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# Artificial boundary conditions for axisymmetric eddy current probe problems

Houssem HADDAR\*, Zixian JIANG†, Armin LECHLEITER‡

## Abstract

We study different strategies for the truncation of computational domains in the simulation of eddy current probes of elongated axisymmetric tubes. For axial fictitious boundaries, an exact Dirichlet-to-Neumann (DtN) map is proposed and mathematically justified via the analysis of a non-selfadjoint spectral problem. We rely of results from the spectral theory of compact perturbations of selfadjoint operators. Convergence for truncated DtN map is also analyzed. We validate our theoretical results through numerical experiments for a realistic physical setting inspired by eddy current probes of nuclear reactor core tubes.

**Keywords:** axisymmetric eddy current equations, artificial boundary conditions, DtN maps for non self-adjoint problems.

## 1 Introduction

### 1.1 Industrial context

The present work is motivated by eddy current non-destructive testing of steam generators (SG), see Figure 1, which are critical components in nuclear power plants. Heat produced in a nuclear reactor core is transferred into the primary loop of a steam generator, consisting of tubes in U-shape, and boils coolant water in the secondary loop on the shell side into steam. This steam is then delivered to turbines generating electrical power. Conductive magnetic deposits usually observed on the shell side of the U-tubes could, however, affect the power production and even the structure security. The upper part of the tubes of U-shape is accessible to normal inspection from the top of the steam generator. But it is difficult to reach the lower part of the tubes, which are straight and long, without disassembling the SG. Therefore, a non-destructive examination procedure using signals of eddy currents is applied to detect the presence, the shape and/or the physical nature of deposits on the lower part of the U-tubes.

In eddy current testing (ECT), we use a probe consisting of two coils of wire. Each of these coils is connected to a current generator producing an alternating current and to a voltmeter measuring the voltage change across the coil. Once the probe is introduced in the lower part of some U-tube, the generator coil excited by the current creates a primary electromagnetic field which in turn induces a current flow in the electrically conductive material nearby, such as the tube itself. The presence of conductive magnetic deposits will distort the flow of the eddy currents. They induce a current change in the receiver coil which is measured in terms of impedance and is called the ECT signal (c.f. [4, 11]).

In order to simulate an eddy current testing experiment, one needs to solve the forward problem for any probe position one wants to incorporate into the measurements. For an iterative inversion method

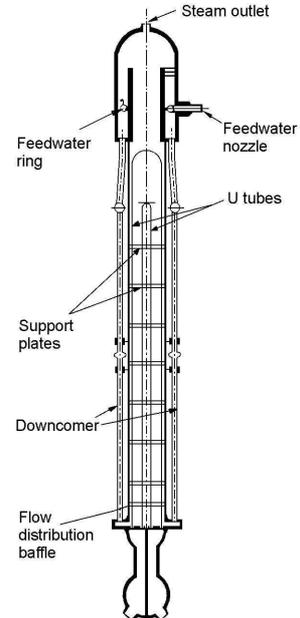


Figure 1: Sketch of a steam generator.

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based on the exploitation of this forward problem, the number of required simulation is also proportional to the number of iterations. Given the large number of tubes to be probed, one easily understands the crucial importance of designing a fast (and reliable) numerical simulation of the forward problem. We consider here the eddy current problem under axisymmetric assumption (see for instance [6]) and investigate strategies to bound the computational domain. While for the radial direction, truncation with brute model for the boundary condition such as Neumann boundary condition would be sufficient due to the conductivity of the tube and the decay of the solution, in the axial direction this strategy requires some fictitious boundaries far from the sources. We rather propose to compute the exact Dirichlet-to-Neumann (DtN) operator for the region outside the source term and apply it as an exact boundary condition on the fictitious boundaries. This would allow the latter to be as close as needed to the source term. DtN boundary conditions for domain cut-off are widely studied in waveguides and gratings [5, 9, 14–16, 22]. The main difficulty here is in the justification of the DtN analytical expansion using spectral decomposition of a non-self adjoint operator. We shall rely on results from perturbation theory for the spectrum of compactly perturbed selfadjoint operators [18]. We also study the error due to truncation in the expression of the DtN operator and relate this to the regularity of the problem parameters. Indeed the latter is important from the computational point of view since this truncation is needed in practice. The DtN expansion relies on some eigenvalues and eigenfunctions that are not known analytically and should be numerically approximated. This may be expensive if a high degree of precision is required. However these calculations can be done off-line and therefore would not affect the speed of solving the problem.

There is a large literature on eddy current problems and without being exhaustive we may refer to [2] for a recent survey on the problem, including an introduction to the eddy current phenomenon, the mathematical justification of the eddy current approximation and different formulations and numerical approaches for the three-dimensional problems. For axisymmetric configurations we refer to the work of [3] for the study of the theoretic tools for the Maxwell's equations in three dimensions, and to the works of [6, 12] for the discussion of the eddy current problem with bounded conductive components in the meridian half-plane, the numerical analysis and some numerical experiments applied to the induction heating system.

The paper is organized as follows. In Section 2, we briefly recall the eddy current model in the cylindrical coordinate system corresponding to the rotational symmetry with respect to the axis of the tube (see Figure 2) and discuss existence and uniqueness of solution to this problem in its equivalent variational formulation in properly defined weighted function spaces.

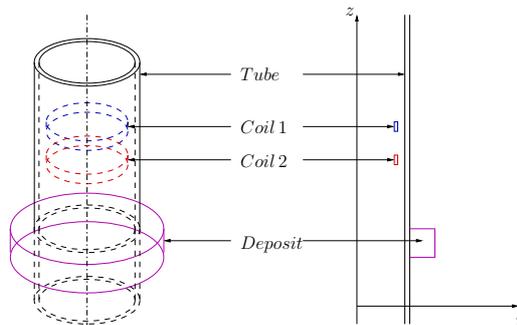


Figure 2: Three- and two-dimensional geometric representations of a steam generator tube covered with deposits and a probe consisting of two coils.

We then introduce truncations of the domain in the radial-direction by introducing some local boundary conditions (see Section 2.1) and then in the axial-direction by constructing the DtN boundary operator (see Section 3). We validate our analytical theory by several numerical tests that are motivated by ECT experiments as done in practice and present these numerical results in Section 4.

## 2 Axisymmetric model

Let us briefly outline the origin of the considered model. We consider the time-harmonic Maxwell's equations for the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{H}$

$$\begin{cases} \operatorname{curl} \mathbf{H} + (i\omega\epsilon - \sigma)\mathbf{E} = \mathbf{J} & \text{in } \mathbb{R}^3, \\ \operatorname{curl} \mathbf{E} - i\omega\mu\mathbf{H} = 0 & \text{in } \mathbb{R}^3, \end{cases} \quad (1)$$

where  $\mathbf{J}$  is the applied electric current density with compact support and satisfies  $\operatorname{div} \mathbf{J} = 0$ , and  $\omega$ ,  $\epsilon$ ,  $\mu$ ,  $\sigma$  respectively denote the frequency, the electrical permittivity, the magnetic permeability and the conductivity. In an axisymmetric (i.e., rotationally invariant) setting, for a vector field  $\mathbf{a}$  we denote by  $\mathbf{a}_m = a_r \mathbf{e}_r + a_z \mathbf{e}_z$  its meridian component and by  $\mathbf{a}_\theta = a_\theta \mathbf{e}_\theta$  its azimuthal component. A vector field  $\mathbf{a}$  is called axisymmetric if, in the sense of distributions,  $\partial_\theta \mathbf{a}$  vanishes. According to [3, Propostion 2.2], the Maxwell equations (1) decouple into two systems, a first one for  $(\mathbf{H}_\theta, \mathbf{E}_m)$ , and a second one for  $(\mathbf{H}_m, \mathbf{E}_\theta)$ . The solution to the first system vanishes if  $\mathbf{J}$  is axisymmetric. Substituting  $\mathbf{H}_m$  in the second system yields the second-order equation for  $\mathbf{E}_\theta = E_\theta \mathbf{e}_\theta$ ,

$$\frac{\partial}{\partial r} \left( \frac{1}{\mu r} \frac{\partial}{\partial r} (r E_\theta) \right) + \frac{\partial}{\partial z} \left( \frac{1}{\mu} \frac{\partial E_\theta}{\partial z} \right) + \omega^2 (\epsilon + i\sigma/\omega) E_\theta = -i\omega J_\theta \quad \text{in } \mathbb{R}_+^2, \quad (2)$$

with  $\mathbb{R}_+^2 := \{(r, z) : r > 0, z \in \mathbb{R}\}$ . The applied current  $J_\theta$  has compact support in  $\mathbb{R}_+^2$  and we denote in the sequel  $J = J_\theta$ . The eddy current approximation corresponds to low frequency regimes and high conductivities:  $\omega\epsilon \ll \sigma$ . From (2) and the above assumption we get the eddy current model

$$\frac{\partial}{\partial r} \left( \frac{1}{\mu r} \frac{\partial}{\partial r} (r E_\theta) \right) + \frac{\partial}{\partial z} \left( \frac{1}{\mu} \frac{\partial E_\theta}{\partial z} \right) + i\omega\sigma E_\theta = -i\omega J \quad \text{in } \mathbb{R}_+^2, \quad (3)$$

with a Dirichlet boundary condition at  $r = 0$  due to symmetry  $E_\theta|_{r=0} = 0$  and, roughly speaking, a decay condition  $E_\theta \rightarrow 0$  as  $r^2 \rightarrow \infty$  to ensure the finite energy of the electric field. From now on, we denote  $u = E_\theta$ . We introduce operators  $\nabla := (\partial_r, \partial_z)^t$  and  $\operatorname{div} := \nabla \cdot$  on the half-plane  $\mathbb{R}_+^2$  and the axis of symmetry  $\Gamma_0 := \{(r, z) : r = 0, z \in \mathbb{R}\}$ . Then the axisymmetric eddy current model reads

$$\begin{cases} -\operatorname{div} \left( \frac{1}{\mu r} \nabla (ru) \right) - i\omega\sigma u = i\omega J & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \Gamma_0, \\ u \rightarrow 0 & \text{as } r^2 \rightarrow \infty. \end{cases} \quad (4)$$

We shall assume that  $\mu$  and  $\sigma$  are in  $L^\infty(\mathbb{R}_+^2)$  such that  $\mu \geq \mu_0 > 0$  on  $\mathbb{R}_+^2$  and that  $\sigma \geq 0$  and  $\sigma = 0$  for  $r \geq r_0$  sufficiently large. For  $\lambda > 1$  and  $\Omega \subset \mathbb{R}_+^2$ , we define the weighted function spaces  $L_{1/2, \lambda}^2(\Omega)$ ,  $H_{1/2, \lambda}^1(\Omega)$  and the norms

$$\begin{aligned} L_{1/2, \lambda}^2(\Omega) &:= \{v : r^{1/2}(1+r^2)^{-\lambda/2} v \in L^2(\Omega)\}, & H_{1/2, \lambda}^1(\Omega) &:= \{v \in L_{1/2, \lambda}^2(\Omega) : r^{-1/2} \nabla(rv) \in L^2(\Omega)\}, \\ \|v\|_{L_{1/2, \lambda}^2(\Omega)} &= \left\| \sqrt{\frac{r}{(1+r^2)^\lambda}} v \right\|_{L^2(\Omega)}, & \|v\|_{H_{1/2, \lambda}^1(\Omega)}^2 &= \|v\|_{L_{1/2, \lambda}^2(\Omega)}^2 + \left\| r^{-1/2} \nabla(rv) \right\|_{L^2(\Omega)}^2. \end{aligned}$$

The following lemma gives a Poincaré-type inequality related to functions in  $H_{1/2, \lambda}^1(\mathbb{R}_+^2)$ . The proof uses classical arguments and is given in Appendix A.1 for the convenience of the reader. Note that for the axisymmetric problems, similar but slightly different weighted spaces have been studied (see for example [20] and [7, Section II.1]).

**Lemma 2.1.** *If  $v$  is in  $H_{1/2, \lambda}^1(\mathbb{R}_+^2)$  for  $\lambda > 1$ , then  $\lim_{0 < r \rightarrow 0} \|v(r, \cdot)\|_{L^2(\mathbb{R})} = 0$ . Moreover, there exists a constant  $C_\lambda > 0$  such that for all  $v$  in  $H_{1/2, \lambda}^1(\mathbb{R}_+^2)$ ,*

$$\|v\|_{H_{1/2, \lambda}^1(\mathbb{R}_+^2)} \leq C_\lambda \left\| r^{-1/2} \nabla(rv) \right\|_{L^2(\mathbb{R}_+^2)}. \quad (5)$$

One easily verifies by integration by parts that  $u$  in  $H_{1/2,\lambda}^1(\mathbb{R}_+^2)$  is a variational solution of the problem (4) if and only if  $u$  satisfies

$$\alpha(u, v) := \int_{\mathbb{R}_+^2} \frac{1}{\mu r} \nabla(ru) \cdot \nabla(r\bar{v}) \, dr \, dz - \int_{\mathbb{R}_+^2} i\omega\sigma u \bar{v} r \, dr \, dz = \int_{\mathbb{R}_+^2} i\omega J \bar{v} r \, dr \, dz \quad \forall v \in H_{1/2,\lambda}^1(\mathbb{R}_+^2). \quad (6)$$

Note that the homogeneous Dirichlet boundary condition on  $\Gamma_0$  of (4) is already included in the space  $H_{1/2,\lambda}^1(\mathbb{R}_+^2)$  by Lemma 2.1.

**Proposition 2.2.** *Assume that  $J \in L_{1/2,\lambda}^2(\mathbb{R}_+^2)$  has compact support. Then the variational problem (6) admits a unique solution  $u$  in  $H_{1/2,\lambda}^1(\mathbb{R}_+^2)$  for all  $\lambda > 1$ .*

*Proof.* The proof is a direct application of the Lax-Milgram Theorem thanks to (5) which yields the coercivity of the sesquilinear form on the left of (6):

$$\Re \alpha(v, v) = \int_{\mathbb{R}_+^2} \frac{1}{\mu r} |\nabla(rv)|^2 \geq \frac{1}{\|\mu\|_\infty C_\lambda^2} \|v\|_{H_{1/2,\lambda}^1(\mathbb{R}_+^2)}^2,$$

where  $C_\lambda$  is the constant given in (5). □

**Remark 2.3.** *The source  $J$  has compact support bounded away from  $\Gamma_0$  in  $\mathbb{R}_+^2$  in the real problem. We have in particular that  $J$  vanishes for  $r > r_0$  and  $|z| > z_0$ , where  $r_0 > 0$  and  $z_0 > 0$  are large enough.*

## 2.1 Asymptotic behavior for large $r$

We are interested here in a more precise evaluation of the decay of the solution  $u$  for large argument  $r$ . We shall assume in addition to the hypothesis from Proposition 2.2 that the source  $J$  and the conductivity  $\sigma$  vanish and that the permeability  $\mu$  is constant for  $r \geq r_0$  where  $r_0 > 0$  is some constant. One then gets from (4) that

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} - u + r^2 \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{for } r \geq r_0.$$

We then apply the Fourier transform with respect to the variable  $z$  and get

$$r^2 \frac{\partial^2 \hat{u}}{\partial r^2} + r \frac{\partial \hat{u}}{\partial r} - (1 + 4\pi^2 \xi^2 r^2) \hat{u} = 0, \quad \text{where } \hat{u}(\cdot, \xi) := \int_{-\infty}^{+\infty} u(\cdot, z) e^{-2\pi i \xi z} \, dz, \quad \xi \in \mathbb{R}. \quad (7)$$

The fundamental solutions of (7) for fixed  $\xi$  are the two modified Bessel functions  $I_1(2\pi|\xi|r)$  and  $K_1(2\pi|\xi|r)$  when  $\xi \neq 0$ , or the functions  $r$  and  $1/r$  when  $\xi = 0$ . Since  $u \in L_{1/2,\lambda}^2(\mathbb{R}_+^2)$  for all  $\lambda > 1$ , the asymptotic behavior for large argument of the modified Bessel functions (see [1, (9.6.1), (9.7.1), (9.7.2)]) implies that  $u$  has the following expression for  $r > r_0$ ,

$$\hat{u}(r, \xi) = \begin{cases} \hat{u}(r_0, \xi) \frac{K_1(2\pi|\xi|r)}{K_1(2\pi|\xi|r_0)} & \xi \neq 0, \\ \hat{u}(r_0, 0) \frac{r_0}{r} & \xi = 0. \end{cases} \quad (8)$$

Let us also quote that  $z \mapsto u(r, z) \in H^{1/2}(\mathbb{R})$  for  $r > 0$  since  $u \in H_{1/2,\lambda}^1(\mathbb{R}_+^2)$ .

