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Cahier n° 2014-22

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October 1, 2014

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This paper studies the existence of some known equilibrium solution concepts in a large class of economic models with discontinuous payoff functions. The issue is well understood for Nash equilibria, thanks to Reny's better-reply security condition [34], and its recent improvements [3, 25, 35, 36]. We propose new approaches, related to Reny's work, and obtain tight conditions for the existence of an approximate equilibrium and of a sharing rule solution in pure and mixed strategies (Simon and Zame [38]). As byproducts, we prove that many auction games with correlated types admit an approximate equilibrium, and that in any general equilibrium model with discontinuous preferences, there is a sharing rule solution.

Keywords: Discontinuous games, better-reply security, sharing rules, approximate equilibrium, Reny equilibrium, strategic approximation, auctions, timing games, exchange economy.

1. INTRODUCTION

Many economic interactions are modeled as games with discontinuous payoff functions. For example, in timing games, price and spatial competitions, auctions, bargaining, preemption games or wars of attrition, discontinuities occur when firms choose the same price, location, bid or acting time.

Once the model is fixed, a solution concept should be used to analyse the problem. The stronger the epistemic, experimental, computational and behavioral foundations of a solution, the higher its predictive power. But, to be useful, a solution must exist. The objective of this paper is to extend and link conditions under which some well known solutions exist, namely Nash, approximate and sharing rule equilibria.

Nash equilibrium is the most popular solution in economics and beyond. It is a strategy profile where each agent is reacting optimally to other players' plans. Mathematically, it is

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*We thank Panayotis Mertikopoulos and Christina Pawlowitsch for their useful comments. We thank the anonymous reviewers and the associated editor for many insightful comments and suggestions. We thank many participants of the following seminars or conferences for their valuable comments: Conference on Economic Theory, June 2012 (Cowles, Yale), International Conference on Game Theory, July 2012 (Stony Brook), SAET conference, July 2012 (Brisbane), Workshop on Dynamic Game Theory, December 2012 (Valparaiso), Roy-Adres Seminar, September 2011 (Paris), CSEF Seminar, November 2011 (Napoli). This research was supported by grant ANR-10-BLAN 0112 (Laraki).

a fixed point of the best-response correspondence. When the game is continuous, the fixed point technique works well (thanks to Brouwer's and Kakutani's theorems) and leads to the famous Nash-Glicksberg theorem [13, 26, 27]. In discontinuous games, the fixed point approach cannot be directly applied because a player may have no optimal reply or because his best choice jumps as a function of the choices of the other players.

A natural issue then is to identify regularity conditions on payoffs, which combined with a limited form of quasiconcavity of the utility functions, guarantee the existence of a Nash equilibrium. The first existence conditions are given by the seminal papers of Dasgupta and Maskin [22, 9]. The significant breakthrough extending the previous results[1] is achieved by Reny [34] via the better-reply security approach.

Quoting Reny, *"A game is better-reply secure if for every nonequilibrium strategy x^* and every payoff vector limit u^* resulting from strategies approaching x^*, some player i has a strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from x^*"*.

Reny's paper generated a large and still extremely active research agenda. For instance, Barelli and Meneghel [3] and McLennan et al. [25] proposed relaxations that cover non-transitive and non-quasiconcave preferences. Reny [35, 36] proposed new refinements for games in mixed strategies using a strategic approximation methodology. In practice, many discontinuous models cited above are better-reply secure or admit a strategic approximation, and consequently admit a Nash equilibrium. Recently, Barelli et al. [2] applied Reny's better-reply security and strategic approximation techniques to prove existence of the value in a large class of zero-sum games including the Colonel Blotto game.

But what if a Nash equilibrium does not exist? How can the analyst predict the outcome? Two related relaxations of Nash equilibrium have been analysed and discussed in the literature: endogenous sharing rules and approximate equilibria.

In many discontinuous games, the *exogenously* given tie-breaking rule leads to games without pure Nash equilibria (e.g. asymmetric Bertrand duopoly, Hotelling location game) or without mixed Nash equilibria (e.g. 3-player preemption games [20], auctions with correlated types or values [11, 16]). However, the existence of a Nash equilibrium is restored if the tie-breaking rule is chosen *endogenously* [1, 23, 38]. For example, in an asymmetric Bertrand duopoly, a pure Nash equilibrium exists if ties are broken in favor of the lower-cost firm. In first-price auctions with complete information, a pure Nash equilibrium exists if ties are broken in favor of the firm with the highest value. Under mild topological conditions, Simon and Zame [38] proved that to any game, one could associate an auxiliary game that admits a Nash equilibrium in *mixed strategies* and where payoffs only differ at discontinuity points (see Section 2 for a formal definition). Jackson et al. [17] remark that their "*results concern only the existence of solutions [sharing rule] in mixed strategies*" and that they "*have little to say about the existence*

[1] Carmona [4] gives an extension of Dasgupta's and Maskin's results, which is unrelated to Reny's approach.

of solutions in pure strategies".

We prove existence of a sharing rule solution in pure strategies in every quasiconcave and compact game. Technically, the proof requires the introduction of a new concept –which we call Reny equilibrium–. This answers positively to the open question of Jackson et al. [17].

An alternative solution for games without a Nash equilibrium is the notion of approximate equilibrium. It is a limit strategy profile x^* and a limit payoff vector u^* of ε-Nash equilibria x^ε with associated payoff vector $u(x^\varepsilon)$, as ε goes to 0. Quoting Levine [21], "*Any theory is an idealization. Players' exact preferences, beliefs, and so forth are never going to be known exactly to the modeler. As a result, the only meaningful theory of Nash equilibrium is Radner's [1980] notion of epsilon equilibrium. This requires only that no player lose more than epsilon compared to the true optimum – which in practice can never be known by the players.*"

Let us give some arguments to highlight the strong foundations of approximate equilibrium:

- Levine [21] argues convincingly that ε-Nash equilibria fit experimental data better than exact Nash equilibria.
- There are many games without a Nash equilibrium but with a reasonable approximate equilibrium prediction. For example, in first-price auctions with complete information, an approximate equilibrium is: the player with the highest value proposes a bid slightly above the second highest value. In Bertrand duopoly with asymmetric costs, the most efficient firm proposes a price slightly below the marginal cost of the opponent.
- Exact Nash equilibria may be impossible to learn and hard to compute. Indeed, computationally, Nash equilibria exhibit higher complexity than ε-Nash equilibria (Papadimitriou in Nisan et al [28]). In a behavioral perspective, Kalai and Lehrer [19] proved that, under the grain of truth hypothesis, ε-Nash equilibria could be learned by rational agents.
- In stochastic games, there exist simple ε-Nash equilibria (i.e. in stationary strategies) while all Nash equilibria are history dependent (e.g. Thuijsman and Vrieze [39]).
- In extensive form games with infinite action sets or infinite horizon, sub-game perfect Nash equilibria fail to exist while sub-game perfect ε-Nash do exist (Harris-Reny-Robson [14], Flesh et al [12], Purves-Sudderth [31]).
- In zero-sum games, the existence of an approximate equilibrium is equivalent to the existence of the value, which is a well accepted solution concept.

Given the above arguments, one needs to establish tight conditions under which approximate equilibria exist. There are few results in the literature, one of which is due[2] to Reny [33] and Prokopovych [30]. While theoretically interesting, it requires assumptions on payoffs that are not satisfied in many applications as will be seen in this paper.

We define a game G to be approximately better-reply secure if for every non-approximate

[2]Historically, Reny proved this result in a working and unpublished paper [33]. Independently, Prokopovych [30] proved the same result with a different technique.

equilibrium strategy profile x^* and every payoff vector limit u^* resulting from strategies approaching x^*, some player i has a strategy yielding a payoff strictly above u_i^*, even if the others deviate slightly from x^*. This is a natural extension of Reny's better-reply security condition and of Reny [33] and Prokopovych [30].

We prove that any approximately better-reply secure quasiconcave compact game admits an approximate equilibrium. An example is given by the class of diagonal games that encompasses many models of competition (in price, time, location or quantity). Each player, $i = 1, ..., N$, chooses a real number x_i in $[0, 1]$. The payoff $u_i(x_i, x_{-i})$ of player i is $f_i(x_i, \phi(x_{-i}))$ if $x_i < \phi(x_{-i})$, $g_i(x_i, \phi(x_{-i}))$ if $x_i > \phi(x_{-i})$, and $h_i(x_i, x_{-i})$ if $x_i = \phi(x_{-i})$, where f_i, g_i and ϕ are continuous. For example, in first-price auctions, $f_i = 0$, $g_i = v_i - x_i$, and $\phi = \max_{j \neq i} x_j$. In second-price auctions, $g_i = v_i - \max_{j \neq i} x_j$ (where v_i is the value of the object for player i).

The paper is organized as follow. In Section 2, we recall the main results for the existence of a solution in discontinuous games: Reny's [34] better-reply security for existence of Nash equilibria, the sharing rule solution of Simon and Zame [38], and Reny [33] and Prokopovych's [30] conditions for the existence of an approximate equilibrium.

Section 3 is dedicated to quasiconcave compact games in *pure strategies*. We introduce the new concept of Reny equilibrium and prove its existence. The concept is used –as a substitute for a fixed point theorem– to prove the existence of a pure sharing rule solution and to provide conditions for the existence of an approximate equilibrium. Reny equilibrium is then applied to construct a solution in a number of economic models. For example, we prove that in any exchange economy with quasiconcave discontinuous preferences, one can modify utilities at discontinuity points such that the new economy admits a general equilibrium (in the usual Arrow-Debreu sense).

Section 4 is dedicated to compact metric games in *mixed strategies*. We prove that the intersection of the sets of Reny equilibria and sharing rule equilibria is nonempty and contains the set of approximate equilibria. This provides a possible answer to a question of Jackson and Swinkel [18] who ask whether "*these approaches [Reny and Simon-Zame] turn out to be related*".[3] In addition, we prove that in any approximately better-reply secure game, approximate equilibria may be obtained as limits of Nash equilibria of an *endogenously* chosen sequence of discretizations of the game. This is a natural extension of a similar result established by Reny [35, 36] for Nash equilibria. As an application, we prove the existence of a mixed approximate equilibrium in a large class of auctions with correlated types and values.

2. THREE STANDARD APPROACHES TO DISCONTINUOUS GAMES

A game in strategic form $G = ((X_i)_{i \in N}, (u_i)_{i \in N})$ is given by a finite set N of players, and for each player $i \in N$, a set X_i of pure strategies, and a payoff function $u_i : X = \prod_{i \in N} X_i \to$ R.

[3]De Castro [10] and Carmona and Podzceck [7] propose different answers to the question of Jackson et al.

This paper assumes G to be *compact*, that is, for every i, X_i is a compact subset of a topological vector space, and u_i is bounded.[4] We let $V_i(x_{-i}) := \sup_{d_i \in X_i} u_i(d_i, x_{-i})$ denote the highest payoff that player i can get against $x_{-i} = (x_j)_{j \neq i} \in X_{-i} := \Pi_{j \neq i} X_j$.

Definition 2.1 A pair $(x, v) \in X \times \mathbf{R}$ is a *Nash equilibrium* of G (and x is a Nash equilibrium profile) if $v = u(x)$ and for every player $i \in I$, $V_i(x_{-i}) \leq v_i$.

The game G is *quasiconcave* if for every player $i \in I$, X_i is convex and for every $x_{-i} \in X_{-i}$, the mapping $u_i(\cdot, x_{-i})$ is quasiconcave. The game is *continuous* if for every $i \in I$, u_i is a continuous function[5].

Theorem 2.2 (Glicksberg's Theorem [13] in pure strategies) Any continuous, quasiconcave and compact game admits a Nash equilibrium.

