



**HAL**  
open science

# Elliptic equations involving general subcritical source nonlinearity and measures

Laurent Veron, Huyuan Chen, Patricio Felmer

► **To cite this version:**

Laurent Veron, Huyuan Chen, Patricio Felmer. Elliptic equations involving general subcritical source nonlinearity and measures. 2014. hal-01072227

**HAL Id: hal-01072227**

**<https://hal.science/hal-01072227v1>**

Preprint submitted on 7 Oct 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Elliptic equations involving general subcritical source nonlinearity and measures

Huyuan Chen<sup>1</sup>   Patricio Felmer<sup>2</sup>   Laurent Véron<sup>3</sup>

## Abstract

In this article, we study the existence of positive solutions to elliptic equation (E1)

$$(-\Delta)^\alpha u = g(u) + \sigma\nu \quad \text{in } \Omega,$$

subject to the condition (E2)

$$u = \varrho\mu \quad \text{on } \partial\Omega \text{ if } \alpha = 1 \quad \text{or in } \Omega^c \text{ if } \alpha \in (0, 1),$$

where  $\sigma, \varrho \geq 0$ ,  $\Omega$  is an open bounded  $C^2$  domain in  $\mathbb{R}^N$ ,  $(-\Delta)^\alpha$  denotes the fractional Laplacian with  $\alpha \in (0, 1)$  or Laplacian operator if  $\alpha = 1$ ,  $\nu, \mu$  are suitable Radon measures and  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a continuous function.

We introduce an approach to obtain weak solutions for problem (E1)-(E2) when  $g$  is integral subcritical and  $\sigma, \varrho \geq 0$  small enough.

**Key words:** Fractional Laplacian; Radon measure; Green kernel; Poisson kernel; Schauder's fixed point theorem.

**MSC2010:** 35R11, 35J61, 35R06

## 1 Introduction

Let  $\alpha \in (0, 1]$ ,  $\Omega$  be an open bounded  $C^2$  domain in  $\mathbb{R}^N$  with  $N > 2\alpha$ ,  $\rho(x) = \text{dist}(x, \partial\Omega)$ ,  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  be a continuous function and denote by  $(-\Delta)^\alpha$  the Laplacian operator if  $\alpha = 1$  or the fractional Laplacian with  $\alpha \in (0, 1)$  defined as

$$(-\Delta)^\alpha u(x) = \lim_{\varepsilon \rightarrow 0^+} (-\Delta)_\varepsilon^\alpha u(x),$$

where for  $\varepsilon > 0$ ,

$$(-\Delta)_\varepsilon^\alpha u(x) = - \int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\varepsilon(|x - z|) dz$$

and

$$\chi_\varepsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \varepsilon], \\ 1, & \text{if } t > \varepsilon. \end{cases}$$

---

<sup>1</sup>hc64@nyu.edu

<sup>2</sup>pfelmer@dim.uchile.cl

<sup>3</sup>Laurent.Veron@lmpt.univ-tours.fr

Our first purpose of this paper is to study the existence of weak solutions to the semilinear elliptic problem

$$(-\Delta)^\alpha u = g(u) + \sigma\nu \quad \text{in } \Omega, \quad (1.1)$$

subject to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \text{ if } \alpha = 1 \quad \text{or in } \Omega^c \text{ if } \alpha \in (0, 1), \quad (1.2)$$

where  $\sigma > 0$ ,  $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$  with  $\beta \in [0, \alpha]$  and  $\mathfrak{M}(\Omega, \rho^\beta)$  being the space of Radon measures in  $\Omega$  satisfying

$$\int_{\Omega} \rho^\beta d|\nu| < +\infty.$$

In particular, we denote  $\mathfrak{M}^b(\Omega) = \mathfrak{M}(\Omega, \rho^0)$ . The associated positive cones are respectively  $\mathfrak{M}_+(\Omega, \rho^\beta)$  and  $\mathfrak{M}_+^b(\Omega)$ .

When  $\alpha = 1$ , problem (1.1)-(1.2) has been studied for some decades. The basic method developed by Ni [21] and Ratto-Rigoli-Véron [22] is to iterate

$$u_{n+1} = \mathbb{G}_1[g(u_n)] + \sigma\mathbb{G}_1[\nu], \quad \forall n \in \mathbb{N}.$$

The crucial ingredient in this approach is to derive a function  $v$  satisfying

$$v \geq \mathbb{G}_1[g(v)] + \sigma\mathbb{G}_1[\nu].$$

Later on, Baras-Pierre [3] applied duality argument to derive weak solution of problem (1.1)-(1.2) with  $\alpha = 1$  under the hypotheses:

- (i) the mapping  $r \mapsto g(r)$  is nondecreasing, convex and continuous;
- (ii) there exist  $c_0 > 0$  and  $\xi_0 \in C_0^{1,1}(\Omega)$ ,  $\xi_0 \neq 0$  such that

$$g^* \left( c_0 \frac{-\Delta \xi_0}{\xi_0} \right) \in L^1(\Omega),$$

where  $g^*$  is the conjugate function of  $g$ ;

(iii)

$$\int_{\Omega} \xi d\nu \leq \int_{\Omega} g^* \left( \frac{-\Delta \xi}{\xi} \right) dx, \quad \forall \xi \in C_0^{1,1}(\Omega).$$

When  $g$  is pure power source, Brezis-Cabré [2] and Kalton-Verbitsky [16] pointed out that the necessary condition for existence of weak solution to

$$\begin{aligned} -\Delta u &= u^p + \sigma\nu & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (1.3)$$

is that

$$\mathbb{G}_1[(\mathbb{G}_1[\nu])^p] \leq c_1 \mathbb{G}_1[\nu], \quad (1.4)$$

for some  $c_1 > 0$ . Bidaut-Véron and Vivier in [5] proved that (1.4) holds for  $p < \frac{N+\beta}{N+\beta-2}$  and problem (1.3) admits a weak solution if  $\sigma > 0$  small. While it is not easy to get explicit condition for general nonlinearity by above methods.

In this article, we introduce a new method to obtain the weak solution of problem (1.1)-(1.2) involving general nonlinearity without convex and nondecreasing properties, which is inspired by the Marcinkiewicz spaces approach.

Let us first make precise the definition of weak solution to (1.1)-(1.2).

**Definition 1.1** *We say that  $u$  is a weak solution of (1.1)-(1.2), if  $u \in L^1(\Omega)$ ,  $g(u) \in L^1(\Omega, \rho^\alpha dx)$  and*

$$\int_{\Omega} u(-\Delta)^\alpha \xi dx = \int_{\Omega} g(u)\xi dx + \sigma \int_{\Omega} \xi d\nu, \quad \forall \xi \in \mathbb{X}_\alpha,$$

where  $\mathbb{X}_\alpha = C_0^{1,1}(\Omega)$  if  $\alpha = 1$  or  $\mathbb{X}_\alpha \subset C(\mathbb{R}^N)$  with  $\alpha \in (0, 1)$  is the space of functions  $\xi$  satisfying:

- (i)  $\text{supp}(\xi) \subset \bar{\Omega}$ ,
- (ii)  $(-\Delta)^\alpha \xi(x)$  exists for all  $x \in \Omega$  and  $|(-\Delta)^\alpha \xi(x)| \leq C$  for some  $C > 0$ ,
- (iii) there exist  $\varphi \in L^1(\Omega, \rho^\alpha dx)$  and  $\varepsilon_0 > 0$  such that  $|(-\Delta)_\varepsilon^\alpha \xi| \leq \varphi$  a.e. in  $\Omega$ , for all  $\varepsilon \in (0, \varepsilon_0]$ .

We denote by  $G_\alpha$  the Green kernel of  $(-\Delta)^\alpha$  in  $\Omega \times \Omega$  and by  $\mathbb{G}_\alpha[\cdot]$  the associated Green operator defined by

$$\mathbb{G}_\alpha[\nu](x) = \int_{\Omega} G_\alpha(x, y) d\nu(y), \quad \forall \nu \in \mathfrak{M}(\Omega, \rho^\alpha).$$

Our first result states as follows.

**Theorem 1.1** *Let  $\alpha \in (0, 1]$ ,  $\sigma > 0$  and  $\nu \in \mathfrak{M}_+(\Omega, \rho^\beta)$  with  $\beta \in [0, \alpha]$ .*

(i) *Suppose that*

$$g(s) \leq c_2 s^{p_0} + \epsilon, \quad \forall s \geq 0, \tag{1.5}$$

for some  $p_0 \in (0, 1]$ ,  $c_2 > 0$  and  $\epsilon > 0$ . Assume more that  $c_2$  is small enough when  $p_0 = 1$ .

Then problem (1.1)-(1.2) admits a weak nonnegative solution  $u_\nu$  which satisfies

$$u_\nu \geq \sigma \mathbb{G}_\alpha[\nu]. \tag{1.6}$$

(ii) *Suppose that*

$$g(s) \leq c_3 s^{p^*} + \epsilon, \quad \forall s \in [0, 1] \tag{1.7}$$

and

$$g_\infty := \int_1^{+\infty} g(s) s^{-1-p_\beta^*} ds < +\infty, \tag{1.8}$$

where  $c_3, \epsilon > 0$ ,  $p_* > 1$  and  $p_\beta^* = \frac{N+\beta}{N-2\alpha+\beta}$ .

Then there exist  $\sigma_0, \epsilon_0 > 0$  depending on  $c_3, p_*, g_\infty$  and  $p_\beta^*$  such that for  $\sigma \in [0, \sigma_0)$  and  $\epsilon \in (0, \epsilon_0)$ , problem (1.1)-(1.2) admits a nonnegative weak solution  $u_\nu$  which satisfies (1.6).

We remark that (i) we do not require any restriction on parameters  $c_2, \epsilon, \sigma$  when  $p_0 \in (0, 1)$  or on parameters  $\epsilon, \sigma$  when  $p_0 = 1$ ; (ii) the assumption (1.8) is called as integral subcritical condition, which is usually used in dealing with elliptic problem with absorption nonlinearity and measures, see the references [5, 9, 10, 24].