**Proposition 2.4.** *The solution  $u \in H_{1/2,\lambda}^1(\mathbb{R}_+^2)$  to (6) satisfies*

$$\|u(r, \cdot)\|_{L^2(\mathbb{R})} \leq \frac{r_0}{r} \|u(r_0, \cdot)\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|u(r, \cdot)\|_{H^{1/2}(\mathbb{R})} \leq \frac{r_0}{r} \|u(r_0, \cdot)\|_{H^{1/2}(\mathbb{R})} \quad \forall r \geq r_0.$$

*Proof.* By the Plancherel theorem and the Cauchy-Schwarz inequality, we have

$$\|u(r, \cdot)\|_{L^2(\mathbb{R})}^2 = \|\hat{u}(r, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \|\hat{u}(r_0, \cdot)\|_{L^2(\mathbb{R})}^2 \left\| \frac{K_1(2\pi|\cdot|r)}{K_1(2\pi|\cdot|r_0)} \right\|_{L^\infty(\mathbb{R})}^2.$$

We note that  $K_1(x) \sim 1/x$  as  $0 < x \rightarrow 0$  (see [1, (9.6.9)]). Therefore,

$$g_d(\xi; r_0, r) := \frac{K_1(2\pi\xi r)}{K_1(2\pi\xi r_0)} \rightarrow \frac{r_0}{r} \quad \text{as } 0 < \xi \rightarrow 0. \quad (9)$$

On the other hand, the derivative of  $g_d$  with respect to  $\xi$  is

$$\begin{aligned} g'_d(\xi; r_0, r) &= \frac{2\pi r K'_1(2\pi\xi r) K_1(2\pi\xi r_0) - 2\pi r_0 K_1(2\pi\xi r) K'_1(2\pi\xi r_0)}{K_1^2(2\pi\xi r_0)} \\ &= \frac{2\pi [-r K_0(2\pi\xi r) K_1(2\pi\xi r_0) + r_0 K_1(2\pi\xi r) K_0(2\pi\xi r_0)]}{K_1^2(2\pi\xi r_0)}, \end{aligned}$$

where the last equality follows from the recurrence formulas for Bessel functions (see [1, (9.6.26)]). From the integral representation (see [17, (2.1)] and the references therein)

$$\frac{x K_0(x)}{K_1(x)} = \frac{4}{\pi^2} \int_0^{+\infty} \frac{x^2}{x^2 + t^2} \frac{t^{-1} dt}{J_1^2(t) + Y_1^2(t)} \quad x > 0, \quad (10)$$

and one concludes that the function  $x K_0(x)/K_1(x)$  is increasing in  $x > 0$ , which implies in our case that

$$\frac{r_0 K_0(2\pi\xi r_0)}{K_1(2\pi\xi r_0)} \leq \frac{r K_0(2\pi\xi r)}{K_1(2\pi\xi r)} \quad \text{and therefore} \quad g'_d(\xi; r_0, r) \leq 0, \quad \xi > 0.$$

Consequently

$$g_d(\xi; r_0, r) \leq g_d(0+; r_0, r) = \frac{r_0}{r}, \quad \forall \xi > 0,$$

which gives the first inequality of the Proposition using the Plancherel theorem. The second one can be proved with the same arguments.  $\square$

**Proposition 2.5.** *The trace on vertical lines of the solution  $u \in H^1_{1/2, \lambda}(\mathbb{R}^2_+)$  to (6) satisfies*

$$|\partial_r(ru)(r, \cdot)|_{H^{-1/2}(\mathbb{R})} := \left( \int_{\mathbb{R}} |\xi|^{-1} |\partial_r(r\hat{u})(r, \xi)|^2 d\xi \right)^{1/2} < \infty \quad \forall r \geq r_0, \quad (11)$$

and one has the following bound for  $r > r_0$  sufficiently large

$$|\partial_r(ru)(r, \cdot)|_{H^{-1/2}(\mathbb{R})} \leq C \frac{1}{\sqrt{r}} \|u(r_0, \cdot)\|_{L^2(\mathbb{R})} \quad (12)$$

for some constant  $C > 0$  independent of  $r$  and  $u$ .

*Proof.* We use the following integral representations of modified Bessel's functions [1, (9.6.23)]

$$\begin{aligned} K_0(x) &= \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2})} \int_1^{+\infty} e^{-xt} \frac{dt}{\sqrt{t^2 - 1}}, \\ K_1(x) &= \frac{\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{3}{2})} \int_1^{+\infty} x e^{-xt} \sqrt{t^2 - 1} dt = \frac{\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{3}{2})} \int_1^{+\infty} e^{-xt} \frac{t dt}{\sqrt{t^2 - 1}}. \end{aligned}$$

They imply in particular

$$\frac{K_0(a)}{K_0(b)} \leq e^{-(a-b)} \quad \forall a > b > 0 \quad \text{and} \quad \frac{K_0(b)}{K_1(b)} \leq \frac{2\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})} \quad \forall b > 0.$$

Therefore

$$\frac{K_0(a)}{K_1(b)} = \frac{K_0(a)}{K_0(b)} \frac{K_0(b)}{K_1(b)} \leq C e^{-(a-b)} \quad \forall a > b > 0. \quad (13)$$

Differentiating the Fourier representation (8) with respect to  $r$  then applying the obtained expression to the couple  $(r, r/2)$  instead of  $(r, r_0)$ , implies for sufficiently large  $r$ ,

$$\partial_r(r\hat{u})(r, \xi) = \begin{cases} \hat{u}(r/2, \xi)(-2\pi|\xi|r) \frac{K_0(2\pi|\xi|r)}{K_1(\pi|\xi|r)} & \xi \neq 0, \\ 0 & \xi = 0. \end{cases} \quad (14)$$

Therefore, using the definition of the norm (11), one gets

$$|\partial_r(ru)(r, \cdot)|_{H^{-1/2}(\mathbb{R})}^2 = \int_{\mathbb{R}} (2\pi r)^2 |\xi| |\hat{u}(r/2, \xi)|^2 \left| \frac{K_0(2\pi|\xi|r)}{K_1(\pi|\xi|r)} \right|^2 d\xi \leq \|u(r/2, \cdot)\|_{L^2(\mathbb{R})}^2 \|g_n\|_{L^\infty(\mathbb{R})},$$

with  $g_n(\xi) := (2\pi r)^2 |\xi| \left| \frac{K_0(2\pi|\xi|r)}{K_1(\pi|\xi|r)} \right|^2$ .

Using (13) we have

$$g_n(\xi) \leq C(2\pi r)^2 |\xi| e^{-\pi|\xi|r} \leq C \frac{4\pi r}{e}.$$

Therefore, for  $r/2 > r_0$ ,

$$|\partial_r(ru)(r, \cdot)|_{H^{-1/2}(\mathbb{R})} \leq \sqrt{C \frac{4\pi r}{e}} \|u(r/2, \cdot)\|_{L^2(\mathbb{R})} \leq 2r_0 \sqrt{C \frac{4\pi}{er}} \|u(r_0, \cdot)\|_{L^2(\mathbb{R})},$$

where the last inequality is due to Proposition 2.4. This proves the bound (12).  $\square$

## 2.2 Radial cut-off for eddy current simulations

The decay in radial direction suggests that reasonable accuracy can be obtained by truncating the computational domain at  $r = r_*$  sufficiently large. In fact, for the application we are interested in, this is also justified by the high conductivity of the tube that would absorb most of the energy delivered by the coil inside the tube (and therefore the value of  $r_*$  that would be convenient in practice is not too large). We shall analyze in the sequel the error resulting from radial truncation independently from the absorption. It turns out in this case that the boundary conditions that lead to reasonable error estimates are Neumann boundary conditions. The case of Dirichlet boundary conditions lead to slower convergence rates that will be confirmed by our numerical examples. We present in this section only the case of Neumann boundary conditions.

For  $R \geq 0$  we denote

$$B_R := \{(r, z) : 0 < r < R, z \in \mathbb{R}\} \quad \text{and} \quad \Gamma_R := \{(r, z) : r = R, z \in \mathbb{R}\},$$

and shall use the short notation

$$L_{1/2}^2(\Omega) := L_{1/2,0}^2(\Omega) = \{v : v\sqrt{r} \in L^2(\Omega)\},$$

$$H_{1/2}^1(\Omega) := H_{1/2,0}^1(\Omega) = \{v \in L_{1/2}^2(\Omega) : r^{-1}\nabla(rv) \in L_{1/2}^2(\Omega)\}.$$

We remark already that the semi-norm of  $H_{1/2,\lambda}^1(\Omega)$  written as  $\|r^{-1/2}\nabla(rv)\|_{L^2(\Omega)}$  is independent of  $\lambda$ . Then from the proof of Lemma 2.1, this implies that for any  $v \in H_{1/2}^1$ , we can define its trace on  $\Gamma_0$  with the same way and the trace vanishes in  $L^2(\mathbb{R})$  norm. Moreover, with  $H^s(\mathbb{R})$  denoting the usual Sobolev space on  $\mathbb{R}$  and for sufficiently regular function  $v$  defined in a neighborhood of  $\Gamma_R$  we set

$$\|v\|_{H^s(\Gamma_R)} := \|v(R, \cdot)\|_{H^s(\mathbb{R})}.$$

Let  $r_* > 0$  be sufficiently large such that the support of the source term  $J$  is included in  $B_{r_*}$ . Then the truncated problem with Neumann boundary conditions on  $\Gamma_{r_*}$  consists into seeking  $u_n \in H_{1/2}^1(B_{r_*})$  satisfying

$$\begin{cases} -\operatorname{div} \left( \frac{1}{\mu r} \nabla(ru_n) \right) - i\omega\sigma u_n = i\omega J & \text{in } B_{r_*}, \\ \frac{\partial}{\partial r}(ru_n) = 0 & \text{on } \Gamma_{r_*}. \end{cases} \quad (15)$$

Note again that the function space  $H_{1/2}^1(B_{r_*})$  implicitly accounts for homogeneous boundary condition on  $\Gamma_0$ . The well-posedness of this problem is guaranteed thanks to the following lemma which will also be useful in quantifying error estimates. The proof of this Lemma is given in Appendix A.2.

**Lemma 2.6.** *Let  $r_* > 0$ . For all  $v \in H_{1/2}^1(B_{r_*})$  we have the Poincaré-type inequalities*

$$\|v\|_{L^2(B_{r_*})} \leq \sqrt{r_*} \|r^{-1} \nabla(rv)\|_{L_{1/2}^2(B_{r_*})} \quad \text{and} \quad \|v\|_{L_{1/2}^2(B_{r_*})} \leq \frac{r_*}{\sqrt{2}} \|r^{-1} \nabla(rv)\|_{L_{1/2}^2(B_{r_*})}, \quad (16)$$

and a trace estimate

$$|v|_{H^{1/2}(\Gamma_{r_*})} \leq r_*^{-1/2} \|r^{-1} \nabla(rv)\|_{L_{1/2}^2(B_{r_*})}, \quad (17)$$

where the semi-norm  $|\cdot|_{H^{1/2}}$  is defined by, using the Fourier transform  $\hat{v}(r_*, \cdot)$  of  $v(r_*, \cdot)$ ,

$$|v|_{H^{1/2}(\Gamma_{r_*})} := \left\| |\cdot|^{-1/2} \hat{v}(r_*, \cdot) \right\|_{L^2(\mathbb{R})}. \quad (18)$$

**Proposition 2.7.** *Assume that the source  $J \in L^2(\mathbb{R}_+^2)$  has compact support and let  $r_* > 0$  be sufficiently large so that the support of  $J$  is included in  $B_{r_*}$ . Then problem (15) has a unique solution  $u_n \in H_{1/2}^1(B_{r_*})$ . Assume in addition that there exists  $0 < r_0 < r_*$  such that  $J$  and the conductivity  $\sigma$  vanish and the permeability  $\mu$  is constant for  $r > r_0$ . Then there exists a constant  $C$  that depends only on  $J$ ,  $r_0$ ,  $\mu$  and  $\sigma$  such that*

$$\|r^{-1} \nabla(r(u_n - u))\|_{L_{1/2}^2(B_{r_*})} \leq Cr_*^{-1} \quad \text{and} \quad \|u_n - u\|_{L^2(B_{r_*})} \leq Cr_*^{-1/2},$$

where  $u$  is the solution to (6).

*Proof.* The proof of the first part is similar to the proof of Proposition 2.2 thanks to Lemma 2.6. Let us set  $w_n := u - u_n \in H_{1/2}^1(B_{r_*})$  such that

$$\int_{B_{r_*}} \frac{1}{\mu r} \nabla(rw_n) \cdot \nabla(r\bar{v}) - i\omega\sigma w_n \bar{v} r = \int_{\Gamma_{r_*}} \frac{1}{\mu} \frac{\partial}{\partial r}(ru) \bar{v} dz \quad \forall v \in H_{1/2}^1(B_{r_*}),$$

where the integral on  $\Gamma_{r_*}$  should be understood as a  $H^{-1/2} - H^{1/2}$  duality pairing. Taking  $v = \bar{w}_n$ , we obtain

$$\left| \int_{B_{r_*}} \frac{1}{\mu r} |\nabla(rw_n)|^2 - i\omega\sigma |w_n|^2 r \right| = \left| \int_{\Gamma_{r_*}} \frac{1}{\mu} \frac{\partial}{\partial r}(ru) \bar{w}_n dz \right| \leq \frac{1}{\mu(r_*)} \left| \frac{\partial}{\partial r}(ru) \right|_{H^{-1/2}(\Gamma_{r_*})} |w_n|_{H^{1/2}(\Gamma_{r_*})}.$$

Using (17) we deduce

$$\begin{aligned} \frac{1}{\|\mu\|_\infty} \left\| \frac{1}{r} \nabla(rw_n) \right\|_{L_{1/2}^2(B_{r_*})}^2 &\leq \left| \int_{B_{r_*}} \frac{1}{\mu r} |\nabla(rw_n)|^2 - i\omega\sigma |w_n|^2 r \right| \\ &\leq \frac{1}{\mu(r_*)} \left| \frac{\partial}{\partial r}(ru) \right|_{H^{-1/2}(\Gamma_{r_*})} \sqrt{\frac{1}{r_*}} \left\| \frac{1}{r} \nabla(rw_n) \right\|_{L_{1/2}^2(B_{r_*})}. \end{aligned}$$

Therefore,

$$\left\| \frac{1}{r} \nabla(rw_n) \right\|_{L_{1/2}^2(B_{r_*})} \leq \frac{\|\mu\|_\infty}{\mu(r_*)} \sqrt{\frac{1}{r_*}} \left| \frac{\partial}{\partial r}(ru) \right|_{H^{-1/2}(\Gamma_{r_*})}.$$

The first estimate in the Lemma then follows from Proposition 2.5 and the second one can be deduced using the first inequality in (16).  $\square$

**Remark 2.8.** *As indicated in the beginning of this section, if one uses Dirichlet boundary conditions on  $\Gamma_{r_*}$  then one loses half an order of magnitude for the convergence rate in terms of  $1/r_*$  (see Appendix C). This is in fact corroborated by our numerical experiments in Section 4.1.*

### 3 DtN operator and cut-off in the longitudinal direction

We discuss in this section the truncation of the domain in the longitudinal direction, i.e., the  $z$ -direction, whenever a truncation has been applied before in the radial direction. We therefore consider the solution  $u_n$  of (15) and in order to shorten notation we abusively denote this solution by  $u$ . Recall that the variational formulation of problem (15) is to find  $u \in H_{1/2}^1(B_{r_*})$  such that

$$\int_{B_{r_*}} \frac{1}{\mu} \frac{1}{r} \nabla(ru) \cdot \nabla(r\bar{v}) \, dr \, dz - \int_{B_{r_*}} i\omega\sigma u \bar{v} r \, dr \, dz = \int_{B_{r_*}} i\omega J \bar{v} r \, dr \, dz \quad \forall v \in H_{1/2}^1(B_{r_*}), \quad (19)$$

where  $r_* > 0$  is as in Proposition 2.7. The idea how to truncate the domain in the  $z$ -direction is to explicitly compute the DtN map for the regions above and below the source and inhomogeneities in the coefficients  $\mu$  and  $\sigma$  using the method of separation of variables. The main difficulty to cope with here is to prove that this is feasible even though the main operator is not selfadjoint.

We truncate the domain by two horizontal boundaries  $\Gamma_{\pm} := \{z = \pm z_*\}$  for some  $z_* > 0$  large enough such that the source is compactly supported in

$$B_{r_*, z_*} := \{(r, z) \in B_{r_*} : |z| < z_*\}.$$

We then assume in addition that  $\mu$  and  $\sigma$  only depends on the variable  $r$  in the complementary region

$$B_{r_*, z_*}^{\pm} := \{(r, z) \in B_{r_*} : z \gtrless \pm z_*\}.$$

Since in  $B_{r_*, z_*}^{\pm}$  it holds that

$$-\operatorname{div} \left( \frac{1}{\mu r} \nabla(ru) \right) - i\omega\sigma u = 0,$$

a solution of the form  $u(r, z) = \rho(r)\zeta(z)$  has to satisfy

$$\frac{1}{\zeta} \frac{d^2\zeta}{dz^2} = -\mu \frac{1}{\rho} \frac{d}{dr} \left( \frac{1}{\mu r} \frac{d}{dr}(r\rho) \right) - i\omega\mu\sigma = \nu, \quad (20)$$

where  $\nu \in \mathbb{C}$  is some eigenvalue that we will estimate. For the first equation, we obtain

$$\frac{d^2\zeta}{dz^2} - \nu\zeta = 0,$$

which has as solutions  $\zeta(z) = c \exp(\pm\sqrt{\nu}z)$  if  $\nu \neq 0$ , while for the second equation we are led to consider the eigenvalue problem

$$\begin{aligned} \mathcal{S}_{\mu}^{\sigma} \rho &:= -\mu \frac{d}{dr} \left( \frac{1}{\mu r} \frac{d}{dr}(r\rho) \right) - i\omega\mu\sigma\rho = \nu\rho \quad \text{in } I := \{r \in \mathbb{R} : 0 < r < r_*\}, \\ \rho(0) &= 0, \quad \left. \frac{d}{dr}(r\rho) \right|_{r=r_*} = 0. \end{aligned} \quad (21)$$

We first formally observe from (21) (after multiplication by  $r\bar{\rho}$  and integration by parts) that  $\Im(\nu) \leq 0$  and  $\Re(\nu) > 0$ . Choosing  $\sqrt{\nu}$  such that  $\Re\sqrt{\nu} > 0$ , we get that  $\zeta(z) = c \exp(\mp\sqrt{\nu}z)$  on  $B_{r_*, z_*}^{\pm}$  are the only admissible solutions due to their boundedness at infinity. The only missing point, that would allow the construction of analytic expansions of solutions to (19) restricted to  $B_{r_*, z_*}^{\pm}$ , is to prove that the set of eigenfunctions associated with (21) forms a complete set for the traces on  $\Gamma_{\pm}$  of solutions to problem (19).

#### 3.1 Analysis of the non-selfadjoint eigenvalue problem

We consider the spaces

$$L_{1/2}^2(I) := \{\phi : \phi\sqrt{r} \in L^2(I)\}, \quad H_{1/2}^1(I) := \{\phi \in L_{1/2}^2(I) : \frac{1}{r} \frac{\partial}{\partial r}(r\phi) \in L_{1/2}^2(I)\}.$$

For convenience, we shall denote in the sequel by  $(\cdot, \cdot)_\mu$  the weighted scalar product

$$(a, b)_\mu := (\mu^{-1}a, b)_1,$$

where  $(\cdot, \cdot)_1$  is the  $L^2_{1/2}(I)$  scalar product. Since  $\mu^{-1} > 0$  is bounded  $(\cdot, \cdot)_\mu$  is equivalent to  $(\cdot, \cdot)_1$ . We also denote by  $L^{2,\mu}_{1/2}(I)$  the  $L^2_{1/2}(I)$  space equipped with the weighted scalar product  $(\cdot, \cdot)_\mu$  and by  $H^{1,\mu}_{1/2}(I)$  the space defined similarly to  $H^1_{1/2}(I)$  but with  $L^2_{1/2}(I)$  replaced with  $L^{2,\mu}_{1/2}(I)$ . We remark that the norm in  $H^{1,\mu}_{1/2}(I)$  is also equivalent to the norm in  $H^1_{1/2}(I)$ . The main results of this section (Lemma 3.2 and Proposition 3.3) will be proved under the assumption that  $\mu$  is of the following form, which corresponds with the practical problem we are interested in,

$$\mu = \mu(r) = \begin{cases} \mu_0 & 0 < r < r_1 \quad \text{and} \quad r_2 < r < r_*, \\ \mu_t & r_1 < r < r_2, \end{cases} \quad (22)$$

where  $0 < r_1 < r_2$  correspond with the two horizontal positions of the tube interfaces and  $\mu_0$  and  $\mu_t$  are constants and respectively refer to the magnetic permeabilities of the vacuum and the tube.

