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# Thin-shell concentration for convex measures

Matthieu Fradelizi, Olivier Guédon and Alain Pajor

## Abstract

We prove that for  $s < 0$ ,  $s$ -concave measures on  $\mathbb{R}^n$  satisfy a thin-shell concentration similar to the log-concave case. It leads to a Berry-Esseen type estimate for most of their one dimensional marginal distributions. We also establish sharp reverse Hölder inequalities for  $s$ -concave measures.

## 1 Introduction

For any subsets  $A, B \subset \mathbb{R}^n$ , the Minkowski sum is defined by

$$A + B = \{a + b : a \in A, b \in B\}.$$

Let  $s \in [-\infty, 1]$ . A measure  $\mu$  on  $\mathbb{R}^n$  is called  $s$ -concave whenever

$$\mu((1 - \lambda)A + \lambda B) \geq ((1 - \lambda)\mu(A)^s + \lambda\mu(B)^s)^{1/s},$$

for every  $\lambda \in [0, 1]$  and every compact subsets  $A, B \subset \mathbb{R}^n$  such that  $\mu(A)\mu(B) > 0$ . When  $s = 0$ , this inequality should be read as

$$\mu((1 - \lambda)A + \lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^\lambda$$

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and it defines  $\mu$  as a log-concave measure. When  $s = -\infty$ , the measure is said to be convex and the inequality is replaced by

$$\mu((1-\lambda)A + \lambda B) \geq \min(\mu(A), \mu(B)).$$

Notice that the class of  $s$ -concave measures on  $\mathbb{R}^n$  is decreasing in  $s$  so that any  $s$ -concave measure is a convex measure. Any  $s$ -concave measure with  $s \geq 0$  is log-concave and the thin-shell concentration for log-concave measures has been studied in [16, 17, 19, 22, 23]. The purpose of this paper is to prove a thin-shell concentration for  $s$ -concave measures in the case  $s < 0$ , which we consider from now on. By measure, we always mean probability measure.

The class of  $s$ -concave measures was introduced and studied in [10, 11], where a complete characterization was established. An  $s$ -concave measure is supported on some convex subset of an affine subspace where it has a density (see Section 2 for more details). When the support of an  $s$ -concave measure  $\mu$  generates the whole space, we say that  $\mu$  is full-dimensional.

A random vector with an  $s$ -concave distribution is called  $s$ -concave. The linear image of an  $s$ -concave random vector is also  $s$ -concave. We say that a random vector is full-dimensional if its distribution is full-dimensional. It is known that any semi-norm of an  $s$ -concave random vector with  $s < 0$  has moments of all order  $p \in (0, -1/s)$  (see [10] and [1]). The Euclidean norm of an  $s$ -concave random vector  $X$  has a finite moment of order 2 if and only if  $s > -1/2$ . Since we are interested in comparison of moments of the Euclidean norm with the moment of order 2, we will always assume that  $-1/2 < s < 0$ .

Let  $n \geq 1$  be an integer. The Euclidean space  $\mathbb{R}^n$  is equipped with its Euclidean norm  $|\cdot|_2$  and its scalar product  $\langle \cdot, \cdot \rangle$ . Its unit sphere is denoted by  $S^{n-1}$  and its unit ball by  $B_2^n$ . We say that a random vector  $X$  is isotropic if  $\mathbb{E}X = 0$  and for every  $\theta \in S^{n-1}$ ,  $\mathbb{E}\langle X, \theta \rangle^2 = 1$ . Observe that if  $X$  is an  $s$ -concave full-dimensional random vector and  $-1/2 < s$ , we can always find an affine transformation  $A$  such that  $AX$  is isotropic.

Let  $p \in \mathbb{R}$  and  $X \in \mathbb{R}^n$  be a random vector. Assume that  $|X|_2$  has finite moments of order 2 and  $p$  with the convention that  $(\mathbb{E}|X|_2^p)^{1/p} = \exp(\mathbb{E} \ln |X|_2)$  for  $p = 0$ . We define

$$\alpha_p(X) := \left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right|.$$

Our main result is the following

**Theorem 1.** *Let  $r > 2$ . Let  $X \in \mathbb{R}^n$  be a full-dimensional  $(-1/r)$ -concave random vector.*

*If  $X$  is isotropic, then for any  $p$  such that  $|p| \leq c \min(r, n^{1/3})$ , we have*

$$\alpha_p(X) \leq \frac{C |p - 2|}{r} + \left( \frac{C |p - 2|}{n^{1/3}} \right)^{3/5},$$

where  $C$  and  $c$  are universal constants.

*In the general case (when  $X$  is not isotropic), let  $A$  be an affine transformation such that  $AX$  is full-dimensional and isotropic. Then for any  $p \in \mathbb{R}$  such that  $|p| \leq c \min\left(r, \frac{n^{1/3}}{\|A\|^{2/3} \|A^{-1}\|^{2/3}}\right)$ , we have*

$$\alpha_p(X) \leq \frac{C |p - 2|}{r} + \left( \frac{C |p - 2| (\|A\| \|A^{-1}\|)^{2/3}}{n^{1/3}} \right)^{3/5},$$

where  $C$  and  $c$  are universal constants.

We also show (see Remark 15) that for  $r > n + \sqrt{n}$ , the estimate of  $\alpha_p(X)$  in Theorem 1 can be improved and recovers the estimate of the log-concave case from [19].

To present the connections between moment inequalities, concentration in a thin-shell and the Berry-Esseen theorem for one dimensional marginals, let us introduce some notations.

Let  $X \in \mathbb{R}^n$  be an isotropic random vector. Thus  $\mathbb{E}|X|_2^2 = n$ . Define  $\varepsilon(X)$  to be the smallest number  $\varepsilon > 0$  such that

$$\mathbb{P}\left(\left|\frac{|X|_2}{\sqrt{n}} - 1\right| \geq \varepsilon\right) \leq \varepsilon. \quad (1)$$

If  $\varepsilon(X) = o(1)$  with respect to the dimension  $n$ , we say that  $X$  is concentrated in a thin-shell. This is the usual jargon of the subject. More rigorously, it suggests that we are considering a sequence of random vectors  $(X_n)$  with  $X_n \in \mathbb{R}^n$  and that  $\varepsilon(X_n) = o(1)$  as  $n$  goes to  $\infty$ . It was shown in [2] (see also [13, 14]) that if an isotropic random vector  $X$  uniformly distributed on a convex body in  $\mathbb{R}^n$  is such that  $\varepsilon(X) = o(1)$ , then almost all one dimensional marginal distributions of  $X$  satisfy a Berry-Esseen theorem. More generally, let  $X \in \mathbb{R}^n$  be an isotropic random vector, it was proved in [7] that

$$\sigma_{n-1}\left(\theta \in S^{n-1} : \sup_{t \in \mathbb{R}} |\mathbb{P}(\langle X, \theta \rangle \leq t) - \Phi(t)| \geq 4\varepsilon(X) + \delta\right) \leq 4n^{3/8} e^{-cn\delta^4},$$

where  $\sigma_{n-1}$  denotes the rotation invariant probability measure on the unit sphere  $S^{n-1}$ ,  $\Phi$  is the standard normal distribution function and  $c > 0$  is a universal constant. It is worth noticing that the result from [7] does not assume log-concavity. Assuming only that  $X$  is isotropic, we get that if  $\varepsilon(X)$  is  $o(1)$  then almost all the one dimension marginal distributions of  $X$  are approximately Gaussian. The fact that indeed for all log-concave random vector  $\varepsilon(X) = o(1)$  was proved later in [22, 17] and the best estimate at this date [19] is that

$$\varepsilon(X) = O(n^{-1/6} \log n).$$

Now let  $p > 2$  and assume that  $X$  is isotropic and that  $|X|_2$  has a finite moment of order  $p$ . Then  $\varepsilon(X)$  is  $o(1)$  if and only if  $\alpha_p(X)$  is  $o(1)$ , see Remark 4 below. Hence Theorem 1 ensures that if  $r \rightarrow +\infty$  with the dimension  $n$  then any isotropic  $(-1/r)$ -concave random vector satisfies a thin-shell concentration and therefore almost all its one dimensional marginals verify a Berry-Esseen theorem. As a matter of fact, this condition on  $r$  is necessary. If  $r$  is fixed and does not depend on the dimension  $n$ , Proposition 5 gives an example of an isotropic  $(-1/r)$ -concave random vector  $X \in \mathbb{R}^n$  which does not satisfy a thin-shell concentration. Remark 6 also shows the asymptotic sharpness of Theorem 1, since for this example, for a fixed  $p > 2$ ,  $\alpha_p(X) \geq C(p-2)/r$  for  $r$  and  $n$  large enough, where  $C > 0$  is a universal constant.

To build the proof of Theorem 1, we need to extend to the case of  $s$ -concave measures several tools coming from the study of log-concave measures. This is the purpose of Section 2. Some of them were already achieved by Bobkov [8], like analog of the Ball's bodies [5] in the  $s$ -concave setting. Some others were also noticed previously (see e.g. [8], [1]) but not with the most accurate point of view. These new ingredients are analog to the results of [12] in the log-concave setting and are at the heart of our proof. As in the approach of [16] or [19], an important ingredient is the log-Sobolev inequality on  $SO(n)$ . It follows e.g. from the work of Bakry and Émery [4] and the calculus of the Ricci curvature of  $SO(n)$  (see [21, Formula (F6)] for example) that for any Lipschitz function  $f : SO(n) \rightarrow \mathbb{R}^+$  (see sections 3 and 4 for definitions)

$$\mathbb{E}(f(U) \log f(U)) - \mathbb{E}f(U) \log(\mathbb{E}f(U)) \leq \frac{c}{n} \mathbb{E}(|\nabla \log f(U)|^2 f(U)), \quad (2)$$

where  $U$  is uniformly distributed on  $SO(n)$ . It allows to get reverse Hölder inequalities (see inequality (15) in [16]): for every  $f : SO(n) \rightarrow \mathbb{R}$ , let  $L$  be

the log-Lipschitz constant of  $f$  (that is the Lipschitz constant of  $\log f$ ), then for every  $q > r > 0$ ,

$$(\mathbb{E}|f(U)|^q)^{1/q} \leq \exp\left(\frac{cL^2}{n}(q-r)\right) (\mathbb{E}|f(U)|^r)^{1/r}, \quad (3)$$

where  $U$  is uniformly distributed on  $SO(n)$ .