Let us sketch the proof of Theorem 1.1. We first approximate the nonlinearity  $g$  and Radon measure  $\nu$  by  $\{g_n\}$  and  $\{\nu_n\}$  respectively, then we make use of the Marcinkiewicz properties and embedding theorems to obtain that for  $n \geq 1$ , problem

$$(-\Delta)^\alpha u_n = g_n(u_n) + \sigma \nu_n \quad \text{in } \Omega,$$

subject to condition (1.2), admits a nonnegative solution  $u_n$  by Schauder's fixed point theorem. The crucial point is to obtain uniformly bound of  $\{u_n\}$  in the Marcinkiewicz space. The proof ends by getting a subsequence of  $\{u_n\}$  that converges in the sense of Definition 1.1.

Our second purpose in this note is to obtain the weak solution to elliptic equations involving boundary measures. Firstly, we study the weak solution of

$$\begin{aligned} -\Delta u &= g(u) \quad \text{in } \Omega, \\ u &= \varrho \mu \quad \text{on } \partial\Omega, \end{aligned} \tag{1.9}$$

where  $\varrho > 0$  and  $\mu \in \mathfrak{M}_+^b(\partial\Omega)$  the space of nonnegative bounded Radon measure on  $\partial\Omega$ . When  $g(s) = s^p$  with  $p < \frac{N+1}{N-1}$ , the weak solution to problem (1.9) is derived by Bidaut-Véron and Vivier in [5] by using iterating procedure. More interests on boundary measures refer to [4, 6, 13, 17, 18, 19].

**Definition 1.2** We say that  $u$  is a weak solution of (1.9), if  $u \in L^1(\Omega)$ ,  $g(u) \in L^1(\Omega, \rho dx)$  and

$$\int_{\Omega} u(-\Delta)\xi dx = \int_{\Omega} g(u)\xi dx + \varrho \int_{\partial\Omega} \frac{\partial\xi(x)}{\partial\vec{n}_x} d\mu(x), \quad \forall \xi \in C_0^{1,1}(\Omega),$$

where  $\vec{n}_x$  is the unit normal vector pointing outside of  $\Omega$  at point  $x$ .

We denote by  $P$  the Poisson kernel of  $-\Delta$  in  $\Omega \times \partial\Omega$  and by  $\mathbb{P}[\cdot]$  the associated Poisson operator defined by

$$\mathbb{P}[\mu](x) = \int_{\partial\Omega} P(x, y) d\mu(y), \quad \forall \mu \in \mathfrak{M}^b(\partial\Omega).$$

Our second result states as follows.

**Theorem 1.2** Let  $\varrho > 0$  and  $\mu \in \mathfrak{M}_+^b(\partial\Omega)$ .

(i) Suppose that

$$g(s) \leq c_4 s^{q_0} + \epsilon, \quad \forall s \geq 0, \quad (1.10)$$

for some  $q_0 \in (0, 1]$ ,  $c_4 > 0$  and  $\epsilon > 0$ . Assume more that  $c_4$  is small enough when  $q_0 = 1$ .

Then problem (1.9) admits a weak nonnegative solution  $u_\mu$  which satisfies

$$u_\mu \geq \varrho \mathbb{P}[\mu]. \quad (1.11)$$

(ii) Suppose that

$$g(s) \leq c_5 s^{q_*} + \epsilon, \quad \forall s \in [0, 1] \quad (1.12)$$

and

$$g_\infty := \int_1^{+\infty} g(s) s^{-1-q_*} ds < +\infty, \quad (1.13)$$

where  $c_5, \epsilon > 0$ ,  $q_* > 1$  and  $q^* = \frac{N+1}{N-1}$ .

Then there exist  $\varrho_0, \epsilon_0 > 0$  depending on  $c_5, q_*, g_\infty$  and  $q^*$  such that for  $\varrho \in [0, \varrho_0)$  and  $\epsilon \in [0, \epsilon_0)$ , problem (1.9) admits a nonnegative weak solution  $u_\mu$  which satisfies (1.11).

We remark that the key-point in the proof of Theorem 1.2 is to derive the uniform bound in Marcinkiewicz quasi-norm to the solutions of

$$\begin{aligned} -\Delta u &= g_n(u + \varrho \mathbb{P}[\mu]) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (1.14)$$

where  $\{g_n\}$  is a sequence of  $C^1$  bounded functions approaching to  $g$  in  $L_{loc}^\infty(\mathbb{R}_+)$ . In fact, the weak solution  $u_\mu$  could be decomposed into

$$u_\mu = v_\mu + \varrho \mathbb{P}[\mu],$$

where  $v_\mu$  is a weak solution to (1.14) replaced  $g_n$  by  $g$ .

Inspired by the fact above, we give the definition of weak solution to

$$\begin{aligned} (-\Delta)^\alpha u &= g(u) & \text{in } \Omega, \\ u &= \varrho \mu & \text{in } \Omega^c \end{aligned} \quad (1.15)$$

as follows.

**Definition 1.3** We say that  $u_\mu$  is a weak solution of (1.15), if

$$u_\mu = v_\mu + \varrho \mathbb{G}_\alpha[w_\mu],$$

where

$$w_\mu(x) = \int_{\Omega^c} \frac{d\mu(z)}{|z-x|^{N+2\alpha}}, \quad x \in \Omega \quad (1.16)$$

and  $v_\mu$  is a solution of

$$\begin{aligned} (-\Delta)^\alpha u &= g(u + \varrho \mathbb{G}_\alpha[w_\mu]) & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \quad (1.17)$$

in the sense of Definition 1.1.

In Definition 1.3, the function  $\mathbb{G}_\alpha[w_\mu]$  plays the role of  $\mathbb{P}[\mu]$  when  $\alpha = 1$ . In order to better classify the measures tackled in follows, we denote

$$\mathfrak{R}_\beta := \{\mu \in \mathfrak{M}_+(\Omega^c) : w_\mu \in L^1(\Omega, \rho^\beta dx)\}, \quad (1.18)$$

where  $\beta \in [0, \alpha]$  and  $w_\mu$  is given by (1.16).

**Theorem 1.3** *Let  $\alpha \in (0, 1)$ ,  $\sigma > 0$  and  $\mu \in \mathfrak{R}_\beta$  with  $\beta \in [0, \alpha]$ .*

(i) *Suppose that*

$$g(s) \leq c_6 s^{q_0} + \epsilon, \quad \forall s \geq 0, \quad (1.19)$$

for some  $q_0 \in (0, 1]$ ,  $c_6 > 0$  and  $\epsilon > 0$ . Assume more that  $c_6$  is small enough when  $q_0 = 1$ .

Then problem (1.15) admits a weak nonnegative solution  $u_\mu$  which satisfies

$$u_\mu \geq \varrho \mathbb{G}_\alpha[w_\mu]. \quad (1.20)$$

(ii) *Suppose that*

$$g(s) \leq c_7 s^{q_*} + \epsilon, \quad \forall s \in [0, 1] \quad (1.21)$$

and

$$g_\infty := \int_1^{+\infty} g(s) s^{-1-p_\beta^*} ds < +\infty, \quad (1.22)$$

where  $c_7, \epsilon > 0$ ,  $q_* > 1$  and  $p_\beta^* = \frac{N+\beta}{N-2\alpha+\beta}$ .

Then there exist  $\sigma_0, \varrho_0 > 0$  depending on  $c_7, q_*, g_\infty$  and  $p_\beta^*$  such that for  $\varrho \in [0, \varrho_0)$  and  $\epsilon \in [0, \epsilon_0)$ , problem (1.15) admits a nonnegative weak solution  $u_\mu$  which satisfies (1.20).

The rest of this paper is organized as follows. In section §2, we recall some basic results on Green kernel and Poisson kernel related to the Marcinkiewicz space. Section §3 is addressed to prove the existence of weak solution to elliptic equation with small forcing measure. Finally, we obtain weak solution to elliptic equation with small boundary type measure.

## 2 Preliminary

In order to obtain the weak solution of (1.1)-(1.2) with integral subcritical nonlinearity, we have to introduce the Marcinkiewicz space and recall some related estimate.

**Definition 2.1** Let  $\Theta \subset \mathbb{R}^N$  be a domain and  $\varpi$  be a positive Borel measure in  $\Theta$ . For  $\kappa > 1$ ,  $\kappa' = \kappa/(\kappa - 1)$  and  $u \in L_{loc}^1(\Theta, d\mu)$ , we set

$$\|u\|_{M^\kappa(\Theta, d\varpi)} = \inf \left\{ c \in [0, \infty] : \int_E |u| d\varpi \leq c \left( \int_E d\varpi \right)^{\frac{1}{\kappa'}}, \forall E \subset \Theta, E \text{ Borel} \right\} \quad (2.1)$$

and

$$M^\kappa(\Theta, d\varpi) = \{u \in L_{loc}^1(\Theta, d\varpi) : \|u\|_{M^\kappa(\Theta, d\varpi)} < +\infty\}. \quad (2.2)$$

$M^\kappa(\Theta, d\varpi)$  is called the Marcinkiewicz space of exponent  $\kappa$ , or weak  $L^\kappa$ -space and  $\|\cdot\|_{M^\kappa(\Theta, d\varpi)}$  is a quasi-norm. We observe that

$$\|u + v\|_{M^\kappa(\Theta, d\varpi)} \leq \|u\|_{M^\kappa(\Theta, d\varpi)} + \|v\|_{M^\kappa(\Theta, d\varpi)} \quad (2.3)$$

and

$$\|tu\|_{M^\kappa(\Theta, d\varpi)} = t\|u\|_{M^\kappa(\Theta, d\varpi)}, \quad \forall t > 0. \quad (2.4)$$

**Proposition 2.1** [1, 11] Assume that  $1 \leq q < \kappa < \infty$  and  $u \in L_{loc}^1(\Theta, d\varpi)$ . Then there exists  $c_8 > 0$  dependent of  $q, \kappa$  such that

$$\int_E |u|^q d\varpi \leq c_8 \|u\|_{M^\kappa(\Theta, d\varpi)} \left( \int_E d\varpi \right)^{1-q/\kappa}$$

for any Borel set  $E$  of  $\Theta$ .