**Lemma 3.1.** *The embedding  $H^{1,\mu}_{1/2}(I) \hookrightarrow L^{2,\mu}_{1/2}(I)$  is dense and compact. Any  $\phi \in H^{1,\mu}_{1/2}(I)$  is continuous in the closure of  $I$  and satisfies  $\phi = 0$  at  $r = 0$ . Moreover, the following Poincaré-type inequalities hold,*

$$\|\phi\|_{L^2(I)} \leq \sqrt{r_*} \left\| \frac{1}{r} \nabla(r\phi) \right\|_{L^2_{1/2}(I)} \quad \text{and} \quad \|\phi\|_{L^{2,\mu}_{1/2}(I)} \leq \frac{r_*}{\sqrt{2}} \left\| \frac{1}{r} \nabla(r\phi) \right\|_{L^2_{1/2}(I)} \quad \forall \phi \in H^{1,\mu}_{1/2}(I). \quad (23)$$

*Proof.* The proof of the compact embedding is a simple application of [10, Corollaire IV.26]. For the detailed proof, see Appendix A.3. The proof of the property  $\phi(0) = 0$  and the inequalities is the same as for Lemma 2.6.  $\square$

This implies that the embedding  $H^{1,\mu}_{1/2}(I) \hookrightarrow L^{2,\mu}_{1/2}(I)$  is dense and compact. Then one can define the unbounded operator  $\mathcal{A}_\mu : D(\mathcal{A}_\mu) \subset L^{2,\mu}_{1/2}(I) \rightarrow L^{2,\mu}_{1/2}(I)$ , where

$$D(\mathcal{A}_\mu) = \left\{ u \in H^{1,\mu}_{1/2}(I) : \mathcal{A}_\mu u = \mu \frac{d}{dr} \left( \frac{1}{\mu r} \frac{d}{dr}(ru) \right) \in L^{2,\mu}_{1/2}(I), \left. \frac{d}{dr}(ru) \right|_{r=r_*} = 0 \right\}.$$

For  $\phi \in D(\mathcal{A}_\mu)$ ,  $\mathcal{A}_\mu \phi$  is defined by

$$(\mathcal{A}_\mu \phi, \psi)_\mu = \int_0^{r_*} \frac{1}{\mu r} \frac{d}{dr}(r\phi) \frac{d}{dr}(r\bar{\psi}) dr \quad \forall \psi \in H^{1,\mu}_{1/2}(I). \quad (24)$$

It is clear from this definition that  $\mathcal{A}_\mu$  is closed and selfadjoint and according to Lemma 3.1, it has a compact resolvent. Moreover, the second inequality in Lemma 3.1 and the property of  $\mu$  show that

$$(\mathcal{A}_\mu \phi, \phi)_\mu \geq c \|\phi\|_{H^{1,\mu}_{1/2}(I)}^2 \quad \forall \phi \in H^{1,\mu}_{1/2}(I)$$

for some positive constant  $c$  independent of  $\phi$ .

We then deduce (see for instance [13, Section VIII.2, Theorem 7]) that  $\mathcal{A}_\mu$  has positive eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}^*}$  with corresponding  $L^{2,\mu}_{1/2}$ -complete orthonormal (under  $(\cdot, \cdot)_\mu$  scalar product) eigenprojectors  $\{P_k\}_{k \in \mathbb{N}^*}$  such that

$$0 < c \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty \quad (k \rightarrow \infty),$$

and  $\forall \phi \in P_k(H^{1,\mu}_{1/2}(I))$

$$(\mathcal{A}_\mu \phi, \psi)_\mu = \lambda_k (\phi, \psi)_\mu \quad \forall \psi \in H^{1,\mu}_{1/2}(I). \quad (25)$$

Since  $\mathcal{S}_\mu^\sigma$  is formally only a compact perturbation of  $\mathcal{A}_\mu$  by using the perturbation theory one can relate the spectrum of  $\mathcal{S}_\mu^\sigma$  to the spectrum of  $\mathcal{A}_\mu$ . We first need to have estimates on the eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}^*}$ . For that purpose we introduce the following lemma.

**Lemma 3.2.** *Let  $\mu = \mu(r)$  as described in (22). If  $\mu_t/\mu_0$  is sufficiently close to 1, for instance  $|\mu_t/\mu_0 - 1| < \frac{1}{8}$ , then the eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}^*}$  of  $\mathcal{A}_\mu$  are simple and grow like  $O(k^2)$  as  $k \rightarrow \infty$ . Moreover, the difference  $\lambda_{k+1} - \lambda_k \rightarrow +\infty$  as  $k \rightarrow \infty$ .*

The proof of this Lemma with involved calculations is given in Appendix B. Let us note that in practice, for steam generator tubes, the condition  $|\mu_t/\mu_0 - 1| < \frac{1}{8}$  is satisfied. In fact  $\mu_t/\mu_0 - 1 \simeq 0.01$ . We also note that all subsequent analysis remains true if we assume that  $\mu \in L^\infty(I)$  with  $0 < \mu_{\inf} \leq \mu \leq \mu_{\sup} < +\infty$  and the corresponding selfadjoint operator  $\mathcal{A}_\mu$  has simple eigenvalues  $\{\lambda_k\}_{k \in \mathbb{N}^*}$  growing up as  $O(k^2)$  and  $\lambda_{k+1} - \lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

Now let us consider the operator  $\mathcal{S}_\mu^\sigma = \mathcal{A}_\mu + \mathcal{M}_\mu^\sigma$  defined in (21). Since the multiplication operator

$$\mathcal{M}_\mu^\sigma : \phi \mapsto -i\omega\mu\sigma\phi \quad \forall \phi \in L_{1/2}^{2,\mu}(I),$$

is bounded on  $L_{1/2}^{2,\mu}(I)$ , the theory for perturbed selfadjoint operators [18, Theorem V-4.15a and Remark V-4.16a] implies:

**Proposition 3.3.** *Under the assumptions of Lemma 3.2, the unbounded operator  $\mathcal{S}_\mu^\sigma : L_{1/2}^{2,\mu}(I) \rightarrow L_{1/2}^{2,\mu}(I)$  is closed with compact resolvent and its eigenvalues and eigenprojectors can be indexed as  $\{\nu_{0j}, \nu_k\}$  and  $\{Q_{0j}, Q_k\}$  respectively, where  $j = 1, \dots, m < \infty$  and  $k = n+1, n+2, \dots$  with  $n \geq m \geq 0$  such that the following results hold:*

1. the sequence  $|\nu_k - \lambda_k|$  is bounded as  $k \rightarrow \infty$ .
2. there exists a bounded operator  $W$  on  $L_{1/2}^{2,\mu}(I)$  with bounded inverse  $W^{-1}$  such that

$$Q_0 := \sum_{j=1}^m Q_{0j} = W^{-1} \left( \sum_{k \leq n} P_k \right) W \quad \text{and} \quad Q_k = W^{-1} P_k W \quad \text{for } k > n. \quad (26)$$

Moreover,  $\{Q_{0j}, Q_k\}$  is a complete family in the sense that

$$\sum_{j=1}^m Q_{0j} + \sum_{k > n} Q_k = 1. \quad (27)$$

We note that although the eigenvalues  $\nu_k$  may be simple (this is for instance the case for piece-wise constant coefficients) we can still have  $m < n$  in the previous proposition. This would correspond with the presence of Jordan's blocs for eigenvalues decompositions of matrices. However, this case has not been observed numerically (as mentioned in the numerical section).

### 3.2 Spectral decomposition of the DtN operator

We are now in position to provide explicit expression for the DtN operator that will be used to truncate the domain in the  $z$ -direction. We first need to specify the space of traces on  $\Gamma_\pm$  of functions in  $H_{1/2}^1(B_{r_*})$ . From the definition of the spectral decomposition of  $\mathcal{A}_\mu$  we immediately deduce that for  $\phi \in H_{1/2}^{1,\mu}(I)$

$$\|\phi\|_{L_{1/2}^{2,\mu}(I)}^2 = \sum_{k \geq 1} \|P_k \phi\|_{L_{1/2}^{2,\mu}(I)}^2 \quad \text{and} \quad \|\phi\|_{H_{1/2}^{1,\mu}(I)}^2 = \sum_{k \geq 1} (1 + \lambda_k) \|P_k \phi\|_{L_{1/2}^{2,\mu}(I)}^2.$$

For  $\theta \in [0, 1]$ , we define  $H_{1/2}^{\theta,\mu}(I)$  as the  $\theta$  interpolation space  $[H_{1/2}^1(I), L_{1/2}^{2,\mu}(I)]_\theta$  (see [19, Définition 2.1] for interpolation spaces) with norm

$$\|\phi\|_{H_{1/2}^{\theta,\mu}(I)}^2 := \sum_{k \geq 1} (1 + \lambda_k)^\theta \|P_k \phi\|_{L_{1/2}^{2,\mu}(I)}^2 \quad (28)$$

and define  $H_{1/2}^{\theta,\mu}(I)^*$  as the dual space of  $H_{1/2}^{\theta,\mu}(I)$  with pivot space  $L_{1/2}^{2,\mu}(I)$  under  $(\cdot, \cdot)_\mu$  scalar product. The definition of the spaces  $H_{1/2}^{\theta,\mu}(\Gamma_\pm)$  and  $H_{1/2}^{\theta,\mu}(\Gamma_\pm)^*$  are obtained from  $H_{1/2}^{\theta,\mu}(I)$  and  $H_{1/2}^{\theta,\mu}(I)^*$  by identifying

$\Gamma_{\pm}$  with  $I$  using the obvious isometry. Let  $v$  be a regular function of  $B_{r_*}$ . We denote the trace mapping by

$$\gamma^{\pm} : v \mapsto v|_{\Gamma_{\pm}}.$$

**Theorem 3.4.** *The trace mapping  $\gamma^{\pm}$  can be extended to a continuous and surjective mapping from  $H_{1/2}^1(B_{r_*, z_*})$  onto  $H_{1/2}^{1/2, \mu}(\Gamma_{\pm})$  and from  $H_{1/2}^1(B_{r_*, z_*}^{\pm})$  onto  $H_{1/2}^{1/2, \mu}(\Gamma_{\pm})$ .*

*Proof.* Obviously we have the equivalent definition

$$H_{1/2}^1(B_{r_*, z_*}) = H_{1/2}^{1, \mu}(B_{r_*, z_*}) = \left\{ v : v \in L^2((-z_*, z_*); H_{1/2}^{1, \mu}(I)), \frac{\partial v}{\partial z} \in L^2((-z_*, z_*); L_{1/2}^{2, \mu}(I)) \right\},$$

with the same norm. Therefore the trace mapping properties for  $H_{1/2}^{1, \mu}(B_{r_*, z_*})$  is a direct application of classical theory for trace spaces: [19, Théorème 3.2]. Similar considerations apply for  $H_{1/2}^1(B_{r_*, z_*}^{\pm})$ .  $\square$

Let us also mention the following result that will be useful later

**Lemma 3.5.** *Let  $\theta \in [0, 1]$ . The norm*

$$\|\phi\|_{H_{1/2}^{\theta, \mu}} := \sum_{k=1}^{\infty} (1 + \lambda_k)^{\theta} \|P_k \phi\|_{L_{1/2}^{2, \mu}(I)}$$

*defines an equivalent norm on  $H_{1/2}^{\theta, \mu}(I)$ .*

*Proof.* From interpolation theory, it is sufficient to prove the result for  $\theta = 0$  and  $\theta = 1$ . The statement is obviously true for  $\theta = 1$ , while for  $\theta = 0$  one needs only to observe that  $H_{1/2}^{0, \mu}(I) = L_{1/2}^{2, \mu}(I)$ .  $\square$

Let  $\phi^{\pm}$  in  $H_{1/2}^{1/2, \mu}(\Gamma_{\pm})$  and denote by  $\mu^{\pm}$  and  $\sigma^{\pm}$  the restrictions of  $\mu$  and  $\sigma$  to  $B_{r_*, z_*}^{\pm}$ . Thanks to Theorem 3.4 one can uniquely define  $u^{\pm} \in H_{1/2}^1(B_{r_*, z_*}^{\pm})$  solution of

$$\begin{cases} -\operatorname{div} \left( \frac{1}{\mu^{\pm} r} \nabla(r u^{\pm}) \right) - i\omega \sigma^{\pm} u^{\pm} = 0 & \text{in } B_{r_*, z_*}^{\pm}, \\ \frac{1}{\mu^{\pm}} \frac{\partial}{\partial r}(r u^{\pm}) = 0 & \text{on } \Gamma_{r_*}, \\ u^{\pm} = \phi^{\pm} & \text{on } \Gamma_{\pm}. \end{cases} \quad (29)$$

The construction of  $u^{\pm}$  can be done for instance by using some continuous lifting linear operators  $\mathcal{R}^{\pm} : H_{1/2}^{1/2, \mu}(\Gamma_{\pm}) \rightarrow H_{1/2}^1(B_{r_*, z_*}^{\pm})$  such that  $\gamma^{\pm} \mathcal{R}^{\pm}(\phi) = \phi$  (these operators exist according to Theorem 3.4). The  $H_{1/2}^1(B_{r_*, z_*}^{\pm})$  norm of  $u^{\pm}$  indeed continuously depends on the  $H_{1/2}^{1/2, \mu}(\Gamma_{\pm})$  norm of the boundary data  $\phi^{\pm}$  (respectively).

**Definition 3.6.** *We define the DtN operators  $\mathcal{T}^{\pm} : H_{1/2}^{1/2, \mu}(\Gamma_{\pm}) \rightarrow H_{1/2}^{1/2, \mu}(\Gamma_{\pm})^*$  by*

$$\langle \mathcal{T}^{\pm} \phi^{\pm}, \psi^{\pm} \rangle = \int_{B_{r_*, z_*}^{\pm}} \frac{1}{\mu^{\pm} r} \nabla(r u^{\pm}) \nabla(r \overline{\mathcal{R}^{\pm} \psi^{\pm}}) \, dr \, dz + \int_{B_{r_*, z_*}^{\pm}} \operatorname{div} \left( \frac{1}{\mu r} \nabla(r u^{\pm}) \right) \overline{\mathcal{R}^{\pm} \psi^{\pm}} r \, dr \, dz$$

*for all  $\psi^{\pm} \in H_{1/2}^{1/2, \mu}(\Gamma_{\pm})$ , where  $u^{\pm} \in H^1(B_{r_*, z_*}^{\pm})$  is the unique solution of problem (29) and where  $\langle \cdot, \cdot \rangle$  denotes the  $H_{1/2}^{1/2, \mu*} - H_{1/2}^{1/2, \mu}$  duality product that coincides with  $(\cdot, \cdot)_{\mu}$  for  $L_{1/2}^{2, \mu}$  functions.*

Indeed

$$\langle \mathcal{T}^{\pm} \phi^{\pm}, \psi^{\pm} \rangle = \int_{B_{r_*, z_*}^{\pm}} \frac{1}{\mu^{\pm} r} \nabla(r u^{\pm}) \nabla(r \overline{\mathcal{R}^{\pm} \psi^{\pm}}) \, dr \, dz - \int_{B_{r_*, z_*}^{\pm}} i\omega \sigma^{\pm} u^{\pm} \overline{\mathcal{R}^{\pm} \psi^{\pm}} r \, dr \, dz \quad (30)$$

and therefore, from the definition of  $\mathcal{R}^\pm$  and the continuity property for the solutions  $u^\pm$ ,

$$\langle \mathcal{T}^\pm \phi^\pm, \psi^\pm \rangle \leq C \|\phi^\pm\|_{H_{1/2}^{1/2, \mu}(\Gamma_\pm)} \|\psi^\pm\|_{H_{1/2}^{1/2, \mu}(\Gamma_\pm)}$$

for some constant  $C$  independent from  $\phi$  and  $\psi$ . This proves that  $\mathcal{T}^\pm : H_{1/2}^{1/2, \mu}(\Gamma_\pm) \rightarrow H_{1/2}^{1/2, \mu}(\Gamma_\pm)^*$  are well-defined and are continuous. We remark that for sufficiently regular  $u^\pm$ , we have (using Green's formula)

$$\mathcal{T}^\pm \phi^\pm = \mp \frac{1}{\mu^\pm} \frac{\partial u^\pm}{\partial z} \Big|_{\Gamma_\pm}. \quad (31)$$

Since  $\gamma^\pm - \gamma^\pm \mathcal{R}^\pm \gamma^\pm = 0$ , we also observe, using Green's formula (and a density argument) that

$$\langle \mathcal{T}^\pm \phi^\pm, \gamma^\pm v \rangle = \int_{B_{r_*, z_*}^\pm} \frac{1}{\mu^\pm r} \nabla(r u^\pm) \nabla(r \bar{v}) \, dr \, dz + \int_{B_{r_*, z_*}^\pm} \operatorname{div} \left( \frac{1}{\mu r} \nabla(r u^\pm) \right) \bar{v} \, dr \, dz^\pm$$

for all  $v \in H_{1/2}^1(B_{r_*, z_*}^\pm)$ . Therefore we also have

$$\langle \mathcal{T}^\pm \phi^\pm, \gamma^\pm v \rangle = \int_{B_{r_*, z_*}^\pm} \frac{1}{\mu^\pm r} \nabla(r u^\pm) \nabla(r \bar{v}) \, dr \, dz - \int_{B_{r_*, z_*}^\pm} i \omega \sigma^\pm u^\pm \bar{v} \, dr \, dz$$

for all  $v \in H_{1/2}^1(B_{r_*, z_*}^\pm)$ . Then it becomes clear from the variational formulation (19) that  $u|_{B_{r_*, z_*}^\pm} \in H_{1/2}^1(B_{r_*, z_*}^\pm)$  and satisfies

$$\begin{aligned} & \int_{B_{r_*, z_*}^\pm} \frac{1}{\mu} \nabla(r u) \cdot \frac{1}{r} \nabla(r \bar{v}) \, dr \, dz - \int_{B_{r_*, z_*}^\pm} i \omega \sigma u \bar{v} \, dr \, dz \\ & + \langle \mathcal{T}^+ \gamma^+ u, \gamma^+ v \rangle + \langle \mathcal{T}^- \gamma^- u, \gamma^- v \rangle = \int_{B_{r_*, z_*}^\pm} i \omega J \bar{v} \, dr \, dz \quad \forall v \in H_{1/2}^1(B_{r_*, z_*}^\pm). \end{aligned} \quad (32)$$

We immediately get the following equivalence result.

