(t) - \lambda_1(0)|\|\widehat{w_t}\|_{\mathrm{L}^2(\Omega)} + \|\lambda_1(t)\widehat{\pi_t \mathsf{H}_t} - \lambda_1(0)\pi_0\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)}$$

where γ_t is chosen so that $\widehat{w_t} - w_0 - \gamma_t v_0 \in \{v_0\}^\perp$. From there and using the previous step, we obtain

$$\|\widehat{w_t} - w_0 - \gamma_t v_0\|_{\mathrm{H}^1(\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{w_t}\|_{\mathrm{H}^1(\Omega)} + \|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right).$$

But we have:

$$|\gamma_t| = \left|\int_\Omega (\widehat{w_t} - w_0)v_0\right| = \left|\int_\Omega \widehat{w_t}\big[\widehat{v_t}J_t - v_0\big]\right| \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|\widehat{w_t}\|_{\mathrm{L}^2(\Omega)},$$

leading to

$$\|\widehat{w_t} - w_0\|_{\mathrm{H}^1(\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{w_t}\|_{\mathrm{H}^1(\Omega)} + \|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right) \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\left(\|\widehat{w_t} - w_0\|_{\mathrm{H}^1(\Omega)} + \|w_0\|_{\mathrm{H}^1(\Omega)} + \|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}\right).$$

Using now $\|w_0\|_{\mathrm{H}^1(\Omega)} \le C\|\mathsf{H}_0\|_{\mathrm{L}^2(\Omega)} \le C\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$ and again that $\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}$ is small enough, this leads to

$$\|\widehat{w_t} - w_0\|_{\mathrm{H}^1(\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$$

and concludes the proof of this lemma. □

Proof of Theorem 1.4: We deal separately with the terms of the decomposition (22):

Estimate of $\mathcal{T}_1(t) - \mathcal{T}_1(0)$. We first observe that

$$\mathcal{T}_1(t) = \int_{\Omega_t} |\nabla v_t'|^2 - \lambda_1(t)\int_{\Omega_t} v_t^2,$$

and also that $\|v_0'\|_{\mathrm{H}^1(\Omega)} \le \|w_0\|_{\mathrm{H}^1(\Omega)} + \|\mathsf{H}_0\|_{\mathrm{H}^1(\Omega)} \le C\|h\|_{\mathrm{H}^{1/2}(\partial\Omega)}$. Therefore using Lemma 4.9, we get

$$\left|\int_{\Omega_t} |\nabla v_t'|^2 - \int_{\Omega_0} |\nabla v_0'|^2\right| = \left|\int_\Omega (A_t - Id)|\nabla\widehat{v_t'}|^2 + \nabla(\widehat{v_t'} - v_0')\cdot\nabla(\widehat{v_t'} + v_0')\right| \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}$$

and

$$\begin{aligned} \left|\lambda_1(t)\int_{\Omega_t} |v_t'|^2 - \lambda_1(0)\int_{\Omega_0} |v_0'|^2\right| &= \left|(\lambda_1(t) - \lambda_1(0))\int_{\Omega_t} |v_t'|^2 + \lambda_1(0)\int_{\Omega_0} (J_t - 1)|\widehat{v_t'}|^2 + (\widehat{v_t'} - v_0')(\widehat{v_t'} + v_0')\right| \\ &\le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}\|h\|^2_{\mathrm{H}^{1/2}(\partial\Omega)}. \end{aligned}$$

Estimate of $\mathcal{T}_2(t) - \mathcal{T}_2(0)$. After change of variable, we have $\mathcal{T}_2(t) = \int_{\partial\Omega} \sigma_t h^2$ where

$$\sigma_t = (\widehat{\partial_{\boldsymbol{n}_t} v_t})^2\left[\widehat{H_t}\widehat{\alpha_t}^2 - \widehat{\boldsymbol{B}_t}(\widehat{\beta_t}, \widehat{\beta_t}) - 2\widehat{\nabla_{\tau_t}(\alpha_t)}\cdot\widehat{\beta_t}\right] J_{\partial\Omega}(t)$$