The rest of the section presents three extensions of this result when payoffs are discontinuous. Our paper combines them into one general idea.

2.1. *Better-Reply Secure Game*

In many discontinuous games, a Nash equilibrium exists (symmetric Bertrand competition, auctions with private values, wars of attrition, among many). Reny's theorem [34] provides an explanation for this. Formally, we let $\Gamma = \{(x, u(x)) : x \in X\}$ denote the graph of G and $\overline{\Gamma}$ be the closure of Γ. Since G is compact, $\overline{\Gamma}$ is compact as well. Define the "secure payoff level" of player i at $(d_i, x_{-i}) \in X$ as follows:

$$\underline{u_i}(d_i, x_{-i}) = \liminf_{x'_{-i} \to x_{-i}} u_i(d_i, x'_{-i}).$$

This is the payoff that d_i can almost guarantee to player i if his opponents play any profile close enough to x_{-i}.

Definition 2.3 A game G is *better-reply secure* if whenever $(x, v) \in \overline{\Gamma}$ and x is not a Nash equilibrium profile, some player $i \in N$ can secure a payoff strictly above v_i, i.e. there exists $d_i \in X_i$ such that $\underline{u_i}(d_i, x_{-i}) > v_i$.

Theorem 2.4 (Reny's Theorem [34] in pure strategies) Any better-reply secure, quasiconcave and compact game admits a Nash equilibrium.

[4] Some results require the strategy sets to be metric or locally convex and Hausdorff.

[5] X is endowed with the product topology.

Since any continuous game is obviously better-reply secure, this extends Glicksberg's theorem. In his paper, Reny gives two practical sufficient conditions under which a game is better-reply secure (see Proposition 2.6 below). In the following, we let $\underline{V_i}(x_{-i}) := \sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i})$ denote the largest payoff that player i can secure against x_{-i}.

Definition 2.5
(i) G is payoff secure if for every $x \in X$, for every $\varepsilon > 0$, every player $i \in N$ can secure[6] a payoff above $u_i(x) - \varepsilon$, which can be equivalently[7] written: $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) = V_i(x_{-i}) = \underline{V_i}(x_{-i})$.
(ii) G is reciprocally upper semicontinuous if, whenever $(x, v) \in \overline{\Gamma}$ and $u(x) \leq v$, then $u(x) = v$.

Proposition 2.6 A payoff secure and reciprocally upper semicontinuous game is better-reply secure.

2.2. *Approximate Equilibrium*

In first-price auctions with complete information, bidding slightly above the second highest evaluation for the bidder with the highest evaluation yields an approximate equilibrium. One of the main goals of this paper is to develop theoretical tools to prove the existence of an approximate equilibrium in a large class of auctions.

Definition 2.7 A pair $(x, v) \in \overline{\Gamma}$ is an *approximate equilibrium* (and x is an approximate equilibrium profile) if there exists a sequence $(x^n)_{n\in \mathrm{N}}$ of X and a sequence $(\varepsilon_n)_{n\in \mathrm{N}}$ of positive real numbers, converging to 0, such that:
(i) for every $n \in \mathrm{N}^*$, x^n is an ε_n-equilibrium: $\forall i \in N$, $\forall d_i \in X_i$, $u_i(d_i, x^n_{-i}) \leq u_i(x^n) + \varepsilon_n$,
(ii) the sequence $(x^n, u(x^n))$ converges to (x, v).

Any Nash equilibrium is obviously an approximate equilibrium. It can be seen[8] as a limit of exact Nash equilibria $(x^n, u(x^n))$ of a sequence of games $G^n = ((X_i)_{i\in N}, (u^n_i)_{i\in N})$ which converges to $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$ for the uniform norm. Another interpretation of approximate equilibria will be given later and is related to the concept of endogenous sharing rule of Simon and Zame [38] (see the next subsection). Let us state one of the very few existing result in the literature.[9]

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[6] Player i can secure a payoff above $\alpha \in \mathrm{R}$ if there exists $d_i \in X_i$ and a neighborhood V_{-i} of x_{-i} such that for every $x'_{-i} \in V_{-i}$, $u_i(d_i, x'_{-i}) \geq \alpha$.

[7] The equivalence is straightforward.

[8] For every integer n, define $u^n_i(y) = u_i(y)$ whenever $y \neq x^n$ and $u^n_i(x^n) = u_i(x^n) + \varepsilon_n$.

[9] Ziad [40] proposes another existence theorem of approximate equilibria, unrelated to our work. See also Carmona [5], [6] and Reny [33].

Definition 2.8 A game G has the marginal continuity property (resp. the marginal continuity property at $x \in X$) if $V_i(x_{-i}) = \sup_{d_i \in X_i} u_i(d_i, x_{-i})$ is a continuous function for all $x_{-i} \in X_{-i}$ and every $i \in I$ (resp. a continuous function at x_{-i} for every player $i \in I$).

Theorem 2.9 (Reny [33] and Prokopovych [30]) Any payoff secure, quasiconcave compact game that has the marginal continuity property admits an approximate equilibrium.

This theorem applies for first-price auctions and asymmetric Bertrand's duopoly. However, the following location game [38] is not payoff secure, but admits an approximate equilibrium.

Example 2.10 *California Location Game*

The length interval $[0, 4]$ represents an interstate highway. The strategy set of player 1 (a psychologist from California) is $X = [0, 3]$ (representing the Californian highway stretch). The strategy set of player 2 (a psychologist from Oregon) is $Y = [3, 4]$ (the Oregon part of the highway). The payoff function of player 1 is $u_1(x, y) = \frac{x+y}{2}$ if $x < y$ and $u_1(3, 3) = 2$. The payoff function of player 2 is $u_2(x, y) = 4 - u_1(x, y)$. The strategy profile $x_n = (3 - \frac{1}{n}, 3)$, corresponding to the vector payoff $v_n = (3 - \frac{1}{2n}, 1 + \frac{1}{2n})$, is a $\frac{1}{2n}$-equilibrium. Consequently, $(x = (3, 3), v = (3, 1))$ is an approximate equilibrium. However, the game is not payoff secure for player 2 at $x = (3, 3)$.

2.3. *Sharing Rule Solutions*

The California location game example above was given by Simon and Zame [38] to illustrate their "endogenous tie-breaking rule" solution. They show that even if a game does not have a Nash equilibrium, it is always possible to slightly change the payoffs (at discontinuity points) so that the new game has a Nash equilibrium.

Example 2.11 *California Location Game, Continued.*

In the California location game, define a new payoff function q as follows: $q(x) = u(x)$ for every $x \neq (3, 3)$ and $q(3, 3) = (3, 1)$. The pure strategy profile $(3, 3)$ with payoff $(3, 1)$ is a Nash equilibrium of the game defined by q. The new sharing rule at $x = (3, 3)$ has a simple interpretation: it corresponds to giving each psychologist his/her natural market share. Moreover, this is exactly the prediction of the approximate equilibrium in Example 2.10. We will prove that this property is very general: any approximate equilibrium is a sharing rule equilibrium (see Theorem 3.10).

To prove the existence of a solution, Simon and Zame do not require the game to be quasiconcave. However, they allow the use of *mixed strategies*. Formally, G is *metric* if strategy sets are Hausdorff and metrizable and payoff functions are measurable. Denote by $M_i = \Delta(X_i)$

the set of Borel probability measures on X_i (usually called the set of mixed strategies of player i). This is a compact Hausdorff metrizable set under the weak* topology. Let $M = \Pi_i M_i$.

Definition 2.12 A *mixed Nash equilibrium* of G is a pure Nash equilibrium of its mixed extension $G' = ((M_i)_{i\in N}, (u_i)_{i\in N})$, where payoff functions are extended multi-linearly to M.

Definition 2.13 A pair (m, q) is a *mixed sharing rule solution* of G if $m \in M$ is a mixed Nash equilibrium of the auxiliary game $\tilde{G} = ((X_i)_{i\in N}, (q_i)_{i\in N})$, where the auxiliary payoff functions $q = (q_i)_{i\in N}$ must satisfy the condition:

$$\text{(SR)}: \forall x \in X,\ q(x) \in co\overline{\Gamma}_x,$$

where, $\overline{\Gamma}_x = \{v \in \mathrm{R}^N : (x, v) \in \overline{\Gamma}\}$ is the x-section of $\overline{\Gamma}$, and *co* stands for the convex hull.

Condition (SR) has two implications: if u is continuous at x, $q(x) = u(x)$; if $\sum_{i\in N} u_i(x)$ is continuous, then $\sum_{i\in N} q_i(x) = \sum_{i\in N} u_i(x)$ (justifying the terminology "sharing rule").

Theorem 2.14 (Simon and Zame [38]) Any compact metric game admits a mixed sharing rule solution.

Jackson et al. [17] extend Simon and Zame's theorem to games with incomplete information. In their paper, they interpret a tie-breaking rule as a proxy for the outcome of an unmodeled second stage game. As example, they recall the analysis of first-price auctions with incomplete information for a single indivisible object. Maskin and Riley [23] add to the sealed-bid stage a second stage where bidders with the highest bid in the first stage play a Vickrey auction. In the private value setting, their dominant strategy is to bid their true values. Consequently, the second stage induces a tie-breaking rule where the bidder with the highest value wins the object. More generally, a tie-breaking rule may be implemented by asking players to send a cheap message (their private values in auctions), in addition to their strategies (bids). The messages will be used only to break ties (as in the second stage of Maskin and Riley's mechanism).

When the game is continuous, the new and the original games coincide, and so we recover the Nash-Glicksberg's theorem in mixed strategies.

Theorem 2.15 (Nash-Glicksberg's Theorem in mixed strategies) Any continuous, metric compact game admits a mixed Nash equilibrium.

Jackson et al. [17] remark that their "*results concern only the existence of solutions [sharing rule equilibrium] in mixed strategies*" and that they "*have little to say about the existence of solutions in pure strategies*". In the next section, we prove the existence of a pure sharing rule solution, defined now.

DEFINITION 2.16 A pair (x, q) is a *pure sharing rule solution* if $x \in X$ is a pure Nash equilibrium of the auxiliary game $\tilde{G} = ((X_i)_{i\in N}, (q_i)_{i\in N})$, where the auxiliary payoff functions $q = (q_i)_{i\in N}$ must satisfy the following stronger condition:

$$\text{(SR strong): } \forall y \in X,\ q(y) \in \overline{\Gamma}_y,$$

Our condition (SR strong) requires that for every strategy profile y, there exists a sequence (y_n) converging to y such that $q(y) = \lim_{n\to\infty} u(y_n)$. On one hand, our condition is stronger than the original condition (SR) because one always has $\overline{\Gamma}_y \subset co\overline{\Gamma}_y$. On the other hand, to prove the existence of a pure sharing rule solution, we need payoff functions to be quasiconcave.

To allow comparison between sharing rule solution and approximate equilibrium, we introduce the following terminology.

DEFINITION 2.17 A pair $(m, v) \in M \times \mathbf{R}^N$ is called a mixed (resp. pure) sharing rule equilibrium if (m, q) is a mixed (resp. pure) sharing rule solution and $q(m) = v$.

The proof of the existence of a pure sharing rule equilibrium is a direct consequence of the existence of a Reny equilibrium, defined in the next subsection.

3. EXISTENCE OF A SOLUTION FOR GAMES IN PURE STRATEGIES

As discussed above, sharing rule and approximate equilibrium concepts are important alternative solutions for games without a Nash equilibrium. Both are defined on $\overline{\Gamma}$ (the closure of the graph of the game). Consequently, to prove their existence, it seems necessary to use some general existence result on $\overline{\Gamma}$ that plays the role of a fixed point theorem: Reny equilibrium is the key concept.