**Proposition 3.7.** *A function  $u \in H_{1/2}^1(B_{r_*})$  is solution of (19) if and only if  $u|_{B_{r_*, z_*}^\pm} \in H_{1/2}^1(B_{r_*, z_*}^\pm)$  and is solution of (32) and  $u = u^\pm$  on  $B_{r_*, z_*}^\pm$  where  $u^\pm \in H_{1/2}^1(B_{r_*, z_*}^\pm)$  are solution of (29) with  $\phi^\pm = \gamma^\pm(u|_{B_{r_*, z_*}^\pm})$ .*

Formulation (32) is the one that we would like to use in practice. Proposition 3.7 and the well-posedness of (19) show that (32) is also well-posed. To be numerically effective one needs explicit expressions for  $\mathcal{T}^\pm$ . We shall use for that purpose Proposition 3.3. We are then led to consider the spectral decompositions of  $\mathcal{S}_{\mu^+}^{\sigma^+}$  and  $\mathcal{S}_{\mu^-}^{\sigma^-}$  that correspond to the one in Proposition 3.3 for  $(\mu, \sigma) = (\mu^+, \sigma^+)$  and  $(\mu, \sigma) = (\mu^-, \sigma^-)$  respectively. Since the treatment of both cases is the same and in order to simplify the notation we shall use the same notation for the spectral decomposition of  $\mathcal{S}_{\mu^+}^{\sigma^+}$  and  $\mathcal{S}_{\mu^-}^{\sigma^-}$ .

For  $\phi^\pm \in H_{1/2}^{1/2, \mu}(\Gamma_\pm)$  we have the spectral decomposition

$$\phi^\pm = \sum_{j=1}^m Q_{0j}(\phi^\pm) + \sum_{k>n} Q_k(\phi^\pm).$$

By definition of  $Q_{0j}$  and  $Q_k$  the functions  $u^\pm$  defined on  $B_{r_*, z_*}^\pm$  by

$$u^\pm(r, z) = \sum_{j=1}^m Q_{0j}(\phi^\pm)(r) \exp(\mp \sqrt{\nu_{0j}}(z \mp z_*)) + \sum_{k>n} Q_k(\phi^\pm)(r) \exp(\mp \sqrt{\nu_k}(z \mp z_*)) \quad \text{in } B_{r_*, z_*}^\pm, \quad (33)$$

(the square root is determined as the one with positive real part) formally satisfy (29). In order to rigorously prove this, one only needs to verify that this function is in  $H_{1/2}^1(B_{r_*, z_*}^\pm)$ . Since the eigenfunctions  $Q_{0j}(\phi^\pm)$  and  $Q_k(\phi^\pm)$  are in  $H_{1/2}^{1, \mu}(I)$ , one easily checks that

$$u_N^\pm(r, z) := \sum_{j=1}^m Q_{0j}(\phi^\pm)(r) \exp(\mp \sqrt{\nu_{0j}}(z \mp z_*)) + \sum_{k=n+1}^N Q_k(\phi^\pm)(r) \exp(\mp \sqrt{\nu_k}(z \mp z_*)) \quad \text{in } B_{r_*, z_*}^\pm, \quad (34)$$

is in  $H_{1/2}^1(B_{r_*, z_*}^\pm)$  and verifies (29) with boundary data on  $\Gamma_\pm$  equal

$$\phi_N^\pm = \sum_{j=1}^m Q_{0j}(\phi^\pm) + \sum_{k=n+1}^N Q_k(\phi^\pm).$$

We then can conclude using the following lemma.

**Lemma 3.8.** *Let  $\phi \in H_{1/2}^{1/2, \mu}(I)$  and set for  $N > n$ ,*

$$\phi_N = \sum_{j=1}^m Q_{0j}(\phi) + \sum_{k=n+1}^N Q_k(\phi).$$

*Then,  $\|\phi_N - \phi\|_{H_{1/2}^{1/2, \mu}(I)} \rightarrow 0$  as  $N \rightarrow \infty$ .*

The proof of this Lemma is itself a straightforward consequence of the following result since, using the notation of the Lemma below,

$$\|\phi_N - \phi\|_{H_{1/2}^{1/2, \mu}(I)}^2 = \sum_{k>N} (1 + |\nu_k|)^{1/2} \|P_k W \phi\|_{L_{1/2}^{2, \mu}(I)}^2.$$

**Lemma 3.9.** *Let  $\theta \in [0, 1]$  and let  $\nu_* \in \mathbb{R}$ . The norm defined by*

$$\|\phi\|_{H_{1/2}^{\theta, \mu}(I)}^2 := \sum_{k=1}^{\infty} (1 + |\nu_k|)^\theta \|P_k W \phi\|_{L_{1/2}^{2, \mu}(I)}^2$$

*where  $\nu_k, k > n$ , are the eigenvalues of  $\mathcal{S}_\mu^\sigma$  as defined in Proposition 3.3 and  $\nu_k = \nu_*$  for  $k \leq n$ , defines an equivalent norm on  $H_{1/2}^{\theta, \mu}(I)$ .*

*Proof.* We first observe that thanks to Lemma 3.5, the result is obvious for  $\theta = 0$  since  $H_{1/2}^{0, \mu}(I) = L_{1/2}^{2, \mu}(I)$  and

$$\|\phi\|_{H_{1/2}^{0, \mu}(I)}^2 = \|W \phi\|_{H_{1/2}^{0, \mu}(I)}^2$$

where  $W : L_{1/2}^{2, \mu}(I) \rightarrow L_{1/2}^{2, \mu}(I)$  is an isomorphism. Using interpolation theory one then only needs to prove the result for  $\theta = 1$ . The case of  $\theta = 1$  will also be proved using interpolation theory since, using again Lemma 3.5 and the definition of  $\mathcal{A}_\mu$ , we have  $H_{1/2}^{1, \mu}(I) = [D(\mathcal{A}_\mu), L_{1/2}^{2, \mu}(I)]_{1/2}$ . Therefore it is sufficient to prove that

$$\|\phi\|_{D(\mathcal{A}_\mu)}^2 = \|\phi\|_{L_{1/2}^{2, \mu}(I)}^2 + \|\mathcal{A}_\mu \phi\|_{L_{1/2}^{2, \mu}(I)}^2 = \sum_{k=1}^{\infty} (1 + \lambda_k^2) \|P_k \phi\|_{L_{1/2}^{2, \mu}(I)}^2$$

is equivalent to

$$\|\phi\|_{H_{1/2}^{2, \mu}(I)}^2 = \sum_{k=1}^{\infty} (1 + |\nu_k|^2) \|P_k W \phi\|_{L_{1/2}^{2, \mu}(I)}^2.$$

Using the identity  $P_k(W \mathcal{S}_\mu^\sigma \phi) = \nu_k P_k W \phi$  for  $k > n$ , we observe that

$$\|\phi\|_{H_{1/2}^{2, \mu}(I)}^2 = \|\phi\|_{H_{1/2}^{0, \mu}(I)}^2 + \|\mathcal{S}_\mu^\sigma(\phi - Q_0 \phi)\|_{H_{1/2}^{0, \mu}(I)}^2 + |\nu_*|^2 \|Q_0 \phi\|_{H_{1/2}^{0, \mu}(I)}^2. \quad (35)$$

Since

$$\|\mathcal{S}_\mu^\sigma(Q_0 \phi)\|_{L_{1/2}^{2, \mu}(I)}^2 \leq C \|Q_0 \phi\|_{L_{1/2}^{2, \mu}(I)}^2$$

with  $C = \sup\{|\nu_{0j}|^2, j = 1, m\}$ , then  $\mathcal{S}_\mu^\sigma(I - Q_0) = \mathcal{A}_\mu + \mathcal{M}_0$ , where  $\mathcal{M}_0 := \mathcal{M}_\mu^\sigma - \mathcal{S}_\mu^\sigma Q_0$  is a bounded operator on  $L_{1/2}^{2,\mu}(I)$ . Therefore, with  $C$  denoting a constant independent of  $\phi$  but whose value may change from a line to another, and using the first part of the proof,

$$\|\phi\|_{D(\mathcal{A}_\mu)}^2 \leq 2\|\mathcal{S}_\mu^\sigma(\phi - Q_0\phi)\|_{L_{1/2}^{2,\mu}(I)}^2 + 2\|\mathcal{M}_0\phi\|_{L_{1/2}^{2,\mu}(I)}^2 + \|\phi\|_{L_{1/2}^{2,\mu}(I)}^2 \leq C\|\phi\|_{\widetilde{H_{1/2}^{2,\mu}(I)}}^2$$

and

$$\|\phi\|_{\widetilde{H_{1/2}^{2,\mu}(I)}}^2 \leq 2\|\mathcal{A}_\mu\phi\|_{\widetilde{H_{1/2}^{0,\mu}(I)}}^2 + 2\|\mathcal{M}_0\phi\|_{\widetilde{H_{1/2}^{0,\mu}(I)}}^2 + (1 + |\nu_*|^2)\|\phi\|_{\widetilde{H_{1/2}^{0,\mu}(I)}}^2 \leq C\|\phi\|_{D(\mathcal{A}_\mu)}^2,$$

which proves the desired equivalence of norms and concludes the proof.  $\square$

The expression of  $u_N^\pm$  in  $B_{r_*, z_*}^\pm$  (34) yields,

$$\mp \frac{\partial u_N^\pm}{\partial z} \Big|_{\Gamma_\pm} = \sum_{j=1}^m \sqrt{\nu_{0j}} Q_{0j}(\phi^\pm) + \sum_{n < k \leq N} \sqrt{\nu_k} Q_k(\phi^\pm).$$

Therefore, using (31) and letting  $N \rightarrow \infty$  we obtain (explicitly specifying in the notation the dependence on  $\Gamma_\pm$  on the spectral decomposition)

$$\mathcal{T}^\pm(\phi^\pm) = \sum_{j=1}^{m^\pm} \sqrt{\nu_{0j}^\pm} Q_{0j}^\pm \phi^\pm + \sum_{k > n^\pm} \sqrt{\nu_k^\pm} Q_k^\pm \phi^\pm. \quad (36)$$

### 3.3 On the analysis of spectral error truncation

For numerical simulations the spectral representation of operators  $\mathcal{T}^\pm$  should be truncated. We shall give here some estimates on the error due to this truncation. For  $N > n^\pm$ , we define the projectors

$$\mathcal{Q}_N^\pm := \sum_{j=1}^{m^\pm} Q_{0j}^\pm + \sum_{n^\pm < k \leq N} Q_k^\pm,$$

and the truncated DtN operators

$$\mathcal{T}_N^\pm := \mathcal{T}^\pm \mathcal{Q}_N^\pm = \sum_{j=1}^{m^\pm} \sqrt{\nu_{0j}^\pm} Q_{0j}^\pm + \sum_{n^\pm < k \leq N} \sqrt{\nu_k^\pm} Q_k^\pm. \quad (37)$$

According to Lemma 3.9,  $\mathcal{Q}_N^\pm : H_{1/2}^{1/2,\mu}(\Gamma_\pm) \rightarrow H_{1/2}^{1/2,\mu}(\Gamma_\pm)^*$  are continuous and therefore  $\mathcal{T}_N^\pm : H_{1/2}^{1/2,\mu}(\Gamma_\pm) \rightarrow H_{1/2}^{1/2,\mu}(\Gamma_\pm)^*$  are also continuous. We are interested in considering  $u_N \in H_{1/2}^1(B_{r_*, z_*})$  solving

$$\begin{aligned} & \int_{B_{r_*, z_*}} \frac{1}{\mu} \nabla(r u_N) \cdot \frac{1}{r} \nabla(r \bar{v}) \, dr \, dz - \int_{B_{r_*, z_*}} i \omega \sigma u_N \bar{v} r \, dr \, dz \\ & + \langle \mathcal{T}_N^+ \gamma^+ u_N, \mathcal{Q}_N^+ \gamma^+ v \rangle + \langle \mathcal{T}_N^- \gamma^- u_N, \mathcal{Q}_N^- \gamma^- v \rangle = \int_{B_{r_*, z_*}} i \omega J \bar{v} r \, dr \, dz \quad \forall v \in H_{1/2}^1(B_{r_*, z_*}). \end{aligned} \quad (38)$$

This variational problem is well-posed for all  $N$  as indicated in the following. Using the Riesz representation theorem, we introduce  $A_0 : H_{1/2}^1(B_{r_*, z_*}) \rightarrow H_{1/2}^1(B_{r_*, z_*})$ ,  $A : H_{1/2}^1(B_{r_*, z_*}) \rightarrow H_{1/2}^1(B_{r_*, z_*})$  and  $A_N : H_{1/2}^1(B_{r_*, z_*}) \rightarrow H_{1/2}^1(B_{r_*, z_*})$  defined by

$$\begin{aligned} (A_0 w, v)_{H_{1/2}^1(B_{r_*, z_*})} &= \int_{B_{r_*, z_*}} \frac{1}{\mu} \nabla(r w) \cdot \frac{1}{r} \nabla(r \bar{v}) - i \omega \sigma w \bar{v} r \, dr \, dz, \\ (A w, v)_{H_{1/2}^1(B_{r_*, z_*})} &= (A_0 w, v)_{H_{1/2}^1(B_{r_*, z_*})} + \langle \mathcal{T}^+ \gamma^+ w, \gamma^+ v \rangle + \langle \mathcal{T}^- \gamma^- w, \gamma^- v \rangle, \quad \text{and} \\ (A_N w, v)_{H_{1/2}^1(B_{r_*, z_*})} &= (A_0 w, v)_{H_{1/2}^1(B_{r_*, z_*})} + \langle \mathcal{T}^+ \mathcal{Q}_N^+ \gamma^+ w, \mathcal{Q}_N^+ \gamma^+ v \rangle + \langle \mathcal{T}^- \mathcal{Q}_N^- \gamma^- w, \mathcal{Q}_N^- \gamma^- v \rangle, \end{aligned}$$

for all  $v \in H_{1/2}^1(B_{r_*, z_*})$ , respectively. We recall that the operator  $A_0$  is coercive, and more precisely

$$\Re(A_0 w, w)_{H_{1/2}^1(B_{r_*, z_*})} \geq a_0 \|w\|_{H_{1/2}^1(B_{r_*, z_*})}^2$$

for some positive constant  $a_0$  independent of  $w$ . We observe from (30) that

$$\Re \langle \mathcal{T}^\pm \phi^\pm, \phi^\pm \rangle \geq 0,$$

and therefore

$$\Re(A_N w, w)_{H_{1/2}^1(B_{r_*, z_*})} \geq a_0 \|w\|_{H_{1/2}^1(B_{r_*, z_*})}^2.$$

This means in particular, thanks to the Lax-Milgram theorem that  $A_N$  is bijective and also

$$\|A_N^{-1}\| \leq 1/a_0.$$

Consequently problem (38) has a unique solution  $u_N \in H_{1/2}^1(B_{r_*, z_*})$  that continuously depends on  $J$  with a modulus of continuity independent of  $N$ .

From the continuity of  $\mathcal{T}^\pm$  we easily obtain

$$\|(A - A_N)w\|_{H_{1/2}^1(B_{r_*, z_*})} \leq C \left( \|\gamma^+ w - \mathcal{Q}_N^+ \gamma^+ w\|_{H_{1/2}^{1, \mu}(\Gamma_+)} + \|\gamma^- w - \mathcal{Q}_N^- \gamma^- w\|_{H_{1/2}^{1, \mu}(\Gamma_-)} \right) \quad (39)$$

for some constant  $C$  independent of  $N$  and  $w \in H_{1/2}^1(B_{r_*, z_*})$ . Therefore, using Lemma 3.9,

$$\lim_{N \rightarrow \infty} \|(A - A_N)w\|_{H_{1/2}^1(B_{r_*, z_*})} = 0 \quad \forall w \in H_{1/2}^1(B_{r_*, z_*}).$$

With  $u \in H_{1/2}^1(B_{r_*, z_*})$  denoting the solution of (32), we observe that

$$Au = A_N u_N.$$

Therefore,

$$u - u_N = A_N^{-1}(A_N u - Au).$$

This proves in particular that

$$\|u - u_N\|_{H_{1/2}^1(B_{r_*, z_*})} \leq 1/a_0 \|(A - A_N)u\|_{H_{1/2}^1(B_{r_*, z_*})} \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (40)$$

We can summarize these results in the following proposition

**Proposition 3.10.** *Under the same assumptions as in Proposition 2.7, the variational problem (38) has a unique solution  $u_N \in H_{1/2}^1(B_{r_*, z_*})$ . Moreover, if  $u \in H_{1/2}^1(B_{r_*, z_*})$  is the solution of (32), then*

$$\|u - u_N\|_{H_{1/2}^1(B_{r_*, z_*})} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

We shall now give some indication on the rate of convergence under some additional regularity assumptions on the source term  $J$  and the coefficients  $\mu$  and  $\sigma$ . Obviously, from (39) and (40)

$$\|u - u_N\|_{H_{1/2}^1(B_{r_*, z_*})} \leq C/a_0 \left( \|\gamma^+ u - \mathcal{Q}_N^+ \gamma^+ u\|_{H_{1/2}^{1, \mu}(\Gamma_+)} + \|\gamma^- u - \mathcal{Q}_N^- \gamma^- u\|_{H_{1/2}^{1, \mu}(\Gamma_-)} \right). \quad (41)$$

Therefore the speed of convergence will depend on the regularity of  $\gamma^\pm u$ . Considering problem (15) satisfied by  $u$  in the unbounded domain  $B_{r_*}$  and differentiating the equations with respect to  $z$  (i.e. considering the equation satisfied  $(u(r, z + \Delta z) - u(r, z))/\Delta z$ , then letting  $\Delta z \rightarrow 0$ ) one easily observes from the well-posedness of problem (15) that if in addition

$$\frac{\partial^m J}{\partial z^m} \in L^2(B_{r_*}), \quad \frac{\partial^m \sigma}{\partial z^m} \in L^\infty(B_{r_*}) \text{ and } \frac{\partial^m \mu^{-1}}{\partial z^m} \in L^\infty(B_{r_*}),$$

for some integer  $m \geq 0$ , then

$$\frac{\partial^m u}{\partial z^m} \in H_{1/2}^1(B_{r_*}).$$

Consequently, if this holds with  $m = 2$ , then the first equation in (15) yields

$$\gamma^\pm \mathcal{S}_{\mu^\pm}^{\sigma^\pm} u = -\frac{1}{\mu^\pm} \gamma^\pm \frac{\partial^2 u}{\partial z^2} \in L_{1/2}^{2,\mu}(\Gamma_\pm).$$

With the help of Lemma 3.9, we can then estimate,

$$\begin{aligned} \|\gamma^\pm u - \mathcal{Q}_N \pm \gamma^\pm u\|_{H_{1/2}^{1/2,\mu}(\Gamma_\pm)}^2 &\leq C_1 \sum_{k=N+1}^{\infty} (1 + |\nu_k^\pm|)^{1/2} \|P_k^\pm W^\pm \phi\|_{L_{1/2}^{2,\mu}(I)}^2 \\ &\leq \frac{C_2}{(1 + |\nu_N^\pm|)^{3/2}} \sum_{k=N+1}^{\infty} (1 + |\nu_k^\pm|)^2 \|P_k^\pm W^\pm \phi\|_{L_{1/2}^{2,\mu}(I)}^2 \\ &\leq \frac{C_3}{(1 + |\nu_N^\pm|)^{3/2}} \left( \|\phi\|_{\widetilde{H_{1/2}^{0,\mu}(\Gamma_\pm)}}^2 + \|\mathcal{S}_{\mu^\pm}^{\sigma^\pm} u\|_{\widetilde{H_{1/2}^{0,\mu}(\Gamma_\pm)}}^2 \right) \end{aligned}$$

where the constants  $C_1$ ,  $C_2$  and  $C_3$  are independent of  $N$  and where we used in the second inequality the fact that  $|\nu_k| \rightarrow \infty$  as  $k \rightarrow \infty$ . According to Proposition 3.3 and Lemma 3.2,

$$|\nu_N| \geq CN^2$$

for some constant  $C > 0$  independent from  $N$ . From the discussion above we then can deduce the following theorem.