and from Lemmas 4.7 and 4.8, we easily get $\|\sigma_t - \sigma_0\|_{\mathrm{L}^p(\partial\Omega)} \le C\|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}$. Notice that the control holds only in L^p and not in L^∞ as in [12] or [7, Appendix], hence we do not obtain a control with the L^2 norm of h. However, by Hölder inequality, it comes $|\mathcal{T}_2(t) - \mathcal{T}_2(0)| \le \|\sigma_t - \sigma_0\|_{\mathrm{L}^p}\|h\|^2_{\mathrm{L}^{\tilde p}}$ for any $\tilde p \ge 2p/(p-1)$. Since $\|h\|_{\mathrm{L}^{\tilde p}} \le C\|h\|_{\mathrm{H}^{1/2}}$ when $\tilde p < 2d/(d-1)$ by Sobolev embeddings, such a $\tilde p$ can be chosen provided $p > d$. Then, it holds

$$|\mathcal{T}_2(t) - \mathcal{T}_2(0)| \le \|\sigma_t - \sigma_0\|_{\mathrm{L}^p}\|h\|^2_{\mathrm{H}^{1/2}} \le C\|h\|_{\mathrm{W}^{2,p}}\|h\|^2_{\mathrm{H}^{1/2}}.$$

Estimate of $\mathcal{T}_3(t) - \mathcal{T}_3(0)$. After change of variable, we have $\mathcal{T}_3(t) = \int_{\partial\Omega} \rho_t\cdot(\nabla_{\widehat{\tau_t}} h)h$ where

$$\rho_t = (\widehat{\partial_{\boldsymbol{n}_t} v_t})^2\widehat{\alpha_t}\widehat{\beta_t}J_{\partial\Omega}(t), \qquad \text{and} \qquad \nabla_{\widehat{\tau_t}} h = \nabla h - (\nabla h\cdot\widehat{\boldsymbol{n}_t})\widehat{\boldsymbol{n}_t}$$

and we obtain (recall that $\nabla h \cdot \boldsymbol{n} = 0$):

$$\begin{aligned} |\mathcal{T}_3(t) - \mathcal{T}_3(0)| &\leq \left| \int_{\partial\Omega} \rho_t \cdot (\nabla_{\widehat{\tau}_t} h - \nabla_\tau h)\, h \right| + \left| \int_{\partial\Omega} (\rho_t - \rho_0) \cdot \nabla_\tau h)\, h \right| \\ &\leq \|\nabla_{\widehat{\tau}_t} h - \nabla_\tau h\|_{\mathrm{H}^{-1/2}} \|\rho_t h\|_{\mathrm{H}^{1/2}} + \|(\rho_t - \rho_0) h\|_{\mathrm{H}^{1/2}} \|\nabla_\tau h\|_{\mathrm{H}^{-1/2}} \\ &\leq \|\nabla h \cdot (\widehat{\boldsymbol{n}_t} - \boldsymbol{n})\|_{\mathrm{H}^{-1/2}} \|\rho_t h\|_{\mathrm{H}^{1/2}} + \|(\rho_t - \rho_0) h\|_{\mathrm{H}^{1/2}} \|h\|_{\mathrm{H}^{1/2}} \end{aligned}$$

In addition to (25), we also have from [37, Theorem 2 p 173] (see the comments on (25)):

$$\|uv\|_{\mathrm{H}^{-1/2}(\partial\Omega)} \leq C \|u\|_{\mathrm{W}^{s,p}(\partial\Omega)} \|v\|_{\mathrm{H}^{-1/2}(\partial\Omega)} \tag{29}$$

if $\max\{1/2, (d-1)/p\} < s \leq 1$. Using again Lemmas 4.3, 4.7 and 4.8, we get

$$\|\rho_t - \rho_0\|_{\mathrm{W}^{1-1/p,p}(\partial\Omega)} \leq C \|h\|_{\mathrm{W}^{2,p}(\partial\Omega)}, \qquad \|\widehat{\boldsymbol{n}_t} - \boldsymbol{n}_0\|_{\mathrm{W}^{1,p}(\partial\Omega)} \leq C \|h\|_{\mathrm{W}^{2,p}(\partial\Omega)},$$

which combined with (28), (25) and (29), concludes the estimate of this term and hence the proof. □

5 Applications

5.1 Retrieving some examples from the literature

In this paragraph, we apply our results to retrieve previous results from the literature:

Isoperimetric inequalities: According to the previous sections, the perimeter satisfy conditions $(\mathbf{C}_{\mathrm{H}^1})$ and $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ at any smooth enough set, and in particular for the ball. Moreover, as shows Section 2.3, we have

$$\ell_1[P](B_1) = (d-1)\ell_1[\mathrm{Vol}](B_1), \qquad \text{and} \qquad \ell_2[P - (d-1)\mathrm{Vol}](B_1)(\varphi,\varphi) = \sum_{k=0}^{\infty} \sum_{l=1}^{d_k} (k-1)(k+d-1)\ \alpha_{k,l}(\varphi)^2.$$

Moreover, $\varphi \in T(\partial B_1)$ if and only if $\alpha_{0,1}(\varphi) = \alpha_{1,i}(\varphi) = 0$ for $i \in \{1, \ldots, d\}$. Therefore B_1 is a critical and strictly stable shape for P under volume constraint, and up to translations: Theorem 1.3 applies, and we retrieve Fuglede's result from [19] about nearly spherical domains.

Recently in [32], different improved versions (even with a better distance than the Fraenkel asymmetry for d_1 in (2)) of the quantitative isoperimetric inequality has been achieved for the anisotropic perimeter

$$P_f(\Omega) = \int_{\partial\Omega} f(\boldsymbol{n}_{\partial\Omega})$$

where $f : \mathbb{R}^d \to \mathbb{R}_+$ is a convex positively 1-homogeneous function, whose minimizer under volume constraint is an homothetic version of the Wulff shape $K = \{f_* < 1\}$ where f_* is the gauge function of f. In particular in [32, Theorem 1.3 and Section 4] focused on the case where K is assumed to be C^2 and uniformly convex, a strategy based on the second variation is used: the author proves in [32, Lemma 4.1] that P_f satisfies conditions $(\mathbf{C}_{\mathrm{H}^1})$ and $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$. Therefore, this falls into the hypothesis of our Theorem 1.3, so if we prove that K satisfies (4), then we retrieve [32, Proposition 1.9] (we assumed the shape to be C^3, but here this can be reduced to C^2, as noticed in Remark 2.3). It is interesting to notice though that in order to show that K satisfies (4), the author in [32] uses the quantitative Wulff isoperimetric inequality from [18] (obtained with optimal transport method). Therefore, up to our knowledge, there is no proof "from scratch" of the quantitative anisotropic isoperimetric inequality using a result similar to Theorem 1.3.

The Ohta-Kawasaki model: In [2], both steps of the strategy described page 1 are achieved in order to deal with the following functional, formulated in $\mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$ and which includes a non-local term:

$$J(\Omega) = P_{\mathbb{T}^N}(\Omega) + \gamma G(\Omega) \quad \text{where } G(\Omega) = \int_{\mathbb{T}^N} |\nabla w_\Omega|^2 \quad \text{and} \quad \begin{cases} -\Delta w_\Omega &= \mathbb{1}_\Omega - \mathbb{1}_{\Omega^c} - m \ \text{ in } \mathbb{T}^N \\ \displaystyle\int_{\mathbb{T}^N} w_\Omega &= 0 \end{cases} \tag{30}$$

where $m = |\Omega| - |\Omega^c| \in (-1, 1)$ is fixed. Again, there is an invariance with translation and a volume constraint.

In order to handle the first step of the strategy, the authors in [2] prove a stability result for the $\mathrm{W}^{2,p}$-topology, for p large enough. The strategy is very similar to [12], but in the framework of $\mathrm{W}^{2,p}$-spaces rather than $\mathrm{C}^{2,\alpha}$-

spaces. Note that this difference in the choice of spaces is not just a detail as it is relevant for the second step of the strategy when proving stability in an L^1-neighborhood as it is done in [2, Section 4]: their regularization procedure needs to allow discontinuity of the mean curvature, see equation (4.9) in the proof of [2, Theorem 4.3]. From the computations of [10], we obtain