3.1. *Existence of a Reny Equilibrium*

In the following definition, recall that $\underline{V_i}(x_{-i}) := \sup_{d_i\in X_i} \underline{u_i}(d_i, x_{-i})$.

DEFINITION 3.1 A pair $(x, v) \in \overline{\Gamma}$ is a *Reny equilibrium* if for every $i \in N$, $\underline{V_i}(x_{-i}) \leq v_i$.

EXAMPLE 3.2 *Two-player first-price auctions*

Both players $i = 1, 2$ choose a bid $x_i \in [0, 1]$, and receive a payoff:

$$u_i(x_i, x_j) = \begin{cases} w_i - x_i & \text{if } x_i > x_j, \\ \frac{w_i - x_i}{2} & \text{if } x_i = x_j, \\ 0 & \text{if } x_i < x_j. \end{cases}$$

If $w_1 \in]0,1[$ (the value of player 1) is higher than $w_2 \in]0,1[$ (the value of player 2), then the above game is quasiconcave, and every $(x_1, x_2, v_1, v_2) = (y, y, w_1 - y, 0)$ is a Reny equilibrium whenever $y \in [w_2, w_1]$. To see this, note first that the game is payoff secure, thus a Reny-equilibrium $(x, v) = (x_1, x_2, v_1, v_2) \in \overline{\Gamma}$ satisfies

(1) $\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \; i = 1, 2$

Since this game has no Nash equilibrium, x_1 is equal to x_2 (otherwise u_i would be continuous at $x = (x_1, x_2)$, and Equation 1 would imply that x is a Nash equilibrium). Moreover, each player can get a payoff of at least 0 by playing 0. Consequently, v_1 and v_2 are non-negative. From $(x, v) \in \overline{\Gamma}$, $(v_1, v_2) = \lim_{n \to +\infty}(u_1(x^n), u_2(x^n))$ for some sequence of profiles $x^n = (x_1^n, x_2^n)$ converging to (x_1, x_1). There are three cases (up to a subsequence), depending on whether the sequence converges to x from above, along the diagonal, or from below. In the two first cases, $v = (0, w_2 - x_1)$ or $v = (\frac{w_1 - x_1}{2}, \frac{w_2 - x_1}{2})$, thus $x_2 = x_1 \leq w_2$. Then, playing slightly above x_1 gives a payoff strictly above v_1 for player 1, which contradicts Equation 1. In the last case, $v = (w_1 - x_1, 0)$, thus $x_1 \leq w_1$. Then, Equation 1 implies that $x_1 \geq w_2$ (otherwise player 2 could do better than 0 by playing slightly above x_1.)

In this example, the set of Reny equilibria coincides with the set of approximate equilibria (playing $y \in [w_2, w_1]$ for player 2 and slightly above for player 1 is an ε-equilibrium). Note that there are several Reny and approximate equilibria, but multiplicity of equilibria can happen also for Nash equilibrium concept: in this example with a second-price auction mechanism, playing y for player 2 and w_1 for player 1 is a Nash equilibrium for all $y \in [0, w_1)$.

Theorem 3.3 For any quasiconcave and compact game G, the set of Reny equilibria is nonempty and compact, and it contains the set of Nash equilibria. Moreover, G is better-reply secure if and only if Nash and Reny equilibrium profiles coincide.

Observe that a Nash equilibrium $(x, u(x))$ is a Reny equilibrium because $\underline{V_i}(x_{-i}) \leq V_i(x_{-i})$, and by Nash conditions, $V_i(x_{-i}) \leq u_i(x)$. Moreover, if the game is continuous, then Reny and Nash equilibria coincide because $\underline{V_i}(x_{-i}) = V_i(x_{-i})$ and $u(x) = v$.

The existence of a Reny equilibrium is a straightforward consequence of Reny's [34] theorem (this explains why we call it Reny equilibrium). Indeed, assume, by contradiction, that there is no Reny equilibrium. This implies that the game is better-reply secure. Consequently, by Reny's theorem [34], there exists a Nash equilibrium, which is a Reny equilibrium: a contradiction. Compactness of the set of Reny equilibria is due to the lower semi-continuity of $\underline{V_i}$, and the last assertion of Theorem 3.3 is a consequence of the definition of better-reply security.

As an illustration, we can revisit the result of Reny in Proposition 2.6 and prove that whenever a game is payoff secure and reciprocally upper semi continuous, it is better-reply secure. Actually, assume (x, v) to be a Reny equilibrium. Thus, for every $i \in N$, $\underline{V_i}(x_{-i}) \leq v_i$. Since the game is payoff secure, $V_i(x_{-i}) = \underline{V_i}(x_{-i}) \leq v_i$. Since $u_i(x) \leq V_i(x_{-i})$, one has $u_i(x) \leq$

v_i for every $i \in N$. By reciprocal upper semicontinuity, $v = u(x)$, and so $V_i(x_{-i}) \leq u_i(x)$ for every $i \in N$. Consequently, (x, v) is a Nash equilibrium.

Two major applications of Reny equilibrium are presented in the next subsections. In Subsection 3.2., Reny equilibrium allows to prove the existence of a pure sharing rule equilibrium in any quasiconcave compact game. In Subsection 3.3., Reny equilibrium is used to prove the existence of approximate equilibria in a number of economic applications.

3.2. *Existence of a Sharing Rule Equilibrium*

The existence of a Reny equilibrium allows to solve the open problem in Jackson et al. [17].

Theorem 3.4 Every Reny equilibrium is a pure sharing rule equilibrium. In particular, any quasiconcave and compact game G admits a pure sharing rule equilibrium.

Remark 3.5 Observe that a pure sharing rule solution (m, q') of G' (the mixed extension of G) is not a mixed sharing rule solution of G because the new payoff profile q' is defined on M and not on X, and q' is not necessarily the multilinear extension of a pure strategy payoff profile. Thus, our result does not imply Simon-Zame's theorem.

To prove Theorem 3.4, consider a Reny equilibrium $(x, v) \in \overline{\Gamma}$. Then, we can build a sharing rule solution as follows. For every $i \in N$ and $d_i \in X_i$, denote by $\underline{S}(d_i, x_{-i})$ the space of sequences $(x^n_{-i})_{n \in \mathrm{N}}$ of X_{-i} converging to x_{-i} such that $\lim_{n \to +\infty} u_i(d_i, x^n_{-i}) = \underline{u}_i(d_i, x_{-i})$. Then, define $q : X \to \mathrm{R}^N$ by

$$q(y) = \begin{cases} v & \text{if } y = x, \\ \text{any limit point of } (u(d_i, x^n_{-i}))_{n \in \mathrm{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N, d_i \neq x_i, (x^n_{-i})_{n \in \mathrm{N}} \in \underline{S}(d_i, x_{-i}), \\ q(y) = u(y) & \text{otherwise.} \end{cases}$$

Now, let us prove that (x, q) is a pure sharing rule solution. Since $(x, v) \in \overline{\Gamma}$, and by definition of q, condition (SR strong) of Definition 2.16 is satisfied at x. Obviously, it is satisfied at every y different from x for at least two components, and also at every (d_i, x_{-i}) with $d_i \neq x_i$, from the definition of $q(d_i, x_{-i})$ in this case. The proof is complete.

Remark 3.6 Thus, Reny equilibrium refines pure sharing rule equilibrium, and the refinement is strict as the following better-reply secure game shows. A player maximizes over $[0, 1]$ the following discontinuous payoff function: $u(x) = 0$ if $x < 1$, and $u(1) = 1$. If $q(y) = 0$ for every y, then (x, q) is a pure sharing rule solution for any $x \in [0, 1]$. Yet, the only Reny equilibrium is $(x, v) = (1, 1)$, and it coincides with the unique Nash equilibrium.

Remark 3.7 Actually, the pure sharing rule solution (x, q) built in the proof of Theorem 3.4 satisfies the additional property:[10] $q_i(d_i, x_{-i}) \geq \underline{u_i}(d_i, x_{-i})$ for every $i \in N$ and every $d_i \in X_i$. This property says that q remains above the secure payoff level in the original game. This additional property is very useful, for example, to prove the existence of a pure sharing rule solution in an exchange economy model with discontinuous preferences (see Application 3.18).

Example 3.8 *Bertrand Duopoly*

In a Bertrand duopoly, two firms $i = 1, 2$ choose prices $p_i \in [0, a]$ $(a > 0)$. Assume a linear demand $a - min(p_1, p_2)$ and marginal costs $c_1 < c_2 < \frac{a+c_1}{2}$. If the firms charge an equal price, then the market demand is equally shared by both the firms. If we assume that the firm charging the lowest price supplies the entire market, then the game has no pure Nash equilibrium. Nevertheless, the game is quasiconcave and compact. This game has a pure sharing rule solution, with a strategy profile (c_2, c_2) and with payoff function $q(c_2, c_2) = ((a - c_2)(c_2 - c_1), 0)$, while $q(x) = u(x)$ elsewhere.

Application 3.9 *Shared Resource Games*

The payoff of each player $i \in N$ can be written as $u_i(x_i, x_{-i}) = F_i(x_i, S_i(x_i, x_{-i}))$, where $F_i : X_i \times \mathbf{R} \to \mathbf{R}$ and $S_i : X \to \mathbf{R}$ (the shared resource of player i). The total amount of the resource $\sum_{i=1}^N S_i$ is a (possibly discontinuous) function of the strategy profile $x \in X$. This game G was introduced to model fiscal competition for mobile capital (Rothstein's [37]).

A sharing rule of G is defined to be a family $(\tilde{S}_i)_{i \in I}$ of functions from X to $\mathbf{R}$ such that for every strategy profile $x \in X$, there is a sequence (x_n) converging to x such that for every player i, $\tilde{S}_i(x) = \lim_{n \to \infty} S_i(x_n)$. Theorem 3.4 implies the following extension of Rothstein's results. Assuming G to be quasiconcave and compact, F_i continuous and S_i bounded for every player i, we get the existence of a new sharing rule $(\tilde{S}_i)_{i \in I}$ whose associated game $\tilde{u}_i(x_i, x_{-i}) = F_i(x_i, \tilde{S}_i(x_i, x_{-i}))$ admits a pure Nash equilibrium. Moreover, under the following assumptions, any Nash equilibrium of $\tilde{G}$ is a Nash equilibrium[11] of G:

A1) For all $x_i \in X_i$, $F_i(x_i, s_i)$ is nondecreasing in s_i.

A2) For all $x_{-i} \in X_{-i}$, $\sup\limits_{d_i \in X_i : (d_i, x_{-i}) \in \mathcal{C}_i} u_i(d_i, x_{-i}) = \sup\limits_{d_i \in X_i} u_i(d_i, x_{-i})$, where $\mathcal{C}_i$ is the set of continuity points of S_i.

A3) If $x \notin \cap_{i \in N} \mathcal{C}_i$, $\sup\limits_{d_i \in X_i} u_i(d_i, x_{-i}) > F_i\left(x_i, \dfrac{\limsup_{x' \to x} \sum_{i=1}^N S_i(x')}{N}\right)$ for every $i \in N$.

[10] Indeed, if $d_i = x_i$, then $q_i(d_i, x_{-i}) = q_i(x) = v_i \geq \underline{u_i}(x)$ because (x, v) is a Reny equilibrium. If $d_i \neq x_i$ then $q_i(d_i, x_{-i}) = \underline{u}_i(d_i, x_{-i})$.