Let  $X$  be a  $(-1/r)$ -concave random vector in  $\mathbb{R}^n$  with full-dimensional support and distributed according to a measure with a density function  $w : \mathbb{R}^n \rightarrow \mathbb{R}_+$ . For any linear subspace  $E$ , denote by  $P_E$  the orthogonal projection onto  $E$  and for any  $x \in E$  denote by

$$\pi_E w(x) = \int_{x+E^\perp} w(y) dy$$

the marginal of  $w$  on  $E$ . Given an integer  $k$  between 1 and  $n$ , a real number  $p \in (-k, r)$ , a linear subspace  $E_0$  of  $\mathbb{R}^n$  of dimension  $k$  and  $\theta_0 \in S(E_0)$ , where  $S(E_0)$  denotes the unit sphere of  $E_0$ , we define the function  $h_{k,p} : SO(n) \rightarrow \mathbb{R}_+$  by

$$h_{k,p}(u) := |S^{k-1}| \int_0^\infty t^{p+k-1} \pi_{u(E_0)} w(tu(\theta_0)) dt, \quad (4)$$

for every  $u \in SO(n)$ , where  $|S^{k-1}|$  denotes the area of the sphere.

Following the approach of [23, 16], we observe that for any  $p \in (-k, r)$

$$\mathbb{E}|X|_2^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \mathbb{E}h_{k,p}(U), \quad (5)$$

where  $U$  is uniformly distributed on  $SO(n)$ . In view of (5) and the definition of  $h_{k,p}$ , we notice that it is of importance to work with family of measures which are stable after taking the marginals and it is clear from the definition that for any subspace  $E$ , if  $X$  is  $(-1/r)$ -concave, then  $P_E X$  is also  $(-1/r)$ -concave.

In the next section 2, we first introduce more notation and recall important facts concerning convex measures. Then we give an example of an isotropic  $(-1/r)$ -concave random vector  $X \in \mathbb{R}^n$  that does not satisfy a thin-shell concentration, when  $r$  is fixed with respect to the dimension. Finally, we extend to the case of  $s$ -concave measures several tools coming from the study of log-concave measures that will be essential in the proof of Theorem 1. Section 3 is devoted to the proof of Theorem 1. Some of the results of

these two sections are either classical or variation of known results; their proofs are shifted to the appendix.

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## 2 Preliminary results for $s$ -concave measures

We first recall some properties of  $s$ -concave measures and their relation to  $\beta$ -concave functions.

The class of  $s$ -concave measures was introduced and studied in [10, 11], where the following complete characterization was established. An  $s$ -concave measure  $\mu$  on  $\mathbb{R}^n$  is supported on some convex subset of an affine subspace where it has a density. When this subspace is the whole space, we say that  $\mu$  is full-dimensional. In this case, its density  $w$  is  $\beta$ -concave with  $\beta = s/(1 - ns)$ . Recall that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called  $\beta$ -concave whenever

$$f((1 - \lambda)x + \lambda y) \geq ((1 - \lambda)f(x)^\beta + \lambda f(y)^\beta)^{1/\beta}$$

for every  $\lambda \in [0, 1]$  and every  $x, y \in \mathbb{R}^n$  such that  $f(x)f(y) > 0$ , where the right hand side is replaced by  $f(x)^{1-\lambda}f(y)^\lambda$  for  $\beta = 0$ . Note that when  $\beta < 0$  which will be the case below,  $\beta$ -concavity means that  $f^\beta$  is convex on its convex support  $\{f > 0\}$ .

We will use a similar language for probability measure, random vector and function which are related here as distribution, law of a random vector and density of probability. It is important to remember that when  $X \in \mathbb{R}^n$  is  $(-1/r)$ -concave full-dimensional, then the result recalled above states that its distribution has a support that generates  $\mathbb{R}^n$  and has a density which is  $(-1/(n + r))$ -concave.

Recall that for every  $x > 0$ ,  $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u} du$  and for every  $x, y > 0$ ,  $B(x, y) = \int_0^1 u^{x-1}(1 - u)^{y-1} du = \int_0^{+\infty} u^{x-1}(u + 1)^{-(x+y)} du$ .

The following inequality of Paley-Zygmund type is well known.

**Lemma 2.** *Let  $2 < p < s$ . Let  $Y$  be a non-negative random variable with finite  $s$ -moment. Then for every  $0 \leq t \leq (\mathbb{E}Y^p)^{1/p}$  we have*

$$\mathbb{P}(Y \geq t) \geq \left( \frac{\mathbb{E}Y^p - t^p}{(\mathbb{E}Y^s)^{p/s}} \right)^{s/(s-p)}.$$

*Proof.* Using Hölder inequality, we have

$$\mathbb{E}Y^p = \mathbb{E}Y^p 1_{Y < t} + \mathbb{E}Y^p 1_{Y \geq t} \leq t^p + (\mathbb{E}Y^s)^{p/s} \mathbb{P}(Y \geq t)^{1-p/s}.$$

Thus

$$\mathbb{P}(Y \geq t) \geq \left( \frac{\mathbb{E}Y^p - t^p}{(\mathbb{E}Y^s)^{p/s}} \right)^{s/(s-p)}.$$

□

**Proposition 3.** *Let  $2 < p < s$ . Let  $X \in \mathbb{R}^n$  be an isotropic random vector such that  $|X|_2$  has a finite  $s$ -moment. Then*

$$\min \left( \frac{\alpha_p(X)}{2}, \left( \frac{p\alpha_p(X)/2}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)} \right) \leq \varepsilon(X) \leq ((\alpha_p(X) + 1)^p - 1)^{1/3}.$$

*Proof.* Let  $\varepsilon > 0$ . Applying Lemma 2 to  $Y = |X|_2 / (\mathbb{E}|X|_2^2)^{1/2}$ ,  $t = \varepsilon + 1$  and noticing that  $\mathbb{E}Y^p = (\alpha_p(X) + 1)^p$ ,  $\mathbb{E}Y^s = (\alpha_s(X) + 1)^s$ , we get that

$$\mathbb{P} \left( \frac{|X|_2}{(\mathbb{E}|X|_2^2)^{1/2}} \geq 1 + \varepsilon \right) \geq \left( \frac{(\alpha_p(X) + 1)^p - (\varepsilon + 1)^p}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}$$

whenever  $0 < \varepsilon \leq \alpha_p(X)$ . Since for  $p \geq 1$  and  $x \geq y \geq 1$ ,  $x^p - y^p \geq p(x - y)$ , we have

$$\mathbb{P} \left( \frac{|X|_2}{(\mathbb{E}|X|_2^2)^{1/2}} \geq 1 + \varepsilon \right) \geq \left( \frac{p(\alpha_p(X) - \varepsilon)}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}.$$

Therefore

$$\mathbb{P} \left( \frac{|X|_2}{(\mathbb{E}|X|_2^2)^{1/2}} \geq 1 + \varepsilon \right) \geq \left( \frac{p\alpha_p(X)/2}{(\alpha_s(X) + 1)^p} \right)^{s/(s-p)}$$

whenever  $0 < \varepsilon \leq \alpha_p(X)/2$ . The left-hand side inequality follows.

Since for  $q \geq 1$ ,  $|x - 1| \leq |x^q - 1|$  for every  $x \geq 0$ , Markov inequality gives

$$\mathbb{P} \left( \left| \frac{|X|_2}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \geq \varepsilon \right) \leq \mathbb{P} \left( \left| \frac{|X|_2^q}{(\mathbb{E}|X|_2^2)^{q/2}} - 1 \right| \geq \varepsilon \right) \leq \frac{\mathbb{E} \left| \frac{|X|_2^q}{(\mathbb{E}|X|_2^2)^{q/2}} - 1 \right|^2}{\varepsilon^2}.$$

To conclude the right-hand side inequality, take  $q = p/2$  and observe that

$$\mathbb{E} \left| \frac{|X|_2^q}{(\mathbb{E}|X|_2^2)^{q/2}} - 1 \right|^2 = (\alpha_{2q}(X) + 1)^{2q} + 1 - 2(\alpha_q(X) + 1)^q \leq (\alpha_{2q}(X) + 1)^{2q} - 1.$$

□



**Remark 4.** Let  $2 < p < s$ . Let  $X \in \mathbb{R}^n$  be an isotropic random vector such that  $|X|_2$  has a finite  $s$ -moment. Proposition 3 shows that  $\varepsilon(X)$  is  $o(1)$  if and only if  $\alpha_p(X)$  is  $o(1)$  when  $n \rightarrow \infty$ .

Now we estimate  $\varepsilon(X)$  for an example which shows that an isotropic  $(-1/r)$ -concave random vector  $X \in \mathbb{R}^n$  may not satisfy a thin-shell concentration. In the proposition below, the notation  $\underline{\lim}$  refers to the limit inferior.

**Proposition 5.** Let  $r > 2$ . There exists a sequence  $(X_n)_n$  of isotropic  $(-1/r)$ -concave random vectors  $X_n \in \mathbb{R}^n$  such that

$$\underline{\lim}_{n \rightarrow \infty} \varepsilon(X_n) \geq c(r) > 0,$$

where  $c(r) > 0$  depends only on  $r$ .

*Proof.* Let  $r > 2$  and  $2 < p < r$  and let  $X_n \in \mathbb{R}^n$  be an isotropic random vector with density

$$f_{n,r}(x) = \frac{c_1}{(1 + c_2|x|_2)^{r+n}},$$

where  $c_1$  and  $c_2$  are normalization factors. From [10, 11], such a random vector is  $(-1/r)$ -concave. An immediate computation gives that

$$\frac{(\mathbb{E}|X_n|_2^p)^{1/p}}{(\mathbb{E}|X_n|_2^2)^{1/2}} = \left( \frac{B(n+p, r-p)}{B(n, r)} \right)^{1/p} \left( \frac{B(n+2, r-2)}{B(n, r)} \right)^{-1/2}.$$

For fixed  $r$  and  $2 < p < r$ , we have

$$\lim_{n \rightarrow +\infty} \frac{(\mathbb{E}|X_n|_2^p)^{1/p}}{(\mathbb{E}|X_n|_2^2)^{1/2}} = \left( \frac{\Gamma(r-p)}{\Gamma(r)} \right)^{1/p} \left( \frac{\Gamma(r-2)}{\Gamma(r)} \right)^{-1/2} \quad (6)$$

and by the strict log-convexity of the Gamma function, we have

$$\lim_{n \rightarrow +\infty} (\alpha_p(X_n) + 1) = \lim_{n \rightarrow +\infty} \frac{(\mathbb{E}|X_n|_2^p)^{1/p}}{(\mathbb{E}|X_n|_2^2)^{1/2}} > 1.$$

As a consequence for any  $2 < p < r$ ,  $\lim_{n \rightarrow +\infty} \alpha_p(X_n) > 0$ .