$$\ell_1[G](\Omega)(\varphi) = 4\int_{\partial\Omega} w_\Omega \varphi,$$

$$\ell_2[G](\Omega)(\varphi,\varphi) = 8\int_{\mathbb{T}^N} |\nabla z_\varphi|^2 dx + 4\int_{\partial\Omega} (\partial_{\boldsymbol{n}} w_\Omega + H)\varphi^2, \text{ where } -\Delta z_\varphi = \varphi \mathcal{H}^{N-1}\lfloor \partial\Omega$$

therefore G satisfies $(\mathbf{C}_{\mathrm{H}^{1/2}})$ and J satisfies $(\mathbf{C}_{\mathrm{H}^1})$, the dominant term being contained in the perimeter term. As we have seen that the perimeter satisfies $(\mathbf{IT}_{\mathrm{H}^1,\mathrm{W}^{1,\infty}})$ condition, it just remains to handle functional G, which is proven to satisfy $(\mathbf{IC}_{\mathrm{H}^1,\mathrm{W}^{2,p}})$ for $p > d$ in [2]. Therefore Theorem 1.3 applies, and we retrieve [2, Theorem 3.9].

The Faber-Krahn inequality: In [7] (see also [21]) a quantitative version of the Faber-Krahn inequality is achieved, using again the two steps described page 1: in order to achieve the first step, they use the Kohler-Jobin inequality ([28]), which implies that the Faber-Krahn deficit is controlled by the deficit of the Dirichlet energy E. We show here that it is possible to achieve this step without this "trick": we have seen that λ_1 satisfies $(\mathbf{C}_{\mathrm{H}^{1/2}})$ and $(\mathbf{IC}_{\mathrm{H}^{1/2},\mathrm{W}^{2,p}})$ for $p > d$, and for any $\varphi \in C^\infty(\partial B_1)$ such that $\int_{\partial B_1} \varphi = 0$, we have

$$\ell_1[\lambda_1](B_1) = -\beta_d^2 \ell_1[\mathrm{Vol}](B_1), \qquad \text{and} \qquad \ell_2[\lambda_1 + \beta_d^2 \mathrm{Vol}](B_1)(\varphi,\varphi) = 2\beta_d^2 \sum_{k=0}^{\infty}\sum_{l=1}^{d_k} Q_k\ \alpha_{k,l}(\varphi)^2.$$

where (using [1, Section 9.1.27, p 361])

$$Q_k = j_{d/2-1}\frac{J'_{k+d/2-1}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} + \frac{d}{2} = k + d - 1 - j_{d/2-1}\frac{J_{k+d/2}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} = j_{d/2-1}\frac{J_{k+d/2-2}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} - k + 1.$$

With the last formula, we easily notice that $Q_1 = 0$. The sign of Q_k can be obtained using [35, section 6.5 page 133] (done when $d = 2$, but as noticed in [26], valid for any d): indeed, their computations imply

$$j_{d/2-1}\frac{J'_{k+d/2-1}(j_{d/2-1})}{J_{k+d/2-1}(j_{d/2-1})} \geq k - d/2 - 1, \forall n \in \mathbb{N}^*,$$

which leads to $\forall k \geq 2, Q_k \geq k - 1$. Therefore Theorem 1.3 applies, and we retrieve a Faber-Krahn quantitative inequality in a $\mathrm{W}^{2,p}$-neighborhood of the ball.

5.2 Examples with competition

In this section, B is a ball, $X = \mathrm{W}^{2,p}(\partial B)$ for $p > d$ and we denote for $\eta > 0$ (see (6) for a definition of d_X):

$$\mathcal{V}_\eta = \{\Omega, d_X(\Omega, B) \leq \eta \text{ and } |\Omega| = |B|\}.$$

Combining Theorem 1.3 to the computations from Section 2.1, we easily obtain the following result:

Proposition 5.1 *There exists $\gamma_0 \in (0,\infty)$ such that for every $\gamma \in [-\gamma_0, \infty)$, there exists $\eta = \eta(\gamma) > 0$ and $c = c(\gamma) > 0$ such that for every $\Omega \in \mathcal{V}_\eta$,*

$$(P + \gamma E)(\Omega) \ \geq\ (P + \gamma E)(B) + c d_{\mathrm{H}^1}(\Omega, B)^2, \quad (P + \gamma\lambda_1)(\Omega) \ \geq\ (P + \gamma\lambda_1)(B) + c d_{\mathrm{H}^1}(\Omega, B)^2$$

$$(E + \gamma\lambda_1)(\Omega) \ \geq\ (E + \gamma\lambda_1)(B) + c d_{\mathrm{H}^{1/2}}(\Omega, B)^2, \quad (\lambda_1 + \gamma E)(\Omega) \ \geq\ (\lambda_1 + \gamma E)(B) + c d_{\mathrm{H}^{1/2}}(\Omega, B)^2.$$