**Theorem 3.11.** *Under the assumptions as in Proposition 2.7 and the additional assumptions that*

$$\frac{\partial^m J}{\partial z^m} \in L^2(B_{r_*}), \quad \frac{\partial^m \sigma}{\partial z^m} \in L^\infty(B_{r_*}) \text{ and } \frac{\partial^m \mu^{-1}}{\partial z^m} \in L^\infty(B_{r_*}),$$

for  $m = 0, 1, 2$ , there exists a constant  $C$  that only depends on  $J$ ,  $\mu$ ,  $\sigma$ ,  $r_*$  and  $z_*$  such that

$$\|u - u_N\|_{H_{1/2}^1(B_{r_*,z_*})} \leq \frac{C}{N^{3/2}},$$

where  $u \in H_{1/2}^1(B_{r_*,z_*})$  and  $u_N \in H_{1/2}^1(B_{r_*,z_*})$  are the respective solutions of (32) and (38).

We end this section with a two remarks. The first one is on exponential convergence rates and was brought to our attention by one of the referees.

**Remark 3.12.** *If we change the variational formulation into  $u_N \in H_{1/2}^1(B_{r_*,z_*})$  solving*

$$\begin{aligned} &\int_{B_{r_*,z_*}} \frac{1}{\mu} \nabla(r u_N) \cdot \frac{1}{r} \nabla(r \bar{v}) \, dr \, dz - \int_{B_{r_*,z_*}} i \omega \sigma u_N \bar{v} \, dr \, dz \\ &+ \langle \mathcal{T}_N^+ \gamma^+ u_N, \gamma^+ v \rangle + \langle \mathcal{T}_N^- \gamma^- u_N, \gamma^- v \rangle = \int_{B_{r_*,z_*}} i \omega J \bar{v} \, dr \, dz \quad \forall v \in H_{1/2}^1(B_{r_*,z_*}), \end{aligned} \quad (42)$$

then, following the approach in [21] for waveguides (see also [8, Chapter 3]), one can prove convergence at exponential rates by exploiting the exponential decay of the (truncated) analytic expression (33) in the  $z$  direction. However, for this formulation, one is able to prove well-posedness of (42) only for  $N$  sufficiently large.

The second one is related to the case of Dirichlet boundary conditions.

**Remark 3.13.** *The results and proofs of this section apply also to the case where the Neumann boundary conditions on  $r = r_*$  are replaced with Dirichlet boundary conditions. The only modification would be the replacement of  $H_{1/2}^1(B)$  by  $H_{1/2,0}^1(B) := \{u \in H_{1/2}^1(B); u = 0 \text{ on } r = r_*\}$  where  $B$  stands for  $B_{r_*}$  or  $B_{r_*,z_*}$ . The eigenvalues  $\lambda_k$  are in this case*

$$\lambda_k = \left( \frac{\rho_1(t(k))}{r_*} \right)^2$$

where  $\rho_1(t)$  is a positive zero of the cylinder function  $\mathcal{C}_1(\cdot; t)$ .

## 4 Numerical validation

We recall the two-dimensional geometric representation of the eddy current testing procedure in the  $Orz$  plan from Figure 2 or, more precisely, Figure 3a. In the following examples, the two coils involved are represented by two rectangles with  $0.67mm$  in length (radial direction) and  $2mm$  in height (longitudinal direction). They are located  $7.83mm$  away from the  $z$ -axis and have a distance of  $0.5mm$  between them. The SG tube measures  $9.84mm$  in radius for the interior interface and  $11.11mm$  for the exterior interface. We assume some deposit with a rectangular shape on the shell side of the tube with  $2mm$  in length and  $6cm$  in height. The probe coils and the deposit are placed symmetrically with regard to the  $r$ -axis. The permeability and conductivity of different materials are given in Table 1. The background permeability  $\mu_0$  is the permeability of vacuum.

	vacuum	tube	deposit
permeability	$\mu_v = \mu_0$	$\mu_t = 1.01\mu_0$	$\mu_d = 10\mu_0$
conductivity (in $S \cdot m^{-1}$ )	$\sigma_v = 0$	$\sigma_t = 1 \times 10^3$	$\sigma_d = 1 \times 10^4$

Table 1: Values of the physical parameters for the numerical examples.

To approximate solutions to the original eddy current problem (6) on the unbounded domain  $\mathbb{R}_+^2$  by numerical simulations, we use a domain  $B_{R,Z}$  with very large truncation parameters  $R = 300mm$ ,  $Z = 100mm$  and we set Neumann conditions on these boundaries. These values of  $R$  and  $Z$  are large enough to ensure that the corresponding reference solution is close enough to the true solution to be able to study the (non-)convergence of the different domain truncations presented above. All numerical examples are done using the open-source finite element software *FreeFem++*. The computation of the reference solution uses a mesh that is adaptively refined with respect to this solution with a maximum edge size  $h_{\max} = 2mm$  as well as P1 finite elements on the mesh. The degrees of freedom of the finite element space are about 49900 for  $B_{R,Z}$  with  $R = 300mm$  and  $Z = 100mm$ .

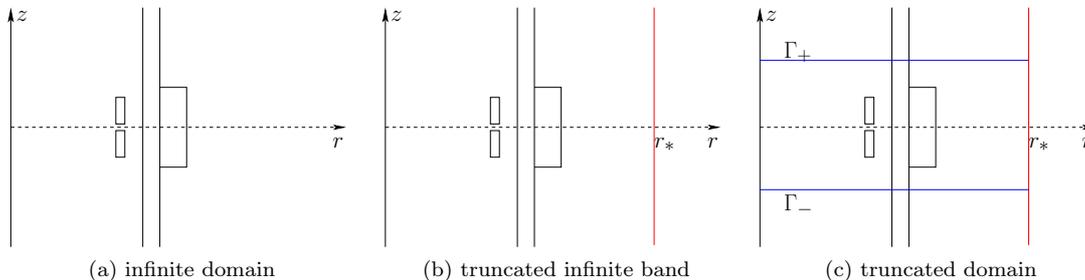


Figure 3: Truncation in the radial and longitudinal directions.

### 4.1 Truncation error in the radial direction

Next we truncate the computational domain much closer to the tube at  $r = r_*$ , see Figure 3b by setting Dirichlet or Neumann boundary conditions on  $\Gamma_{r_*} = \{r = r_*\}$ . Using the same physical parameters as above and setting again  $Z = 100mm$  to approximate solutions to the truncated problem on  $B_{r_*,Z}$ ; again, the value for  $Z$  gave sufficient numerical accuracy in our tests. In Figure 4 we show the numerical results corresponding to the convergence results of Proposition 2.7 and Proposition C.3. As  $r_*$  increases, the relative error of the the Neumann problem (15) tend to zero in the semi-norm  $|\cdot|_{H_{1/2}^1(B_{r_*,Z})}$  and in the norm  $\|\cdot\|_{L^2(B_{r_*,Z})}$  with higher rates than the relative error of the Dirichlet problem (61) does. (Figure 4a). This observation precisely corresponds to our theoretical results (see Remark 2.8). Note that it is reasonable that the convergence rates observed in numerical tests are better than theoretical ones which give just a lower bound of estimate. The advantage of truncating the computational domain in the radial direction using a Neumann instead of a Dirichlet boundary condition is clearly confirmed by these examples.

In eddy current testing, one is interested in particular in measurements of impedances, which only depend on the solution inside the deposit domain  $\Omega_d$ . To this end, we also compare the relative error due to truncation in the radial direction on  $\Omega_d$ . From Figures 4c and 4d, at a truncation position  $r_* = 50mm$  ( $\log_{10} r_* \approx -1.3$ ), the relative error issued from the Neumann problem in the semi-norm of  $H_{1/2}^1(\Omega_d)$  is less than 0.1% and that in the norm of  $L^2(\Omega_d)$  are less than 1%. Therefore we conclude that simulations computed in a domain truncated at  $r = r_* = 50mm$  using Neumann boundary conditions are sufficiently precise for iterative reconstruction algorithms, since the noise level in the measurements would most probably be higher than the numerical error. Concerning the finite element space on  $B_{r_*,Z}$ , this truncation reduced the degrees of freedom in our experiments to about 16000.

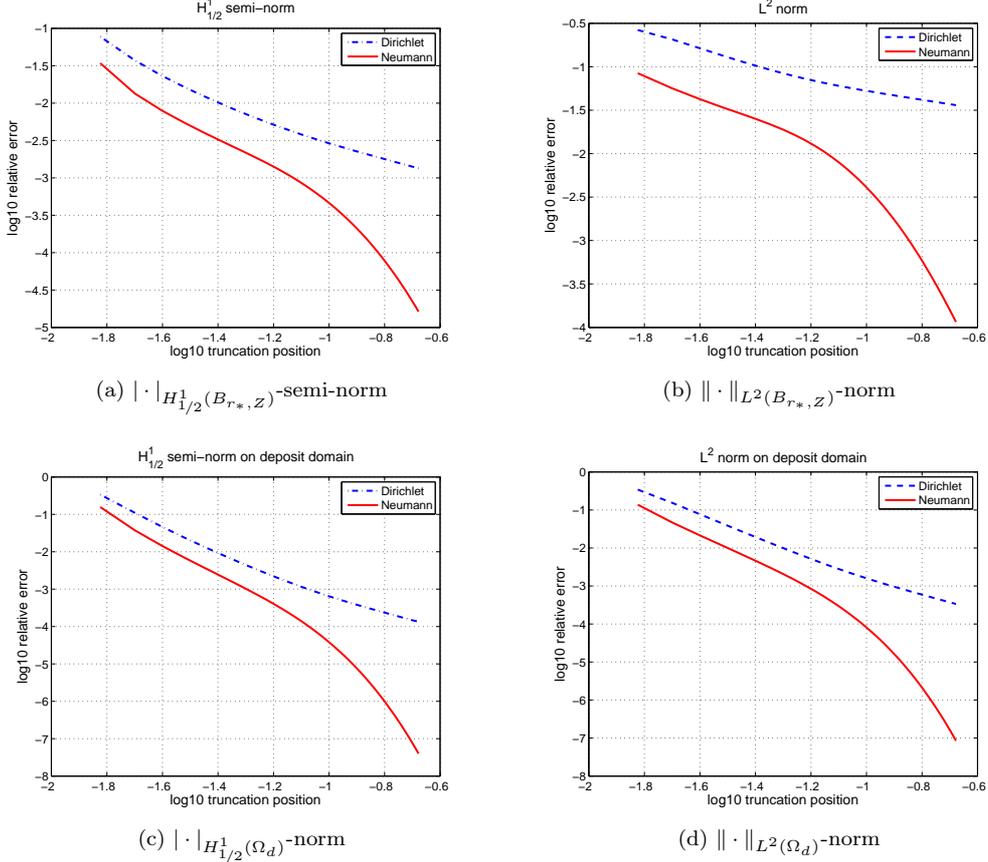


Figure 4: Relative errors with different truncation positions in the radial direction and different boundary conditions. Theoretical convergence rates: with Dirichlet b.c.,  $r^{-1/2}$  in  $H_{1/2}^1$  semi-norm and no convergence in  $L^2$  norm; with Neumann b.c.,  $r^{-1}$  in  $H_{1/2}^1$  semi-norm and  $r^{-1/2}$  in  $L^2$  norm.

## 4.2 Error introduced by the DtN maps

In the following, we denote by  $u_{\text{exact}}$  a reference solution for the eddy current problem computed on the truncated infinite band  $B_{r_*}$  with  $r = r_* = 50mm$  using Neumann boundary conditions on  $\Gamma_{r_*}$ , compare Figure 3b. To compute  $u_{\text{exact}}$  numerically, we resolve the problem in a domain bounded  $B_{r_*,Z}$  with  $Z = 100mm$ , as explained above. Then  $B_{r_*,Z}$  is truncated into the bounded domain  $B_{r_*,z_*}$  with  $\Gamma_{\pm} = \{0 < r < r_*, z = \pm z_* = \pm 5mm\}$ , compare Figure 3c. The degrees of freedom of the P1 finite element space reduced by this truncation to about 3500 elements. We set different boundary conditions – Dirichlet, Neumann or DtN boundary conditions – on the top and bottom boundaries  $\Gamma_{\pm}$  and solve the corresponding variational problems again using the finite element software package *FreeFem++*. The solutions are denoted by  $u_{\text{Dirichlet}}$ ,  $u_{\text{Neumann}}$  and  $u_{\text{DtN}}$  in the following.

To build the DtN maps, we first discretize the interval  $I = (0, r_*)$  (that has the same length as

$\Gamma_{\pm}$ ) into 5000 equi-length segments and use an eigenvalue solver (more precisely, the function `eigs` in *Matlab*) to compute the first eigenvalues  $\{\nu_{0j}^{\pm}, \nu_k^{\pm}\}$  and the corresponding eigenvectors corresponding to the physical parameters  $\mu^{\pm}$  and  $\sigma^{\pm}$ . In our numerical example, we observe that the eigenvalues are simple. This means in particular that no Jordan block is present for this set of eigenvalues since otherwise the function `eigs` would return an eigenvalue with a multiplicity equal to the dimension of the Jordan bloc sub-space (and also return linearly dependent eigenvectors). The eigenprojections  $\{Q_{0j}^{\pm}, Q_k^{\pm}\}$  are approximated by matrices which we obtain by interpolating the complex conjugate of these eigenvectors with the boundary elements on  $\Gamma_{\pm}$  (141 elements on each boundary), using the bi-orthogonality between the eigenvectors of the operator and those of its adjoint. The sum of these matrices weighted with the square root of the corresponding eigenvalues yields numerical approximations of the truncated DtN maps introduced in (37).

Figure 5 illustrates the relative errors of  $u_{\text{DtN}}$  using different truncation parameters  $N$  for the DtN operator  $\mathcal{T}_N^{\pm}$ , see (37), with respect to  $u_{\text{exact}}$  in the  $\|\cdot\|_{L^2_{1/2}(B_{r^*, z^*})}$ -norm and the  $|\cdot|_{H^1_{1/2}(B_{r^*, z^*})}$ -semi-norms. The relative error decreases as the truncation order  $N$  increases before saturating at about  $N = 16$ . For  $N = 20$ , the errors are sufficiently small. Let us note that in the case  $\sigma_t = 0$ , *i.e.* the operator  $\mathcal{S}_{\mu}^{\sigma} = \mathcal{A}_{\mu}$  is selfadjoint, using  $N = 1$  would be sufficient to achieve the same accuracy.

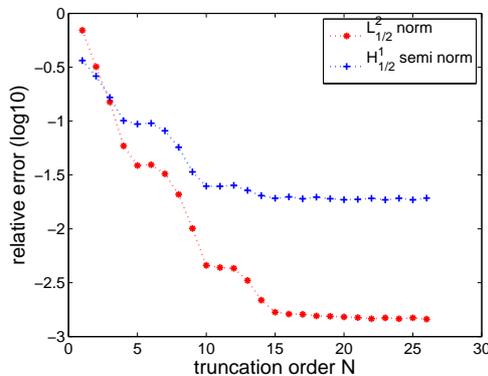


Figure 5: Relative errors for eddy current simulations using DtN maps with different truncation orders  $N$ .

Figure 6 illustrates real and imaginary parts of solutions for the three different horizontal truncation techniques (Dirichlet, Neumann, and DtN) we investigated above. It shows in particular that the truncation using DtN maps constructed with the first 20 eigenvalues and eigenprojections approaches the most the exact model. Moreover, Table 2 indicates the relative errors of the truncated models in the  $\|\cdot\|_{L^2_{1/2}(B_{r^*, z^*})}$ -norm and the  $|\cdot|_{H^1_{1/2}(B_{r^*, z^*})}$ -semi-norm compared to the reference solution. Again, one clearly observes that using the DtN maps for the horizontal truncation introduces a reasonably small error compared to the reference solution  $u_{\text{exact}}$  while truncating using Dirichlet- or Neumann boundary conditions on horizontal boundaries close to the coils and the deposit yields unacceptable errors. In particular, merely pre-computed DtN maps can ensure fast simulations of non-destructive eddy current measurements when many forward problems need to be solved. As mentioned in the introduction, such fast simulations are crucial for, e.g., iterative solution methods for the inversion of these measurements.

b. c.	Dirichlet	Neumann	DtN
$\ \cdot\ _{L^2_{1/2}(B_{r^*, z^*})}$	55.73%	181.57%	0.15%
$ \cdot _{H^1_{1/2}(B_{r^*, z^*})}$	28.79%	49.26%	1.86%

Table 2: Errors of longitudinal domain truncation with different boundary conditions. The DtN maps are truncated at  $N = 20$ .