[11] Indeed, for every Nash equilibrium x of $\tilde{G}$, one has $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) \leq \sup_{d_i \in X_i} F_i(d_i, \tilde{S}_i(d_i, x_{-i})) \leq F_i(x_i, \tilde{S}_i(x))$ for every player i. The first inequality is a consequence of A2 and the definition of $\tilde{S}_i$. If $x \in \mathcal{C}_i$ for every i, then $\tilde{S}_i(x) = S_i(x)$, and x is a Nash equilibrium of the initial game G. Otherwise, from A1 and A3, we get $\frac{\limsup_{x' \to x} \sum_{i=1}^N S_i(x')}{N} < \tilde{S}_i(x)$ for every $i \in N$. Summing these inequalities contradicts the definition of $\tilde{S}_i$.

Consequently, Theorem 3.4 permits to strengthen Rothstein's results. Moreover, it answers to the following remark of Rothstein[12] "the work of Simon and Zame [38] is directly applicable, but only to establish the existence of a Nash equilibrium in mixed strategies with an endogenous sharing rule." By Theorem 3.4, the game has a pure endogenous sharing rule equilibrium.

3.3. *Existence of an Approximate Equilibrium*

The following theorem proves that Reny equilibrium can detect approximate equilibrium.

THEOREM 3.10 Every approximate equilibrium is a Reny equilibrium, and so is a pure sharing rule equilibrium.

The proof is as follows. Let $(x^n)_{n\in \mathrm{N}}$ be a sequence of ε_n-equilibria such that $(x^n, u(x^n))$ converges to (x, v). By definition, $u_i(d_i, x^n_{-i}) \leq u_i(x^n) + \varepsilon_n$ for every $n \in \mathrm{N}$, every player $i \in N$ and every deviation $d_i \in X_i$. Passing to the infimum limit when n tends to infinity, we obtain $\underline{u_i}(d_i, x_{-i}) \leq v_i$. Thus, (x, v) is a Reny equilibrium, and also a pure sharing rule equilibrium by Theorem 3.4. This leads to the following definition.

DEFINITION 3.11 A game G is *approximately better-reply secure* if whenever $(x, v) \in \overline{\Gamma}$ and x is not an approximate equilibrium profile, some player i can secure a payoff strictly above v_i.

The existence of a Reny equilibrium implies the following result.

THEOREM 3.12 Any approximately better-reply secure quasiconcave and compact game admits an approximate equilibrium.

California location game is approximately better-reply secure. This theorem provides a local version of Reny-Prokopovych's theorem (described in Subsection 2.2).

PROPOSITION 3.13 If (x, v) is a Reny equilibrium and if, at x, the game is payoff secure and marginally continuous, then (x, v) is an approximate equilibrium.

Actually, if (x, v) is a Reny equilibrium, then $\sup_{d_i\in X_i} \underline{u_i}(d_i, x_{-i}) = \sup_{d_i\in X_i} u_i(d_i, x_{-i}) \leq v_i$ (the equality being a consequence of payoff security at x). Since $v = \lim_{x^n\to x} u(x^n)$ for some sequence x^n, the local continuity of $\sup_{d_i\in X_i} u_i(d_i, x_{-i})$ with respect to x guarantees that (x, v) is an approximate equilibrium. This proposition implies Reny-Prokopovych's theorem, and is useful in practice as the following application shows.

[12] See [37], footnote 3.

Application 3.14 *Diagonal Games.*

For every $i \in N$, we let f_i, g_i be *continuous* mappings from $[0,1] \times [0,1]$ to $\mathbf{R}$, and $h_i : [0,1]^N \to \mathbf{R}$ be a bounded mapping. The payoff of player i is:

$$u_i(x_i, x_{-i}) = \begin{cases} f_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) > x_i, \\ g_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) < x_i, \\ h_i(x_i, x_{-i}) & \text{if } \phi(x_{-i}) = x_i, \end{cases}$$

where $\phi : [0,1]^{N-1} \to [0,1]$ is a continuous (aggregation) function that satisfies:

Monotonicity[13]*:* if $(y_1, ..., y_{N-1}) \leq (z_1, ..., z_{N-1})$ then $\phi(y_1, ..., y_{N-1}) \leq \phi(z_1, ..., z_{N-1})$ and if $(y_1, ..., y_{N-1}) << (z_1, ..., z_{N-1})$ then $\phi(y_1, ..., y_{N-1}) << \phi(z_1, ..., z_{N-1})$.

Anonymity: for any permutation σ of $\{1, ..., N-1\}$, $\phi(y_1, ..., y_{N-1}) = \phi(y_{\sigma(1)}, ..., y_{\sigma(N-1)})$.

Unanimity: $\phi(y, ..., y) = y$ for every $y \in [0,1]$.

Representativity: $\phi(y_1, ..., y_i, ..., y_{N-1}) > 0$ and $y_i > 0$ imply $\phi(y_1, ..., z_i, ..., y_{N-1}) > 0$ for every $z_i > 0$. Similarly, $\phi(y_1, ..., y_i, ..., y_{N-1}) < 1$ and $y_i < 1$ imply $\phi(y_1, ..z_i, ..., y_{N-1}) < 1$ for every $z_i < 1$.

The four properties are satisfied[14] by functions such as $\max_j y_j$, $\min_j y_j$, $\frac{1}{N-1} \sum_j y_j$, or the k-th highest value of $\{y_1, ..., y_{N-1}\}$ for $k = 1, ..., N-1$. Diagonal games include many models of competition with complete information. For example, in some auctions, $\phi(y_1, ..., y_{N-1}) = \max_j y_j$, in wars of attrition, preemption or Bertrand competition $\phi(y_1, ..., y_{N-1}) = \min_j y_j$.

Proposition 3.15 Any quasiconcave diagonal game satisfying condition (C) below is approximately better-reply secure, thus it possesses an approximate equilibrium.

(C) there is $\alpha > 0$ such that for every $x \in [0,1]^N$ and $i \in N$, if $x_i = \phi(x_{-i})$ then there is $\alpha_i(x) \in]\alpha, 1-\alpha[$ such that $h_i(x) = \alpha_i(x) f_i(x_i, \phi(x_{-i})) + (1 - \alpha_i(x)) g_i(x_i, \phi(x_{-i}))$.

Condition (C) just means that h_i is a strict convex combination of g_i and f_i with weights that are bounded below. The assumption is satisfied in many models. In auctions, the winner is usually decided uniformly among highest bidders, thus the payoff of a player in case of ties is a strict convex combination between his payoff if he wins, g_i, and if he looses, f_i. The coefficient of the convex combination depends on how many players are tied, inducing a discontinuity on h_i. The probability of being selected or not selected is bounded below by $\frac{1}{N} = \alpha$.

Sketch of the proof. (See the appendix for a detailed proof.) Under Assumption (C), the

[13] In the following, for every $x = (x_1, ..., x_n) \in \mathbf{R}^n$ and $y = (y_1, ..., y_n) \in \mathbf{R}^n$, $x << y$ means $x_i < y_i$ for every i.

[14] Our paper gives several existence results, in pure and mixed strategies, where an aggregation function is used, but the proofs of some of these results do not use all the properties. For example, in Proposition 4.12 (private value setting), only monotonicity of ϕ is used.

game is payoff secure. Consequently, if $(x,v) \in \overline{\Gamma}$ is a Reny equilibrium then

$$\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \ i \in N.$$

To prove that x is an approximate equilibrium profile, one has to check four different cases: first, if $x_i \neq \phi(x_{-i})$ for every i, then the payoff functions are continuous at x, $v = u(x)$, and the Reny equilibrium equation above implies that (x,v) is a Nash equilibrium. Second, if there exists i such that $x_i = \phi(x_{-i}) \in]0,1[$, then anonymity, representativity, and monotonicity give $\phi(x_{-j}) \in]0,1[$ for every j. Then, the marginal continuity property is satisfied at x, and from Proposition 3.13, (x,v) is an approximate equilibrium. Third, assume there is i such that $x_i = \phi(x_{-i}) = 0$. By anonymity and monotonicity, $\phi(x_{-j}) = 0$ for every $j \in N$. Let x^n be such that $u(x^n) \to v$. Define a sequence of profiles (y^n) as follows: we let j be any player; if $v_j \leq f_j(0)$, define $y_j^n := 0$ for every n, otherwise, define $y_j^n := x_j^n$ for every n. One can check that y^n is an ε^n-equilibrium for some $\varepsilon^n \to 0$. In the last case, there is i such that $x_i = \phi(x_{-i}) = 1$: this is similar to the third case.

3.4. *Refining Reny Equilibria*

In this subsection, we use a recent work of Barelli and Meneghel [3] to improve the previous subsections. This requires the strategy spaces to be Hausdorff, locally convex subsets of a topological vector space. It is assumed in this subsection.[15] Equivalently, this can be formalized by the use of the mapping $\underline{\underline{u_i}}$ defined as follows:[16]

(2) for every $x \in X$, $\underline{\underline{u_i}}(x) := \sup_{U \in \mathcal{V}(x_{-i})} \sup_{d_i \in W_U(x)} \inf_{x'_{-i} \in U} u_i(d_i(x'_{-i}), x'_{-i})$,

where $\mathcal{V}(x_{-i})$ denotes the set of neighborhoods of x_{-i} and $W_U(x)$ is the set of continuous mappings d_i from U to X_i such that $d_i(x_{-i}) = x_i$. Barelli and Meneghel existence result [3] implies easily the following extension of Theorem 3.3, the proof being identical.

THEOREM 3.16 Every quasiconcave and compact game admits $(x,v) \in \overline{\Gamma}$ s.t. for all $i \in N$:

$$\sup_{d_i \in X_i} \underline{\underline{u_i}}(d_i, x_{-i}) \leq v_i.$$

[15] In fact, Barelli and Meneghel go further replacing $d_i(.)$ by some well behaved multivalued "Kakutani-type" mapping. The extension we propose can easily be adapted to their more general framework. For simplicity, we keep on continuous functions. Observe that Barelli and Meneghel construction requires assumptions on X so that Brouwer fixed point theorem holds on X. From Cauty [8]. Reny's better-reply security condition asks for the existence of a deviation $d_i \in X_i$ with the property that $u_i(d_i, x'_{-i}) > v_i + \varepsilon$ for some $\varepsilon > 0$ and for every x' in some neighborhood of x. Barelli and Meneghel propose a natural extension: they allow d_i to depend continuously on x'_{-i}, meaning that we should now have $u_i(d_i(x'_{-i}), x'_{-i}) > v_i + \varepsilon$, where $d_i(.)$ is a continuous function from a neighborhood of x_{-i} to X_i, we know it is sufficient for X to be a convex and compact subset of a Hausdorff topological vector space. Thus we could get rid of the local convexity assumption in this subsection.

[16] This function was first introduced by Carmona [6], in the more general case where $d_i(.)$ is multivalued.

Call such a pair (x, v) a *strong Reny equilibrium*. It refines Reny equilibrium because $\underline{u_i} \leq \underline{\underline{u_i}}$ (simply take the constant mapping $d_i = x_i$ in the supremum of Equation (2) above)[17]. The existence of a strong Reny equilibrium permits to refine pure sharing rule solutions in Theorem 3.4 as follows:

THEOREM 3.17 Every quasiconcave compact game admits a pure sharing rule solution (x, q), with the additional property: for every player i and for every $d_i \in X_i$, $q_i(d_i, x_{-i}) \geq \underline{\underline{u_i}}(d_i, x_{-i})$.

The construction follows the proof of Theorem 3.4, by using $\underline{\underline{u_i}}$ instead of $\underline{u_i}$. The additional property is obtained as explained in Remark 3.7. The following example illustrates the importance of this condition in practice.