Proof of Proposition 5.1: We show that we can apply Theorem 1.3 to $\Omega^* = B$ and

$$J \in \{P + \gamma E, P + \gamma\lambda_1, E + \gamma\lambda_1, \lambda_1 + \gamma E)\}.$$

It is shown in Sections 3.1 and 4 that (P, E, λ_1) satisfy $(\mathbf{C}_{\mathrm{H}^{s_2}})$ and $(\mathbf{IT}_{\mathrm{H}^{s_2},X})$ for suitable values of s_2, and with Lemmata 2.9 and 2.8 we easily check that the ball is a critical and strictly stable domain for J under volume constraint and up to translations, either if $\gamma \geq 0$ or if $\gamma < 0$ is small enough. □

Corollary 5.2 *With the same notations as in Proposition 5.1, we have, with $\eta_0 = \eta(\gamma_0)$:*

$$\forall \Omega \in \mathcal{V}_{\eta_0}, \quad \frac{P(\Omega) - P(B)}{E(\Omega) - E(B)} \geq \gamma_0, \qquad \frac{P(\Omega) - P(B)}{\lambda_1(\Omega) - \lambda_1(B)} \geq \gamma_0$$

$$\gamma_0 \leq \frac{\lambda_1(\Omega) - \lambda_1(B)}{E(\Omega) - E(B)} \leq \gamma_0^{-1}.$$

Remark 5.3 In [33], the second inequality in Corollary 5.2 is also investigated, but we provide here a uniform neighborhood so that this estimate applies. We also refer to [35] for some result of this kind.

Remark 5.4 To the contrary to the last two-sided inequality, it is not possible to bound the first two ratio from above. Indeed we can observe that for any given $\gamma < 0$, the ball is not a local minimizer of $E + \gamma P$ or $\lambda_1 + \gamma P$ in $X = W^{2,p}(\partial B)$, under volume constraint. More precisely, one obtains from Proposition 4.5 and Theorem 1.4 that for $\eta > 0$ small enough, there exists $c(\eta) > 0$ such that for any $h \in C^\infty(\partial B)$ such that $\|h\|_X \leq \eta$,

$$\frac{P(B_h) - P(B)}{E(B_h) - E(B)} \geq c(\eta) \frac{\|h\|_{H^1}^2}{\|h\|_{H^{1/2}}^2}.$$

One can now choose h oscillating so that $\|h\|_X \leq \eta$ and $\|h\|_{H^{1/2}}$ is very small compare to $\|h\|_{H^1}$ (for example in dimension 2, we can choose $h_n(x) = \alpha_n \sin(nx)$ where α_n is chosen to satisfy $\|h\|_{W^{2,p}} \leq \eta$. We can do this construction with a preservation of the volume (??DETAILS???). The same argument works also for the ration $(P(\Omega) - P(B))/(\lambda_1(\Omega) - \lambda_1(B))$.

This phenomenon is due to the fact that the functionals P and (E, λ_1) satisfy conditions $(\mathbf{C}_{\mathrm{H}^{s_2}})$ for different values of s_2.

Explicit constants: We want to go further and compute explicit numbers γ such that the inequalities of Proposition 5.1 holds. To simplify the expressions, we restrict ourselves to the case of the unit ball. In the first two cases, we find the optimal constant, see Remark 5.6 about the other cases.