The next estimate is the key-stone in the proof of Theorem 1.1 to control the nonlinearity in  $\{g \geq 1\}$ .

**Proposition 2.2** Let  $\alpha \in (0, 1]$ ,  $\beta \in [0, \alpha]$  and  $p_\beta^* = \frac{N+\beta}{N-2\alpha+\beta}$ , then there exists  $c_9 > 0$  such that

$$\|\mathbb{G}_\alpha[\nu]\|_{M^{p_\beta^*}(\Omega, \rho^\beta dx)} \leq c_9 \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)}. \quad (2.5)$$

**Proof.** When  $\alpha \in (0, 1)$ , it follows by [9, Proposition 2.2] that for  $\gamma \in [0, \alpha]$ , there exists  $c_{10} > 0$  such that

$$\|\mathbb{G}_\alpha[\nu]\|_{M^{k_{\alpha, \beta, \gamma}}(\Omega, \rho^\gamma dx)} \leq c_{10} \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)},$$

where

$$k_{\alpha, \beta, \gamma} = \begin{cases} \frac{N+\gamma}{N-2\alpha+\beta}, & \text{if } \gamma < \frac{N\beta}{N-2\alpha}, \\ \frac{N}{N-2\alpha}, & \text{if not.} \end{cases}$$



We just take  $\gamma = \beta$ , then  $k_{\alpha,\beta,\gamma} = p_\beta^*$  and (2.5) holds.

When  $\alpha = 1$ , (2.5) follows by [24, Theorem 3.5].  $\square$

The following proposition does not just provide regularity but also plays an essential role to control in  $\{g < 1\}$ .

**Proposition 2.3** *Let  $\alpha \in (0, 1]$  and  $\beta \in [0, \alpha]$ , then the mapping  $f \mapsto \mathbb{G}_\alpha[f]$  is compact from  $L^1(\Omega, \rho^\beta dx)$  into  $L^q(\Omega)$  for any  $q \in [1, \frac{N}{N+\beta-2\alpha})$ . Moreover, for  $q \in [1, \frac{N}{N+\beta-2\alpha})$ , there exists  $c_{11} > 0$  such that for any  $f \in L^1(\Omega, \rho^\beta dx)$*

$$\|\mathbb{G}_\alpha[f]\|_{L^q(\Omega)} \leq c_{11} \|f\|_{L^1(\Omega, \rho^\beta dx)}. \quad (2.6)$$

**Proof.** When  $\alpha \in (0, 1)$  and  $\beta \in [0, \alpha]$ , it follows by [9, Proposition 2.5] that for  $p \in (1, \frac{N}{N-2\alpha+\beta})$ , there exists  $c_{12} > 0$  such that for any  $f \in L^1(\Omega, \rho^\beta dx)$

$$\|\mathbb{G}_\alpha[f]\|_{W^{2\alpha-\gamma,p}(\Omega)} \leq c_{12} \|f\|_{L^1(\Omega, \rho^\beta dx)}, \quad (2.7)$$

where  $\gamma = \beta + \frac{N(p-1)}{p}$  if  $\beta > 0$  and  $\gamma > \frac{N(p-1)}{p}$  if  $\beta = 0$ . By [20, Theorem 6.5], the embedding of  $W^{2\alpha-\gamma,p}(\Omega)$  into  $L^q(\Omega)$  is compact, then the mapping  $f \mapsto \mathbb{G}_\alpha[f]$  is compact from  $L^1(\Omega, \rho^\beta dx)$  into  $L^q(\Omega)$  for any  $q \in [1, \frac{N}{N+\beta-2\alpha})$ . We observe that (2.6) follows by (2.7) and the embedding inequality.

When  $\alpha = 1$  and  $\beta \in [0, 1]$ , it follows by [5, Theorem 2.7] that

$$\|\mathbb{G}_\alpha[f]\|_{W_0^{1, \frac{N}{N-1+\beta}}(\Omega)} \leq c_{13} \|f\|_{L^1(\Omega, \rho^\beta dx)}, \quad (2.8)$$

where  $c_{13} > 0$ . By the compactness of the embedding from  $W_0^{1, \frac{N}{N-1+\beta}}(\Omega)$  into  $L^q(\Omega)$  with  $q \in [1, \frac{N}{N+\beta-2})$ , we have that the mapping  $f \mapsto \mathbb{G}_\alpha[f]$  is compact from  $L^1(\Omega, \rho^\beta dx)$  into  $L^q(\Omega)$  for  $q \in [1, \frac{N}{N+\beta-2})$ . Similarly, (2.6) follows by (2.8) and the related embedding inequality.  $\square$

When we deal with problem (1.9), the Poisson kernel changes the boundary measure to forcing term and the following proposition plays an important role in obtaining the weak solution to (1.14) replaced  $g_n$  by  $g$ .

**Proposition 2.4** [5, Theorem 2.5] *Let  $\gamma > -1$  and  $p_\gamma = \frac{N+\gamma}{N-1}$ , then there exists  $c_{14} > 0$  such that*

$$\|\mathbb{P}[\nu]\|_{M^{p_\gamma}(\Omega, \rho^\gamma dx)} \leq c_{14} \|\nu\|_{\mathfrak{M}^b(\partial\Omega)}. \quad (2.9)$$

## 3 Forcing measure

### 3.1 Sub-linear

In this subsection, we are devoted to prove the existence of weak solution to (1.1) when the nonlinearity is sub-linear.

**Proof of Theorem 1.1 part (i).** Let  $\beta \in [0, \alpha]$ , we define the space

$$C_\beta(\bar{\Omega}) = \{\zeta \in C(\bar{\Omega}) : \rho^{-\beta}\zeta \in C(\bar{\Omega})\}$$

endowed with the norm

$$\|\zeta\|_{C_\beta(\bar{\Omega})} = \|\rho^{-\beta}\zeta\|_{C(\bar{\Omega})}.$$

Let  $\{\nu_n\} \subset C^1(\bar{\Omega})$  be a sequence of nonnegative functions such that  $\nu_n \rightarrow \nu$  in sense of duality with  $C_\beta(\bar{\Omega})$ , that is,

$$\lim_{n \rightarrow \infty} \int_{\bar{\Omega}} \zeta \nu_n dx = \int_{\bar{\Omega}} \zeta d\nu, \quad \forall \zeta \in C_\beta(\bar{\Omega}). \quad (3.1)$$

By the Banach-Steinhaus Theorem,  $\|\nu_n\|_{\mathfrak{M}(\Omega, \rho^\beta)}$  is bounded independently of  $n$ . We may assume that  $\|\nu_n\|_{L^1(\Omega, \rho^\beta dx)} \leq \|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} = 1$  for all  $n \geq 1$ . We consider a sequence  $\{g_n\}$  of  $C^1$  nonnegative functions defined on  $\mathbb{R}_+$  such that  $g_n(0) = g(0)$ ,

$$g_n \leq g_{n+1} \leq g, \quad \sup_{s \in \mathbb{R}_+} g_n(s) = n \quad \text{and} \quad \lim_{n \rightarrow \infty} \|g_n - g\|_{L_{loc}^\infty(\mathbb{R}_+)} = 0. \quad (3.2)$$

We set

$$M(v) = \|v\|_{L^1(\Omega)}.$$

*Step 1. To prove that for  $n \geq 1$ ,*

$$\begin{aligned} (-\Delta)^\alpha u &= g_n(u) + \sigma \nu_n & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \quad (3.3)$$

*admits a nonnegative solution  $u_n$  such that*

$$M(u_n) \leq \bar{\lambda},$$

*where  $\bar{\lambda} > 0$  independent of  $n$ .*

To this end, we define the operators  $\{\mathcal{T}_n\}$  by

$$\mathcal{T}_n u = \mathbb{G}_\alpha [g_n(u) + \sigma \nu_n], \quad \forall u \in L_+^1(\Omega),$$

where  $L_+^1(\Omega)$  is the positive cone of  $L^1(\Omega)$ . By (2.6) and (1.19), we have that

$$\begin{aligned} M(\mathcal{T}_n u) &\leq c_{11} \|g_n(u) + \sigma \nu_n\|_{L^1(\Omega, \rho^\beta dx)} \\ &\leq c_2 c_{11} \int_{\Omega} u^{p_0} \rho^\beta(x) dx + c_6(\sigma + \epsilon) \\ &\leq c_2 c_{15} \int_{\Omega} u^{p_0} dx + c_6(\sigma + \epsilon) \\ &\leq c_2 c_{16} (\int_{\Omega} u dx)^{p_0} + c_6(\sigma + \epsilon) \\ &= c_2 c_{16} M(u)^{p_0} + c_6(\sigma + \epsilon), \end{aligned} \quad (3.4)$$

where  $c_{15}, c_{16} > 0$  independent of  $n$ . Therefore, we derive that

$$M(\mathcal{T}_n u) \leq c_2 c_{16} M(u)^{p_0} + c_{11}(\sigma + \epsilon).$$

If we assume that  $M(u) \leq \lambda$  for some  $\lambda > 0$ , it implies

$$M(\mathcal{T}_n u) \leq c_2 c_{16} \lambda^{p_0} + c_{11}(\sigma + \epsilon).$$

In the case of  $p_0 < 1$ , the equation

$$c_2 c_{16} \lambda^{p_0} + c_{11}(\sigma + \epsilon) = \lambda$$

admits a unique positive root  $\bar{\lambda}$ . In the case of  $p_0 = 1$ , for  $c_2 > 0$  satisfying  $c_2 c_{16} < 1$ , the equation

$$c_2 c_{16} \lambda + c_{11}(\sigma + \epsilon) = \lambda$$

admits a unique positive root  $\bar{\lambda}$ . For  $M(u) \leq \bar{\lambda}$ , we obtain that

$$M(\mathcal{T}_n u) \leq c_2 c_{16} \bar{\lambda}^{p_0} + c_{11}(\sigma + \epsilon) = \bar{\lambda}. \quad (3.5)$$

Thus,  $\mathcal{T}_n$  maps  $L^1(\Omega)$  into itself. Clearly, if  $u_m \rightarrow u$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ , then  $g_n(u_m) \rightarrow g_n(u)$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ , thus  $\mathcal{T}_n$  is continuous. For any fixed  $n \in \mathbb{N}$ ,  $\mathcal{T}_n u_m = \mathbb{G}_\alpha [g_n(u_m) + \sigma \nu_n]$  and  $\{g_n(u_m) + \sigma \nu_n\}_m$  is uniformly bounded in  $L^1(\Omega, \rho^\beta dx)$ , then it follows by Proposition 2.3 that  $\{\mathbb{G}_\alpha [g_n(u_m) + \sigma \nu_n]\}_m$  is pre-compact in  $L^1(\Omega)$ , which implies that  $\mathcal{T}_n$  is a compact operator.