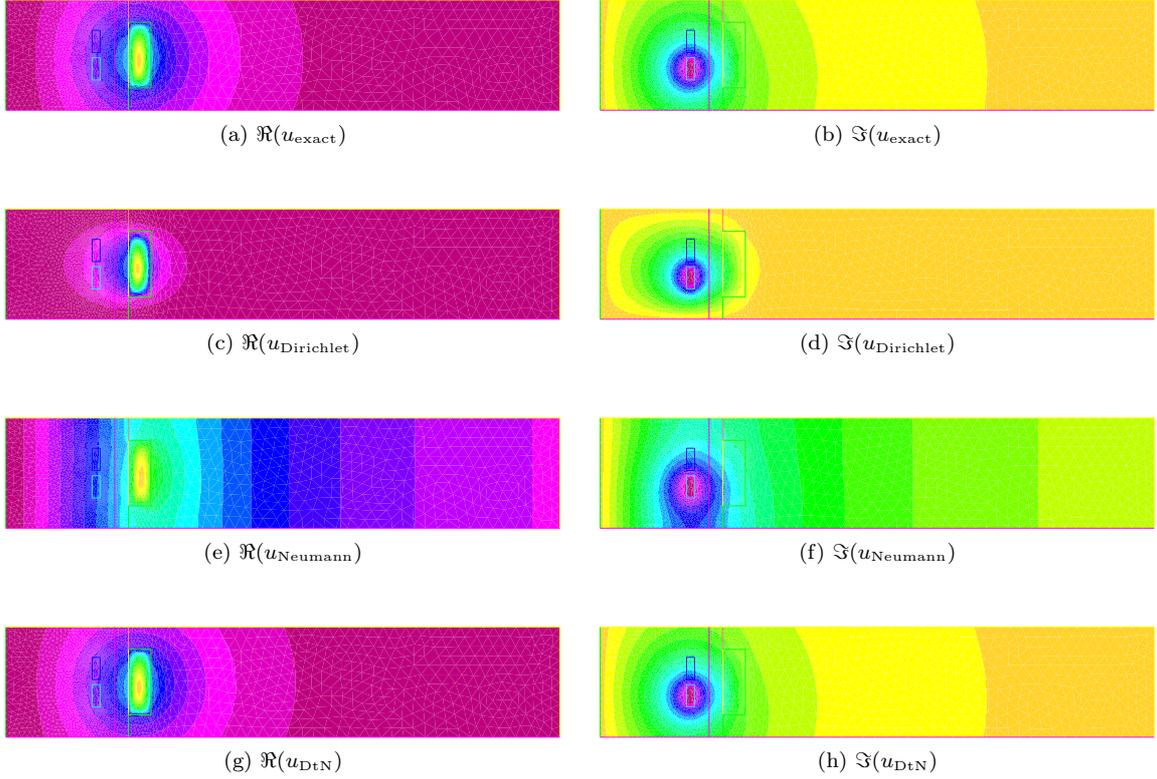


Figure 6: Real and imaginary parts of  $u$  fields of the truncated model with different boundary conditions. DtN maps of truncation order  $N = 20$ .

## A Some properties of the weighted spaces

We recall the definition of the following spaces and corresponding norms. Let  $\Omega \subset \mathbb{R}_+^2 := \{(r, z) : r > 0, z \in \mathbb{R}\}$  an open set. For  $\lambda > 1$ ,

$$L_{1/2, \lambda}^2(\Omega) := \{v : r^{1/2}(1+r^2)^{-\lambda/2}v \in L^2(\Omega)\}, \quad H_{1/2, \lambda}^1(\Omega) := \{v \in L_{1/2, \lambda}^2(\Omega) : r^{-1/2}\nabla(rv) \in L^2(\Omega)\},$$

$$\|v\|_{L_{1/2, \lambda}^2(\Omega)} = \left\| \sqrt{\frac{r}{(1+r^2)^\lambda}} v \right\|_{L^2(\Omega)}, \quad \|v\|_{H_{1/2, \lambda}^1(\Omega)}^2 = \|v\|_{L_{1/2, \lambda}^2(\Omega)}^2 + \left\| r^{-1/2}\nabla(rv) \right\|_{L^2(\Omega)}^2.$$

For  $\lambda = 0$ , we define

$$L_{1/2}^2(\Omega) := L_{1/2, 0}^2(\Omega) = \{v : v\sqrt{r} \in L^2(\Omega)\},$$

$$H_{1/2}^1(\Omega) := H_{1/2, 0}^1(\Omega) = \{v \in L_{1/2}^2(\Omega) : r^{-1}\nabla(rv) \in L^2(\Omega)\}.$$

We shall also use the short notation

$$\|v\|_{H_{1/2}^1(\Omega)}^2 = \left\| r^{-1/2}\nabla(rv) \right\|_{L^2(\Omega)}^2.$$

For  $r_* > 0$  and an interval  $I = \{r \in \mathbb{R} : 0 < r < r_*\}$  we define

$$L_{1/2}^2(I) := \{\phi : \phi\sqrt{r} \in L^2(I)\}, \quad H_{1/2}^1(I) := \{\phi \in L_{1/2}^2(I) : r^{-1}\partial_r(r\phi) \in L^2(I)\}.$$

### A.1 Proof of Lemma 2.1

*Proof.* First of all, we prove that the trace on  $\Gamma_0$  as the limit of traces on  $\Gamma_\epsilon$  exists and its  $L^2$ -norm vanishes. Given  $0 < \epsilon < r_*$ , we set  $B_{r_*}^\epsilon := \{(r, z) \in B_{r_*} : r \geq \epsilon\}$  and  $I^\epsilon := \{r \in \mathbb{R} : \epsilon < r < r_*\}$ .

One easily observes that  $L^2_{1/2,\lambda}(B_{r_*}^\epsilon) = L^2(B_{r_*}^\epsilon)$  and  $H^1_{1/2,\lambda}(B_{r_*}^\epsilon) = H^1(B_{r_*}^\epsilon) \subset L^2(\mathbb{R}, H^1((\epsilon, r_*)$ ). Since  $H^1((\epsilon, r_*)) \subset \mathcal{C}((\epsilon, r_*))$ , for  $0 < \epsilon < r < r' < r_*$  and for almost all  $z \in \mathbb{R}$ , we can write for  $v \in H^1_{1/2,\lambda}(B_{r_*}^\epsilon)$ ,

$$\begin{aligned} |r'v(r', z) - rv(r, z)| &= \left| \int_r^{r'} \frac{\partial}{\partial s}(sv(s, z)) ds \right| \leq |r' - r|^{1/2} \sqrt{r_*} |v(\cdot, z)|_{H^1_{1/2}(I^\epsilon)}, \\ \int_{\mathbb{R}} |r'v(r', z) - rv(r, z)|^2 dz &\leq |r' - r| r_* \int_{\mathbb{R}} |v(\cdot, z)|^2_{H^1_{1/2}(I^\epsilon)} dz \leq |r' - r| r_* |v|^2_{H^1_{1/2}(B_{r_*}^\epsilon)}. \end{aligned}$$

Thus, for  $r_n \rightarrow 0$  ( $n \rightarrow \infty$ ),  $\{r_n v(r_n, \cdot)\}_{n \in \mathbb{N}^*}$  is a Cauchy sequence in  $L^2(\mathbb{R})$ . Since  $L^2(\mathbb{R})$  is complete, the sequence converges and we denote the  $L^2(\mathbb{R})$ -norm of its limit by  $l \geq 0$ . Now we will show that  $l = 0$ . If not, due to the continuity of  $rv$  on  $r$  for almost all  $z$ , one should have

$$\exists \delta > 0 \quad \forall 0 < r < \delta \quad \int_{\mathbb{R}} |rv(r, z)|^2 dz \geq \frac{l^2}{2}.$$

For  $0 < \epsilon < \delta < r_*$ , with Fubini's theorem,

$$\|v\|^2_{L^2_{1/2,\lambda}(B_{r_*}^\epsilon)} \geq \|v\|^2_{L^2_{1/2,\lambda}(B_\delta^\epsilon)} = \int_{\mathbb{R}} \left( \int_\epsilon^\delta \frac{1}{r(1+r^2)^{1/2,\lambda}} |rv(r, z)|^2 dr \right) dz \geq \frac{l^2}{2} \frac{1}{(1+\delta^2)^\lambda} \int_\epsilon^\delta \frac{1}{r} dr \xrightarrow{\epsilon \rightarrow 0} \infty,$$

which contradicts the fact that  $v \in L^2_{1/2,\lambda}(\mathbb{R}_+^2)$ . Hence we have proved that  $l = 0$ . Therefore, for almost all  $z \in \mathbb{R}$  and  $v \in H^1_{1/2,\lambda}(B_{r_*}^\epsilon) \subset L^2(\mathbb{R}, H^1((\epsilon, r_*))$ ,

$$|v(r, z)|^2 = \frac{1}{r^2} |rv|^2 = \frac{1}{r^2} \left| \int_0^r \frac{\partial}{\partial s}(sv(s, z)) ds \right|^2 \leq \int_0^r \left| \frac{1}{\sqrt{s}} \frac{\partial}{\partial s}(sv(s, z)) \right|^2 ds \quad \forall r > 0. \quad (43)$$

For any  $r > 0$ , we integrate the above inequality for  $z$  over  $\mathbb{R}$  and get

$$\|v(r, \cdot)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |v(r, z)|^2 dz \leq \int_{\mathbb{R}} \int_0^r \left| \frac{1}{\sqrt{s}} \frac{\partial}{\partial s}(sv(s, z)) \right|^2 ds dz. \quad (44)$$

We conclude that

$$\lim_{r \rightarrow 0} \|v(r, \cdot)\|^2_{L^2(\mathbb{R})} = 0. \quad (45)$$

Now using (44),

$$\begin{aligned} \int_{\mathbb{R}_+^2} \frac{r}{(1+r^2)^\lambda} |v|^2 dr dz &\leq \int_{-\infty}^{\infty} \left( \int_0^\infty \frac{r}{(1+r^2)^\lambda} dr \int_0^\infty \left| \frac{1}{\sqrt{r}} \frac{\partial}{\partial r}(rv(r, z)) \right|^2 dr \right) dz \\ &= \left( \int_0^\infty \frac{r}{(1+r^2)^\lambda} dr \right) |v|^2_{H^1_{1/2}(\mathbb{R}_+^2)}. \end{aligned}$$

Therefore, the inequality is proved by setting

$$C = \sqrt{1 + \int_0^\infty \frac{r}{(1+r^2)^\lambda} dr}.$$

□

## A.2 Proof of Lemma 2.6

*Proof.* Let  $r \in I = (0, r_*)$ . From inequality (43) in the proof of Lemma 2.1, one has  $|v(r, z)|^2 \leq |v(\cdot, z)|^2_{H^1_{1/2}(I)}$ . We integrate the above inequality for  $r$  over  $I$  and obtain

$$\begin{aligned} \|v(\cdot, z)\|^2_{L^2(I)} &\leq \int_0^{r_*} dr |v(\cdot, z)|^2_{H^1_{1/2}(I)} = r_* |v(\cdot, z)|^2_{H^1_{1/2}(I)}, \\ \|v(\cdot, z)\|^2_{L^2_{1/2}(I)} &\leq \int_0^{r_*} r dr |v(\cdot, z)|^2_{H^1_{1/2}(I)} = \frac{r_*^2}{2} |v(\cdot, z)|^2_{H^1_{1/2}(I)}. \end{aligned}$$

Therefore, integration over  $z \in \mathbb{R}$  yields

$$\begin{aligned} \|v\|_{L^2(B_{r_*})}^2 &= \left\| \|v(\cdot, z)\|_{L^2(I)} \right\|_{L^2(\mathbb{R})}^2 \leq r_* \left\| |v(\cdot, z)|_{H_{1/2}^1(I)}^2 \right\|_{L^2(\mathbb{R})}^2 \leq r_* |v|_{H_{1/2}^1(B_{r_*})}^2, \\ \|v\|_{L_{1/2}^2(B_{r_*})}^2 &= \left\| \|v(\cdot, z)\|_{L_{1/2}^2(I)} \right\|_{L^2(\mathbb{R})}^2 \leq \frac{r_*^2}{2} \left\| |v(\cdot, z)|_{H_{1/2}^1(I)}^2 \right\|_{L^2(\mathbb{R})}^2 \leq \frac{r_*^2}{2} |v|_{H_{1/2}^1(B_{r_*})}^2. \end{aligned}$$

By setting  $C_p = \sqrt{r_*}$  in the first inequality and  $C_p = r_*/\sqrt{2}$  in the second inequality, the Poincaré-type inequalities (16) are proved. We now prove (17). We recall that

$$|v|_{H^{1/2}(\Gamma_{r_*})}^2 = \int_{\mathbb{R}} |\xi| |\hat{v}(r_*, \xi)|^2 d\xi = \frac{1}{r_*} \left( \int_{\mathbb{R}} |\xi| r_* |\hat{v}(r_*, \xi)|^2 d\xi \right).$$

For  $\epsilon > 0$ ,

$$r_* |\hat{v}(r_*, \xi)|^2 - \epsilon |\hat{v}(\epsilon, \xi)|^2 = \int_{\epsilon}^{r_*} \partial_r (r |\hat{v}|^2) dr = \int_{\epsilon}^{r_*} \left( 2\Re(\hat{v} \overline{\partial_r(r\hat{v})}) - |\hat{v}|^2 \right) dr \leq 2 \int_{\epsilon}^{r_*} |\hat{v}| |\partial_r(r\hat{v})| dr.$$

Multiplying the inequality with  $|\xi|$ , integrating for  $|\xi| < M$ , letting first  $\epsilon \rightarrow 0$  (use (45) to observe that the  $\epsilon$  terms in the left hand side goes to zero) and then letting  $M \rightarrow \infty$ , yields

$$|v|_{H^{1/2}(\Gamma_{r_*})}^2 \leq \frac{2}{r_*} \int_{\mathbb{R}} |\xi| \int_0^{r_*} |\hat{v}| |\partial_r(r\hat{v})| dr.$$

Therefore

$$\begin{aligned} |v|_{H^{1/2}(\Gamma_{r_*})}^2 &\frac{2}{r_*} \int_{\mathbb{R}} \int_0^{r_*} |\xi| \left| r^{1/2} \hat{v} \right| \left| r^{-1/2} \partial_r(r\hat{v}) \right| dr d\xi \\ &\leq \frac{1}{r_*} \int_{\mathbb{R}} \int_0^{r_*} \left( |\xi|^2 \left| r^{1/2} \hat{v} \right|^2 + \left| r^{-1/2} \partial_r(r\hat{v}) \right|^2 \right) dr d\xi = \frac{1}{r_*} |v|_{H_{1/2}^1(B_{r_*})}^2. \end{aligned}$$

This proves (17).  $\square$

### A.3 Proof of Lemma 3.1

*Proof.* We suppose  $\mathcal{B}$  is a unit ball in  $H_{1/2}^1(I)$ . To prove the compactness of  $\mathcal{B}$  in  $L_{1/2}^2(I)$ , it is sufficient to show that  $\tilde{\mathcal{B}} := \{\phi(\cdot)\sqrt{\cdot} : \phi \in \mathcal{B}\}$  is compact in  $L^2(I)$ . We use [10, Corollaire IV.26]. We suppose for arbitrary  $\eta > 0$  small enough,  $\omega \subset ]\eta, r_* - \eta[$  is strictly included in  $I$ , written as  $\omega \subset\subset I$ . We note  $\tau_h$  the translation operator:  $(\tau_h\phi)(r) = \phi(r+h)$ .

First of all, we shall show

$$\forall h \in \mathbb{R} \quad \text{with} \quad |h| < \eta \quad \text{and} \quad \forall \psi = \phi(\cdot)\sqrt{\cdot} \in \tilde{\mathcal{B}}, \quad \|\tau_h\psi - \psi\|_{L^2(\omega)} \xrightarrow{h \rightarrow 0} 0.$$

For  $r \in \omega$ , we have  $|h| < \eta < r$ . For  $h > 0$ ,

$$\begin{aligned} |\psi(r+h) - \psi(r)|^2 &= |\phi(r+h)\sqrt{r+h} - \phi(r)\sqrt{r}|^2 \\ &\leq |\phi(r+h)(r+h) - \phi(r)r|^2 \frac{1}{r} + |\phi(r+h)|^2 (\sqrt{r+h} - \sqrt{r})^2 \frac{r+h}{r} \\ &= \left| \int_r^{r+h} \frac{d}{ds}(s\phi(s)) ds \right|^2 \frac{1}{r} + |\phi(r+h)|^2 \left( \frac{h}{\sqrt{r+h} + \sqrt{r}} \right)^2 \frac{r+h}{r} \\ &\leq \frac{h}{r} \int_r^{r+h} \left| \frac{d}{ds}(s\phi(s)) \right|^2 ds + |\phi(r+h)|^2 \left( \frac{h}{2\sqrt{r}} \right)^2 \frac{2r}{r} \\ &\leq 2h |\phi|_{H_{1/2}^1(I)}^2 + \frac{h}{2} |\phi(r+h)|^2, \end{aligned}$$

thus by Lemma 2.6 and the fact that  $\phi \in \mathcal{B}$

$$\|\tau_h \psi - \psi\|_{L^2(\omega)}^2 \leq 2hr_* |\phi|_{H_{1/2}^1(I)}^2 + \frac{h}{2} \|\phi\|_{L^2(I)}^2 \leq 2hr_* |\phi|_{H_{1/2}^1(I)}^2 + \frac{h}{2} r_* |\phi|_{H_{1/2}^1(I)}^2 \leq \frac{5h}{2} r_* \xrightarrow{h \rightarrow 0} 0.$$

For  $h < 0$ , we note always  $h > 0$  but we calculate

$$\begin{aligned} |\psi(r-h) - \psi(r)|^2 &= |\phi(r-h)\sqrt{r-h} - \phi(r)\sqrt{r}|^2 \\ &\leq |\phi(r)r - \phi(r-h)(r-h)|^2 \frac{1}{r} + |\phi(r-h)|^2 (\sqrt{r} - \sqrt{r-h})^2 \frac{r-h}{r} \\ &= \left| \int_{r-h}^r \frac{d}{ds}(s\phi(s)) ds \right|^2 \frac{1}{r} + |\phi(r-h)|^2 \left( \frac{h}{\sqrt{r} + \sqrt{r-h}} \right)^2 \frac{r-h}{r} \\ &\leq \frac{h}{r} \int_{r-h}^r \left| \frac{d}{ds}(s\phi(s)) \right|^2 ds + |\phi(r-h)|^2 \left( \frac{h}{2\sqrt{r-h}} \right)^2 \frac{r-h}{r} \\ &\leq h |\phi|_{H_{1/2}^1(I)}^2 + \frac{h}{4} |\phi(r-h)|^2 \end{aligned}$$

again by Lemma 2.6 we have

$$\|\tau_{-h} \psi - \psi\|_{L^2(\omega)}^2 \leq hr_* |\phi|_{H_{1/2}^1(I)}^2 + \frac{h}{4} r_* |\phi|_{H_{1/2}^1(I)}^2 \leq \frac{5h}{4} r_* \xrightarrow{h \rightarrow 0} 0.$$

It remains to prove that

$$\forall \epsilon > 0 \quad \exists \omega \subset\subset I \quad \text{such that} \quad \|\phi(\cdot)\sqrt{\cdot}\|_{L^2(I \setminus \omega)} < \epsilon \quad \forall \phi \in \mathcal{B}.$$

If we take  $\omega = (\eta, r_* - \eta)$ , then by Lemma 2.6

$$\|\phi(\cdot)\sqrt{\cdot}\|_{L^2(I \setminus \omega)}^2 = \|\phi\|_{L_{1/2}^2((0, \eta))}^2 + \|\phi\|_{L_{1/2}^2((r_* - \eta, r_*))}^2 \leq \frac{\eta^2}{2} |\phi|_{H_{1/2}^1((0, \eta))}^2 + \frac{\eta^2}{2} |\phi|_{H_{1/2}^1((r_* - \eta, r_*))}^2 \leq \frac{\eta^2}{2}.$$

By setting  $\eta$  small enough we obtain the result.