Proposition 5.5 *Using notations of Proposition 5.1 and β_d defined in* (13),

(i) *if $\gamma > -(d+1)d^2$, then B_1 is a local strict minimizer of $P + \gamma E$. Moreover, when $\gamma = -(d+1)d^2$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -(d+1)d^2$, the ball is a saddle shape for $P + \gamma E$.*

(ii) *if $\gamma > -\dfrac{d(d+1)}{2\beta_d^2(j_{d/2-1}^2 - d)}$, then B_1 is a local strict minimizer of $P + \gamma\lambda_1$. Moreover, when $\gamma = -\dfrac{d(d+1)}{2\beta_d^2(j_{d/2-1}^2 - d)}$, the second derivative of the Lagrangian cancels in some directions and when $\gamma < -\dfrac{d(d+1)}{2\beta_d^2(j_{d/2-1}^2 - d)}$, the ball is a saddle shape for $P + \gamma\lambda_1$.*

(iii) *if $\gamma > -\dfrac{1}{d^2(d+1)\beta_d^2}$, then B_1 is a local strict minimizer of $E + \gamma\lambda_1$.*

(iv) *if $\gamma > -\beta_d^2 d^2$, then B_1 is a local strict minimizer of $\lambda_1 + \gamma E$.*

Remark 5.6 In the cases (iii) and (iv), the constants we compute are not optimal, in particular we do not claim the ball is a saddle point once we go beyond the computed value. Nevertheless computing the optimal value only requires to compute $\sup_{k\geq 2} \tau_k'$ and $\sup_{k\geq 2} \tau_k''$ (see the notations in the proof below) as it is done in the cases (i) and (ii). As it is seen in the second case (ii) handled by Nitsch in [33], these computations can be rather technical. Let us notice also that we simplify the expression of the optimal constant given by Nitsch.

Proof of Proposition 5.5:
(i) We first compute the Lagrange multiplier $\mu(t)$ associated to the volume constraint at B_1: it is defined as $\ell_1[P + tE + \mu(t)\mathrm{Vol}] = 0$ that is from the expression of the shape gradients of Vol, P and E:

$$\mu(t) = \frac{1}{2d^2}\, t \; - (d-1).$$

Let us now turn our attention to hessian of the function $P + tE + \mu(t)\mathrm{Vol}$ on the ball B_1. As a consequence of Lemma 2.9, the shape hessian of the lagrangian $P + tE + \mu(t)\mathrm{Vol}$ at balls is

$$\ell_2[P + tE + \mu(t)\mathrm{Vol}](B_1).(\varphi, \varphi) = \sum_{k=0}^{\infty} c_k(t) \sum_{l=1}^{d_k} \alpha_{k,l}(\varphi)^2$$

where we have set

$$c_k(t) = k^2 + \left[(d-2) + \frac{1}{d^2}\, t\right] k - \left[(d-1) + \frac{1}{d^2}\, t\right] = (k-1)\left[k + (d-1) + \frac{1}{d^2}\, t\right].$$

Therefore, the hessian of the Lagrangian $\ell_2[P + tE + \mu(t)\mathrm{Vol}](B_1)$ is coercive in $T(\partial B_1)$ if and only if t solves the inequalities

$$k + (d-1) + \frac{1}{d^2}\, t > 0$$

for all $k \geq 2$. Of course, it suffices to solves that inequality in the special case $k = 2$ that provides $t > -(d+1)d^2$.

(ii) With the same notations as in (i) with $P + t\lambda_1 + \mu(t)\mathrm{Vol}$, we obtain :

$$\mu(t) = \beta_d^2\, t \; - (d-1), \qquad c_k(t) = k^2 + (d-2)k - (d-1) + 2t\beta_d^2 \left[k + (d-1) - j_{d/2-1} \frac{J_{k+d/2}(j_{d/2-1})}{J_{k-1+d/2}(j_{d/2-1})}\right]. \quad (31)$$

We introduce the sequences $a_k = J_{k-1+d/2}(j_{d/2-1})$ and $b_k = a_{k+1}/a_k$ so that (31) can be written:

$$c_k(t) = k^2 + (d-2)k - (d-1) + 2t\beta_d^2 \left[k + d - 1 - j_{d/2-1} b_k\right].$$