Let

$$\mathcal{G} = \{u \in L^1_+(\Omega) : M(u) \leq \bar{\lambda}\},$$

which is a closed and convex set of  $L^1(\Omega)$ . It infers by (3.5) that

$$\mathcal{T}_n(\mathcal{G}) \subset \mathcal{G}.$$

It follows by Schauder's fixed point theorem that there exists some  $u_n \in L^1_+(\Omega)$  such that  $\mathcal{T}_n u_n = u_n$  and  $M(u_n) \leq \bar{\lambda}$ , where  $\bar{\lambda} > 0$  independent of  $n$ .

We observe that  $u_n$  is a classical solution of (3.3). For  $\alpha = 1$ , since  $g_n$  bounded and  $C^1$ , then it is natural to see that. When  $\alpha \in (0, 1)$ , let open set  $O$  satisfy  $O \subset \bar{O} \subset \Omega$ . By [23, Proposition 2.3], for  $\theta \in (0, 2\alpha)$ , there exists  $c_{17} > 0$  such that

$$\|u_n\|_{C^\theta(O)} \leq c_{17} \{\|g(u_n)\|_{L^\infty(\Omega)} + \sigma \|\nu_n\|_{L^\infty(\Omega)}\},$$

then applied [23, Corollary 2.4],  $u_n$  is  $C^{2\alpha+\epsilon_0}$  locally in  $\Omega$  for some  $\epsilon_0 > 0$ . Then  $u_n$  is a classical solution of (3.3). Moreover, from [10, Lemma 2.2], we derive that

$$\int_{\Omega} u_n (-\Delta)^\alpha \xi dx = \int_{\Omega} g_n(u_n) \xi dx + \sigma \int_{\Omega} \xi \nu_n dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (3.6)$$

*Step 2. Convergence.* We observe that  $\{g_n(u_n)\}$  is uniformly bounded in  $L^1(\Omega, \rho^\beta dx)$ , so is  $\{\nu_n\}$ . By Proposition 2.3, there exist a subsequence  $\{u_{n_k}\}$  and  $u$  such that  $u_{n_k} \rightarrow u$  a.e. in  $\Omega$  and in  $L^1(\Omega)$ , then by (1.19), we derive that  $g_{n_k}(u_{n_k}) \rightarrow g(u)$  in  $L^1(\Omega)$ . Pass the limit of (3.6) as  $n_k \rightarrow \infty$  to derive that

$$\int_{\Omega} u(-\Delta)^{\alpha} \xi = \int_{\Omega} g(u) \xi dx + \sigma \int_{\Omega} \xi d\nu, \quad \forall \xi \in \mathbb{X}_{\alpha},$$

thus  $u$  is a weak solution of (1.1)-(1.2) and  $u$  is nonnegative since  $\{u_n\}$  are nonnegative.  $\square$

### 3.2 Integral subcritical

In this subsection, we prove the existence of weak solution to (1.1) when the nonlinearity is integral subcritical. We first introduce an auxiliary lemma.

**Lemma 3.1** *Assume that  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is a continuous function satisfying*

$$\int_1^{+\infty} g(s) s^{-1-p} ds < +\infty \quad (3.7)$$

for some  $p > 0$ . Then there is a sequence real positive numbers  $\{T_n\}$  such that

$$\lim_{n \rightarrow \infty} T_n = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} g(T_n) T_n^{-p} = 0.$$

*Proof.* Let  $\{s_n\}$  be a sequence of real positive numbers converging to  $\infty$ . We observe

$$\begin{aligned} \int_{s_n}^{2s_n} g(t) t^{-1-p} dt &\geq \min_{t \in [s_n, 2s_n]} g(t) (2s_n)^{-1-p} \int_{s_n}^{2s_n} dt \\ &= 2^{-1-p} s_n^{-p} \min_{t \in [s_n, 2s_n]} g(t) \end{aligned}$$

and by (3.7),

$$\lim_{n \rightarrow \infty} \int_{s_n}^{2s_n} g(t) t^{-1-p} dt = 0.$$

Then we choose  $T_n \in [s_n, 2s_n]$  such that  $g(T_n) = \min_{t \in [s_n, 2s_n]} g(t)$  and then the claim follows.  $\square$

**Proof of Theorem 1.1 part (ii).** Let  $\{\nu_n\} \subset C^1(\bar{\Omega})$  be a sequence of nonnegative functions such that  $\nu_n \rightarrow \nu$  in sense of duality with  $C_{\beta}(\Omega)$  and we may assume that  $\|\nu_n\|_{L^1(\Omega, \rho^\beta dx)} \leq 2\|\nu\|_{\mathfrak{M}(\Omega, \rho^\beta)} = 1$  for all  $n \geq 1$ . We consider a sequence  $\{g_n\}$  of  $C^1$  nonnegative functions defined on  $\mathbb{R}_+$  satisfying  $g_n(0) = g(0)$  and (3.2). We set

$$M_1(v) = \|v\|_{M^{p\beta}_{(\Omega, \rho^\beta dx)}}^* \quad \text{and} \quad M_2(v) = \|v\|_{L^{p^*}(\Omega)},$$

where  $p_\beta^*$  and  $p_*$  are from (1.7) and (1.8). We may assume that  $p_* \in (1, \frac{N}{N-2\alpha+\beta})$ . In fact, if  $p_* \geq \frac{N}{N-2\alpha+\beta}$ , then for any given  $p \in (1, \frac{N}{N-2\alpha+\beta})$ , (1.8) implies that

$$g(s) \leq c_3 s^p + \epsilon, \quad \forall s \in [0, 1].$$

*Step 1. To prove that for  $n \geq 1$ ,*

$$\begin{aligned} (-\Delta)^\alpha u &= g_n(u) + \sigma \nu_n & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \quad (3.8)$$

*admits a nonnegative solution  $u_n$  such that*

$$M_1(u_n) + M_2(u_n) \leq \bar{\lambda},$$

*where  $\bar{\lambda} > 0$  independent of  $n$ .*

To this end, we define the operators  $\{\mathcal{T}_n\}$  by

$$\mathcal{T}_n u = \mathbb{G}_\alpha [g_n(u) + \sigma \nu_n], \quad \forall u \in L_+^1(\Omega).$$

By Proposition 2.2, we have

$$\begin{aligned} M_1(\mathcal{T}_n u) &\leq c_9 \|g_n(u) + \sigma \nu_n\|_{L^1(\Omega, \rho^\beta dx)} \\ &\leq c_9 [\|g_n(u)\|_{L^1(\Omega, \rho^\beta dx)} + \sigma]. \end{aligned} \quad (3.9)$$

In order to deal with  $\|g_n(u)\|_{L^1(\Omega, \rho^\beta dx)}$ , for  $\lambda > 0$  we set  $S_\lambda = \{x \in \Omega : u(x) > \lambda\}$  and  $\omega(\lambda) = \int_{S_\lambda} \rho^\beta dx$ ,

$$\|g_n(u)\|_{L^1(\Omega, \rho^\beta dx)} \leq \int_{S_1^c} g(u) \rho^\beta dx + \int_{S_1} g(u) \rho^\beta dx. \quad (3.10)$$

We first deal with  $\int_{S_1} g(u) \rho^\beta dx$ . In fact, we observe that

$$\int_{S_1} g(u) \rho^\beta dx = \omega(1)g(1) + \int_1^\infty \omega(s)dg(s),$$

where

$$\int_1^\infty g(s)d\omega(s) = \lim_{T \rightarrow \infty} \int_1^T g(s)d\omega(s).$$

It infers by Proposition 2.1 and Proposition 2.2 that there exists  $c_{18} > 0$  such that

$$\omega(s) \leq c_{18} M_1(u)^{p_\beta^*} s^{-p_\beta^*} \quad (3.11)$$

and by (1.8) and Lemma 3.1 with  $p = p_\beta^*$ , there exist a sequence of increasing numbers  $\{T_j\}$  such that  $T_1 > 1$  and  $T_j^{-p_\beta^*} g(T_j) \rightarrow 0$  when  $j \rightarrow \infty$ , thus

$$\begin{aligned} \omega(1)g(1) + \int_1^{T_j} \omega(s)dg(s) &\leq c_{18} M_1(u)^{p_\beta^*} g(1) + c_{18} M(u)^{p_\beta^*} \int_1^{T_j} s^{-p_\beta^*} dg(s) \\ &\leq c_{18} M_1(u)^{p_\beta^*} T_j^{-p_\beta^*} g(T_j) + \frac{c_{18} M_1(u)^{p_\beta^*}}{p_\beta^* + 1} \int_1^{T_j} s^{-1-p_\beta^*} g(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{S_1} g(u)\rho^\beta dx &= \omega(1)g(1) + \int_1^\infty \omega(s) dg(s) \\
&\leq \frac{c_{18}M_1(u)^{p_\beta^*}}{p_\beta^*+1} \int_1^\infty s^{-1-p_\beta^*} g(s) ds \\
&= c_{18}g_\infty M_1(u)^{p_\beta^*},
\end{aligned} \tag{3.12}$$

where  $c_{18} > 0$  independent of  $n$ .