Now let  $2 < p < s < r$ . From Proposition 3, we get

$$\underline{\lim}_{n \rightarrow +\infty} \varepsilon(X_n) \geq \lim_{n \rightarrow \infty} \min \left( \frac{\alpha_p(X_n)}{2}, \left( \frac{p\alpha_p(X_n)/2}{(\alpha_s(X_n) + 1)^p} \right)^{s/(s-p)} \right) > 0. \quad (7)$$

Choose  $p = (2+r)/2$  and  $s = (p+r)/2$  for which  $2 < p < s < r$  and note that the right hand side term in equation (7) depends only on  $r$ . This concludes the proof. □

**Remark 6.** Let  $2 < p < r$  and let  $r \rightarrow \infty$ . Applying Stirling formula in (6) when  $r \rightarrow \infty$ , a calculation gives that

$$\lim_{r \rightarrow \infty} r \lim_{n \rightarrow \infty} \alpha_p(X_n) = (p - 2)/2.$$

This asymptotic estimate shows that for a fixed  $p > 2$  and  $r$  and  $n$  large enough, then  $\alpha_p(X_n) \geq C(p - 2)/r$  where  $C > 0$  is a universal constant. This proves the sharpness of Theorem 1 under these conditions.

We now prove some inequalities for  $s$ -concave measures that will be useful tools in the next section.

**Theorem 7.** (1) Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a measurable function such that  $\|f\|_\infty > 0$ . Then

$$p \mapsto \left( \int_0^\infty pt^{p-1} f(t) dt / \|f\|_\infty \right)^{1/p}$$

is non-decreasing on its domain of definition.

(2) Let  $\alpha > 0$  and  $f : [0, \infty) \rightarrow [0, \infty)$  be  $(-1/\alpha)$ -concave, continuous and integrable. Define  $H_f : [0, \alpha) \rightarrow \mathbb{R}_+$  by

$$H_f(p) = \begin{cases} \frac{1}{B(p, \alpha - p)} \int_0^{+\infty} t^{p-1} f(t) dt & \text{for } 0 < p < \alpha \\ f(0) & \text{for } p = 0. \end{cases}$$

Then  $H_f$  is log-concave on  $[0, \alpha)$ .

The proof of the first part may be treated as in Lemma 2.1 in [25] and the proof of the second part is identical to the well known  $(1/n)$ -concave case [12]. We postpone the proof of Theorem 7 to the appendix.

We present several consequences of this result such as some reverse Hölder inequalities with sharp constants in the spirit of Borell's [12] and Berwald's [6] inequalities.

**Corollary 8.** *Let  $r > 0$  and  $\mu$  be a  $(-1/r)$ -concave measure on  $\mathbb{R}^n$ . Let  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that  $\{\phi > 0\}$  is convex and  $\phi$  is concave on  $\{\phi > 0\}$ . Then the function*

$$p \mapsto \begin{cases} \frac{1}{pB(p, r-p)} \int \phi(x)^p d\mu(x) & \text{for } 0 < p < r \\ \mu(\{\phi > 0\}) & \text{for } p = 0 \end{cases}$$

is log-concave on  $[0, r)$ .

Moreover, if  $\mu(\{\phi > 0\}) > 0$  then for any  $0 < p \leq q < r$ ,

$$\left( \int_{\mathbb{R}^n} \phi(x)^q \frac{d\mu(x)}{\mu(\{\phi > 0\})} \right)^{1/q} \leq \frac{(qB(q, r-q))^{1/q}}{(pB(p, r-p))^{1/p}} \left( \int_{\mathbb{R}^n} \phi(x)^p \frac{d\mu(x)}{\mu(\{\phi > 0\})} \right)^{1/p}.$$

*Proof.* By the concavity of  $\phi$ , for every  $u, v \geq 0$  and every  $\lambda \in [0, 1]$

$$(1 - \lambda)\{\phi > u\} + \lambda\{\phi > v\} \subset \{\phi > (1 - \lambda)u + \lambda v\}.$$

By the  $(-1/r)$ -concavity of  $\mu$ , the function  $f(t) = \mu(\{\phi > t\})$  is  $(-1/r)$ -concave and it is clearly continuous on  $\mathbb{R}_+$ . Observe by Fubini that for any  $p > 0$ ,

$$\int_{\mathbb{R}^n} \phi(x)^p d\mu(x) = \int_0^{+\infty} pt^{p-1} f(t) dt.$$

The result follows from the part (2) of Theorem 7. The moreover part follows from the log-concavity since then  $p \mapsto (H_f(p)/f(0))^{1/p}$  is a non-increasing function.  $\square$

The second corollary concerns the function  $h_{k,p}$  defined in (4).

**Corollary 9.** *Let  $r > 0$  and  $u \in SO(n)$ . For any  $(-1/(r+n))$ -concave function  $w : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and any subspace  $E_0$  of dimension  $k \leq n$ , the function*

$$p \mapsto \begin{cases} \frac{h_{k,p}(u)}{B(p+k, r-p)} & \text{for } p > -k+1 \\ |S^{k-1}| \pi_{u(E_0)} w(0) & \text{for } p = -k+1 \end{cases}$$

is log-concave on  $[-k+1, r)$ .

*Proof.* Since  $w$  is  $(-1/(r+n))$ -concave, we note that  $t \mapsto \pi_{U(E_0)}w(tu(\theta_0))$  is  $(-1/(r+k))$ -concave and it is clearly continuous on  $\mathbb{R}_+$ . Theorem 7 proves the result.  $\square$

We finish with some geometric properties of a family of bodies introduced by K. Ball in [5] in the log-concave case.

**Corollary 10.** *Let  $\alpha > 0$ . Let  $w : \mathbb{R}^n \rightarrow \mathbb{R}_+$  be a  $(-1/\alpha)$ -concave function such that  $w(0) > 0$ . For  $0 < a < \alpha$  let*

$$K_a(w) = \left\{ x \in \mathbb{R}^n; a \int_0^{+\infty} t^{a-1} w(tx) dt \geq w(0) \right\}.$$

Then for any  $0 < a \leq b < \alpha$

$$\left( \frac{w(0)}{\|w\|_\infty} \right)^{\frac{1}{a}-\frac{1}{b}} K_a(w) \subset K_b(w) \subset \frac{(bB(b, \alpha - b))^{1/b}}{(aB(a, \alpha - a))^{1/a}} K_a(w).$$

*Proof.* Notice that the sets  $K_a$  are star-shaped with respect to the origin, that is for every  $x \in K_a$  and every  $\lambda \in [0, 1]$ ,  $\lambda x \in K_a$ . The radial function of  $K_a$  is

$$\rho_{K_a}(x) := \sup\{r : rx \in K_a\} = \left( a \int_0^{+\infty} t^{a-1} \frac{w(tx)}{w(0)} dt \right)^{\frac{1}{a}}.$$

For any  $x \in \mathbb{R}^n$ , let  $f$  be the continuous  $(-1/\alpha)$ -concave function defined on  $\mathbb{R}^+$  by  $f(t) = w(tx)/w(0)$ . By (1) of Theorem 7, the function  $a \mapsto \left( \int_0^{+\infty} t^{a-1} \frac{f(t)}{\|f\|_\infty} dt \right)^{\frac{1}{a}}$  is non-decreasing. The left hand side inclusion follows. Moreover, from (2) of Theorem 7, the function  $H_f : [0, \alpha) \rightarrow \mathbb{R}_+$  is log-concave on  $[0, \alpha)$  with  $H_f(0) = 1$ . For  $0 < a \leq b < \alpha$ , we have thus  $H_f(b)^{1/b} \leq H_f(a)^{1/a}$ . The right hand side inclusion follows.  $\square$

### 3 Thin shell for convex measures

The purpose of this section is to prove Theorem 1. We follow the strategy of the log-concave case initiated in [22, 17, 23] and further developed in [16, 19]. The support function  $h_K$  of a non-empty compact set  $K \subset \mathbb{R}^n$  is defined by

$$\forall \theta \in \mathbb{R}^n, \quad h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle.$$

To any random vector  $X$  in  $\mathbb{R}^n$  and any  $p \geq 1$ , we associate its  $Z_p^+$ -body defined by its support function

$$\forall \theta \in \mathbb{R}^n, \quad h_{Z_p^+(X)}(\theta) = (\mathbb{E} \langle X, \theta \rangle_+^p)^{1/p}.$$

When the distribution of  $X$  has a density  $g$ , we write  $Z_p^+(g) = Z_p^+(X)$ . Extending a theorem of Ball [5] for log-concave functions, Bobkov proved in [8, Remark 2.6] (see also [15, Theorem 3.1]) that if  $w$  is  $(-1/(r+n))$ -concave on  $\mathbb{R}^n$  such that  $w(0) > 0$ , then

$$K_a(w) \text{ is convex and compact for any } 0 < a \leq r + n - 1. \quad (8)$$

In the case of log-concave measures [26, 27, 19, 20], several relations between the  $Z_p^+$  bodies and the convex sets  $K_a$  are known. We need their analogue in the setting of  $s$ -concave measures for negative  $s$ . We start with two technical lemmas. We postpone their proofs to the appendix.

**Lemma 11.** *Let  $x, y \geq 1$ , then*

$$c \frac{x}{x+y} \leq (xB(x, y))^{1/x} \leq C \frac{x}{x+y}, \quad (9)$$

where  $c, C$  are positive universal constants. Moreover, for  $k, r > 1$ , the extension by continuity at 0 of the function  $p \mapsto \frac{1}{p} \log \frac{B(k+p, r-p)}{B(k, r)}$  is differentiable on  $[-(\frac{k-1}{2}), \frac{r-1}{2}]$  and satisfies

$$0 \leq \frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k+p, r-p)}{B(k, r)} \right) \leq \frac{1}{r-1} + \frac{1}{k-1} \quad (10)$$

for  $p \in [-(\frac{k-1}{2}), \frac{r-1}{2}]$ .