APPLICATION 3.18 *Exchange Economy with Discontinuous Preferences*

Consider the following n-consumer, m-commodity exchange economy. Consumer i's consumption set X_i, $i \in N$, is a nonempty, convex and compact subset of $\mathbf{R}^m_+$, his utility function $u_i : X_i \to \mathbf{R}$ is non negative, bounded and quasiconcave, and his initial endowment e_i is assumed to be in the interior of X_i. Call $\{X_i, u_i, e_i\}_{i \in N}$ a quasiconcave compact economy if it satisfies these assumptions. A Walrasian equilibrium is $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}^m_+)$ such that $\sum_{i \in N} x_i = \sum_{i \in N} e_i$ and x_i maximizes the utility u_i of agent i on its budget set $p \cdot (x_i - e_i) \leq 0$, where $\Delta(\mathbf{R}^m_+)$ is the unit simplex of $\mathbf{R}^m_+$. We define $(x, p) \in \Pi_{i \in N} X_i \times \Delta(\mathbf{R}^m_+)$ to be a sharing rule Walrasian equilibrium if there exists a new utility profile $\{\tilde{u}_i\}_{i \in N}$ such that (x, p) is a Walrasian equilibrium of $\{X_i, \tilde{u}_i, e_i\}_{i \in N}$, where $\tilde{u}_i : X_i \to \mathbf{R}$ satisfies: for every i and $y_i \in X_i$, there is $y_i^n \to y_i$ such that $\tilde{u}_i(y_i) = \lim_{n \to +\infty} u_i(y_i^n)$.

PROPOSITION 3.19 Every quasiconcave and compact economy $\{X_i, u_i, e_i\}_{i \in N}$ admits a sharing rule Walrasian equilibrium.

The proof can be found in the appendix, and uses in a crucial way the above refinement of sharing rule solution.

REMARK 3.20 In general equilibrium theory, preferences are usually assumed to be continuous. Discontinuities may arise, for example, in presence of externalities (preferences of a player depend on consumptions of other players). Note that the proof of Proposition 3.19 could be adapted to economies with externalities. This sharing rule existence result opens the door to other interesting questions such as existence of approximate or even exact Walrasian equilibria.

[17] It can be proved that if $W_U(x)$ in the definition of $\underline{\underline{u_i}}$ is replaced by the set of all mappings without any continuity restriction, then existence in the above theorem is lost. Thus, Barelli and Meneghel refinement is tight.

Another consequence of Theorem 3.16 is the following extension of Proposition 3.15.

PROPOSITION 3.21 In every quasiconcave diagonal game where h is continuous and $\phi(x_{-i}) = \max_{j\neq i} x_j$, approximate and strong Reny equilibrium profiles coincide, consequently the game admits an approximate equilibrium.

4. EXISTENCE OF SOLUTIONS FOR GAMES IN MIXED STRATEGIES

This section improves the previous one by providing a constructive approach for approximate and Reny equilibria in mixed strategies. In a first part we establish a formal link between the set of Reny equilibria of G' and the set of mixed sharing rule equilibria by proving that the intersection of these sets is nonempty. In a second part, we prove that if G' is approximately better-reply secure then some approximate equilibria can be obtained as limits of Nash equilibria of finite discretizations of the initial game. The approximation methodology is illustrated in auctions with correlated types or values.

In this section, we let G be a metric compact game, G' its mixed extension and $\overline{\Gamma'}$ the closure of the graph of G'. As previously defined, $M_i = \Delta(X_i)$ is the set of mixed strategies of player i. This is a compact Hausdorff metrizable set under the weak* topology. Let $M = \Pi_i M_i$. In the following, a Reny equilibrium of G' is called a mixed Reny equilibrium. It always exists from Theorem 3.3 because G' is compact and quasiconcave.

4.1. *Linking Approximate, Reny and Simon-Zame Equilibria*

The following proves that the intersection of mixed Reny and mixed sharing rule equilibria plays a central role.

THEOREM 4.1 Mixed approximate equilibria are always in the intersection of mixed Reny and mixed sharing rule equilibria[18].

The first inclusion is a consequence of Theorem 3.10 applied to G'. The second inclusion is a straightforward adaptation of the proof of Simon and Zame [38] (simply take $\varepsilon-$equilibria of finite approximations of G instead of exact equilibria).

Importantly, this intersection is always nonempty, and is a consequence of a simple limit argument we now explain. We let $\mathcal{D}_0$ be the set of all finite subsets $\Pi_{i\in N} D_i$ of M. Consider the inclusion relationship on $\mathcal{D}_0$: it is reflexive, transitive and binary. Then, each pair $\Pi_{i\in N} D_i$ and $\Pi_{i\in N} D'_i$ in $\mathcal{D}_0$ has an upper bound $\Pi_{i\in N}(D_i \cup D'_i)$ in $\mathcal{D}_0$. The pair $(\mathcal{D}_0, \subset)$ is called a directed

[18] From Section 3, this is also true in pure strategies. Nevertheless, it cannot be directly proved applying Section 3 to G', simply because a pure sharing rule equilibrium of G' may not be a mixed sharing rule equilibrium of G.

set. To every $D = \Pi_{i \in N} D_i \in \mathcal{D}_0$, we can associate $(m^D, u(m^D))$, where m^D is a mixed Nash equilibrium of the finite game restricted to D. This defines a mapping[19] from $\mathcal{D}_0$ to $\overline{\Gamma'}$, called a net (of $\overline{\Gamma'}$). A limit point $(m, v) \in \overline{\Gamma'}$ of this net, denoted $(m^D, u(m^D))_{D \in \mathcal{D}_0}$, is defined by the following property: for every neighborhood $V_{m,v}$ of (m, v) and every $D = \Pi_{i \in N} D_i \in \mathcal{D}_0$, there exists $D' \in \mathcal{D}_0$ with $D \subset D'$ such that $(m^{D'}, u(m^{D'})) \in V_{m,v}$.

Definition 4.2 A pair $(m, v) \in \overline{\Gamma'}$ is a limit-equilibrium of G' if it is a limit point[20] of a net $(m^D, u(m^D))_{D \in \mathcal{D}_0}$ of mixed Nash equilibria of the finite game restricted to D.

Theorem 4.3 Every compact metric game has a limit-equilibrium. Any limit-equilibrium (m, v) is a mixed Reny and a mixed sharing rule equilibrium. Consequently, the intersection between mixed Reny and mixed sharing rule equilibria is nonempty, and if G' is better-reply secure then any limit-equilibrium is a mixed Nash equilibrium.

Existence of a limit-equilibrium is a consequence of the compactness of $\overline{\Gamma'}$. The rest of the proof is presented in the appendix and is an adaptation of the arguments of Simon and Zame.

Remark 4.4 In particular, we get a short and constructive proof of Reny's existence result for games in mixed strategies [34] by a limit argument (as in the proof of the theorem of Simon and Zame).

4.2. *Weak Strategic Approximation*

The idea of using a sequence of finite games to detect Nash equilibria goes back to Dagsputa and Maskin [22]. This has been formalized by Reny [36] in the class of better-reply secure games via the notion of strategic approximation. We can extend this method to approximately better-reply secure games.

Definition 4.5 A game G admits a weak strategic approximation if there is a sequence of finite sets $D_n \subset M$ such that all accumulation points of mixed Nash equilibria[21] of the game restricted to D_n are approximate equilibria of G'.

Theorem 4.6 If the game G' is approximately better-reply secure, then it has a weak strategic approximation.

[19] By definition, $m^D = (m_i^D)_{i \in N}$ is an element of $\Pi_{i \in I} \Delta(\Delta(X_i))$. More precisely, it is a profile of probability measures on finite subsets of $\Delta(X_i)$, where $i \in N$. Given $i \in N$, let $\{\sigma_1, ..., \sigma_K\}$ be the support of m_i^D and $p_1, ..., p_K$ the associated weights. By abuse of notation, we can define $m_i^D = \sum_{k=1}^{K} p_k.\sigma_k$, which is now an element of $\Delta(X_i)$. Up to this identification, $(m^D, u(m^D))$ can be seen as an element of $\overline{\Gamma'}$.

[20] The pair $(m^D, u(m^D))$ can be identified with an element of $\overline{\Gamma'}$, see footnote 19.

[21] By taking a sequence of Selten perfect equilibria rather than Nash equilibria, one may obtain approximate equilibria who's support are included in the closure of the set of undominated strategies.

The proof (in the appendix) is an adaptation of Reny's arguments [36], thanks to the notion of limit-equilibrium.

4.3. *Applications*

Non Quasiconcave Two Player Diagonal Games

Proposition 4.7 Any two-player diagonal game in which h is continuous admits a weak strategic approximation (and so an approximate equilibrium in mixed strategies).

The construction of a weak strategic approximation is proved in Appendix 6.6. Interestingly, the approximation is endogenous (i.e. game dependent). Some particular cases covered by Proposition 4.7 follow.

Example 4.8 *Bertrand Duopoly with Discontinuous Costs*

Hoerning [15] introduced the following modification of Bertrand's game: each firm $i = 1, 2$ chooses a price $p_i \in [0, 1]$; the demand is $D(p_1, p_2) = \max\{0, 1 - \min(p_1, p_2)\}$; the total (symmetric) cost for each firm is $C(q) = \tilde{C} \in (0, \frac{1}{4})$ if the production q is positive, and $C(0) = 0$ otherwise. Assuming equal sharing at ties, Hoerning [15] proved that the game has no mixed Nash equilibrium. By Proposition 4.7, it has an approximate equilibrium.

Example 4.9 *Bertrand-Edgeworth Duopoly with Capacity constraints*

There are two firms. Firm i has an endowment of C_i units of the commodity (the capacity of a zero-cost technology). Firms choose prices (p_1 and p_2). The firm choosing the lowest price (say p) serves the entire market $D(p)$ up to its capacity. The residual demand $D(p) - C_i$ is met by the other firm (up to its capacity as well). If the duopolists set the same price they share the market according to some rule h. If h shares the market in proportion to the capacities, Dasgupta and Maskin [9] proved the existence of a mixed equilibrium. Proposition 4.7 proves the existence of an approximate equilibrium for any continuous h.

Example 4.10 *Timing Games*

Two players $i = 1, 2$ must choose a time $t_i \in [0, 1]$ and the game is over at the first stop $t = \min\{t_1, t_2\}$. If player 1 stops first, the payoff vector is $a(t)$, if it is player 2, the payoff is $b(t)$, and if both stop simultaneously, the payoff is $c(t)$. This is a diagonal game (under the change of variable $x_i = 1 - t_i$). Well-known examples are wars of attrition and preemption games. When a, b and c are continuous, Proposition 4.7 states that there is a mixed approximate equilibrium, a result already known from Laraki et al. [20].

Example 4.11 *Silent Duels*

These are two-player zero-sum timing games. In the simplest version [32], both players are endowed with one bullet, and have to choose when to fire. As time goes, the two players get closer. More precisely, $p_i(t)$ denotes the probability that player i wins if he shoots the opponent at time t. This probability $p_i(t)$ is assumed to be continuous and strictly increasing as t goes from 0 to 1, with $p_i(0) = 0$ and $p_i(1) = 1$. We suppose that when a player fires, the other does not know it (the game continues for that player). Thus, a strategy of a player is when to fire, conditionally that he is still alive. Thus $f_i(t_i, t_j) = p_i(t_i) - (1 - p_i(t_i))p_j(t_j)$, $g_i(t_i, t_j) = -p_j(t_j) + (1 - p_j(t_j)p_i(t_i))$, and $h_i(t_i, t_j) = p_i(t_i)(1 - p_j(t_j)) - p_j(t_j)(1 - p_i(t_i))$.

It is well know that this game does not have a value in pure strategies, but does have one in mixed strategies when $p_i(t) = p_j(t) = t$. From Proposition 4.7, there is an approximate equilibrium, and so a value for every continuous functions p_i and p_j (even if they are not monotonic as usually assumed). In fact, the proposition implies the existence of an approximate equilibrium in every two-player silent timing game with complete information (see [29]).