We next deal with  $\int_{S_1^c} g(u)\rho^\beta dx$ . For  $p_* \in (1, \frac{N}{N-2\alpha+\beta})$ , we have that

$$\begin{aligned}
\int_{S_1^c} g(u)\rho^\beta dx &\leq c_3 \int_{S_1^c} u^{p_*} \rho^\beta dx + \epsilon \int_{S_1^c} \rho^\beta dx \\
&\leq c_3 c_{19} \int_\Omega u^{p_*} dx + c_{19} \epsilon \\
&\leq c_3 c_{19} M_2(u)^{p_*} + c_{19} \epsilon,
\end{aligned} \tag{3.13}$$

where  $c_{19} > 0$  independent of  $n$ .

Along with (3.9), (3.10), (3.12) and (3.13), we derive

$$M_1(\mathcal{T}_n u) \leq c_9 c_{18} g_\infty M_1(u)^{p_\beta^*} + c_9 c_3 c_{19} M_2(u)^{p_*} + c_9 c_{19} \epsilon + c_9 \sigma. \tag{3.14}$$

By [20, Theorem 6.5] and (2.6), we derive that

$$M_2(\mathcal{T}_n u) \leq c_{11} \|g_n(u) + \sigma \nu_n\|_{L^1(\Omega, \rho^\beta dx)},$$

which along with (3.10), (3.12) and (3.13), implies that

$$M_2(\mathcal{T}_n u) \leq c_{11} c_{18} g_\infty M_1(u)^{p_\beta^*} + c_{11} c_3 c_{19} M_2(u)^{p_*} + c_{11} c_{19} \epsilon + c_{11} \sigma. \tag{3.15}$$

Therefore, inequality (4.7) and (4.8) imply that

$$M_1(\mathcal{T}_n u) + M_2(\mathcal{T}_n u) \leq c_{20} g_\infty M_1(u)^{p_\beta^*} + c_{13} M_2(u)^{p_*} + c_{21} \epsilon + c_{22} \sigma,$$

where  $c_{20} = (c_9 + c_{11})c_{18}$ ,  $c_{21} = (c_9 + c_{11})c_{19}$  and  $c_{22} = c_9 + c_{11}$ . If we assume that  $M_1(u) + M_2(u) \leq \lambda$ , implies

$$M_1(\mathcal{T}_n u) + M_2(\mathcal{T}_n u) \leq c_{20} g_\infty \lambda^{p_\beta^*} + c_{21} \lambda^{p_*} + c_{21} \epsilon + c_{22} \sigma.$$

Since  $p_\beta^*, p_* > 1$ , then there exist  $\sigma_0 > 0$  and  $\epsilon_0 > 0$  such that for any  $\sigma \in (0, \sigma_0]$  and  $\epsilon \in (0, \epsilon_0]$ , the equation

$$c_{20} g_\infty \lambda^{p_\beta^*} + c_{21} \lambda^{p_*} + c_{21} \epsilon + c_{22} \sigma = \lambda$$

admits the largest root  $\bar{\lambda} > 0$ .

We redefine  $M(u) = M_1(u) + M_2(u)$ , then for  $M(u) \leq \bar{\lambda}$ , we obtain that

$$M(\mathcal{T}_n u) \leq c_{20} g_\infty \bar{\lambda}^{p_\beta^*} + c_{21} \bar{\lambda}^{p_*} + c_{21} \epsilon + c_{22} \sigma = \bar{\lambda}. \tag{3.16}$$

Especially, we have that

$$\|\mathcal{T}_n u\|_{L^1(\Omega)} \leq c_8 M_1(\mathcal{T}_n u) |\Omega|^{1 - \frac{1}{p_\beta^*}} \leq c_{23} \bar{\lambda} \quad \text{if } M(u) \leq \bar{\lambda}.$$

Thus,  $\mathcal{T}_n$  maps  $L^1(\Omega)$  into itself. Clearly, if  $u_m \rightarrow u$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ , then  $g_n(u_m) \rightarrow g_n(u)$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ , thus  $\mathcal{T}_n$  is continuous. For any fixed  $n \in \mathbb{N}$ ,  $\mathcal{T}_n u_m = \mathbb{G}_\alpha [g_n(u_m) + \sigma \nu_n]$  and  $\{g_n(u_m) + \sigma \nu_n\}_m$  is uniformly bounded in  $L^1(\Omega, \rho^\beta dx)$ , then it follows by Proposition 2.3 that  $\{\mathbb{G}_\alpha [g_n(u_m) + \sigma \nu_n]\}_m$  is pre-compact in  $L^1(\Omega)$ , which implies that  $\mathcal{T}_n$  is a compact operator.

Let

$$\mathcal{G} = \{u \in L^1_+(\Omega) : M(u) \leq \bar{\lambda}\}$$

which is a closed and convex set of  $L^1(\Omega)$ . It infers by (4.9) that

$$\mathcal{T}_n(\mathcal{G}) \subset \mathcal{G}.$$

It follows by Schauder's fixed point theorem that there exists some  $u_n \in L^1_+(\Omega)$  such that  $\mathcal{T}_n u_n = u_n$  and  $M(u_n) \leq \bar{\lambda}$ , where  $\bar{\lambda} > 0$  independent of  $n$ .

In fact,  $u_n$  is a classical solution of (3.8). For  $\alpha = 1$ , since  $g_n$  bounded and  $C^1$ , then it is natural to see that. When  $\alpha \in (0, 1)$ , let open set  $O$  satisfy  $O \subset \bar{O} \subset \Omega$ . By [23, Proposition 2.3], for  $\theta \in (0, 2\alpha)$ , there exists  $c_{24} > 0$  such that

$$\|u_n\|_{C^\theta(O)} \leq c_{24} \{\|g(u_n)\|_{L^\infty(\Omega)} + \sigma \|\nu_n\|_{L^\infty(\Omega)}\},$$

then applied [23, Corollary 2.4],  $u_n$  is  $C^{2\alpha+\epsilon_0}$  locally in  $\Omega$  for some  $\epsilon_0 > 0$ . Then  $u_n$  is a classical solution of (3.8). Moreover,

$$\int_\Omega u_n (-\Delta)^\alpha \xi dx = \int_\Omega g_n(u_n) \xi dx + \sigma \int_\Omega \xi \nu_n dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (3.17)$$

*Step 2. Convergence.* Since  $\{g_n(u_n)\}$  and  $\{\nu_n\}$  are uniformly bounded in  $L^1(\Omega, \rho^\beta dx)$ , then by Propostion 2.3, there exist a subsequence  $\{u_{n_k}\}$  and  $u$  such that  $u_{n_k} \rightarrow u$  a.e. in  $\Omega$  and in  $L^1(\Omega)$ , and  $g_{n_k}(u_{n_k}) \rightarrow g(u)$  a.e. in  $\Omega$ .

Finally we prove that  $g_{n_k}(u_{n_k}) \rightarrow g(u)$  in  $L^1(\Omega, \rho^\beta dx)$ . For  $\lambda > 0$ , we set  $S_\lambda = \{x \in \Omega : |u_{n_k}(x)| > \lambda\}$  and  $\omega(\lambda) = \int_{S_\lambda} \rho^\beta dx$ , then for any Borel set  $E \subset \Omega$ , we have that

$$\begin{aligned} \int_E |g_{n_k}(u_{n_k})| \rho^\beta dx &= \int_{E \cap S_\lambda^c} g(u_{n_k}) \rho^\beta dx + \int_{E \cap S_\lambda} g(u_{n_k}) \rho^\beta dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^\beta dx + \int_{S_\lambda} g(u_{n_k}) \rho^\beta dx \\ &\leq \tilde{g}(\lambda) \int_E \rho^\beta dx + \omega(\lambda) g(\lambda) + \int_\lambda^\infty \omega(s) dg(s), \end{aligned} \quad (3.18)$$

where  $\tilde{g}(\lambda) = \max_{s \in [0, \lambda]} g(s)$ .

On the other hand,

$$\int_{\lambda}^{\infty} g(s)d\omega(s) = \lim_{T_m \rightarrow \infty} \int_{\lambda}^{T_m} g(s)d\omega(s).$$

where  $\{T_m\}$  is a sequence increasing number such that  $T_m^{-p_{\beta}^*}g(T_m) \rightarrow 0$  as  $m \rightarrow \infty$ , which could obtained by assumption (1.8) and Lemma 3.1 with  $p = p_{\beta}^*$ .

It infers by (3.11) that

$$\begin{aligned} \omega(\lambda)g(\lambda) + \int_{\lambda}^{T_m} \omega(s)dg(s) &\leq c_{18}g(\lambda)\lambda^{-p_{\beta}^*} + c_{25} \int_{\lambda}^{T_m} s^{-p_{\beta}^*}dg(s) \\ &\leq c_{25}T_m^{-p_{\beta}^*}g(T_m) + \frac{c_{25}}{p_{\beta}^* + 1} \int_{\lambda}^{T_m} s^{-1-p_{\beta}^*}g(s)ds, \end{aligned}$$

where  $c_{25} = c_{18}p_{\beta}^*$ . Pass the limit of  $m \rightarrow \infty$ , we have that

$$\omega(\lambda)g(\lambda) + \int_{\lambda}^{\infty} \omega(s) dg(s) \leq \frac{c_{25}}{p_{\beta}^* + 1} \int_{\lambda}^{\infty} s^{-1-p_{\beta}^*}g(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when  $\lambda \rightarrow \infty$ . The conclusion follows: for any  $\epsilon > 0$  there exists  $\lambda > 0$  such that

$$\frac{c_{17}}{p_{\beta}^* + 1} \int_{\lambda}^{\infty} s^{-1-p_{\beta}^*}g(s)ds \leq \frac{\epsilon}{2}.$$

Since  $\lambda$  is fixed, together with (3.10), there exists  $\delta > 0$  such that

$$\int_E \rho^{\beta} dx \leq \delta \implies g(\lambda) \int_E \rho^{\beta} dx \leq \frac{\epsilon}{2}.$$

This proves that  $\{g \circ u_{n_k}\}$  is uniformly integrable in  $L^1(\Omega, \rho^{\beta} dx)$ . Then  $g \circ u_{n_k} \rightarrow g \circ u$  in  $L^1(\Omega, \rho^{\beta} dx)$  by Vitali convergence theorem.