In this paper, we use the notion of geometric distance between sets, defined for every compact subsets  $K, L \subset \mathbb{R}^n$  containing 0 in their interior by

$$d(K, L) = \inf \{ t_2/t_1 : t_1 L \subset K \subset t_2 L, t_1, t_2 > 0 \}.$$

Let  $n \geq 1$ ,  $r \geq 2$  and  $w$  be the  $(-1/(r+n))$ -concave density of a probability measure  $\mu$  on  $\mathbb{R}^n$ . Then by Corollary 8 and Lemma 11, for  $1 \leq p \leq q \leq r-1$ , one has

$$Z_p^+(w) \subset Z_q^+(w) \subset c \frac{q}{p} \left( \inf_{\theta \in S^{n-1}} \mu(\{x : \langle x, \theta \rangle > 0\}) \right)^{\frac{1}{q} - \frac{1}{p}} Z_p^+(w).$$

Fix  $\theta \in S^{n-1}$  and define  $F(t) = \mu(\{x : \langle x, \theta \rangle \leq t\})$ , for  $t \in \mathbb{R}$ . One has  $\int_{\mathbb{R}} tF'(t)dt = \int_{\mathbb{R}^n} \langle x, \theta \rangle w(x)dx = 0$  and  $F$  is  $(-1/r)$ -concave. Using Jensen's inequality, we get

$$F(0)^{-\frac{1}{r}} = F\left(\int_{\mathbb{R}} tF'(t)dt\right)^{-\frac{1}{r}} \leq \int_{\mathbb{R}} F(t)^{-\frac{1}{r}} F'(t)dt = \left[\frac{F(t)^{1-\frac{1}{r}}}{1-\frac{1}{r}}\right]_{-\infty}^{+\infty} = \frac{1}{1-\frac{1}{r}}.$$

Hence  $\mu(\{x : \langle x, \theta \rangle > 0\}) \geq (1 - \frac{1}{r})^r \geq 1/4$  for  $r \geq 2$ . We have recovered here in a simple way a Grünbaum's type inequality for convex measures due to Bobkov [8, Theorem 5.2]. We deduce that, for  $1 \leq p \leq q \leq r - 1$ ,

$$Z_p^+(w) \subset Z_q^+(w) \subset C \frac{q}{p} Z_p^+(w) \quad \text{and} \quad d(Z_p^+(w), Z_q^+(w)) \leq C \frac{q}{p}. \quad (11)$$

**Lemma 12.** *Let  $r, m$  and  $p$  be such that  $m$  is a positive integer,  $r \geq m + 1$  and  $-\frac{m}{2} \leq p \leq r - 1$ . Let  $F$  be a subspace of  $\mathbb{R}^n$  of dimension  $m$  and let  $g$  be a  $(-1/(r + m))$ -concave density of a probability measure on  $F$  such that  $\int_F xg(x)dx = 0$ . Then we have*

$$d(K_{m+p}(g), Z_{\max(m,p)}^+(g)) \leq c,$$

where  $c$  is a universal constant.

As in [19], an important ingredient in the proof of the thin-shell concentration inequality is an estimate from above of the log-Lipschitz constant of the map on  $SO(n) : u \mapsto h_{k,p}(u)$ . Let  $\mathcal{M}_n(\mathbb{R})$  be the set of square  $n \times n$  matrices. We equip

$$SO(n) = \{u \in \mathcal{M}_n(\mathbb{R}) : u^t u = Id, \det(u) = 1\}$$

with its standard invariant Riemannian metric, which we specify for concreteness on  $T_{Id}SO(n)$ , the tangent space at the identity element  $Id \in SO(n)$ . Since  $u^t u = Id$ , this tangent space may be identified with the set of anti-symmetric matrices  $\{B \in \mathcal{M}_n(\mathbb{R}) : B^t + B = 0\}$ . We define the scalar product  $\langle B, B \rangle = \frac{1}{2}\text{tr}(B^t B)$  on  $T_{Id}SO(n)$ .

**Proposition 13.** *Let  $n \geq 1$ ,  $r > 10$  and  $w$  be the  $(-1/(r + n))$ -concave density of a probability measure on  $\mathbb{R}^n$  such that  $\int_{\mathbb{R}^n} xw(x)dx = 0$ . Let  $k$  be an integer such that  $k \geq 2$ ,  $2k - 1 \leq n$  and  $2k \leq r$ . Let  $p$  such that*

$-\frac{k}{2} \leq p \leq r - 1$ . Denote by  $L_{k,p}$  the log-Lipschitz constant of the map on  $SO(n) : u \mapsto h_{k,p}(u)$ . Then

$$L_{k,p} \leq C \max(k, p) d(Z_{\max(k,p)}^+(w), B_2^n),$$

where  $C$  is a universal constant.

*Proof.* For any subspace  $F$  of dimension  $m$ , the marginal  $\pi_F(w)$  is a  $(-1/(r+m))$ -concave function on  $F$  and from (8), for any  $a \in [0, r+m-1]$ , we associate the convex body  $K_a(\pi_F(w))$  in  $F$ . Then the proof of Theorem 2.1 in [19, section 2.2] gives the upper bound:

$$L_{k,p} \leq \max_F \{(m+p) d(K_{m+p}(\pi_F(w)), B_2(F))\}$$

over all subspaces  $F$  of dimension  $m = k, k+1, 2k-1$ , where  $B_2(F)$  is the Euclidean unit ball in  $F$ . By assumptions on  $k$ , we get that for these values of  $m$ ,  $m \leq 2k-1 \leq n$  and  $r \geq 2k \geq m+1$  and  $p \geq -k/2 \geq -m/2$ . Hence from Lemma 12, we have

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \leq c d(Z_{\max(m,p)}^+(\pi_F(w)), B_2(F)).$$

By definition, if  $X$  is the random vector with density  $w$  on  $\mathbb{R}^n$ , the marginal  $\pi_F(w)$  is the density of the projection of  $X$  onto  $F$ , namely  $P_F X$ . By identification of the support functions, we have that, for any  $\theta \in F$ ,

$$h_{Z_{\max(m,p)}^+(\pi_F(w))}^p(\theta) = \mathbb{E} \langle P_F X, \theta \rangle_+^p = \mathbb{E} \langle X, \theta \rangle_+^p.$$

This means that  $Z_{\max(m,p)}^+(\pi_F(w)) = P_F(Z_{\max(m,p)}^+(w))$ . Since the distance to the Euclidean ball cannot increase after projections, we conclude that

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \leq c d(Z_{\max(m,p)}^+(w), B_2^n).$$

By equation (11), for  $m = k, k+1, 2k-1$ , one has

$$d(Z_{\max(m,p)}^+(w), Z_{\max(k,p)}^+(w)) \leq c.$$

This finishes the proof.  $\square$

We define the  $q$ -condition number of a random vector  $X$  to be

$$\rho_q(X) = \frac{\sup_{|\theta|_2=1} (\mathbb{E} \langle X, \theta \rangle_+^q)^{1/q}}{\inf_{|\theta|_2=1} (\mathbb{E} \langle X, \theta \rangle_+^q)^{1/q}}.$$

Obviously, if  $w$  is the density of a full-dimensional random vector  $X$  in  $\mathbb{R}^n$  then  $\rho_q(X) = d(Z_q^+(w), B_2^n)$ .

**Proposition 14.** *With the same assumptions as in Proposition 13, if a random vector  $X$  with density  $w$  is isotropic then*

$$L_{k,p} \leq C \max(k, p)^2.$$

*More generally if  $A$  is such that  $AX$  is isotropic then*

$$L_{k,p} \leq C \max(k, p)^2 \|A\| \|A^{-1}\|. \quad (12)$$

*Proof.* Let  $q = \max(k, p)$ , then one has  $1 \leq q \leq r - 1$ . Using the triangular inequality we get

$$\rho_q(X) = d(Z_q^+(w), B_2^n) \leq d(Z_q^+(w), Z_2^+(w)) d(Z_2^+(w), B_2^n).$$

From equation (11) we deduce that  $d(Z_q^+(w), Z_2^+(w)) \leq cq$ . For any  $\theta \in S^{n-1}$ ,  $\mathbb{E}\langle X, \theta \rangle = 0$ , hence  $\mathbb{E}\langle X, \theta \rangle_+ = \mathbb{E}\langle -X, \theta \rangle_+$ . Using this equality and equation (11) we deduce that

$$(\mathbb{E}\langle -X, \theta \rangle_+^2)^{\frac{1}{2}} \leq c \mathbb{E}\langle -X, \theta \rangle_+ = c \mathbb{E}\langle X, \theta \rangle_+ \leq c (\mathbb{E}\langle X, \theta \rangle_+^2)^{\frac{1}{2}}.$$

Thus

$$\mathbb{E}\langle X, \theta \rangle_+^2 \leq \mathbb{E}\langle X, \theta \rangle^2 = \mathbb{E}\langle X, \theta \rangle_+^2 + \mathbb{E}\langle -X, \theta \rangle_+^2 \leq C \mathbb{E}\langle X, \theta \rangle_+^2.$$

Hence if  $X$  is isotropic we deduce that  $d(Z_2^+(w), B_2^n) \leq c'$ . We conclude that

$$\rho_q(X) = d(Z_q^+(w), B_2^n) \leq C'q.$$

The conclusion follows from Proposition 13. In the general case, notice that  $Z_q^+(AX) = AZ_q^+(X)$  and  $d(AB_2^n, B_2^n) = \|A\| \|A^{-1}\|$ , thus

$$\rho_q(X) \leq \rho_q(AX) \|A\| \|A^{-1}\|.$$

□

*Proof of Theorem 1.* Without loss of generality, we can assume that  $r > 32$ . Indeed, if  $r \leq 32$  then the statement in Theorem 1 is valid for  $|p| \leq cr$  and it gives only a comparison of  $(\mathbb{E}|X|_2^p)^{1/p}$  with  $(\mathbb{E}|X|_2^2)^{1/2}$  up to a constant factor. The result is a consequence of Theorem 5.2 in [1].

From now on, we assume that  $r > 32$  and that  $|p| \leq \frac{r}{8}$ . We start by presenting a complete argument following [16]. This will give a complete



proof of Theorem 1 with a slightly weaker result. In the second part, we just indicate the needed modifications of the argument of [19] to get the complete conclusion.