For a given integer $k \geq 2$, $c_k(t) > 0$ holds when $t > \tau_k$ defined as

$$\tau_k = -\frac{(k-1)(k+d-1)}{2\beta_d^2(k + d - 1 - j_{d/2-1} b_k)}.$$

In order to find the optimal value of t so that these inequalities are satisfied for every $k \geq 2$, we need to compute the supremum of $\{\tau_k, k \geq 2\}$. It is proven by Nitsch in [33, proof of Lemma 2.3, p 332] that for all $k \geq 2, \tau_k \leq \tau_2$, so the ball is strictly stable if and only if $t > \tau_2$. We describe here how one can obtain a more explicit version of τ_2: from the recurrence formula for Bessel function ([1, section 9.1.27, p 361])

$$(2\nu/z) J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z)$$

applied to $\nu = k - 1 + d/2$ and $z = j_{d/2-1}$, the sequences a_k and b_k satisfy the recurrence property

$$a_{k+1} = \frac{2(k-1) + d}{j_{d/2-1}}\, a_k - a_{k-1} \text{ and } b_k = \frac{2(k-1) + d}{j_{d/2-1}} - \frac{1}{b_{k-1}}$$

with the initial terms $a_0 = 0$ and $a_1 = J_{d/2}(j_{d/2-1})$ so that $b_1 = a_2/a_1 = d/j_{d/2-1}$ (which explains $c_1(t) = 0$ for any t, as known for the invariance by translations of all the involved functions). Therefore, we have:

$$b_2 = \frac{2+d}{j_{d/2-1}} - \frac{j_{d/2-1}}{d} = \frac{d(d+2) - j_{d/2-1}^2}{d j_{d/2-1}}$$

and as a consequence, we obtain that

$$\tau_2 = -\frac{d(d+1)}{2\beta_d^2(j_{d/2-1}^2 - d)}.$$

(iii) With the same notions as in (i) with $E + t\lambda_1 + \mu(t)\mathrm{Vol}$, we obtain :

$$\mu(t) = (1/d^2) + t\beta_d^2, \qquad c_k(t) = \left(\frac{1}{d^2} + t\beta_d^2\right) k - \frac{1}{d^2} + t\beta_d^2 \left[d - 1 - j_{d/2-1} b_k\right].$$

Again $c_1(t) = 0$ and $c_k(t) > 0$ if and only if

$$t > \tau_k' = -\frac{k-1}{d^2\beta_d^2(k + d - 1 - j_{d/2-1} b_k)}.$$

Using that $b_1 \geq b_k > 0$, we obtain

$$\tau_k' < -\frac{1}{d^2\beta_d^2}\, \frac{k-1}{k+d-1} = -\frac{1}{d^2\beta_d^2}\left(1 - \frac{d}{k+d-1}\right) \leq -\frac{1}{d^2(d+1)\beta_d^2}.$$

Therefore, if $t > -\dfrac{1}{d^2(d+1)\beta_d^2}$ then for any $k \geq 2$, $t > \tau_k'$, which leads to the result.

(iv) With the same notions as in (i) with $\lambda_1 + tE + \mu(t)\mathrm{Vol}$, we obtain :

$$\mu(t) = (t/d^2) + \beta_d^2, \quad c_k(t) = \left(\frac{t}{d^2} + \beta_d^2\right) k - \frac{t}{d^2} + \beta_d^2 \left[d - 1 - j_{d/2-1} b_k\right].$$

We check $c_1(t) = 0$, and $c_k(t) > 0$ if and only if

$$t > \tau_k'' = -\beta_d^2 d^2 \left(1 + \frac{d - j_{d/2-1} b_k}{k-1}\right).$$

Using that $b_1 \geq b_k > 0$, we obtain $\tau_k'' \leq -\beta_d^2 d^2$, and therefore, if $t > -\beta_d^2 d^2$ then for any $k \geq 2$, $t > \tau_k''$, which leads to the result. □

6 Counterexample for non smooth perturbations

We show in this section that even if the ball is a local minimum in a smooth neighborhood, it may not be a local minimum in a non-smooth neighborhood.

Consider $\Omega^* = B$ a ball of volume V_0. We have seen in Proposition 5.1 that there is a real number $\gamma_0 \in (0, \infty)$ such that for every $\gamma \in (-\gamma_0, \infty)$, B is a stable local minimum for $P + \gamma E$.