Pass the limit of (3.17) as  $n_k \rightarrow \infty$  to derive that

$$\int_{\Omega} u(-\Delta)^{\alpha} \xi = \int_{\Omega} g(u)\xi dx + \sigma \int_{\Omega} \xi d\nu, \quad \forall \xi \in \mathbb{X}_{\alpha},$$

thus  $u$  is a weak solution of (1.1)-(1.2) and  $u$  is nonnegative since  $\{u_n\}$  are nonnegative.  $\square$

## 4 Boundary type measure

In order to prove the elliptic problem involving boundary type measure, the idea is to change the boundary type measure to a forcing source.



**Lemma 4.1** For  $\mu \in \mathfrak{M}_+^b(\partial\Omega)$ , we have that

$$\mathbb{P}[\mu] \in C^1(\Omega).$$

**Proof.** It infers by [5, Proposition 2.1] that for  $(x, y) \in \Omega \times \partial\Omega$ ,

$$P(x, y) \leq c_N |x - y|^{1-N} \quad \text{and} \quad |\nabla_x P(x, y)| \leq c_N |x - y|^{-N},$$

then by the formulation of  $\mathbb{P}[\mu]$  we have that  $\mathbb{P}[\mu] \in C^1(\Omega)$ .  $\square$

**Lemma 4.2** Assume that  $\varrho > 0$ ,  $\mu \in \mathfrak{M}_+^b(\partial\Omega)$ ,  $g$  is a nonnegative function satisfying (1.12) and (1.13),  $\{g_n\}$  are a sequence of  $C^1$  nonnegative functions defined on  $\mathbb{R}_+$  satisfying  $g_n(0) = g(0)$  and (3.2).

Then there exists  $\varrho_0 > 0$  and  $\epsilon_0 > 0$  such that for  $\varrho \in [0, \varrho_0]$  and  $\epsilon \in [0, \epsilon_0]$ ,

$$\begin{aligned} -\Delta u &= g_n(u + \varrho \mathbb{P}[\mu]) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{aligned} \tag{4.1}$$

admits a nonnegative solution  $w_n$  such that

$$M_1(w_n) + M_2(w_n) \leq \bar{\lambda}$$

for some  $\bar{\lambda} > 0$  independent of  $n$ , where

$$M_1(v) = \|v\|_{M^{q^*}(\Omega, \rho dx)} \quad \text{and} \quad M_2(v) = \|v\|_{L^{q^*}(\Omega)},$$

with  $q_*$  and  $q^*$  given in (1.12) and (1.13) respectively.

**Proof.** Without loss generality, we assume  $\|\mu\|_{\mathfrak{M}^b(\partial\Omega)} = 1$  and  $q_* \in (1, \frac{N+1}{N-1})$ . Redenote the operators  $\{\mathcal{T}_n\}$  by

$$\mathcal{T}_n u = \mathbb{G}_1 [g_n(u + \varrho \mathbb{P}[\mu])], \quad \forall u \in L_+^1(\Omega).$$

By Proposition 2.2, we have

$$\begin{aligned} M_1(\mathcal{T}_n u) &\leq c_9 \|g_n(u + \varrho \mathbb{P}[\mu])\|_{L^1(\Omega, \rho dx)} \\ &\leq c_9 \|g(u + \varrho \mathbb{P}[\mu])\|_{L^1(\Omega, \rho dx)} \end{aligned} \tag{4.2}$$

For  $\lambda > 0$ , we set  $S_\lambda = \{x \in \Omega : u + \varrho \mathbb{P}[\mu] > \lambda\}$  and  $\omega(\lambda) = \int_{S_\lambda} \rho dx$ ,

$$\|g(u + \varrho \mathbb{P}[\mu])\|_{L^1(\Omega, \rho dx)} \leq \int_{S_1^c} g(u + \varrho \mathbb{P}[\mu]) \rho dx + \int_{S_1} g(u + \varrho \mathbb{P}[\mu]) \rho dx. \tag{4.3}$$

We first deal with  $\int_{S_1} g(u + \varrho \mathbb{P}[\mu]) \rho dx$ . In fact, we observe that

$$\int_{S_1} g(u + \varrho \mathbb{P}[\mu]) \rho dx = \omega(1)g(1) + \int_1^\infty \omega(s) dg(s),$$

where

$$\int_1^\infty g(s)d\omega(s) = \lim_{T \rightarrow \infty} \int_1^T g(s)d\omega(s).$$

It infers by Proposition 2.2 and Proposition 2.4 with  $\gamma = 1$  that there exists such that

$$\begin{aligned} \omega(s) &\leq c_{26} \|u + \varrho \mathbb{P}[\mu]\|_{M^{q^*}(\Omega, \rho dx)}^{q^*} s^{-q^*} \\ &\leq c_{27} \left( \|u\|_{M^{q^*}(\Omega, \rho dx)} + \|\varrho \mathbb{P}[\mu]\|_{M^{q^*}(\Omega, \rho dx)} \right)^{q^*} s^{-q^*} \\ &\leq c_{27} (M_1(u) + c_{14} \varrho)^{q^*} s^{-q^*} \end{aligned} \quad (4.4)$$

where  $c_{26}, c_{27} > 0$  independent of  $n$ . By (1.13) and Lemma 3.1 with  $p = q^*$ , there exist a sequence of increasing numbers  $\{T_j\}$  such that  $T_1 > 1$  and  $T_j^{-q^*} g(T_j) \rightarrow 0$  when  $j \rightarrow \infty$ , thus

$$\begin{aligned} \omega(1)g(1) + \int_1^{T_j} \omega(s)dg(s) &\leq c_{27} (M_1(u) + c_{14} \varrho)^{p_\beta^*} g(1) + c_{27} (M_1(u) + c_{14} \varrho)^{q^*} \int_1^{T_j} s^{-q^*} dg(s) \\ &\leq c_{27} (M_1(u) + c_{14} \varrho)^{q^*} T_j^{-q^*} g(T_j) \\ &\quad + \frac{c_{27}(M_1(u) + c_{14} \varrho)^{q^*}}{q^* + 1} \int_1^{T_j} s^{-1-q^*} g(s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{S_1} g(u) \rho dx &= \omega(1)g(1) + \int_1^\infty \omega(s) dg(s) \\ &\leq \frac{c_{27}(M_1(u) + c_{14} \varrho)^{q^*}}{q^* + 1} \int_1^\infty s^{-1-q^*} g(s) ds \\ &\leq c_{28} g_\infty M_1(u)^{q^*} + c_{28} g_\infty \varrho^{q^*}, \end{aligned} \quad (4.5)$$

where  $c_{28} > 0$  independent of  $n$ .

We next deal with  $\int_{S_1^c} g(u + \varrho \mathbb{P}[\mu]) \rho dx$ . For  $q_* \in (1, \frac{N+1}{N-1})$ , we have that

$$\begin{aligned} \int_{S_1^c} g(u + \varrho \mathbb{P}[\mu]) \rho dx &\leq c_5 \int_{S_1^c} (u + \varrho \mathbb{P}[\mu])^{q_*} \rho dx + \epsilon \int_{S_1^c} \rho dx \\ &\leq c_5 c_{29} \int_\Omega u^{q_*} dx + c_5 c_{29} \varrho^{q_*} + c_{29} \epsilon \\ &\leq c_5 c_{29} M_2(u)^{q_*} + c_5 c_{29} \varrho^{q_*} + c_{29} \epsilon, \end{aligned} \quad (4.6)$$

where  $c_{29} > 0$  independent of  $n$ .

Along with (4.2), (4.3), (4.5) and (4.6), we derive that

$$M_1(\mathcal{T}_n u) \leq c_9 c_{26} g_\infty M_1(u)^{q^*} + c_9 c_5 c_{29} M_2(u)^{q_*} + c_9 c_{29} \epsilon + c_9 l_\varrho, \quad (4.7)$$

where  $l_\varrho = c_{28}g_\infty\varrho^{p^*} + c_5c_{29}\varrho^{p^*}$ . By [20, Theorem 6.5] and (2.6), we derive that

$$M_2(\mathcal{T}_n u) \leq c_{11} \|g(u + \varrho\mathbb{P}[\mu])\|_{L^1(\Omega, \rho dx)},$$

which along with (4.3), (4.5) and (4.6), implies that

$$M_2(\mathcal{T}_n u) \leq c_{11}c_{26}g_\infty M_1(u)^{q^*} + c_{11}c_5c_{29}M_2(u)^{q^*} + c_{11}c_{29}\epsilon + c_{11}l_\varrho. \quad (4.8)$$

Therefore, inequality (4.7) and (4.8) imply that

$$M_1(\mathcal{T}_n u) + M_2(\mathcal{T}_n u) \leq c_{30}g_\infty M_1(u)^{q^*} + c_{31}M_2(u)^{q^*} + c_{31}\epsilon + c_{32}l_\varrho,$$

where  $c_{30} = (c_9 + c_{11})c_{26}$ ,  $c_{31} = (c_9 + c_{11})c_5c_{29}$  and  $c_{32} = c_9 + c_{11}$ . If we assume that  $M_1(u) + M_2(u) \leq \lambda$ , implies

$$M_1(\mathcal{T}_n u) + M_2(\mathcal{T}_n u) \leq c_{30}g_\infty \lambda^{q^*} + c_{13}\lambda^{q^*} + c_{31}\epsilon + c_{32}l_\varrho.$$

Since  $q^*, q_* > 1$ , then there exist  $\varrho_0 > 0$  and  $\epsilon_0 > 0$  such that for any  $\varrho \in (0, \varrho_0]$  and  $\epsilon \in (0, \epsilon_0]$ , the equation

$$c_{30}g_\infty \lambda^{q^*} + c_{31}\lambda^{q^*} + c_{31}c_5\epsilon + c_{32}l_\varrho = \lambda$$

admits the largest root  $\bar{\lambda} > 0$ .