In this first part, we will prove that for any  $p \in [\frac{1}{\sqrt{n}}, \min(cn^{1/8}, \frac{r}{8})]$

$$(\mathbb{E}|X|_2^p \mathbb{E}|X|_2^{-p})^{1/p} \leq 1 + \frac{Cp}{r} + \left(\frac{Cp}{n^{1/3}}\right)^{3/5}. \quad (13)$$

Assuming (13), few elementary steps are needed to prove that for any  $p$  such that  $|p| \leq \min(cn^{1/8}, \frac{r}{8})$ ,

$$\left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \leq \frac{C(1+|p|)}{r} + \left(\frac{C(1+|p|)}{n^{1/3}}\right)^{3/5}, \quad (14)$$

which is already enough to get a thin-shell concentration. Indeed, for  $p \geq 2$ , by Hölder inequality, we have

$$0 \leq \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \leq \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^{-p})^{-1/p}} - 1$$

and we conclude by (13). For  $p \leq -2$ , we have  $|p| = -p \geq 2$  and from Hölder inequality and (13),

$$0 \leq \frac{(\mathbb{E}|X|_2^2)^{1/2}}{(\mathbb{E}|X|_2^p)^{1/p}} - 1 \leq \frac{(\mathbb{E}|X|_2^{|p|})^{1/|p|}}{(\mathbb{E}|X|_2^{-|p|})^{-1/|p|}} - 1 \leq \frac{C|p|}{r} + \left(\frac{C|p|}{n^{1/3}}\right)^{3/5}.$$

An elementary computation shows that

$$\left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \leq \frac{C|p|}{r} + \left(\frac{C|p|}{n^{1/3}}\right)^{3/5}.$$

For  $p \in [-2, 2]$ , by Hölder inequality, we have

$$0 \leq 1 - \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} \leq 1 - \frac{(\mathbb{E}|X|_2^{-2})^{-1/2}}{(\mathbb{E}|X|_2^2)^{1/2}}$$

and we conclude by the previous estimate for  $p = -2$ . This concludes the proof of (14).

Let us start the proof of (13). Let  $p \in [\frac{1}{\sqrt{n}}, \min(cn^{1/8}, \frac{r}{8})]$  and  $k$  be an integer greater or equal than 2 such that  $p < k \leq n$ . We will optimize the choice of  $k$  at the end of the proof. Recall that by (5),

$$\mathbb{E}|X|_2^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \mathbb{E}h_{k,p}(U),$$

where  $U$  is uniformly distributed on  $SO(n)$ . Using that the function  $\frac{d}{dp} \log \Gamma(p)$  is concave (see for example the proof of Lemma 11 in the appendix), we deduce that

$$\frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma((p+k)/2)\Gamma(n/2)} \right) \leq 0. \quad (15)$$

It follows that for any  $0 < p < k$ ,

$$\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \frac{\Gamma((-p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((-p+k)/2)} \leq 1.$$

Then for all  $0 < p < r$  and  $n \geq k > p$  we have

$$\mathbb{E}|X|_2^p \mathbb{E}|X|_2^{-p} \leq \mathbb{E}h_{k,p}(U) \mathbb{E}h_{k,-p}(U). \quad (16)$$

Applying log-Sobolev inequality (3) to  $h_{k,p}$  and  $h_{k,-p}$  we get

$$\mathbb{E}h_{k,p}(U)^2 \leq e^{\frac{cL_{k,p}^2}{n}} (\mathbb{E}h_{k,p}(U))^2, \quad \mathbb{E}h_{k,-p}(U)^2 \leq e^{\frac{cL_{k,-p}^2}{n}} (\mathbb{E}h_{k,-p}(U))^2. \quad (17)$$

Since  $\text{Var} f = \mathbb{E}f^2 - (\mathbb{E}f)^2$  we deduce that

$$\begin{cases} \text{Var} h_{k,p}(U) \leq \left( e^{\frac{cL_{k,p}^2}{n}} - 1 \right) (\mathbb{E}h_{k,p}(U))^2, \\ \text{Var} h_{k,-p}(U) \leq \left( e^{\frac{cL_{k,-p}^2}{n}} - 1 \right) (\mathbb{E}h_{k,-p}(U))^2. \end{cases} \quad (18)$$

By Corollary 9, we know that  $p \mapsto h_{k,p}(u)/B(k+p, r-p)$  is log-concave on  $[-k+1, r)$  hence

$$h_{k,p}(u) h_{k,-p}(u) \leq \left( \frac{B(k+p, r-p)}{B(k, r)} \frac{B(k-p, r+p)}{B(k, r)} \right) h_{k,0}^2(u).$$

Taking the expectation with respect to  $SO(n)$ , we get that

$$\mathbb{E}h_{k,p}(U)h_{k,-p}(U) \leq \left( \frac{B(k+p, r-p)}{B(k, r)} \frac{B(k-p, r+p)}{B(k, r)} \right) \mathbb{E}h_{k,0}^2(U).$$

Since  $\mathbb{E}h_{k,0}(U) = 1$  we deduce from (17) that

$$\mathbb{E}h_{k,0}^2(U) \leq e^{\frac{cL_{k,0}^2}{n}}.$$

Assume that  $k$  is such that  $k \leq r$  then by (10), we know that for  $p \leq (k-1)/2$ ,

$$\left( \frac{B(k+p, r-p) B(k-p, r+p)}{B(k, r)} \right)^{1/p} \leq e^{2p(\frac{1}{k-1} + \frac{1}{r-1})} \leq e^{4p(\frac{1}{k} + \frac{1}{r})}$$

since  $k, r \geq 2$ . Hence

$$\mathbb{E}h_{k,p}(U)h_{k,-p}(U) \leq e^{\frac{cL_{k,0}^2}{n} + 4p^2(\frac{1}{k} + \frac{1}{r})}. \quad (19)$$

Moreover

$$\begin{aligned} \mathbb{E}h_{k,p}(U)h_{k,-p}(U) &= \mathbb{E}h_{k,p}(U)\mathbb{E}h_{k,-p}(U) + \text{Cov}(h_{k,p}(U), h_{k,-p}(U)) \\ &\geq \mathbb{E}h_{k,p}(U)\mathbb{E}h_{k,-p}(U) - \sqrt{\text{Var}h_{k,p}(U)\text{Var}h_{k,-p}(U)} \\ &\geq \mathbb{E}h_{k,p}(U)\mathbb{E}h_{k,-p}(U) \left( 1 - \sqrt{\left( e^{\frac{cL_{k,p}^2}{n}} - 1 \right) \left( e^{\frac{cL_{k,-p}^2}{n}} - 1 \right)} \right) \end{aligned} \quad (20)$$

where the last inequality follows from (18). Assume moreover that  $k$  is such that  $2k-1 \leq n$  and  $2k \leq r$  then for  $p \leq (k-1)/2$ , we can evaluate  $L_{k,p}$ ,  $L_{k,-p}$  and  $L_{k,0}$  from Proposition 14 since the assumptions are fulfilled. We get that if  $X$  is isotropic then  $\max(L_{k,p}, L_{k,-p}, L_{k,0}) \leq Ck^2$ . If  $k \leq c_0n^{1/4}$  for a small enough numerical constant  $c_0$ , we have

$$\sqrt{\left( e^{\frac{cL_{k,p}^2}{n}} - 1 \right) \left( e^{\frac{cL_{k,-p}^2}{n}} - 1 \right)} \leq c' \frac{k^4}{n} \leq \frac{1}{10}.$$

Combining this estimate with (20) and (19), we have proved that if  $k$  is an integer such that  $k \geq 2$ ,  $2k-1 \leq n$ ,  $2k \leq r$ ,  $k \leq c_0n^{1/4}$  and  $2p+1 \leq k$  (this set of integers is not empty since  $r > 32$  and  $p \leq r/8$ ) then

$$\mathbb{E}h_{k,p}(U)\mathbb{E}h_{k,-p}(U) \leq \frac{e^{4p^2(\frac{1}{k} + \frac{1}{r}) + c\frac{k^4}{n}}}{1 - c' \frac{k^4}{n}} \leq e^{4p^2(\frac{1}{k} + \frac{1}{r}) + C\frac{k^4}{n}}.$$

For  $p \leq 1$ , we also force  $k$  to satisfy  $k \leq C_0 p^{1/4} n^{1/4}$ . Hence taking the power  $1/p$  in the last expression, we conclude from (16) that

$$(\mathbb{E}|X|_2^p \mathbb{E}|X|_2^{-p})^{1/p} \leq e^{4p(\frac{1}{k} + \frac{1}{r}) + C \frac{k^4}{pn}} \leq 1 + cp \left( \frac{1}{k} + \frac{1}{r} \right) + c \frac{k^4}{pn},$$

since  $p/k, p/r$  and  $k^4/pn$  are bounded by universal constants. It remains to optimize the choice of  $k$ . Let  $p_0 = n^{-1/2}$ . In this case we choose  $k = 2$  and get

$$(\mathbb{E}|X|_2^{p_0} \mathbb{E}|X|_2^{-p_0})^{1/p_0} \leq 1 + \frac{C}{\sqrt{n}}. \quad (21)$$

If  $p \geq n^{-1/2}$  we choose  $k$  to be an integer such that  $\min(r/4, (p^2 n)^{1/5}) \leq k \leq 2 \min(r/4, (p^2 n)^{1/5})$  with the restriction  $2p + 1 \leq k \leq cn^{1/4}$  and that  $k \leq cp^{1/4} n^{1/4}$ . For any  $p$  such that  $p_0 \leq p \leq \min(cn^{1/8}, r/8)$ , the integer  $k$  satisfies  $k \geq 2$ ,  $2k - 1 \leq n$ ,  $2k \leq r$ ,  $k \leq c_0 n^{1/4}$  and  $2p + 1 \leq k$  and we get that

$$(\mathbb{E}|X|_2^p \mathbb{E}|X|_2^{-p})^{1/p} \leq 1 + \frac{Cp}{r} + \left( \frac{Cp}{n^{1/3}} \right)^{3/5}.$$

This ends the proof of (13).