So the conditions of [10, Corollaire IV.26] are satisfied, and  $\mathcal{B}$  is relatively compact in  $L_{1/2}^2(I)$ . The embedding  $H_{1/2}^1(I) \hookrightarrow L_{1/2}^2(I)$  is hence compact.  $\square$

## B Proof of Lemma 3.2

*Proof.* We observe from (25) (after interpreting in the distributional sense) that if a couple  $(\lambda, \phi)$  is an eigenpair of  $\mathcal{A}_\mu$  then

$$-\frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr}(r\phi) \right) = \lambda \phi \quad r \in (0, r_1), (r_1, r_2) \text{ and } (r_2, r_*), \quad (46a)$$

$$[\phi] = \left[ \mu^{-1} \frac{d}{dr}(r\phi) \right] = 0 \quad r = r_1, r_2, \quad (46b)$$

$$\phi(0) = 0 \quad \text{and} \quad \frac{d}{dr}(r\phi) \Big|_{r_*} = 0. \quad (46c)$$

Here  $[\cdot]$  is the jump operator. By setting  $\zeta = \sqrt{\lambda}$ , we rewrite (46a) in the form of a Bessel's equation

$$r^2 \phi'' + r \phi' + (\zeta^2 r^2 - 1) \phi = 0.$$

Using the first condition in (46c), we get  $\phi$  in the following form up to a constant factor

$$\phi(r) = \check{\phi}(\zeta r) := \begin{cases} J_1(\zeta r) & 0 < r < r_1, \\ a\mathcal{C}_1(\zeta r; s) & r_1 < r < r_2, \\ b\mathcal{C}_1(\zeta r; t) & r_2 < r < r_*, \end{cases}$$

where  $J_1(\cdot)$  is the first kind Bessel function of order 1,  $\mathcal{C}_1(\cdot; s)$  is the cylinder function of order 1. In general, a cylinder function is defined as

$$\mathcal{C}_\nu(x; s) := J_\nu(x) \cos(\pi s) + Y_\nu(x) \sin(\pi s), \quad (47)$$

where  $Y_\nu$  is the second kind Bessel function of order  $\nu$ . Due to the second boundary condition in (46c),

$$\zeta = \frac{\rho_0(t)}{r_*}$$

where  $\rho_0(t)$  is a zero of cylinder function  $\mathcal{C}_0(\cdot; t)$ . Finally,  $a, b \in \mathbb{R}$  and  $s, t > 0$  are the four constants to be determined by the four jump conditions (46b) at  $r = r_1$  and  $r = r_2$ . To simplify the notation, we will note for the moment

$$x_1 = \frac{\rho_0(t)}{r_*} r_1, \quad x_2 = \frac{\rho_0(t)}{r_*} r_2.$$

Considering the recurrence relations of cylinder functions [1, (9.1.27)], the jumps conditions (46b) write

$$J_1(x_1) = a\mathcal{C}_1(x_1; s), \quad \frac{1}{\mu_0} J_0(x_1) = \frac{a}{\mu_t} \mathcal{C}_1(x_1; s), \quad (48a)$$

$$a\mathcal{C}_1(x_2; s) = b\mathcal{C}_1(x_2; t), \quad \frac{a}{\mu_t} \mathcal{C}_0(x_2; s) = \frac{b}{\mu_0} \mathcal{C}_0(x_2; t). \quad (48b)$$

Using the definition of cylinder functions (47) and the Wronskians of Bessel functions [1, (9.1.16)], one obtains from (48a)

$$\tan(\pi s) = -\frac{\left(\frac{\mu_t}{\mu_0} - 1\right) J_0(x_1) J_1(x_1)}{\frac{2}{\pi x_1} - \left(\frac{\mu_t}{\mu_0} - 1\right) J_0(x_1) Y_1(x_1)}, \quad (49)$$

and from (48b)

$$\tan(\pi t) = -\frac{\left(\frac{\mu_t}{\mu_0} - 1\right) J_0(x_2) J_1(x_2) + \left(\left(\frac{\mu_t}{\mu_0} - 1\right) J_0(x_2) Y_1(x_2) - \frac{2}{\pi x_2}\right) \tan(\pi s)}{\left(\frac{2}{\pi x_2} + \left(\frac{\mu_t}{\mu_0} - 1\right) Y_0(x_2) J_1(x_2)\right) + \left(\frac{\mu_t}{\mu_0} - 1\right) Y_0(x_2) Y_1(x_2) \tan(\pi s)}. \quad (50)$$

If we denote  $\left(\frac{\mu_t}{\mu_0} - 1\right)$  by  $\delta$ , we get from (49) - (50)

$$\tan(\pi t) = -\frac{\delta \left( J_0(x_2) J_1(x_2) \left( \frac{2}{\pi x_1} - \delta J_0(x_1) Y_1(x_1) \right) + \left( \frac{2}{\pi x_2} - \delta J_0(x_2) Y_1(x_2) \right) J_0(x_1) J_1(x_1) \right)}{\frac{4}{\pi^2 x_1 x_2} + \delta \left( \frac{2}{\pi x_1} Y_0(x_2) J_1(x_2) - \frac{2}{\pi x_2} J_0(x_1) Y_1(x_1) \right) - \delta^2 Y_0(x_2) Y_1(x_2) J_0(x_1) J_1(x_1)}. \quad (51)$$

Hence, if  $\lambda = \zeta^2 = \left(\frac{\rho_0(t)}{r_*}\right)^2$  is an eigenvalue of  $\mathcal{A}_\mu$ , then  $t$  is a zero of the following function

$$\vartheta(t) := \tan(\pi t) + \chi\left(\frac{\rho_0(t)}{r_*}\right) \quad \text{with}$$

$$\chi(\zeta; \delta) := \frac{\delta \left( J_0(\zeta r_2) J_1(\zeta r_2) \left( \frac{2}{\pi \zeta r_1} - \delta J_0(\zeta r_1) Y_1(\zeta r_1) \right) + \left( \frac{2}{\pi \zeta r_2} - \delta J_0(\zeta r_2) Y_1(\zeta r_2) \right) J_0(\zeta r_1) J_1(\zeta r_1) \right)}{\frac{4}{\pi^2 \zeta r_1 \zeta r_2} + \delta \left( \frac{2}{\pi \zeta r_1} Y_0(\zeta r_2) J_1(\zeta r_2) - \frac{2}{\pi \zeta r_2} J_0(\zeta r_1) Y_1(\zeta r_1) \right) - \delta^2 Y_0(\zeta r_2) Y_1(\zeta r_2) J_0(\zeta r_1) J_1(\zeta r_1)}. \quad (52)$$

From [1, Section 9.5], if we fix  $\rho_0(0) = 0$ , then we can write  $\rho_0(t)$  as a continuous and increasing function of the continuous variable  $t$  and we have the McMahon's expansions for large  $t$  (see [1, (9.5.12)]):

$$\rho_0(t) = \beta + \frac{1}{8\beta} + O(\beta^{-3}) \quad (t \rightarrow +\infty), \quad (53)$$

$$\text{where } \beta = \left(t - \frac{1}{4}\right) \pi,$$

Now we shall show that for all  $k \in \mathbb{N}_*$ , there is one and only one zero of  $\vartheta$  in the interval  $(k - \frac{1}{2}, k + \frac{1}{2})$ . Firstly, we note that  $t \mapsto \tan(\pi t)$  is bijective from  $(k - \frac{1}{2}, k + \frac{1}{2})$  to  $\mathbb{R}$ . Therefore the existence can be obtained by the uniform boundedness of the function  $t \mapsto \chi(\frac{\rho_0(t)}{r_*})$  for all  $t > 0$ . Considering the fact that  $\rho_0(\cdot)$  is continuously increasing and (53), we only need to prove that the function  $\chi$  is uniformly bounded for all  $\zeta > 0$ . Secondly, we note that

$$\frac{d}{dt} \tan(\pi t) = \pi \sec^2 t \geq \pi \quad \forall t \in (k - \frac{1}{2}, k + \frac{1}{2}), \forall k \in \mathbb{N}_*.$$

If we can prove

$$\frac{d}{dt} \chi\left(\frac{\rho_0(t)}{r_*}\right) = \chi'\left(\frac{\rho_0(t)}{r_*}; \delta\right) \frac{\rho_0'(t)}{r_*} < \pi \quad \forall t \in (k - \frac{1}{2}, k + \frac{1}{2}), \forall k \in \mathbb{N}_*,$$

then the uniqueness can be concluded from the monotonicity of  $\vartheta(t)$  on these intervals.

Using the McMahon's expansion (53), we get

$$0 < \rho_0'(t) \rightarrow \pi \quad \text{as } t \rightarrow +\infty \quad \text{and therefore} \quad \|\rho_0'(\cdot)\|_\infty < \infty. \quad (54)$$

We rewrite the function  $\chi$  given by (51) in the following form

$$\chi(\zeta; \delta) = \frac{A(\zeta; \delta)}{B(\zeta; \delta)} = \frac{\delta A_1(\zeta) + \delta^2 A_2(\zeta)}{B_0(\zeta) + \delta B_1(\zeta) + \delta^2 B_2(\zeta)}$$

with

$$\begin{aligned} A_1(\zeta) &:= \frac{2}{\pi \zeta r_1} J_0(\zeta r_2) J_1(\zeta r_2) + \frac{2}{\pi \zeta r_2} J_0(\zeta r_1) J_1(\zeta r_1), \\ A_2(\zeta) &:= -J_0(\zeta r_2) J_1(\zeta r_2) J_0(\zeta r_1) Y_1(\zeta r_1) - J_0(\zeta r_2) Y_1(\zeta r_2) J_0(\zeta r_1) J_1(\zeta r_1), \\ B_0(\zeta) &:= \frac{4}{\pi^2 \zeta^2 r_1 r_2}, \\ B_1(\zeta) &:= \frac{2}{\pi \zeta r_1} Y_0(\zeta r_2) J_1(\zeta r_2) - \frac{2}{\pi \zeta r_2} J_0(\zeta r_1) Y_1(\zeta r_1), \\ B_2(\zeta) &:= -Y_0(\zeta r_2) Y_1(\zeta r_2) J_0(\zeta r_1) J_1(\zeta r_1). \end{aligned}$$

We also have

$$\chi'(\zeta; \delta) = \frac{\delta C_1(\zeta) + \delta^2 C_2(\zeta) + \delta^3 C_3(\zeta) + \delta^4 C_4(\zeta)}{B_0^2(\zeta) + 2\delta B_0(\zeta) B_1(\zeta) + \delta^2 (B_1^2(\zeta) + 2B_0(\zeta) B_2(\zeta)) + 2\delta^3 B_1(\zeta) B_2(\zeta) + \delta^4 B_2^2(\zeta)},$$

with

$$\begin{aligned} C_1(\zeta) &:= A_1'(\zeta) B_0(\zeta) - A_1(\zeta) B_0'(\zeta), \\ C_2(\zeta) &:= A_2'(\zeta) B_0(\zeta) - A_2(\zeta) B_0'(\zeta) + A_1'(\zeta) B_1(\zeta) - A_1(\zeta) B_1'(\zeta), \\ C_3(\zeta) &:= A_2'(\zeta) B_1(\zeta) - A_2(\zeta) B_1'(\zeta) + A_2'(\zeta) B_1(\zeta) - A_2(\zeta) B_1'(\zeta), \\ C_4(\zeta) &:= A_2'(\zeta) B_2(\zeta) - A_2(\zeta) B_2'(\zeta). \end{aligned}$$

We calculate the derivatives of these functions using the properties of Bessel functions and get

$$\begin{aligned} A_1'(\zeta) &= -\frac{4}{\pi \zeta^2 r_1} J_0(\zeta r_2) J_1(\zeta r_2) - \frac{2r_2}{\pi \zeta r_1} (J_1^2(\zeta r_2) + J_0^2(\zeta r_2)) \\ &\quad - \frac{4}{\pi \zeta^2 r_2} J_0(\zeta r_1) J_1(\zeta r_1) - \frac{2r_1}{\pi \zeta r_2} (J_1^2(\zeta r_1) + J_0^2(\zeta r_1)), \\ A_2'(\zeta) &= r_2 \left( \left( J_1^2(\zeta r_2) - J_0^2(\zeta r_2) \right) J_0(\zeta r_1) Y_1(\zeta r_1) + \left( J_1(\zeta r_2) Y_1(\zeta r_2) - J_0(\zeta r_2) Y_0(\zeta r_2) \right) J_0(\zeta r_1) J_1(\zeta r_1) \right) \\ &\quad + r_1 \left( \left( J_1^2(\zeta r_1) - J_0^2(\zeta r_1) \right) J_0(\zeta r_2) Y_1(\zeta r_2) + \left( J_1(\zeta r_1) Y_1(\zeta r_1) - J_0(\zeta r_1) Y_0(\zeta r_1) \right) J_0(\zeta r_2) J_1(\zeta r_2) \right) \\ &\quad + \frac{2}{\zeta} \left( J_0(\zeta r_2) J_1(\zeta r_2) J_0(\zeta r_1) Y_1(\zeta r_1) + J_0(\zeta r_2) Y_1(\zeta r_2) J_0(\zeta r_1) J_1(\zeta r_1) \right), \end{aligned}$$

$$B'_0(\zeta) = -\frac{8}{\pi^2 \zeta^3 r_1 r_2},$$

$$B'_1(\zeta) = -\frac{4}{\pi \zeta^2 r_1} Y_0(\zeta r_2) J_1(\zeta r_2) + \frac{2r_2}{\pi \zeta r_1} \left( Y_0(\zeta r_2) J_0(\zeta r_2) - Y_1(\zeta r_2) J_1(\zeta r_2) \right) \\ + \frac{4}{\pi \zeta^2 r_2} J_0(\zeta r_1) Y_1(\zeta r_1) + \frac{2r_1}{\pi \zeta r_2} \left( Y_1(\zeta r_1) J_1(\zeta r_1) - Y_0(\zeta r_1) J_0(\zeta r_1) \right),$$

$$B'_2(\zeta) = r_2 \left( Y_1^2(\zeta r_2) - Y_0^2(\zeta r_2) \right) J_0(\zeta r_1) J_1(\zeta r_1) + r_1 \left( J_1^2(\zeta r_1) - J_0^2(\zeta r_1) \right) Y_0(\zeta r_2) Y_1(\zeta r_2) \\ + \frac{2}{\zeta} Y_0(\zeta r_2) Y_1(\zeta r_2) J_0(\zeta r_1) J_1(\zeta r_1),$$

With the limiting forms of Bessel functions for small arguments [1, (9.1.7)-(9.1.9)], we get as  $\zeta \rightarrow 0$ ,

$$\begin{aligned} A_1(\zeta) &\rightarrow \frac{1}{\pi} \left( \frac{r_2}{r_1} + \frac{r_1}{r_2} \right), & A'_1(\zeta) &\sim -\frac{4}{\pi \zeta} \left( \frac{r_2}{r_1} + \frac{r_1}{r_2} \right), \\ A_2(\zeta) &\rightarrow \frac{1}{\pi} \left( \frac{r_2}{r_1} + \frac{r_1}{r_2} \right), & A'_2(\zeta) &\sim -\frac{2r_1 r_2}{\pi} \zeta \ln(\zeta), \\ B_0(\zeta) &= \frac{4}{\pi^2 \zeta^2 r_1 r_2}, & B'_0(\zeta) &= -\frac{8}{\pi^2 \zeta^3 r_1 r_2}, \\ B_1(\zeta) &\sim \frac{2}{\pi^2 \zeta^2 r_1 r_2}, & B'_1(\zeta) &\sim -\frac{8}{\pi^2 \zeta^3 r_1 r_2}, \\ B_2(\zeta) &\sim \frac{2r_1}{\pi^2 r_2} \ln(\zeta), & B'_2(\zeta) &\sim \frac{2r_1}{\pi^2 r_2 \zeta}. \end{aligned}$$

Hence, as  $\zeta \rightarrow 0$

$$C_1 \sim \frac{8(r_1^2 + r_2^2)}{\pi^3 \zeta^3 r_1^2 r_2^2}, \quad C_2 \sim \frac{8(r_1^2 + r_2^2)}{\pi^3 \zeta^3 r_1^2 r_2^2}, \quad C_3 \sim \frac{8(r_1^2 + r_2^2)}{\pi^3 \zeta^3 r_1^2 r_2^2}, \quad C_4 \sim -\frac{r_1^2 + r_2^2}{\pi^3 r_2^2 \zeta}.$$