We redefine  $M(u) = M_1(u) + M_2(u)$ , then for  $M(u) \leq \bar{\lambda}$ , we obtain that

$$M(\mathcal{T}_n u) \leq c_{30}g_\infty \bar{\lambda}^{q^*} + c_{31}\bar{\lambda}^{q^*} + c_{31}\epsilon + c_{32}l_\varrho = \bar{\lambda}. \quad (4.9)$$

Especially, we have that

$$\|\mathcal{T}_n u\|_{L^1(\Omega)} \leq c_8 M_1(\mathcal{T}_n u) |\Omega|^{1-\frac{1}{q^*}} \leq c_{33}\bar{\lambda} \quad \text{if } M(u) \leq \bar{\lambda}.$$

Thus,  $\mathcal{T}_n$  maps  $L^1(\Omega)$  into itself. Clearly, if  $u_m \rightarrow u$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ , then  $g_n(u_m) \rightarrow g_n(u)$  in  $L^1(\Omega)$  as  $m \rightarrow \infty$ , thus  $\mathcal{T}_n$  is continuous. For any fixed  $n \in \mathbb{N}$ ,  $\mathcal{T}_n u_m = \mathbb{G}_1[g_n(u_m + \varrho\mathbb{P}[\mu])]$  and  $\{g_n(u_m) + \varrho\mathbb{P}[\mu]\}_m$  is uniformly bounded in  $L^1(\Omega, \rho dx)$ , then it follows by Proposition 2.3 that  $\{\mathbb{G}_1[g_n(u_m + \varrho\mathbb{P}[\mu])]\}_m$  is pre-compact in  $L^1(\Omega)$ , which implies that  $\mathcal{T}_n$  is a compact operator.

Let

$$\mathcal{G} = \{u \in L_+^1(\Omega) : M(u) \leq \bar{\lambda}\}$$

which is a closed and convex set of  $L^1(\Omega)$ . It infers by (4.9) that

$$\mathcal{T}_n(\mathcal{G}) \subset \mathcal{G}.$$

It follows by Schauder's fixed point theorem that there exists some  $w_n \in L_+^1(\Omega)$  such that  $\mathcal{T}_n w_n = w_n$  and  $M(w_n) \leq \bar{\lambda}$ , where  $\bar{\lambda} > 0$  does not depend

on  $n$ . Since  $g_n$  and  $\mathbb{P}[\mu]$  are  $C^1$  functions by Lemma 4.1, then  $w_n$  is a classical solution of (4.1) and

$$\int_{\Omega} w_n(-\Delta)\xi dx = \int_{\Omega} g_n(w_n + \varrho\mathbb{P}[\mu])\xi dx, \quad \forall \xi \in C_0^{1,1}(\Omega).$$

**Proof of Theorem 1.2 (ii).** It derives by Lemma 4.1 that  $w_n$  is a classical solution of (4.1). Denote  $u_n = w_n + \varrho\mathbb{P}[\mu]$  and then

$$\int_{\Omega} u_n(-\Delta)\xi = \int_{\Omega} g_n(u_n)\xi dx + \varrho \int_{\partial\Omega} \frac{\partial \xi(x)}{\partial \vec{n}_x} d\mu(x), \quad \forall \xi \in \mathbb{X}_{\alpha}, \quad (4.10)$$

Since  $\{g_n(u_n)\}$  are uniformly bounded in  $L^1(\Omega, \rho dx)$ , then by Propostion 2.3, there exist a subsequence  $\{w_{n_k}\}$  and  $w$  such that  $w_{n_k} \rightarrow w$  a.e. in  $\Omega$  and in  $L^1(\Omega)$  and then  $u_{n_k} \rightarrow u$  a.e. in  $\Omega$  and in  $L^1(\Omega)$  where  $u = w + \varrho\mathbb{P}[\mu]$ . Thus,  $g_{n_k}(u_{n_k}) \rightarrow g(u)$  a.e. in  $\Omega$ .

Similarly to the argument in *Proof of Theorem 1.1 part (ii) in Step 2*, we have that  $g_{n_k}(u_{n_k}) \rightarrow g(u)$  in  $L^1(\Omega, \rho dx)$ .

Pass the limit of (4.10) as  $n_k \rightarrow \infty$  to derive that

$$\int_{\Omega} u(-\Delta)\xi dx = \int_{\Omega} g(u)\xi dx + \varrho \int_{\partial\Omega} \frac{\partial \xi}{\partial \vec{n}} d\mu, \quad \forall \xi \in \mathbb{X}_{\alpha},$$

thus  $u$  is a weak solution of (1.9) and  $u$  is nonnegative since  $\{u_n\}$  are nonnegative.  $\square$

**Proof of Theorem 1.2 (i).** It proceeds similarly to the proof of Theorem 1.1 (i), so we omit here.  $\square$

## 5 Boundary type measure for $\alpha \in (0, 1)$

### 5.1 Basic results

In this subsection, we devoted to study the properties of  $\mathfrak{R}_{\beta}$  with  $\beta \in [0, \alpha]$ , see the definition 1.18. Here and in what follows, we assume that  $\alpha \in (0, 1)$ .

**Lemma 5.1** *Let  $1 \leq \beta' \leq \beta \leq \alpha$ , then*

$$\emptyset \neq \mathfrak{R}_{\beta'} \subset \mathfrak{R}_{\beta} \neq \mathfrak{M}_+(\Omega^c). \quad (5.1)$$

**Proof.** Let  $x_0 \in \partial\Omega$ ,  $x_t = x_0 + t\vec{n}_{x_0}$  and  $\delta_t$  be the dirac mass concentrated at  $x_t$ , where  $\vec{n}_{x_0}$  is the unit normal vector pointing outside at  $x_0$ .

Fixed  $t > 0$ ,  $w_{\delta_t}(x) = |x - x_t|^{-N-2\alpha}$  for  $x \in \Omega$ . It is easy to see that  $w_{\delta_t} \in L^\infty(\Omega)$  and then  $\delta_t \in \mathfrak{R}_{\beta}$  for any  $\beta \in [0, \alpha]$ .

Fixed  $t = 0$ ,  $w_{\delta_0}(x) = |x - x_0|^{-N-2\alpha}$  for  $x \in \Omega$ . We observe that  $w_{\delta_0} \notin L^1(\Omega, \rho^\alpha dx)$  and then  $\delta_0 \notin \mathfrak{R}_{\beta}$  for any  $\beta \in [0, \alpha]$ .  $\square$

**Example.** Let  $x_0 \in \partial\Omega$ ,  $x_t = x_0 + t\vec{n}_{x_0}$  and  $\delta_t$  be the dirac mass concentrated at  $x_t$ . Denote

$$\mu = \sum_{n=1}^{\infty} b_n \delta_{\frac{1}{n}},$$

where  $\{b_n\}$  a sequence nonnegative numbers will be chosen latter. We observe that

$$w_\mu(x) = \sum_{n=1}^{\infty} \frac{b_n}{|x - x_{\frac{1}{n}}|^{N+2\alpha}}, \quad x \in \Omega$$

and  $w_\mu \in L^1(\Omega, \rho^\beta dx)$  if and only if

$$\sum_{n=1}^{\infty} b_n n^{2\alpha-\beta} < +\infty. \quad (5.2)$$

**Lemma 5.2** Let  $\mu \in \mathfrak{R}_\beta$  with  $\beta \in [0, \alpha]$  and  $w_\mu$  is given by (1.16).

(i)  $w_\mu \in C^1(\Omega) \cap L^1(\Omega, \rho^\beta dx)$ .

(ii) Let  $\tilde{w}_\mu = \mathbb{G}_\alpha[w_\mu]$  in  $\Omega$  and  $\tilde{w}_\mu = \mu$  in  $\Omega^c$ , then  $\tilde{w}_\mu$  is a weak solution of

$$\begin{aligned} (-\Delta)^\alpha u &= 0 & \text{in } \Omega, \\ u &= \mu & \text{in } \Omega^c \end{aligned} \quad (5.3)$$

in the sense of

$$\int_{\Omega} u(-\Delta)^\alpha \xi dx = \int_{\Omega} \xi w_\mu dx, \quad \forall \xi \in \mathbb{X}_\alpha.$$

**Proof.** (i)  $\mu \in \mathfrak{R}_\beta$  implies that  $w_\mu \in L^1(\Omega, \rho^\beta dx)$  and since the function:  $x \mapsto |x - y|^{-N-2\alpha}$  is  $C^1(\Omega)$  for any  $y \in \Omega^c$ , then  $w_\mu \in C^1(\Omega)$ .

(ii) For  $\mu \in \mathfrak{R}_\beta$  with  $\beta \in [0, \alpha]$ , let  $\{\mu_n\} \subset C_0^1(\mathbb{R}^N)$  with  $\text{supp}(\mu_n) \subset \bar{\Omega}^c$  be a sequence of nonnegative functions such that  $\mu_n \rightarrow \mu$  in distribution sense.