In the second part, we follow the argument developed in [19] to get a better estimate. We deal now with the case of  $p$  being positive or negative and, as already said, we can assume without loss of generality that  $r > 34$  and  $|p| \leq r/8$ . As in [19], our goal is to estimate

$$\frac{d}{dp} \log((\mathbb{E}|X|_2^p)^{\frac{1}{p}}) = \frac{d}{dp} \log((\mathbb{E}h_{k,p}(U))^{\frac{1}{p}}) + \frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2) \Gamma(k/2)}{\Gamma(n/2) \Gamma((p+k)/2)} \right).$$

Most of the computation of section 3.2 in [19] is identical. All the ingredients needed for the proof have been established and, adapting the argument done in section 3.2 in [19], we get

$$\frac{d}{dp} \log((\mathbb{E}|X|_2^p)^{\frac{1}{p}}) \leq \frac{c}{p^2 n} (2L_{k,p}^2 + 3L_{k,0}^2) + \frac{C}{k-1} + \frac{C}{r-1}. \quad (22)$$

For convenience of the reader, we will shortly reproduce the proof of (22) in the appendix.

Assume that  $X$  is isotropic. For any  $2|p| \leq k \leq r/2$  (this set of integers is not empty since  $r > 32$  and  $|p| \leq r/8$ ), we know by Proposition 14, that

$L_{k,p}$  and  $L_{k,0}$  are smaller than  $Ck^2$ . We get that

$$\frac{d}{dp} \log((\mathbb{E}|X|_2^p)^{\frac{1}{p}}) \leq C \left( \frac{k^4}{p^2 n} + \frac{1}{k} + \frac{1}{r} \right).$$

We have to minimize this expression for  $k$  being an integer greater or equal than 2 and  $k \in [2|p|, r/2]$ . For  $|p| \in [n^{-1/2}, cn^{1/3}]$ , we set  $k$  being an integer such that  $\min(r/4, 2(p^2 n)^{1/5}) \leq k \leq 2 \min(r/4, 2(p^2 n)^{1/5})$ . Therefore  $k$  satisfies the restrictions and we get for any  $p$  such that  $|p| \in [n^{-1/2}, cn^{1/3}]$ ,

$$\frac{d}{dp} \log((\mathbb{E}|X|_2^p)^{\frac{1}{p}}) \leq C \left( \frac{1}{(p^2 n)^{1/5}} + \frac{1}{r} \right). \quad (23)$$

After integration over  $p$ , we get that for all  $p \in [n^{-1/2}, c \min(r, n^{1/3})]$

$$\left| \log \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} \right| \leq \frac{C|p-2|}{r} + \frac{C|p^{3/5} - 2^{3/5}|}{n^{1/5}}.$$

Since  $|p^{3/5} - 2^{3/5}| \leq |p - 2|^{3/5}$  and all terms in the right hand side of the inequality are bounded by a universal constant, we conclude by adjusting that

$$\left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \leq \frac{C|p-2|}{r} + \left( \frac{C|p-2|}{n^{1/3}} \right)^{3/5}, \quad \forall p \in [n^{-1/2}, c \min(r, n^{1/3})].$$

Since (23) holds only for  $|p| \geq n^{-1/2}$ , we use (21) to bridge the gap between  $-n^{-1/2}$  and  $n^{-1/2}$ . Indeed, from (21), the previous inequality for  $p_0 = n^{-1/2}$  and using that  $|p_0 - 2| = 2 - p_0 \leq 2$ , we get that for  $p \in [-p_0, p_0]$ ,

$$\begin{aligned} (\mathbb{E}|X|_2^p)^{1/p} &\geq (\mathbb{E}|X|_2^{-p_0})^{-1/p_0} \geq \frac{1}{1 + \frac{C}{\sqrt{n}}} (\mathbb{E}|X|_2^{p_0})^{1/p_0} \\ &\geq \frac{1 - \frac{2C}{r} - \left(\frac{2C}{n^{1/3}}\right)^{3/5}}{1 + \frac{C}{n^{1/5}}} (\mathbb{E}|X|_2^2)^{1/2}. \end{aligned}$$

An easy adaptation of the constants leads to the conclusion of Theorem 1 for all  $p \in [-n^{-1/2}, n^{-1/2}]$ .

Integrating again (23), we get, for  $p \in [-c \min(r, n^{1/3}), -n^{-1/2}]$ ,

$$\frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^{-p_0})^{-1/p_0}} \geq 1 - \frac{C|p+p_0|}{r} - \left( \frac{C|p+p_0|}{n^{1/3}} \right)^{3/5}.$$

Using that  $|p + p_0| \leq |p - 2|$  and the previous comparison of the moment of order  $-p_0$  with the moment of order 2 and adjusting the constants, this proves that for all  $p \in [-c \min(r, n^{1/3}), -n^{-1/2}]$ ,

$$\left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \leq \frac{C|p-2|}{r} + \left( \frac{C|p-2|}{n^{1/3}} \right)^{3/5}.$$

This concludes the proof of the first part of Theorem 1.

If  $X$  is such that  $AX$  is isotropic, we know from Proposition 14 that for any integer  $k$  such that  $2|p| \leq k \leq r/2$ ,

$$\max(L_{k,p}, L_{k,0}) \leq Ck^2 \|A\| \|A^{-1}\|.$$

The proof is identical to the previous one replacing  $n$  by  $\frac{n}{\|A\|^2 \|A^{-1}\|^2}$ .  $\square$

**Remark 15.** In [19], a preprocessing step consisted in adding a Gaussian isotropic vector to the random vector  $X$  in order to start at the very beginning with a better information on the  $Z_p^+$  bodies associated to the measure. In [23, 16], this convolution argument played a role of regularization. It is natural to ask if such a process could be done in the situation of  $s$ -concave measure. Nothing is doable by adding a Gaussian vector because for  $s < 0$ , the new vector does not belong to any class of  $s$ -concave vectors. However, for  $r > n$ , we can build a similar argument, adding to  $X$  a random vector  $Z$  uniformly distributed on the Euclidean ball, see also [9]. Since  $Z$  is  $(1/n)$ -concave and  $X$  is  $(-1/r)$ -concave, the new vector  $Y = \frac{X+Z}{\sqrt{2}}$  will be  $(-1/(r-n))$ -concave. For any  $p \geq 1$ , we have (see inequality (4.7) in [19])

$$\alpha_p(X) \leq \alpha_{2p}(Y) (2 + \alpha_{2p}(Y))$$

so that it remains to bound  $\alpha_{2p}(Y)$ . It is easy to see that  $Y$  is such that for every  $q \geq 2$  and every  $\theta \in S^{n-1}$ ,  $(\mathbb{E} \langle Y, \theta \rangle_+^q)^{1/q} \geq c\sqrt{q}$ . Adapting the proof of Proposition 14, we get  $L_{k,p} \leq C \max(k, p)^{3/2}$ . As in [19], this improvement leads to the following estimate: if  $r - n > 2$ , then for any  $p$  such that  $1 \leq p \leq c \min(r - n, \sqrt{n})$

$$\alpha_{2p}(Y) \leq \frac{C(2p-2)}{r-n} + \left( \frac{C(2p-2)}{\sqrt{n}} \right)^{1/2}.$$

For  $r > n + \sqrt{n}$ , we recover the same thin-shell concentration as in the log-concave case. It would be interesting to understand in which precise

sense the  $s$ -concave measures are close to the log-concave measures for  $s \in (-1/n, 1/n)$ . Another question is to know what kind of preprocessing argument like in [24] would enable to recover the small ball estimates from [1].

## 4 Appendix

*Proof of Theorem 7.* (1) This result is classical. In the symmetric case, it follows from Lemma 2.1 in [25]. The general case is similar. We provide its proof for completeness. We may assume, without loss of generality, that  $\|f\|_\infty = 1$ . Denote  $I_p(f) = \int_0^{+\infty} t^{p-1} f(t) dt$ . From Hölder inequality, the function  $p \mapsto \log(I_p(f))$  is convex on its convex support, thus the domain of definition of  $I_p(f)$  is an interval. Let  $0 < p < q$  be fixed such that  $I_p(f) < +\infty$  and  $I_q(f) < +\infty$ . Let  $a = (pI_p(f))^{1/p}$  and  $\varphi(t) = t^{p-1}(f(t) - 1_{[0,a]}(t))$ . Notice that  $\varphi \leq 0$  on  $[0, a]$ ,  $\varphi \geq 0$  on  $[a, +\infty)$  and  $\int_0^{+\infty} \varphi(t) dt = 0$ . Thus

$$I_q(f) - I_q(1_{[0,a]}) = \int_0^{+\infty} t^{q-p} \varphi(t) dt = \int_0^{+\infty} (t^{q-p} - a^{q-p}) \varphi(t) dt \geq 0,$$

since the integrand is non negative on  $\mathbb{R}_+$ . We conclude that

$$I_q(f) \geq I_q(1_{[0,a]}) = \frac{a^q}{q} = \frac{1}{q} (pI_p(f))^{\frac{q}{p}}.$$

(2) Since  $f$  is  $(-1/\alpha)$ -concave, there exists a convex function  $\varphi : [0, \infty) \rightarrow (0, \infty)$  such that  $f = \varphi^{-\alpha}$ . Since  $f$  is integrable it follows that  $\varphi$  tends to  $+\infty$  at  $+\infty$ . From the convexity of  $\varphi$ , one deduces that for some constant  $c > 0$ ,  $\varphi(t) \geq c(1+t)$ . Thus  $f(t) \leq (c+ct)^{-\alpha}$ , for every  $t \geq 0$ . Therefore,  $t^{p-1}f$  is integrable for every  $p < \alpha$ , which means that  $H_f(p) < +\infty$  for every  $0 < p < \alpha$ . Let  $p \in (0, \alpha)$  and  $m, M > 0$ . Define  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  by  $g(t) = m \left(1 + \frac{t}{M}\right)^{-\alpha}$ . Then

$$\int_0^{+\infty} t^{p-1} g(t) dt = mM^p \int_0^{+\infty} v^{p-1} (1+v)^{-\alpha} dv = mM^p B(p, \alpha - p).$$

Thus  $H_g(p) = mM^p$ , which implies that  $\log(H_g)$  is affine on  $(0, \alpha)$ . Take  $0 < a < b < c < \alpha$ . Let  $\lambda \in [0, 1]$  be such that  $b = (1-\lambda)a + \lambda c$ . Choose  $m$  and  $M$  such that  $mM^a = H_f(a)$  and  $mM^b = H_f(b)$  so that  $H_g(a) = H_f(a)$  and  $H_g(b) = H_f(b)$ . If we prove that

$$\int_0^{+\infty} t^{c-1} (g - f)(t) dt \geq 0, \tag{24}$$

that is  $H_g(c) \geq H_f(c)$ , then using that  $\log(H_g)$  is affine, we will deduce that

$$H_f(b) = H_g(b) = H_g(a)^{1-\lambda} H_g(c)^\lambda \geq H_f(a)^{1-\lambda} H_f(c)^\lambda$$

and this will prove the log-concavity of  $H$  on  $(0, \alpha)$ . If  $f = g$  then (24) is satisfied so that in the following we assume that the function  $h := g - f \not\equiv 0$ . Let

$$H_1(t) = \int_t^{+\infty} s^{a-1} h(s) ds \quad \text{and} \quad H_2(t) = \int_t^{+\infty} s^{b-a-1} H_1(s) ds.$$