Auctions with Correlated Types

In many economic models, such as auctions, players do not have full knowledge about other player's evaluations. This leads naturally to the following class of Bayesian diagonal games. At stage 0, a type $t = (t_1, ..., t_N) \in T = T_1 \times ... \times T_N$ is drawn according to some joint probability distribution p, and each player i is informed of his own type t_i (correlations between types are allowed). At stage 1, each player i is asked to choose an element $x_i \in [0, 1]$ (interpreted as a bid). The payoff of player i is:

$$u_i(t, x_i, x_{-i}) = \begin{cases} f_i(t, x_i, \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) > x_i, \\ g_i(t, x_i \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) < x_i, \\ h_i(t, x_i, x_{-i}) & \text{if } \phi_i(x_{-i}) = x_i, \end{cases}$$

where, $\bar{x}_{-i} = \max_{j \neq i} x_j$, where $f_i(t, \cdot, \cdot)$ and $g_i(t, \cdot, \cdot)$ are two continuous mappings on $[0, 1] \times [0, 1]$, and $\phi_i : \Pi_{j \neq i}, X_j \to [0, 1]$ is an aggregation function. The mapping h is the tie-breaking rule and may be discontinuous. The game is called a game of private values if for every i, u_i depends only on its own type t_i and does not depend on t_{-i}.

Proposition 4.12 Any Bayesian diagonal game admits a weak strategic approximation (and so a mixed approximate equilibrium)[22] if for every i and t one has:

(a) $f_i(t, 0, 0) \leq h_i(t, 0, .., 0) \leq g_i(t, 0, 0)$;

(b1) $f_i(t, 1) \geq h_i(t, 1) \geq g_i(t, 1)$ **or**[23] (b2) there is $\eta > 0$ such that there is always a best

[22] Importantly, for every $\varepsilon > 0$, the ε-equilibria we build in the proof are tie-breaking rule independent.

[23] Conditions b1 and b2 are boundary conditions at 1. Condition b1 is satisfied by first-price, second-price, multi-unit and double auctions, but not for all-pay auctions. Condition b2 is true for all-pay, first-price, second-price and multi-unit auctions, but not for double auctions (see Examples 4.13, 4.14 and 4.15).

response of each type in $[0, 1-\eta[$;

(c1) there are only two players **or** (c2) values are private

The proposition is formally proved in Appendix 6.7. Let us give some auction models for which conditions (a) and (b) are satisfied.

Example 4.13 *One unit first-pay, Second-pay and All-Pay Auctions*

Take any N-player auction where the winner is the player with maximal bid. More precisely, suppose that player i's value for the object is $v_i(t) \in]0,1[$. If i wins the auction ($x_i > \bar{x}_{-i}$, where $\bar{x}_{-i} = \max_{j \neq i} x_j$) and pays a price $p_i(x_i, \bar{x}_{-i}) \geq 0$, his final utility is $g_i(t,x) = v_i(t) - p_i(x_i, \bar{x}_{-i})$ where p_i is continuous, non decreasing in both variables and $p_i(y,y) = y$ for every y. If player i looses the auction ($x_i < \bar{x}_{-i}$), he pays a transfer $\tau_i(x_i) \geq 0$, and so his utility is $f_i(t,x) = -\tau_i(x_i)$ where $\tau_i(x_i)$ is continuous, non decreasing and $\tau_i(0) = 0$. In case of a tie ($x_i = \bar{x}_{-i}$), the winner is selected uniformly among the set of players with maximum bid. For example, in first-price and second-price auctions, $\tau_i = 0$. In all-pay auctions, $\tau_i = -x_i$. In first-pay and all-pay auctions, $p_i(x_i, \bar{x}_{-i}) = \max\{x_i, \bar{x}_{-i}\}$, in second-price auctions, $p_i(x_i, \bar{x}_{-i}) = \bar{x}_{-i}$. In this general model, $0 = f_i(t,0) < g_i(t,0) = v_i(t)$ and $h_i(t,0) = \frac{v_i(t)}{N}$, so that (a) is satisfied. Condition (b1) is satisfied in first-price and second-price auctions, but not in all-pay auctions. However, Condition (b2) is satisfied in these three type of auctions because player i has always a best response in $[0, max_t v_i(t) + \varepsilon[$. Thus Proposition 4.12 applies when there are two players or values are private.

Example 4.14 *Multi-unit Auctions*

Consider the previous model with the following modification: K homogeneous units of an indivisible good are sold, but each bidder $i = 1, ..., N$ ($N \geq K$) can buy only one unit of the good. Player i wins if his bids is among the K highest bids. In case of a tie, the remainig winners are chosen at random among the tie-players. Proposition 4.12 applies, and here $\phi(x_1, ..., x_{N-1})$ is simply the K-th highest of $x_1, ..., x_{N-1}$.

Example 4.15 *Double Auction*

Suppose player 1 is a buyer with a value $v(t_1, t_2) \in]0,1[$ and player 2 is a seller with a cost $c(t_1, t_2) \in]0,1[$. Player 1 chooses a maximum bid $x_1 \in [0,1]$ and player 2 a minimum price x_2. If $x_1 < x_2$, there is no trade (so that $f_1(t, x_1, x_2) = g_2(t, x_1, x_2) = 0$). If $x_1 \geq x_2$, there is a trade at price $p = \frac{x_1+x_2}{2}$, so that $h_1(t, x_1, x_2) = g_1(t, x_1, x_2) = v(t_1, t_2) - \frac{x_1+x_2}{2}$ and $h_2(t, x_1, x_2) = f_2(t, x_1, x_2) = \frac{x_1+x_2}{2} - c(t_1, t_2)$. Consequently, $f_1(t, 0, 0) = 0 < h_1(t, 0, 0) = g_1(t, 0, 0) = v(t)$ and $f_2(t, 0, 0) = -c(t) = h_2(t, 0, 0) < g_2(t, 0, 0) = 0$: condition (a) is satisfied. Condition (b1) is satisfied similarly, but Condition (b2) is not satisfied for the seller. Since the game has only two players, Proposition 4.12 applies.

Our existence result above is to be compared with the one in Jackson and Swinkels [18]. They show the existence of a Nash equilibrium which is tie-breaking-rule independent in multi-unit auctions with private values and uncorrelated types. Recalling that when types are correlated, a Nash equilibrium may not exist (Fanga and Morris [11]), the existence of an approximate equilibrium is the best one can prove. From the last examples, we conclude that any two player general standard auctions (first-pay, second-pay, all-pay and double auctions) has an approximate equilibrium, even if types are correlated and values are common. An open question is whether Proposition 4.12 holds for N-player without condition (c2). Without conditions (a) or (b) it does not hold as a following example shows.

4.4. *Games without Approximate Equilibria*

The following examples illustrate that the two last propositions are tight.

Example 4.16 Sion-Wolfe's [24] zero-sum game shows that the existence of a mixed approximate equilibrium in Proposition 4.7 might fail whenever a game on $[0,1]\times[0,1]$ admits two lines of discontinuities (in Sion-Wolfe's game: $\{x_2 = \frac{1}{2} + x_1, x_1 \leq \frac{1}{2}\} \cup \{x_2 = x_1\}$) instead of only one line of discontinuity (in our case $\{x_2 = x_1\}$).

Example 4.17 In Proposition 4.12, without condition (a) or (b), an approximate equilibrium may fail to exist, as the following example shows. Consider a zero-sum timing game which may be viewed as a diagonal game with constant payoff functions f, g and h. Each player should decide when to stop the game between 0 and 1. The game stops at the first moment when one of the two players stops. If both players stop simultaneously before the exit time $t = 1$ or no player stops before time $t = 1$, then there is a tie (payoff is given by h).
Player 2 has two types A and B with equal probabilities. Player 1 has only one type. If player 1 stops first, he gets $f = 1$. If player 1 stops second he gets $g = -1$. The payoffs depend on the type of player 2 only when the players stop simultaneously. If his type is A, player 1 has an advantage and gets the payoff $h = 3$, and if his type is B, player 1 has a disadvantage and gets the payoff $h = -2$.
We can prove that $\max\min \leq -\frac{1}{2}$ and that $\min\max \geq -\frac{1}{4}$, so that the game has no value and so no approximate equilibrium (See Appendix 6.8).

5. Conclusion

Our paper proposes a unifying framework to study the existence of approximate and sharing rule equilibria in discontinuous games, which links Simon-Zame and Reny approaches in pure and mixed strategies.

In the first part, we focus on quasiconcave compact games in *pure strategies*. A new relaxation of Nash equilibrium notion (Reny equilibrium) is shown to always exist. It provides tight conditions, in the spirit of Reny's conditions, that guarantee the existence of an approximate equilibrium. Reny equilibrium is also used to solve an open problem in Jackson et al. [17], namely the existence of a sharing rule equilibrium in pure strategies (up to now, existence was known only for games in mixed strategies). As applications, we prove the existence of a sharing rule equilibrium in any exchange economy model with discontinuous preferences and of a pure approximate equilibrium in a large class of multi-player diagonal games.

In the second part, we concentrate on metric compact games in *mixed strategies*. We prove that the intersection between the sets of Simon-Zame's solutions and Reny equilibria contains the set of approximate equilibria. Moreover, this intersection is nonempty. This shows that the three main solution concepts of the paper are strongly connected. As application, we prove the existence of an approximate equilibrium in a large class of auctions.

6. Appendix

6.1. *Proof of Proposition 3.15*

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Under Assumption (C), the game is payoff secure: indeed, if $x_i \neq \phi(x_{-i})$, x_i is secure for player i. If $x_i = \phi(x_{-i})$, then player i can secure his payoff (up to an arbitrary $\varepsilon > 0$), increasing or decreasing x_i slightly. Consequently, if $(x, v) \in \overline{\Gamma}$ is a Reny equilibrium then:

$$(3) \qquad \sup_{d_i \in [0,1]} \underline{u_i}(d_i, x_{-i}) = \sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \ i \in N.$$

Now, we prove that x is an approximate equilibrium profile by checking 4 different cases. In the *first case*, assume for every player i, $x_i \neq \phi(x_{-i})$. Thus, utilities are continuous at x and $v = u(x)$. From Equation (3), $(x, v) = (x, u(x))$ is a Nash equilibrium. In the *second case*, there exists a player i such that $x_i = \phi(x_{-i}) \in]0, 1[$. Then $\phi(x_{-j}) \in]0, 1[$ for every j. Indeed, for every $j \neq i$, either $x_j \geq x_i > 0$, and anonymity and representativity imply that $\phi(x_{-j}) > 0$ and monotonicity that $\phi(x_{-j}) \leq \phi(x_{-i}) < 1$, or $x_j \leq x_i < 1$, and we get similarly $\phi(x_{-j}) \in]0, 1[$. Thus, the marginal continuity property is satisfied at x, since $\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) = \max\{\sup_{d_i < \phi(x_{-i})} f_i(d_i, x_{-i}), \sup_{d_i > \phi(x_{-i})} g_i(d_i, x_{-i})\}$ (from assumption (C)), and from the continuity of ϕ, f_i and g_i. Thus, by Proposition 3.13, (x, v) is an approximate equilibrium. In the *third case*, there exists a player i s.t. $x_i = \phi(x_{-i}) = 0$. Then $\phi(x_{-j}) = 0$ for every player j: indeed, by anonymity, $\phi(x_{-j}) = 0$ for every j such that $x_j = 0$, and by monotonicity of ϕ, $\phi(x_{-j}) = 0$ for every j such that $x_j > 0$. We let (x^n) be a sequence of profiles such that $(x^n, u(x^n)) \to (x, v)$. For every player j such that $x_j = 0$ and $v_j \leq f_j(0)$, we let $y_j^n := 0$ for every integer n, and $y_j^n := x_j^n$ (for every integer n) otherwise. This defines a sequence of profiles y^n converging to x. Let us prove that y^n is an ε^n-equilibrium from some