Then we have

$$\lim_{\zeta \rightarrow 0} \chi(\zeta; \delta) = 0, \tag{55}$$

$$\lim_{\zeta \rightarrow 0} \chi'(\zeta; \delta) = 0. \tag{56}$$

Using the principal asymptotic forms of Bessel functions for large arguments [1, (9.2.1)-(9.2.2)], one has as  $\zeta \rightarrow +\infty$ ,

$$\begin{aligned} A_1(\zeta) &\sim -\frac{2}{\pi^2 \zeta^2 r_1 r_2} \left( \cos(2\zeta r_1) + \cos(2\zeta r_2) \right), \\ A_2(\zeta) &\sim -\frac{1}{\pi^2 \zeta^2 r_1 r_2} \left( \cos(2\zeta r_1) + \cos(2\zeta r_2) + \sin(2\zeta(r_1 + r_2)) \right), \\ B_1(\zeta) &\sim \frac{2}{\pi^2 \zeta^2 r_1 r_2} \left( 2 + \sin(2\zeta r_1) - \sin(2\zeta r_2) \right), \\ B_2(\zeta) &\sim \frac{1}{\pi^2 \zeta^2 r_1 r_2} \cos(2\zeta r_1) \cos(2\zeta r_2), \end{aligned}$$

and

$$\begin{aligned}
A'_1(\zeta) &\sim -\frac{4}{\pi^2\zeta^2 r_1 r_2} (r_1 + r_2), \\
A'_2(\zeta) &\sim \frac{2}{\pi^2\zeta^2 r_1 r_2} \left( r_1 \sin(2\zeta r_1) + r_2 \sin(2\zeta r_2) - (r_1 + r_2) \cos(2\zeta(r_1 + r_2)) \right), \\
B'_1(\zeta) &\sim \frac{4}{\pi^2\zeta^2 r_1 r_2} \left( r_1 \cos(2\zeta r_1) - r_2 \cos(2\zeta r_2) \right), \\
B'_2(\zeta) &\sim -\frac{2}{\pi^2\zeta^2 r_1 r_2} \left( r_1 \sin(2\zeta r_1) \cos(2\zeta r_2) + r_2 \cos(2\zeta r_1) \sin(2\zeta r_2) \right).
\end{aligned}$$

Therefore, as  $\zeta \rightarrow +\infty$ ,

$$\begin{aligned}
C_1(\zeta) &\sim -\frac{16}{\pi^4\zeta^4 r_1^2 r_2^2} (r_1 + r_2), & C_2(\zeta) &\sim \frac{8}{\pi^4\zeta^4 r_1^2 r_2^2} D_2(\zeta; r_1, r_2), \\
C_3(\zeta) &\sim \frac{4}{\pi^4\zeta^4 r_1^2 r_2^2} D_3(\zeta; r_1, r_2), & C_4(\zeta) &\sim \frac{2}{\pi^4\zeta^4 r_1^2 r_2^2} D_4(\zeta; r_1, r_2).
\end{aligned}$$

where

$$\begin{aligned}
D_2(\zeta; r_1, r_2) &:= r_1 \sin(2\zeta r_1) + r_2 \sin(2\zeta r_2) - (r_1 + r_2) \cos(2\zeta(r_1 + r_2)) \\
&\quad - (r_1 + r_2) (2 + \sin(2\zeta r_1) - \sin(2\zeta r_2)) \\
&\quad + (\cos(2\zeta r_1) + \cos(2\zeta r_2)) (r_1 \cos(2\zeta r_1) - r_2 \cos(2\zeta r_2)),
\end{aligned}$$

$$\begin{aligned}
D_3(\zeta; r_1, r_2) &:= (r_1 \sin(2\zeta r_1) + r_2 \sin(2\zeta r_2) - (r_1 + r_2) \cos(2\zeta(r_1 + r_2))) (2 + \sin(2\zeta r_1) - \sin(2\zeta r_2)) \\
&\quad + (\cos(2\zeta r_1) + \cos(2\zeta r_2) + \sin(2\zeta(r_1 + r_2))) (r_1 \cos(2\zeta r_1) - r_2 \cos(2\zeta r_2)) \\
&\quad - (r_1 + r_2) \cos(2\zeta r_1) \cos(2\zeta r_2) \\
&\quad - (\cos(2\zeta r_1) + \cos(2\zeta r_2)) (r_1 \sin(2\zeta r_1) \cos(2\zeta r_2) + r_2 \cos(2\zeta r_1) \sin(2\zeta r_2)),
\end{aligned}$$

$$\begin{aligned}
D_4(\zeta; r_1, r_2) &:= (r_1 \sin(2\zeta r_1) + r_2 \sin(2\zeta r_2) - (r_1 + r_2) \cos(2\zeta(r_1 + r_2))) \cos(2\zeta r_1) \cos(2\zeta r_2) \\
&\quad - (\cos(2\zeta r_1) + \cos(2\zeta r_2) + \sin(2\zeta(r_1 + r_2))) (r_1 \sin(2\zeta r_1) \cos(2\zeta r_2) + r_2 \cos(2\zeta r_1) \sin(2\zeta r_2)).
\end{aligned}$$

Therefore,

$$\|D_2(\cdot; r_1, r_2)\|_\infty \leq 8(r_1 + r_2), \quad \|D_3(\cdot; r_1, r_2)\|_\infty \leq 10(r_1 + r_2), \quad \|D_4(\cdot; r_1, r_2)\|_\infty \leq 6(r_1 + r_2).$$

We have

$$|\chi(\zeta; \delta)| \leq \frac{4|\delta| + 3|\delta|^2}{4 - 8|\delta| - |\delta|^2} \quad \text{as } \zeta \rightarrow \infty, \tag{57}$$

$$|\chi'(\zeta; \delta)| \leq \frac{4|\delta| + 16|\delta|^2 + 10|\delta|^3 + 3|\delta|^4}{4 - 16|\delta| - 2|\delta|^2 - 4|\delta|^3} (r_1 + r_2) \quad \text{as } \zeta \rightarrow \infty, \tag{58}$$

Take for instance  $\delta_1 = \frac{1}{5}$ . From (55) and (57),  $\chi(\zeta; \delta)$  is uniformly bounded for  $\zeta > 0$  and there exists  $Z_1 > 0$  such that

$$|\chi(\zeta; \delta)| < 2 \frac{4\delta_1 + 3\delta_1^2}{4 - 8\delta_1 - \delta_1^2} < 1 \quad \forall \zeta > Z_1 \quad \forall |\delta| < \delta_1.$$

Therefore, we get the existence of a zero of  $\vartheta(t)$  on  $(k - \frac{1}{4}, k + \frac{1}{4})$ .

Let  $\delta_2 = \frac{1}{8}$ . From (56) and (58),  $\chi'(\zeta; \delta)$  is uniformly bounded for  $\zeta > 0$  and there exists  $Z_2 > 0$  such that

$$|\chi'(\zeta; \delta)| < 2 \frac{4|\delta_2| + 16\delta_2^2 + 10\delta_2^3 + 3\delta_2^4}{4 - 16\delta_2 - 2\delta_2^2 - 4\delta_2^3} (r_1 + r_2) < \frac{1}{2} (r_1 + r_2) \quad \forall \zeta > Z_2 \quad \forall |\delta| < \delta_2.$$

By (54) and the fact that  $r_1, r_2 < r_*$ , we have  $|\frac{d}{dt}\chi(\frac{\rho_0(t)}{r_*})| < \pi$ . So we conclude that there is a unique zero of  $\vartheta(t)$  on  $(k - \frac{1}{2}, k + \frac{1}{2})$ .

Without losing the generality, we denote by  $\lambda_k = (\frac{\rho_0(t_k)}{r_*})^2$  the  $k^{\text{th}}$  eigenvalue of  $\mathcal{A}_\mu$  with  $t_k \in (k - \frac{1}{2}, k + \frac{1}{2})$ . For  $|\delta| < \min\{\delta_1, \delta_2\} = \frac{1}{8}$ , we have from the above discussion that there exists  $K > 0$  such that

$$t_k \in (k - \frac{1}{4}, k + \frac{1}{4}) \quad \forall k > K.$$

Using McMahon's expansion (53), one gets that  $\lambda_k$  grows like  $O(k^2)$ ,

$$\lambda_k \sim \frac{\pi^2}{r_*^2} k^2 \quad \text{as } k \rightarrow \infty.$$

Moreover,

$$\begin{aligned} \lambda_{k+1} - \lambda_k &= \frac{1}{r_*^2} \left( \rho_0^2(t_{k+1}) - \rho_0^2(t_k) \right) \\ &> \frac{1}{r_*^2} \left( \left( k + 1 - \frac{1}{4} - \frac{1}{4} \right)^2 \pi^2 - \left( k + \frac{1}{4} - \frac{1}{4} \right)^2 \pi^2 + O(k^{-1}) \right) \\ &= \frac{1}{r_*^2} \left( \frac{1}{2} \left( 2k - \frac{1}{2} \right) \pi^2 + O(k^{-1}) \right) \rightarrow +\infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

We conclude that the difference  $\lambda_{k+1} - \lambda_k \rightarrow +\infty$  as  $k \rightarrow \infty$ . □

## C Dirichlet boundary conditions for radial domain truncation

**Definition C.1.** We build a lifting operator  $\mathcal{R}_{r_*} : H^{1/2}(\mathbb{R}) \rightarrow H_{1/2}^1(B_{r_0})$  such that its Fourier transform satisfies

$$\widehat{(\mathcal{R}_{r_*}\phi)}(r, \xi) = \frac{I_1(2\pi|\xi|r)}{I_1(2\pi|\xi|r_*)} \hat{\phi}(\xi),$$

where  $I_1$  is the modified Bessel function.

We verify easily that  $(\mathcal{R}_{r_*}\phi)|_{r=0} = 0$ ,  $(\mathcal{R}_{r_*}\phi)|_{r=r_*} = \phi$  and  $-\text{div}(r^{-1}\nabla(r\mathcal{R}_{r_*}\phi)) = 0$  in  $B_{r_*}$ . By multiplying the previous divergence-free condition with  $r\overline{\mathcal{R}_{r_*}\phi}$  and integrating by parts, we get

$$\begin{aligned} \|r^{-1}\nabla(r\mathcal{R}_{r_*}\phi)\|_{L_{1/2}^2(B_{r_*})}^2 &= \int_{\mathbb{R}} \left( \frac{\partial}{\partial r}(r\mathcal{R}_{r_*}\phi) \right) (r_*, z) \bar{\phi}(z) dz = \int_{\mathbb{R}} 2\pi r_* \frac{I_0(2\pi|\xi|r_*)}{I_1(2\pi|\xi|r_*)} |\xi| |\hat{\phi}(\xi)|^2 d\xi \\ &\leq 2\pi r_* \frac{I_0(2\pi|\xi|r_*)}{I_1(2\pi|\xi|r_*)} \int_{\mathbb{R}} (1 + |\xi|^2)^{1/2} |\hat{\phi}(\xi)|^2 d\xi = 2\pi r_* \frac{I_0(2\pi|\xi|r_*)}{I_1(2\pi|\xi|r_*)} \|\phi\|_{H^{1/2}(\mathbb{R})}^2. \end{aligned}$$

This implies  $\mathcal{R}_{r_*}\phi \in H_{1/2}^1(B_{r_*})$ , and

$$\|r^{-1}\nabla(r\mathcal{R}_{r_*}\phi)\|_{L_{1/2}^2(B_{r_*})}^2 \leq C(r_*) \|\phi\|_{H^{1/2}(\mathbb{R})}^2 \quad \text{with } C(r_*) = \sqrt{2\pi r_* \frac{I_0(2\pi|\xi|r_*)}{I_1(2\pi|\xi|r_*)}}. \quad (59)$$

Considering the asymptotic behavior of  $I_0, I_1$  with big argument, we have

$$C(r_*) \sim O(\sqrt{r_*}) \quad r_* \rightarrow \infty. \quad (60)$$

So the lifting operator  $\mathcal{R}_{r_*}$  grows with a rate of  $(r_*)^{1/2}$  when  $r_*$  tends to infinity.

**Remark C.2.** Lemma 2.6 and (60) show that the lifting  $\mathcal{R}_{r_*}$  is "minimal" in the sense that its norm grows with the least rate, i.e. as  $(r_*)^{1/2}$  when  $r_*$  tends to infinity.

The problem with Dirichlet boundary condition on  $\Gamma_{r_*} = \{r = r_*\}$  reads

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\mu r}\nabla(ru_d)\right) - i\omega\sigma u_d = i\omega J & \text{in } B_{r_*}, \\ u_d = 0, & \text{on } \Gamma_{r_*}. \end{cases} \quad (61)$$

**Proposition C.3.** *Let  $r_* > 0$  be sufficiently large so that the support of the source term  $J \in L^2(\mathbb{R}_+^2)$  is included in  $B_{r_*}$ . Then problem (61) has a unique solution  $u_d \in H_{1/2,0}^1(B_{r_*})$ . Assume in addition that there exists positive  $r_0 < r_*$  such that the source  $J$  and the conductivity  $\sigma$  vanish and the permeability  $\mu$  is constant for  $r > r_0$ . Then there exists a constant  $C$  that depends only on  $J$ ,  $r_0$ ,  $\mu$  and  $\sigma$  such that*

$$\left\| \frac{1}{r}\nabla(r(u_d - u)) \right\|_{L_{1/2}^2(B_{r_*})} \leq C/r_*^{1/2},$$

where  $u$  is the solution of (6) (in Proposition 2.2).

*Proof.* The proof of the first part is similar to the proof of Proposition 2.2 thanks to Lemma 2.6. Let us set  $w_d = u - u_d$ . Then  $w_d \in H_{1/2}^1(B_{r_*})$  and satisfies

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\mu r}\nabla(rw_d)\right) - i\omega\sigma w_d = 0 & \text{in } B_{r_*}, \\ w_d = u, & \text{on } \Gamma_{r_*}, \end{cases} \quad (62)$$

Using the lifting operator  $\mathcal{R}_{r_*}$ , then  $\tilde{w}_d := w_d - \mathcal{R}_{r_*}(u|_{\Gamma_{r_*}})$  satisfies

$$\int_{B_{r_*}} \frac{1}{\mu r} \nabla(r\tilde{w}_d) \cdot \nabla(r\bar{v}) - i\omega\sigma\tilde{w}_d\bar{v} \, dr \, dz \quad (63)$$

$$= \int_{B_{r_*}} \frac{1}{\mu r} \nabla(r\mathcal{R}_{r_*}(u|_{r_*})) \cdot \nabla(r\bar{v}) - i\omega\sigma\mathcal{R}_{r_*}(u|_{r_*})\bar{v} \, dr \, dz \quad \forall v \in H_{1/2,0}^1(B_{r_*}). \quad (64)$$

A similar argument to the proof of Proposition 2.2 thanks to Lemma 2.6 yields the the existence and uniqueness of  $\tilde{w}_d \in H_{1/2,0}^1(B_{r_*})$ , thus the existence and uniqueness of the solution  $w_d \in H_{1/2}^1(B_{r_*})$ . By taking  $v = \tilde{w}_d$  in the variational formulation (63), we get the following estimates

$$\begin{aligned} \frac{1}{\|\mu\|_\infty} \left\| \frac{1}{r}\nabla(r\tilde{w}_d) \right\|_{L_{1/2}^2(B_{r_*})}^2 &\leq \left| \int_{B_{r_*}} \frac{1}{\mu r} |\nabla(r\tilde{w}_d)|^2 - i\omega\sigma|\tilde{w}_d|^2 \, dr \, dz \right| \\ &= \left| \int_{B_{r_*}} \frac{1}{\mu r} \nabla(r\mathcal{R}_{r_*}(u|_{r_*})) \cdot \nabla(r\bar{\tilde{w}}_d) - i\omega\sigma\mathcal{R}_{r_*}(u|_{r_*})\bar{\tilde{w}}_d \, dr \, dz \right| \\ &\leq \frac{1}{\inf|\mu|} \left\| \frac{1}{r}\nabla(r\mathcal{R}_{r_*}(u|_{r_*})) \right\|_{L_{1/2}^2(B_{r_*})} \left\| \frac{1}{r}\nabla(r\tilde{w}_d) \right\|_{L_{1/2}^2(B_{r_*})} + \omega\|\sigma\|_\infty \|\mathcal{R}_{r_*}(u|_{r_*})\|_{L_{1/2}^2(B_{r_0})} \|\tilde{w}_d\|_{L_{1/2}^2(B_{r_0})}. \end{aligned}$$

The last inequality is due to the fact that  $\sigma$  vanishes for  $r > r_0$ . Thanks to the second Poincaré-type inequality (16), we have

$$\|\tilde{w}_d\|_{L_{1/2}^2(B_{r_0})} \leq \frac{r_0}{\sqrt{2}} \left\| \frac{1}{r}\nabla(r\tilde{w}_d) \right\|_{L_{1/2}^2(B_{r_0})} \leq \frac{r_0}{\sqrt{2}} \left\| \frac{1}{r}\nabla(r\tilde{w}_d) \right\|_{L_{1/2}^2(B_{r_*})}.$$

Thus

$$\|r^{-1}\nabla(r\tilde{w}_d)\|_{L_{1/2}^2(B_{r_*})} \leq C(r_0, \mu, \sigma) \|r^{-1}\nabla(r\mathcal{R}_{r_*}(u|_{r_*}))\|_{L_{1/2}^2(B_{r_*})}.$$

Considering (59), (60) and Proposition 2.4, we have

$$\begin{aligned} \|r^{-1}\nabla(rw_d)\|_{L_{1/2}^2(B_{r_*})} &= \|r^{-1}\nabla(r(\tilde{w}_d + \mathcal{R}_{r_*}u|_{r_*}))\|_{L_{1/2}^2(B_{r_*})} \\ &\leq (1 + C(r_0, \mu, \sigma)) \|r^{-1}\nabla(r\mathcal{R}_{r_*}u|_{r_*})\|_{L_{1/2}^2(B_{r_*})} \leq (1 + C(r_0, \mu, \sigma))C(r_*) \|u|_{r_*}\|_{H^{1/2}(\mathbb{R})} \\ &\leq C/r_*^{1/2}, \end{aligned}$$

where  $C$  depends only on  $r_0, J, \mu, \sigma$ . □

**Remark C.4.** Considering the first Poincaré-type inequality (16) with  $C_p = \sqrt{r_*}$ , we do not have the convergence of  $u_d$  to  $u$  in  $L^2(B_{r_*})$  as  $r_* \rightarrow \infty$ .

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