Since  $h(t) = O(t^{-\alpha})$  at infinity, we deduce that  $H_1(t) = O(t^{a-\alpha})$  and  $H_2(t) = O(t^{b-\alpha})$ . We have  $\int_0^{+\infty} t^{a-1} h(t) dt = 0$  thus  $H_1(\infty) = H_1(0) = 0$ . Obviously  $H_2(\infty) = 0$ . We also observe

$$\begin{aligned} 0 &= \int_0^{+\infty} t^{b-1} h(t) dt = \int_0^{+\infty} t^{b-a} t^{a-1} h(t) dt = - \int_0^{+\infty} t^{b-a} H_1'(t) dt \\ &= [t^{b-a} H_1(t)]_0^{+\infty} + (b-a) \int_0^{+\infty} t^{b-a-1} H_1(t) dt = (b-a) H_2(0), \end{aligned}$$

whence  $H_2(\infty) = H_2(0) = 0$ . Since  $\int_0^{+\infty} t^{b-a-1} H_1(t) dt = 0$  and  $H_1 \not\equiv 0$ , the function  $H_1$  has at least one change of sign. Moreover, using that  $H_1(0) = H_1(\infty) = 0$ , we deduce that  $H_1'$  and therefore  $h$  has at least two sign changes. Since  $h = g - f$  has the same sign as  $f^{-\alpha} - g^{-\alpha}$  which is convex, it cannot have more than two sign changes. Thus it has exactly two sign changes at some  $0 < t_1 < t_2$ . Moreover, from the convexity of  $f^{-\alpha} - g^{-\alpha}$ , the sign of  $h$  has to be negative on  $(t_1, t_2)$  and positive on  $(0, t_1)$  and  $(t_2, +\infty)$ . From an easy study of the function  $H_2$ , we deduce that  $H_2 \geq 0$ . Therefore, using that  $H_1(0) = H_1(\infty) = H_2(0) = H_2(\infty) = 0$ , we get

$$\begin{aligned} \int_0^{+\infty} t^{c-1} h(t) dt &= \int_0^{+\infty} t^{c-a} t^{a-1} h(t) dt = - \int_0^{+\infty} t^{c-a} H_1'(t) dt \\ &= [-t^{c-a} H_1(t)]_0^{+\infty} + (c-a) \int_0^{+\infty} t^{c-a-1} H_1(t) dt \\ &= (c-a) \int_0^{+\infty} t^{c-b} t^{b-a-1} H_1(t) dt \\ &= (c-a) [-t^{c-b} H_2(t)]_0^{+\infty} + (c-a)(c-b) \int_0^{+\infty} t^{c-b-1} H_2(t) dt \\ &= (c-a)(c-b) \int_0^{+\infty} t^{c-b-1} H_2(t) dt \geq 0. \end{aligned}$$



This proves (24) and establish the log-concavity of  $H_f$  on  $(0, \alpha)$ . To get it on  $[0, \alpha)$ , it is enough to prove that  $H_f$  is continuous at 0. This follows from the observation that

$$B(p, \alpha - p) \underset{p \rightarrow 0}{\sim} \Gamma(p) \underset{p \rightarrow 0}{\sim} \frac{1}{p} \quad \text{thus} \quad H_f(p) \underset{p \rightarrow 0}{\sim} p \int_0^{+\infty} t^{p-1} f(t) dt.$$

And it is classical that, for a continuous function  $f$ , the right hand side term tends to  $f(0)$  when  $p \rightarrow 0$ .  $\square$

*Proof of Lemma 11.* Equation (9) follows easily from the classical bounds for the Gamma function (see [3]), valid for  $x \geq 1$ :

$$\sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \leq \Gamma(x) \leq \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x+\frac{1}{12}}.$$

For equation (10), we write that

$$\frac{B(k+p, r-p)}{B(k, r)} = \frac{\Gamma(k+p)\Gamma(r-p)}{\Gamma(k)\Gamma(r)}.$$

Denote  $G(p) = \log \Gamma(p)$ , for  $p > 0$ . We know that  $G''(p) = \sum_{i \geq 0} 1/(p+i)^2$  hence  $G''$  is non-increasing and  $0 \leq G''(p) \leq 1/(p-1)$ , for  $p > 1$ . Denote  $F_k(p) = \frac{G(k+p)-G(k)}{p}$ , for  $k > 0$  and  $p > -k$ . We have  $F_k(p) = \int_0^1 G'(k+up) du$ . Using that  $G''$  is non-increasing, we get that for  $k > 1$  and  $p \geq -(k-1)/2$ ,

$$F'_k(p) = \int_0^1 G''(k+up) u du \leq G''\left(\frac{k+1}{2}\right) \int_0^1 u du = \frac{1}{2} G''\left(\frac{k+1}{2}\right) \leq \frac{1}{k-1}$$

and  $F'_k(p) \geq 0$ . Therefore, if  $k > 1$ ,  $r > 1$  and  $-\frac{k-1}{2} \leq p \leq \frac{r-1}{2}$  then

$$\begin{aligned} 0 \leq \frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k+p, r-p)}{B(k, r)} \right) &= \frac{d}{dp} (F_k(p) - F_r(-p)) \\ &= F'_k(p) + F'_r(-p) \leq \frac{1}{k-1} + \frac{1}{r-1}. \end{aligned}$$

$\square$

*Proof of Lemma 12.* We present here a similar proof than in the appendix of [19]. Applying Corollary 10 to  $w = g$ ,  $n = m$ ,  $\alpha = r + m$ , we deduce that, for  $\frac{m}{2} \leq a \leq b \leq r + m - 1$ , one has

$$\left( \frac{g(0)}{\|g\|_\infty} \right)^{\frac{1}{a}-\frac{1}{b}} K_a(g) \subset K_b(g) \subset \frac{(bB(b, r+m-b))^{1/b}}{(aB(a, r+m-a))^{1/a}} K_a(g).$$

From Lemma 11, we have

$$\frac{(bB(b, r + m - b))^{1/b}}{(aB(a, r + m - a))^{1/a}} \leq c \frac{b}{a}$$

Moreover since  $\int xg(x)dx = 0$ , from Lemma 7.2 of [1], one has

$$\frac{g(0)}{\|g\|_\infty} \geq \left( \frac{r-1}{r+m-1} \right)^{r+m} \geq e^{-2m}.$$

Using that  $\frac{1}{a} - \frac{1}{b} \leq \frac{1}{a} \leq \frac{2}{m}$ , we deduce that  $\left( \frac{g(0)}{\|g\|_\infty} \right)^{\frac{1}{a} - \frac{1}{b}} \geq e^{-4}$ . We conclude that for  $\frac{m}{2} \leq a \leq b \leq r + m - 1$ , one has

$$e^{-4}K_a(g) \subset K_b(g) \subset c \frac{b}{a} K_a(g). \quad (25)$$

By integration in polar coordinates, it is well known [26] (see also [20]) that we have the following relation between the  $Z_q^+$ -bodies associated with  $g$  and the  $Z_q^+$ -bodies associated with one of the convex bodies  $K_a(g)$ : for any  $0 < q < r$

$$Z_q^+(g) = g(0)^{1/q} Z_q^+(K_{m+q}(g)), \quad (26)$$

where for any body  $K$ ,  $Z_q^+(K)$  denotes the convex body whose support function is defined by

$$\forall \theta \in \mathbb{R}^m, \quad h_{Z_q^+(K)}(\theta) = \left( \int_K \langle x, \theta \rangle_+^q dx \right)^{\frac{1}{q}}.$$

Let  $\theta \in \mathbb{R}^m$  and  $K$  be a convex body containing 0. From Berwald's inequalities [6] applied to  $K \cap \{\langle x, \theta \rangle \geq 0\}$  and the function  $x \mapsto \langle x, \theta \rangle_+$  which is concave on  $K \cap \{\langle x, \theta \rangle \geq 0\}$ , the function

$$p \mapsto \left( \frac{\int_K \langle x, \theta \rangle_+^p dx}{mB(p+1, m)\text{Vol}(K \cap \{\langle x, \theta \rangle \geq 0\})} \right)^{\frac{1}{p}}$$

is decreasing. Observe that for every  $\theta \in \mathbb{R}^m$ ,  $\lim_{p \rightarrow \infty} \left( \int_K \langle x, \theta \rangle_+^p dx \right)^{\frac{1}{p}} = h_K(\theta)$  and that

$$(mB(p+1, m))^{\frac{1}{p}} = \left( m \int_0^1 u^p (1-u)^{m-1} du \right)^{\frac{1}{p}} \xrightarrow{p \rightarrow +\infty} 1.$$

We deduce that

$$\left( \frac{\int_K \langle x, \theta \rangle_+^q dx}{mB(q+1, m) \text{Vol}(K \cap \{\langle x, \theta \rangle \geq 0\})} \right)^{\frac{1}{q}} \geq h_K(\theta).$$

Note also that  $\int_K \langle x, \theta \rangle_+^q dx \leq h_K(\theta)^q \text{Vol}(K \cap \{\langle x, \theta \rangle \geq 0\})$  and that  $mB(q+1, m) = qB(q, m+1)$ . Therefore

$$h_K(\theta) \geq \frac{h_{Z_q^+(K)}(\theta)}{\text{Vol}(K \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}} \geq (qB(q, m+1))^{1/q} h_K(\theta). \quad (27)$$