For $\gamma \geq 0$ this is not surprising. However, *for $\gamma < 0$, the fact that the ball is a local minimizer is no longer trivial*: there is a competition between the minimization of the perimeter and maximization of the Dirichlet energy. If γ is small enough, our result shows that B is still a local minimizer in a $\mathrm{W}^{2,p}$-neighborhood. Nevertheless, in that case B is no longer a local minimizer in a L^1-neighborhood :

Proposition 6.1 *Let B be a ball. For every $\gamma < 0$ and any $\varepsilon > 0$ one can find Ω_ε such that*

$$|\Omega_\varepsilon \Delta B| < \varepsilon, \quad |\Omega_\varepsilon| = |B|, \quad \text{and} \quad (P + \gamma E)(\Omega_\varepsilon) < (P + \gamma E)(B).$$

To prove this result, we use the idea of topological derivative: it is well known that if one consider a small hole of size ε in the interior of a fixed shape, the energy will change at order ε^{d-2} if $d \geq 3$ and $1/\log(\varepsilon)$ if $d = 2$, which is strictly bigger than the change of perimeter which is of order ε^{d-1}, and therefore will strictly decrease the energy $P + \gamma E$ when $\gamma < 0$. For the sake of completeness, we provide a proof of this fact for a centered hole.

Proof. We can assume without loss of generality (using translation and scaling properties) that $B = B_1$ is the centered ball of radius 1, and we define $\Omega_\varepsilon = B_1 \setminus B(0, \varepsilon)$. Using that $\Delta u = \partial_{rr} u + \frac{d-1}{r} \partial_r u$ when u is radial, the state function is:

$$u_{\Omega_\varepsilon}(r) = \frac{(\varepsilon^{d-2} - \varepsilon^d) r^{2-d} + \varepsilon^d - 1}{2d(\varepsilon^{d-2} - 1)} - \frac{r^2}{2d}, \quad \text{if } d \geq 3$$

$$u_{\Omega_\varepsilon}(r) = \frac{1 - \varepsilon^2}{-4\log(\varepsilon)} \log(r) + \frac{1 - r^2}{4}, \quad \text{if } d = 2$$

and therefore

$$\begin{aligned}
\text{if } d \geq 3, \quad E(\Omega_\varepsilon) &= -\frac{1}{2} \int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{d(1-\varepsilon^2)^2 \varepsilon^{d-2} - 2(1-\varepsilon^d)^2}{8d^2(1-\varepsilon^{d-2})} + \frac{1 - \varepsilon^{d+2}}{4d(d+2)}\right] P(B_1) \\
&= \left[-\frac{1}{2d^2(d+2)} + \frac{d-2}{8d^2} \varepsilon^{d-2} + o(\varepsilon^{d-2})\right] P(B_1), \\
\text{if } d = 2, \quad E(\Omega_\varepsilon) &= -\frac{1}{2} \int_{\Omega_\varepsilon} u_{\Omega_\varepsilon} = \left[\frac{(1-\varepsilon^2)}{-8\log(\varepsilon)} (1 - \varepsilon^2(1 - 2\log(\varepsilon))) - \frac{1}{16}(1 - \varepsilon^2 + \frac{\varepsilon^4}{2})\right] P(B_1) \\
&= \left[-\frac{1}{16} - \frac{1}{8\log(\varepsilon)} + o\left(\frac{1}{\log(\varepsilon)}\right)\right] P(B_1).
\end{aligned}$$

We now define $\widetilde{\Omega_\varepsilon} = \mu_\varepsilon \Omega_\varepsilon$ where $\mu_\varepsilon = (1 - \varepsilon^d)^{-1/d}$ so that

$$|\widetilde{\Omega_\varepsilon}| = |B_1|, \quad P(\widetilde{\Omega_\varepsilon}) - P(B_1) = \left[\mu_\varepsilon^{d-1}(1 + \varepsilon^{d-1}) - 1\right] P(B_1) \sim_{\varepsilon \to 0} \varepsilon^{d-1} P(B_1)$$

$$E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon \to 0} \frac{(d-2)P(B_1)}{8d^2} \varepsilon^{d-2} > 0, \quad \text{if } d \geq 3, \quad E(\widetilde{\Omega_\varepsilon}) - E(B_1) \sim_{\varepsilon \to 0} \frac{P(B_1)}{-8\log(\varepsilon)} > 0, \quad \text{if } d = 2$$

so that in both cases, for any negative γ, $(P + \gamma E)(\Omega_\varepsilon) - (P + \gamma E)(B_1) < 0$ for small ε. □

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