$\varepsilon^n \to 0$. Observe that from continuity of ϕ, $\phi(y^n_{-j}) \to 0$ for every player j. We now consider 3 sub-cases:

- If $v_j \leq f_j(0)$, from Equation (3) above, $\sup_{d_j} g_j(d_j, 0) \leq v_j \leq f_j(0)$ and in particular we get $g_j(0) \leq f_j(0)$. Let us prove that $\sup_{d_j} g_j(d_j, 0) = g_j(0) \leq v_j \leq f_j(0)$. If $g_j(0) < f_j(0)$, this is a consequence of quasi-concavity of u_i (indeed, $h_j(0) > g_j(0)$, thus $g_j(., 0)$ is non increasing from quasi-concavity of $u_j(., 0)$). If $g_j(0) = f_j(0)$, this is obvious (the weak inequalities are satisfied with equalities). Moreover, using (C) one has $g_j(0) \leq h_j(0) \leq f_j(0)$ and by continuity of g_i and f_i, one has $g_j(\phi(y^n_{-j}), \phi(y^n_{-j})) - \varepsilon_n \leq h_j(\phi(y^n_{-j}), y^n_{-j}) \leq f_j(\phi(y^n_{-j}), \phi(y^n_{-j})) + \varepsilon_n$ for some $\varepsilon_n \to 0$. The above inequalities imply that player j is $4\varepsilon_n$-optimizing by playing $y^n_j = 0$ against y^n_{-j}, because if $\phi(y^n_{-j}) > 0$ player j cannot get more that $f_j(0)$ up to $4\varepsilon_n$ and he can get it by playing 0 and if $\phi(y^n_{-j}) = 0$ player j's best payoff is $h_j(0)$, which he can achieve by playing 0.
- Second, if $v_j > f_j(0)$ and $x_j = 0$, from equation (3) $g_j(0) \leq v_j$ and from (C) we deduce that $f_j(0) < g_j(0) = v_j$. By the strict convex combination assumption in (C) and continuity of f and g, we have $x^n_j > \phi(x^n_{-j})$ along the sequence (otherwise, $v_j < g_j(0)$). By monotonicity of ϕ, $y^n_j = x^n_j > \phi(x^n_{-j}) \geq \phi(y^n_{-j})$. Thus, $u_j(y_n) = g_j(y^n_j, \phi(y^n_{-j})) \to v_j = g_j(0)$. From equation (3), the fact that $f_j(0) < g_j(0)$ and condition (C), we get that y^n_j is an ε_n-best-response against y^n_{-j}.
- Third, if $v_j > f_j(0)$ and $x_j > 0$, since $\phi(x_{-j}) = 0$, $v_j = g_i(x_j, 0) = u_j(x)$. From Equation (3), x_j is a best response against x_{-j} and since $v_j > f_j(0)$, y^n_j is an ε_n-best-response against y^n_{-j}. Consequently, x is a approximate equilibrium profile associated with the sequence y^n.

In the *last case*, we assume there exists $i \in I$ such that $x_i = \phi(x_{-i}) = 1$: this can be treated as in the third case.[24]

6.2. *Proof of Proposition 3.19*

As usual [34], we can associate to these data the following $(N+1)$-player strategic game: for $i = 1, ..., N$, player i's strategy space is X_i, and its payoff is $v_i(x_i, p) = u_i(x_i)$ if $p \cdot x_i \leq p \cdot e_i$ (i.e. if the budget constraint of i is satisfied), and $v_i(x_i, p) = -1$ otherwise. The strategy space of Player $(N+1)$ (the auctioneer) is $X_{N+1} = \Delta(\mathbf{R}^m_+)$, and his payoff function is $v_{N+1} = p.\sum_{i \in N}(x_i - e_i)$. This defines a compact and quasiconcave game which admits a sharing rule solution (x, p, q), where $q_i : \Pi_{j=1}^{n+1} X_j \to \mathbf{R}$ is the new payoff of player i. From Theorem 3.17, we can assume: for every $i \in N$ and every $y_i \in X_i$, $q_i(y_i, x_{-i}, p) \geq \underline{\underline{v_i}}(y_i, x_{-i}, p)$, the last quantity being non negative if and only if $p \cdot (y_i - e_i) \leq 0$ (simply consider a local selection $d_i(\cdot)$ of the budget constraint around y_i, continuous with respect to p, in the Definition of $\underline{\underline{v_i}}$), thus in

[24]In this proof or in the proof of Proposition 3.21, we do not need Unanimity condition $\phi(y, ..., y) = y$.

particular at $y_i = e_i$. Thus, for every consumer i, $q_i(x_i, x_{-i}, p) \geq \sup_{d_i \in X_i} q_i(d_i, x_{-i}, p) \geq 0$. From the definition of a sharing rule, this implies that for every $i \leq n$, there exists a sequence (x^n, p^n) converging to (x, p), with $p^n \cdot (x_i^n - e_i) \leq 0$, and $q_i(x, p) = \lim_{n \to +\infty} u_i(x_i^n)$. Thus, all budget constraints are satisfied at (x, p). Now, define $\tilde{u}_i(y_i) = q_i(y_i, x_{-i}, p)$ whenever $p \cdot (y_i - e_i) \leq 0$, and $\tilde{u}_i(y_i) = u_i(y_i)$ elsewhere. To prove that (x, p) is a Walrasian equilibrium of $\{X_i, \tilde{u}_i, e_i\}_{i \in I}$, we let $y_i \in X_i$ such that $p \cdot (y_i - e_i) \leq 0$. Then $\tilde{u}_i(y_i) = q_i(y_i, x_{-i}, p) \leq q_i(x_i, x_{-i}, p) = \tilde{u}_i(x_i)$. Last, summing the budget constraints of all consumers, we get $p \cdot \sum_{i \in I}(x_i - e_i) \leq 0$, which implies $\sum_{i \in I}(x_i - e_i) = 0$ since p maximizes $p \cdot \sum_{i \in I}(x_i - e_i)$ in $\Delta(\mathbf{R}_+^m)$ (indeed, u_{N+1} is continuous; thus it is equal to $\tilde{u}_{N+1}$).

6.3. *Proof of Proposition 3.21*

First, Theorem 3.16 permits to refine Proposition 3.13 as follows.

PROPOSITION 6.1 If (x, v) is a strong Reny equilibrium, if the game G is weakly payoff secure[25] (meaning $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) = \sup_{d_i \in X_i} \underline{\underline{u_i}}(x_i, x_{-i})$) and if G has the marginal continuity property at x, then (x, v) is an approximate equilibrium.

The steps of the proof are as in Proposition 3.15. Firstly, for every ϕ the game is weakly payoff secure (but not payoff secure): if $x_i \neq \phi(x_{-i})$, then u_i is continuous at x and x_i allows to continuously secure $u_i(x) - \varepsilon$. If $x_i = \phi(x_{-i})$, the deviation mapping $d_i(x) = \phi(x_{-i})$ allows to continuously secure $u_i(x) - \varepsilon$. Consequently, if (x, v) is a strong Reny equilibrium, then:

$$\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \; i \in N \tag{4}$$

The *cases 1 and 2* (i.e. $x_i \neq \phi(x_{-i})$ for every player i, or $x_i = \phi(x_{-i}) \in]0, 1[$ for some i), are solved as in Proposition 3.15 (where Proposition 6.1 is used instead of Proposition 3.13). Consider the *third case* $x_i = \phi(x_{-i}) = 0$ for some $i \in I$, which implies $x = 0$ and $v_i \in \{g_i(0), h_i(0), f_i(0)\}$ for every i (from the continuity of f_i, h_i and g_i). Assuming x is not a Nash equilibrium (otherwise the proof of the third case is done), some player j has a profitable deviation, that is $u_j(0) = h_j(0) < \sup_{d_j} g_j(d_j, 0)$. For such a player, $f_j(0) \leq g_j(0)$, which follows from the quasi-concavity of $u_j(0, \varepsilon)$ for every $\varepsilon > 0$ and the previous inequality. Equation 4 implies that $v_j = g_j(0)$. Considering a sequence $(x^n, u(x^n)) \to (x, v)$, one must have $x_j^n > \phi(x_{-j}^n)$ for n large enough. Then, for any player $k \neq j$, $v_k = f_k(0)$. Thus, from equation (4), we deduce that the strategy profile defined by $x_j^n = \frac{1}{n}$, and $x_k = 0$ for $k \neq j$, is an ε^n-equilibrium, for some $\varepsilon^n \to 0$. Now, in the *last case*, there exists i s.t. $x_i = \phi(x_{-i}) = 1$.

[25] As for payoff security, weak payoff security concept has an alternative definition that is: for every $\varepsilon > 0$, every $x \in X$ and every player $i \in I$, player i can secure continuously $u_i(x) - \varepsilon$, which means there is a continuous mapping d_i from some neighborhood V of x_{-i} to X_i, with $u_i(d_i(y_{-i}), y_{-i}) > u_i(x) - \varepsilon$ for every $y_{-i} \in V$. Again, we restrict to the simpler case of a continuous mapping d_i and not a multivalued mapping.

Consequently, for every k, $\phi(x_{-k}) = 1$ and if $K = \{k \in N : x_k = 1\}$, then K has at least two elements. For a player $j \notin K$, utility is continuous at x and from equation 4 he has no interest to deviate. If no player in K wants to deviate from x, x is a Nash equilibrium profile. Otherwise, as in the third case, at most one player $k \in K$ wants to deviate, and we can then check that $x_k^n = 1 - \frac{1}{n}$ and $x_j^n = x_j$ for $j \neq k$ defines an ε^n-equilibrium for some $\varepsilon^n \to 0$.

6.4. *Proof of Theorem 4.3*

A limit-equilibrium (m, v) exists by compactness of $\overline{\Gamma'}$. First let us prove that it is a Reny equilibrium of Γ': for every deviation $d \in M$, and any neighborhood $V(m, v)$ of (m, v), the definition gives $\exists m^V \in M$ s.t. (1) $\forall i \in N$, $u_i(d_i, m_{-i}^V) \leq u_i(m^V)$ and (2) $(m^V, u(m^V)) \in V$. Shrinking V to (m, v) implies that $\underline{u_i}(d_i, m_{-i}) \leq v_i$ for every $d \in M$, i.e. (m, v) is a Reny equilibrium of Γ'.