Now we establish that for  $q = \max(p, m)$

$$d(K_{m+q}(g), Z_q^+(g)) \leq c. \quad (28)$$

Take  $K = K_{m+q}(g)$ . By Lemma 11, for any  $q \geq m \geq 1$ ,  $(qB(q, m+1))^{1/q} \geq cq/(m+q+1) \geq c/3$  and we deduce from (27) that for every  $\theta \in \mathbb{R}^n$ ,

$$h_{K_{m+q}(g)}(\theta) \geq \frac{h_{Z_q^+(K_{m+q}(g))}(\theta)}{\text{Vol}(K_{m+q}(g) \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}} \geq \frac{c}{3} h_K(\theta).$$

where  $c$  is a universal constant. Together with (26), we conclude that

$$\begin{aligned} d(K_{m+q}(g), Z_q^+(g)) &= d(K_{m+q}(g), Z_q^+(K_{m+q}(g))) \\ &\leq c \frac{\sup_{\theta \in \mathbb{R}^n} \text{Vol}(K_{m+q}(g) \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}}{\inf_{\theta \in \mathbb{R}^n} \text{Vol}(K_{m+q}(g) \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}} \end{aligned} \quad (29)$$

for a universal constant  $c$ . Applying (25) for  $a = m+1$  and  $b = m+q$ , we get

$$e^{-4} K_{m+1}(g) \subset K_{m+q}(g) \subset c \frac{m+q}{m+1} K_{m+1}(g).$$

Since  $q \geq m$  and  $\left(\frac{m+q}{m+1}\right)^{m/q} \leq e$ , we get from (29)

$$d(K_{m+q}(g), Z_q^+(g)) \leq C \frac{\sup_{\theta \in \mathbb{R}^n} \text{Vol}(K_{m+1}(g) \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}}{\inf_{\theta \in \mathbb{R}^n} \text{Vol}(K_{m+1}(g) \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}}$$

for a universal constant  $C$ . Since  $g$  has its barycenter at the origin then  $K_{m+1}(g)$  has also its barycenter at the origin and we deduce from a classical result of Grünbaum [18] that there exists a universal constant  $c$  for which

$$\frac{\sup_{\theta \in \mathbb{R}^n} \text{Vol}(K_{m+1}(g) \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}}{\inf_{\theta \in \mathbb{R}^n} \text{Vol}(K_{m+1}(g) \cap \{\langle x, \theta \rangle \geq 0\})^{1/q}} \leq (e-1)^{1/q} \leq e-1.$$

And (28) is proved.

To conclude the proof of the Lemma, it is enough to establish that  $d(K_{m+q}, K_{m+p}) \leq c$ , where  $q = \max(m, p)$ . For  $q = p$ , this is obvious so we may assume that  $q = m \geq p$ . Then  $\frac{m}{2} \leq m + p \leq m + q = 2m$  and using equation (25) for  $a = m + p \leq b = 2m$ , we deduce that

$$d(K_{m+p}(g), K_{2m}(g)) \leq ce^4 \frac{2m}{m+p} \leq 4ce^4.$$

□

*Proof of inequality (22).* Our goal is to estimate

$$\frac{d}{dp} \log \left( (\mathbb{E}|X|_2^p)^{\frac{1}{p}} \right) = \frac{d}{dp} \log \left( (\mathbb{E}h_{k,p}(U))^{\frac{1}{p}} \right) + \frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \right).$$

As already mentioned in (15), by concavity of  $p \mapsto \frac{d}{dp} \log \Gamma(p)$ , we have

$$\frac{d}{dp} \left( \frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)} \right) \leq 0.$$

We use the following convention: let  $(\Omega, \mu)$  be a measurable space, for any measurable function  $f : \Omega \rightarrow \mathbb{R}^+$ , we set

$$\mathbb{E}_\mu(f) = \int f d\mu \quad \text{and} \quad \text{Ent}_\mu(f) = \mathbb{E}_\mu(f \log f) - \mathbb{E}_\mu(f) \log(\mathbb{E}_\mu(f)).$$

Let  $w$  be the density of the distribution of  $X$  on  $\mathbb{R}^n$ . Since  $X$  is  $(-1/r)$ -concave,  $w$  is  $(-1/(r+n))$ -concave on  $\mathbb{R}^n$ . To any fixed  $u \in SO(n)$ , we associate the measure  $\mu_u$  on  $\mathbb{R}^+$  with density

$$t \mapsto |S^{k-1}| t^{k-1} \pi_{u(E_0)} w(tu(\theta_0))$$

so that

$$h_{k,p}(u) = |S^{k-1}| \int_0^\infty t^{p+k-1} \pi_{u(E_0)} w(tu(\theta_0)) dt = \mathbb{E}_{\mu_u}(t^p).$$

Define also  $\mu_{k,p}$  the measure on  $\mathbb{R}^+$  with density

$$t \mapsto |S^{k-1}| t^{k-1} \mathbb{E} \pi_{U(E_0)} w(tU(\theta_0)).$$

Then  $\mathbb{E}h_{k,p}(U) = \mathbb{E}_U \mathbb{E}_{\mu_U}(t^p) = \mathbb{E}_{\mu_{k,p}}(t^p)$ . Since  $w$  is a density of probability,  $\mu_{k,p}$  is a probability measure on  $\mathbb{R}^+$ . A classical fact, verified by direct computation, is that

$$\frac{d}{dp} \log \left( (\mathbb{E}_{\mu}(f^p))^{1/p} \right) = \frac{1}{p^2} \frac{\text{Ent}_{\mu}(f^p)}{\mathbb{E}_{\mu}(f^p)}.$$

Therefore

$$\begin{aligned} \frac{d}{dp} \log \left( (\mathbb{E}h_{k,p}(U))^{1/p} \right) &= \frac{d}{dp} \log \left( (\mathbb{E}_{\mu_{k,p}}(t^p))^{1/p} \right) \\ &= \frac{1}{p^2} \frac{\text{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}_{\mu_{k,p}}(t^p)} = \frac{1}{p^2} \frac{\text{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}h_{k,p}(U)}. \end{aligned} \quad (30)$$

The numerator can be decomposed into two terms:

$$\text{Ent}_{\mu_{k,p}}(t^p) = \mathbb{E}_U \text{Ent}_{\mu_U}(t^p) + \text{Ent}_U \mathbb{E}_{\mu_U}(t^p) = \mathbb{E}_U \text{Ent}_{\mu_U}(t^p) + \text{Ent}_U h_{k,p}(U).$$

To control the second term, we use the log-Sobolev inequality (2):

$$\frac{1}{p^2} \frac{\text{Ent}_U h_{k,p}(U)}{\mathbb{E}h_{k,p}(U)} \leq \frac{c}{p^2 n} \frac{\mathbb{E}(|\nabla \log h_{k,p}|^2(U) h_{k,p}(U))}{\mathbb{E}h_{k,p}(U)} \leq \frac{cL_{k,p}^2}{p^2 n}. \quad (31)$$

To control the first term, we start by observing that for a fixed  $u \in SO(n)$ ,

$$\begin{aligned} \frac{1}{p^2} \frac{\text{Ent}_{\mu_u}(t^p)}{\mathbb{E}_{\mu_u}(t^p)} &= \frac{d}{dp} \log \left( (\mathbb{E}_{\mu_u}(f^p))^{1/p} \right) = \frac{d}{dp} \left( \frac{1}{p} \log h_{k,p}(u) \right) \\ &= \frac{d}{dp} \frac{1}{p} \left( \log \frac{h_{k,p}(u)}{B(p+k, r-p)} - \log \frac{h_{k,0}(u)}{B(k, r)} + \log \frac{B(p+k, r-p)}{B(k, r)} + \log h_{k,0}(u) \right). \end{aligned}$$

By Corollary 9, the map  $p \mapsto \frac{h_{k,p}(u)}{B(p+k, r-p)}$  is log-concave on  $(-k+1, r)$ . This implies that

$$\frac{d}{dp} \frac{1}{p} \left( \log \frac{h_{k,p}(u)}{B(p+k, r-p)} - \log \frac{h_{k,0}(u)}{B(k, r)} \right) \leq 0.$$

We know from Lemma 11 that, for all  $p \in [-\frac{k-1}{2}, \frac{r-1}{2}]$ ,

$$\frac{d}{dp} \left( \frac{1}{p} \log \frac{B(k+p, r-p)}{B(k, r)} \right) \leq C \left( \frac{1}{k-1} + \frac{1}{r-1} \right).$$

Therefore, for any fixed  $u \in SO(n)$ ,

$$\frac{1}{p^2} \text{Ent}_{\mu_u}(t^p) \leq C h_{k,p}(u) \left( \frac{1}{k-1} + \frac{1}{r-1} \right) - \frac{1}{p^2} h_{k,p}(u) \log h_{k,0}(u).$$

Integrating over  $u \in SO(n)$ , we deduce that

$$\frac{1}{p^2} \frac{\mathbb{E} \text{Ent}_{\mu_U}(t^p)}{\mathbb{E} h_{k,p}(U)} \leq C \left( \frac{1}{k-1} + \frac{1}{r-1} \right) + \frac{1}{p^2} \frac{\mathbb{E} h_{k,p}(U) \log(h_{k,0}(U)^{-1})}{\mathbb{E} h_{k,p}(U)}. \quad (32)$$

From Jensen and Hölder inequalities,

$$\begin{aligned} \frac{\mathbb{E}(h_{k,p}(U) \log h_{k,0}(U)^{-1})}{\mathbb{E} h_{k,p}(U)} &\leq \log \left( \frac{\mathbb{E}(h_{k,p}(U) h_{k,0}(U)^{-1})}{\mathbb{E} h_{k,p}(U)} \right) \\ &\leq \log \left( \frac{(\mathbb{E} h_{k,p}(U)^2)^{1/2}}{\mathbb{E} h_{k,p}(U)} \right) + \log ((\mathbb{E}(h_{k,0}(U)^{-2}))^{1/2}). \end{aligned}$$

From (3), the first term is upper bounded by  $\frac{c}{n} L_{k,p}^2$ . For the second term, we first use (3) with  $f = h_{k,0}^{-1}$ ,  $q = 2$  and  $r = 0$ , then we use (3) again with  $f = h_{k,0}$ ,  $q = 1$  and  $r = 0$ . Since  $\mathbb{E} h_{k,0}(U) = \mathbb{E}_{\mu_{k,0}}(1) = 1$ , we deduce that this term is bounded by  $\frac{3c}{n} L_{k,0}^2$ . Combining this last inequality with (32), (31) and (30), we conclude that

$$\frac{d}{dp} \log((\mathbb{E}|X|_2^p)^{\frac{1}{p}}) \leq \frac{c}{p^2 n} (2L_{k,p}^2 + 3L_{k,0}^2) + \frac{C}{k-1} + \frac{C}{r-1}.$$

□

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