Then we derive that  $w_{\mu_n} \in C^1(\bar{\Omega})$  and there exists a unique classical solution  $\mathbb{G}_\alpha[w_{\mu_n}]$  to

$$\begin{aligned} (-\Delta)^\alpha u &= w_{\mu_n} & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c. \end{aligned} \quad (5.4)$$

Moreover,

$$\int_{\Omega} u(-\Delta)^\alpha \xi dx = \int_{\Omega} \xi w_{\mu_n} dx, \quad \forall \xi \in \mathbb{X}_\alpha. \quad (5.5)$$

Let  $u_n = \mathbb{G}_\alpha[w_{\mu_n}] + \mu_n$ , then we have that

$$(-\Delta)^\alpha u_n = (-\Delta)^\alpha \mathbb{G}_\alpha[w_{\mu_n}] + (-\Delta)^\alpha \mu_n = w_{\mu_n} - w_{\mu_n} = 0$$

and (5.5) holds for  $u_n$ . Passing the limit of  $n \rightarrow \infty$ , we derive that  $\tilde{w}_\mu$  is a weak solution of (5.4).  $\square$

We note that (i) Lemma 5.2(ii) indicates that  $\mathbb{G}_\alpha[w_\mu]$  has the similar role as  $\mathbb{P}[\mu]$  when  $\alpha = 1$ ; (ii) the definition 1.3 is equivalent to

**Definition 5.1**  $u_\mu$  is a weak solution of (1.15), if  $u_\mu \in L^1(\Omega)$ ,  $g(u_\mu) \in L^1(\Omega, \rho^\alpha dx)$  and

$$\int_{\Omega} u_\mu (-\Delta)^\alpha \xi dx = \int_{\Omega} g(u_\mu) \xi dx + \int_{\Omega} w_\mu \xi dx, \quad \xi \in \mathbb{X}_\alpha,$$

where  $w_\mu$  is given by (1.16).

## 5.2 Proof of Theorem 1.3

Inspired by the proof of Theorem 1.2, we first give an important lemma, which is important in dealing with the subcritical case.

**Lemma 5.3** Assume that  $\varrho > 0$ ,  $\mu \in \mathfrak{A}_\beta$ ,  $g$  is a nonnegative function satisfying (1.21) and (1.22),  $\{g_n\}$  are a sequence of  $C^1$  nonnegative functions defined on  $\mathbb{R}_+$  satisfying  $g_n(0) = g(0)$  and (3.2).

Then there exists  $\varrho_0 > 0$  and  $\epsilon_0 > 0$  such that for  $\varrho \in [0, \varrho_0]$  and  $\epsilon \in [0, \epsilon_0]$ ,

$$\begin{aligned} (-\Delta)^\alpha u &= g_n(u + \varrho \mathbb{G}_\alpha[w_\mu]) & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c \end{aligned} \tag{5.6}$$

admits a nonnegative solution  $w_n$  such that

$$M_1(w_n) + M_2(w_n) \leq \bar{\lambda}$$

for some  $\bar{\lambda} > 0$  independent of  $n$ , where

$$M_1(v) = \|v\|_{M^{p_\beta^*}(\Omega, \rho^\beta dx)} \quad \text{and} \quad M_2(v) = \|v\|_{L^{q_*}(\Omega)},$$

with  $q_*$  and  $p_\beta^*$  given in (1.21) and (1.22) respectively.

**Proof.** For  $\mu \in \mathfrak{A}_\beta$ , we have that  $w_\mu \in L^1(\Omega, \rho^\beta dx)$ , which, by Proposition 2.3, implies that  $\mathbb{G}_\alpha[w_\mu] \in M^{p_\beta^*}(\Omega, \rho^\beta dx)$ . It proceeds as Lemma 4.2, replaced  $\mathbb{P}[\mu]$  by  $\mathbb{G}_\alpha[w_\mu]$  to obtain that there exists  $\varrho_0 > 0$  and  $\epsilon_0 > 0$  such that for  $\varrho \in [0, \varrho_0]$  and  $\epsilon \in [0, \epsilon_0]$ , there exists  $w_n$  such that

$$w_n = \mathbb{G}_\alpha[g_n(w_n + \varrho \mathbb{G}_\alpha[w_\mu])]$$

and

$$M_1(w_n) + M_2(w_n) \leq \bar{\lambda}$$

for some  $\bar{\lambda} > 0$  independent of  $n$ .

By Lemma 5.2 (i), we see that  $w_n$  is a classical solution of (5.6). Moreover,

$$\int_{\Omega} w_n (-\Delta)^\alpha \xi dx = \int_{\Omega} g_n(w_n + \varrho \mathbb{G}_\alpha[w_\mu]) \xi dx, \quad \forall \xi \in C_0^{1,1}(\Omega). \tag{5.7}$$

**Proof of Theorem 1.3 (ii).** It derives by Lemma 5.3 that  $w_n$  is a classical solution of (5.6). Denote  $u_n = w_n + \varrho\mathbb{G}_\alpha[w_\mu]$ . Since  $\{g_n(u_n)\}$  are uniformly bounded in  $L^1(\Omega, \rho dx)$ , then by Propostion 2.3, there exist a subsequence  $\{w_{n_k}\}$  and  $w$  such that  $w_{n_k} \rightarrow w$  a.e. in  $\Omega$  and in  $L^1(\Omega)$  and then  $u_{n_k} \rightarrow u$  a.e. in  $\Omega$  and in  $L^1(\Omega)$  where  $u = w + \varrho\mathbb{G}_\alpha[w_\mu]$ . Thus,  $g_{n_k}(u_{n_k}) \rightarrow g(u)$  a.e. in  $\Omega$ .

Similarly to the argument in *Proof of Theorem 1.1 part (ii) in Step 2*, we have that  $g_{n_k}(u_{n_k}) \rightarrow g(u)$  in  $L^1(\Omega, \rho^\beta dx)$ .

Pass the limit of (5.7) as  $n_k \rightarrow \infty$  to derive that

$$\int_{\Omega} w(-\Delta)^\alpha \xi dx = \int_{\Omega} g(w + \varrho\mathbb{G}_\alpha[w_\mu])\xi dx, \quad \forall \xi \in \mathbb{X}_\alpha.$$

Thus  $u = w + \varrho\mathbb{G}_\alpha[w_\mu]$  is a weak solution of (1.15) and  $u$  is nonnegative since  $\{w_n\}$  are nonnegative.  $\square$

**Proof of Theorem 1.3 (i).** It proceeds similarly to the proof of Theorem 1.1 (i), so we omit here.  $\square$

## References

- [1] Ph. Bénylan, H. Brezis and M. Crandall, A semilinear elliptic equation in  $L^1(\mathbb{R}^N)$ , *Ann. Sc. Norm. Sup. Pisa Cl. Sci. 2*, 523-555 (1975).
- [2] H. Brezis and X. Cabré, Some simple PDEs without solutions, *Boll. Unione Mat. Italiana 8*, 223-262, (1998).
- [3] P. Baras and M. Pierre, Critères d'existence de solutions positives pour des équations semi-linéaires non monotones, *Ann. Inst. H. Poincaré, Analyse Non Linéaire 2*, 185-212 (1985).
- [4] M. Bhakta and M. Marcus, Reduced limit for semilinear boundary value problems with measure data, *J. Differential Equations* (2014).
- [5] M. F. Bidaut-Véron and L. Vivier, An elliptic semilinear equation with source term involving boundary measures: the subcritical case, *Rev. Mat. Iberoamericana 16*, 477-513 (2000).
- [6] M. F. Bidaut-Véron and C. Yarur, Semilinear elliptic equations and systems with measure data: existence and a priori estimates. *Advances in Differential Equations*, 7(3), 257-296 (2002).
- [7] H. Brezis, Some variational problems of the Thomas-Fermi type, Variational inequalities and complementarity problems, *Proc. Internat. School, Erice, Wiley, Chichester*, 53-73 (1980).
- [8] Z. Chen, and R. Song, Estimates on Green functions and poisson kernels for symmetric stable process, *Math. Ann. 312*, 465-501 (1998).

- [9] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *J. Differential equations* 257(5) , 1457-1486 (2014).
- [10] H. Chen and L. Véron, Semilinear fractional elliptic equations with gradient nonlinearity involving measures, *J. Funct. Anal.* 266(8), 5467-5492 (2014).
- [11] R. Cignoli and M. Cottlar, An Introduction to Functional Analysis, *North-Holland, Amsterdam* (1974).
- [12] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order, 2nd ed., *Springer-Verlag, Berlin, vol. 224* (1983).
- [13] A. Gmira and L. Véron, Boundary singularities of solutions of some nonlinear elliptic equations. *Duke Math. J.* 64, 271-324 (1991).
- [14] P.L. Lions, Quelques remarques sur les problèmes elliptiques quasilineaires du second ordre, *J. Analyse Math.* 45, 234-254 (1985).
- [15] P.L. Lions, Isolated singularities in semilinear problems, *J. Diff. Equ.*, 38, 441-450 (1980).
- [16] N. J. Kalton and I. E. Verbitsky, Nonlinear equations and weighted norm inequalities, *Trans. A. M. S.* 351, 3341-3397 (1999).
- [17] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, *Arch. rat. Mech. Anal.* 144, 201-231 (1998).
- [18] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case, *J. Math. Pures Appl.* 77, 481-524, 1998.
- [19] M. Marcus and L. Véron, Nonlinear second order elliptic equations involving measures, Series in Nonlinear Analysis and Applications 21, *De Gruyter, Berlin/Boston* (2013).
- [20] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136, 521-573 (2012).
- [21] W. Ni, On the elliptic equation  $\Delta u + K(x)u^{\frac{N+2}{N-2}} = 0$ , *Indiana Univ. Math. J.* 31, 493-539 (1982).
- [22] A. Ratto, M. Rigoli and L. Véron, Scalar curvature and conformal deformation of hyperbolic space, *J. Funct. Anal.* 121, 543-572 (1994).
- [23] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional laplacian: regularity up to the boundary, *J. Math. Pures Appl.* 101(3), 275-302 (2014).



- [24] L. Véron, Elliptic equations involving Measures, Stationary Partial Differential equations, *Vol. I, 593-712, Handb. Differ. Equ., North-Holland, Amsterdam* (2004).
- [25] L. Véron, Existence and Stability of Solutions of General Semilinear Elliptic Equations with Measure Data, *Adv. Nonlin. Stud. 13*, 447-460 (2013).

Huyuan Chen

Department of Mathematics, Jiangxi Normal University,  
Nanchang, Jiangxi 330022, PR China

and

Institute of Mathematical Sciences, New York University Shanghai,  
Shanghai 200120, PR China

Patricio Felmer

Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático  
UMR2071 CNRS-UCHile, Universidad de Chile, Chile

Laurent Véron,

Laboratoire de Mathématiques et Physique Théorique  
Université François Rabelais, Tours, France