Now, let us prove that (m, v) induces a solution à la Simon-Zame. Since M is a compact metric set, there exists a countable decreasing basis of neighborhoods V^n of (m, v) in $\overline{\Gamma'}$. Consider a sequence $D^n = \Pi_{i\in I} D_i^n$ converging to X for the Hausdorff distance. By definition of a limit-equilibrium, for every integer n, there exists a finite set $D^{'n} = \Pi_{i \in I} D_i^{'n}$ of M containing D^n, and a probability m^n, which is a Nash equilibrium of the game restricted to $D^{'n}$ such that $(m^n, u(m^n)) \in V^n$. Recall that Simon and Zame's [38] existence proof consists in approximating the game by a finite game in pure-strategies (here D^n), and in considering a weak limit of a sequence (m^n) of Nash equilibria of this approximation. We cannot apply Simon and Zame's proof directly to the Nash equilibria m^n of the finite games D'^n, because D'^n contains mixed strategies. But $D^{'n} \supset D^n$: thus, no player i has a deviation in D_i^n against m^n, and we shall prove that this property is sufficient to adapt Simon-Zame's proof. Note that the sequence (m^n) converges (strongly and weakly) to m. We let E be the space of R^N-valued vector measures on X, endowed with the weak* topology. Consider the sequence $(u \cdot m^n)_{n \in \mathrm{N}}$ of the compact space E (here, $u \cdot m^n$ denotes the R^N-valued measure on X defined by $u \cdot m^n(F) = \int_F u\, dm^n$ for every Borelian set F of X). Without any loss of generality, up to a subsequence, this sequence converges to some measure ν. From Lemma 2, p.867 (Simon-Zame [38]), there exists a Borel measurable selection q of Q, the multivalued function from X to R^N, defined by $Q(x) =$co $\overline{\Gamma_x}$, such that the $\nu = q \cdot m$ (remark that the proof of this lemma does not use the support of m^n, but only the fact that u is a selection of Q). Define, for every player i, $H_i = \{x \in X_i : \int q_i d(\delta_x \times m_{-i}) > \int q_i d(m_i \times m_{-i})\}$. We prove $m_i(H_i) = 0$: otherwise, consider $K \subset H_i \subset U$, where K is compact, U open, $m_i(U - K) < \varepsilon$ with $\varepsilon > 0$, and with $m_i(K) > 0$. We let $f : X_i \to [0, 1]$ be a continuous function which is identically equal to 1 on K and 0 on the complement of U. Consider the strategy $\beta_i^n = \frac{f m_i^n}{\int f dm_i^n}$: it is a better response to m_{-i}^n for n large enough and $\varepsilon > 0$ small enough, which contradicts the fact that m^n is a Nash equilibrium of the game restricted to $D^{'n}$. From Simon-Zame ([38], Step 5 and Step 6), there exists a modification $\tilde{q}$ of q, such that $q = \tilde{q}$ except on a set of $m-$measure 0, such that m is a Nash equilibrium of the game $\tilde{G} = ((X_i)_{i\in N}, (\tilde{q}_i)_{i\in N})$, and $\tilde{q}(m) = q(m)$. More precisely, take $\tilde{p}^i$ a Borel measurable selection of Q which minimizes the i-th component of Q, define $T = \{x \in X : x_i \in H_i$ for at least two indices $i \in N\}$, define $\tilde{q}(x) = \tilde{p}^i(x)$ if $x \in H_i \times X_{-i}$ but $x \notin T$, and $\tilde{q}(x) = q(x)$ otherwise. To prove that m is a Nash equilibrium of $\tilde{G}$, assume that some player i has a better pure response than m_i, denoted δ_x, to m_{-i}. Then the case $x \notin H_i$ yields an easy contradiction. For the second case, simply consider a sequence x^n converging to x such that $x^n \in D_i^n$ (here, we use that $D^n = \Pi_{i\in I} D_i^n$ converges to X for the Hausdorff distance): an easy limit argument proves that δ_{x^n} is a better response than m_i^n to m_{-i}^n for n large enough, a contradiction with the choice of m^n.

6.5. *Proof of Theorem 4.6*

If $(m, v) \in \overline{\Gamma'}$ is not a limit-equilibrium (and so is not a Reny and so is not an approximate equilibrium, because the game is approximately better-reply secure), then $\exists V^{(m,v)}$ a neighborhood of (m, v) and a finite set D_1 (associated with $V^{(m,v)}$) such that for any D that contains D_1 and any equilibrium m' of the game restricted to D', $(m', u(m'))$ is not in $V^{(m,v)}$. Since the set of limit-equilibria is compact, there is a countable basis $\{V^{(m_n,v_n)}\}_n$ that covers the open set of non-limit-equilibria, to which we can associate a sequence of finite sets $\{D_1^n\}_n$ (as defined above). Define $D^n = \bigcup_{k=1}^n D_1^k$ and let $\{m_n\}_n$ be any sequence of equilibria associated to D^n. Then, by construction, $(m_n, u(m_n))$ is not in $V^{(m_k,v_k)}$ for all $k \leq n$. Consequently, any accumulation point of the sequence $\{(m_n, u(m_n,)\}_n$ is a limit-equilibrium, and, thus a Reny equilibrium and so is an approximate equilibrium.

6.6. *Proof of Proposition 4.7*

The proof of Proposition 4.7 and Proposition 4.12 uses the same principle: for every $\varepsilon > 0$, we construct a finite approximation of G' such that for every mixed strategy of players $j \neq i$, player i has an ε-best response (in G') which belongs to the approximated game. This proves that every mixed Nash equilibrium of the finite approximation is an $\varepsilon-$Nash equilibrium of the initial game. The approximation shall depend on the structure of the game, and in particular on the behaviour of the payoffs in a neighborhood of the boundary.

First note that in two player diagonal games, necessarily $\phi(y) = y$ (by the unanimity condition). Call $x \in [0, 1]$ a right local equilibrium if $h_i(x, x) > g_i(x, x)$ for both $i = 1, 2$ and a left local equilibrium if $h_i(x, x) > f_i(x, x)$ for both $i = 1, 2$. Thus, if players are supposed to play (x, x) and if x is a right local equilibrium, no player has an interest to deviate to some strategy in some right neighborhood of x (but he may have a profitable deviation outside that neighborhood) and similarly for left equilibria.

We let x_0 be the largest element in $[0,1]$ such that all $x < x_0$ are right local equilibria and y_0 be the smallest element in $[0,1]$ such that all $y > y_0$ are left local equilibria. Note that x_0 may be 0 and y_0 could be 1. By continuity of f, g and h, if $x_0 < 1$ then $h_i(x_0,x_0) \leq g_i(x_0,x_0)$ for some $i \in \{1,2\}$ and similarly, if $y_0 > 0$ then $h_j(y_0,y_0) \leq f_j(y_0,y_0)$ for some $j \in \{1,2\}$. Depending on the relative position of x_0 and y_0, we consider the three following cases.

First case. $x_0 > y_0$. In this case, the finite approximated game is simply defined by some finite discretization D of $[0,1]$ containing 0 and 1 and σ a mixed strategy of the game restricted to D . Without any loss of generality, taking the mesh of this discretization smaller than some $\eta > 0$, we can assume that the payoff functions f and g do not change by more than $\frac{\varepsilon}{2}$ if a player moves by no more than η, and such that if $x < x_0$ is in D, then $h_i(x,x) > g_i(y,x)$ for all $x < y < x+\eta$, and if $x > y_0$ is in D, then $h_i(x,x) > f_i(y,x)$ for all $x > y > x+\eta$. We let $y \in [0,1]$ be some $\varepsilon/2$-best reply to σ_j of player i which is not in D (if such strategy does not exist, we are done with the proof). Then either $y < x_0$ or $y > y_0$. In the first case, denote by z the highest element in D smaller than y, so that $h_i(z,z) > g_i(y,z)$ by assumption of the discretization and since z is a right equilibrium. Since player j plays a probability distribution supported on D, moving from y to z for player i induces for him a higher payoff from the event associated to player j playing z and at most a change of $\frac{\varepsilon}{2}$ on the events where player j is playing a strategy in D different from z. Thus, z is an ε-best reply for player i. A similar argument applies to $y > y_0$ (use the left equilibrium property).

Second case. $x_0 < y_0$, which implies that $h_k(x_0,x_0) \leq g_k(x_0,x_0)$ and $h_l(y_0,y_0) \leq f_l(y_0,y_0)$ for some $k \in \{1,2\}$ and $l \in \{1,2\}$. By continuity, we get $\eta > 0$ small enough such that $h_k(x_0,x_0) < g_k(x,x_0) + \frac{\varepsilon}{4}$ for every $x \in]x_0, x_0+\eta[$ and $h_l(y_0,y_0) < f_l(y,y_0) + \frac{\varepsilon}{4}$ for every $y \in]y_0-\eta, y_0[$. Thus, there are four cases to check, depending on the values of k and l. Let us solve explicitly the case $k=1$ and $l=2$. The same idea of construction could be done in the other cases, with a small adaptation in the definition of the weak strategic approximation.

Fix $\varepsilon > 0$ and define $x_0 = t_0 < s_0 < t_1, ... < s_{K-1} < t_K = y_0$, a discretization of $[x_0,y_0]$ with a mesh smaller than some $\eta > 0$ so that payoff functions f and g do not change by more than $\varepsilon/4$ if the pure strategy moves by no more than η. As in the first case, we let D be a finite discretization of $[0,x_0[\cup]y_0,1]$ with a mesh smaller than $\eta > 0$ so that payoff functions f and g do not change by more than $\frac{\varepsilon}{2}$ if the pure strategy moves by no more than η and such that if $x < x_0$ is in D, then $h_i(x,x) > g_i(y,x)$ for all $x < y < x+\eta$ and if $x > y_0$ is in D, then $h_i(x,x) > f_i(y,x)$ for all $x > y > x+\eta$. Now, the finite approximation of G' is defined as follows: player 1 is restricted to play in D or uniformly on one of the intervals $[t_k, s_k]$, $k = 0,...,K-1$, or to choose $t_K = y_0$. Player 2 is restricted to play in D or uniformly on one of the intervals $[s_k, t_{k+1}]$, $k = 0,...,K-1$, or to choose $t_0 = x_0$. Observe that the intervals where players are uniformly mixing are disjoint and alternate. We let σ be some strategy of player 2 in the restricted game. Let y be some $\varepsilon/4$ pure best response of player 1 in G, which is not in the discretization D (again, if it does not exist, this is finished). Several subcases have to be examined. *First subcase*, if $y < x_0$ or $y > y_0$, we proceed as in the first case to construct an ε-best reply in D. *Second subcase*, if y is in some interval $]s_k, t_{k+1}[$ of player 2 ($k \in \{0, 1, ..., K-1\}$), and if player 2 is choosing that interval with positive probability, an easy computation proves that the payoff of player 1 coming from that interval is, up to $\varepsilon/4$, a convex combination of his payoff if he chooses t_{k+1} and his payoff if he chooses s_k. But, the payoff of player 1 coming from Player 2 playing in the other intervals or in D changes by no more that $\varepsilon/4$ when he moves in the interval $[t_k, s_{k+1}]$. Consequently, player 1 has a $\frac{3}{4}\varepsilon$-best response at the extreme points t_k or s_{k+1} of the interval, a case which is treated in the next subcase: *Third subcase*, let $z \in [t_k, s_k]$ being a $\frac{3}{4}\varepsilon$-best response, for some $k \in \{0, 1, ..., K-1\}$. If $k > 0$, by assumption, there is zero probability that player 2 stops in that interval and so player 1's payoff does not move by more that $\varepsilon/4$ if he plays uniformly in $[t_k, s_k]$ (which is authorized for player 1) instead of playing z. This gives an ε-best response. If $k = 0$, if player 2 is playing x_0 with positive probability and player 1 is playing $z = x_0$, then player 1 does not lose more than $\varepsilon/4$ by playing slightly more than x_0 instead of x_0 (since $h_1(x_0, x_0) < g_1(x, x_0) + \frac{\varepsilon}{4}$ for every $x \in]x_0, x_0 + \eta[$.): then remains the case where z belongs to the interval $]t_0, s_0[$. But, again, since his payoff moves continuously in that interval, playing uniformly in it is an ε-best response. The proof for player 2 is similar (we use the fact that $h_2(y_0) < f_2(y, y_0) + \frac{\varepsilon}{4}$ for every $y \in]y_0 - \eta, y_0[$).

The three remaining cases for k and l are solved similarly, by a judicious choice of who of the two players is allowed to stop at x_0 and y_0: if $k = 2$ and $l = 1$, then player 1 can stop at x_0 and player 2 at y_0; if $k = 2$ and $l = 2$, (only) player 1 is allowed to stop at both x_0 and y_0; if $k = 1$ and $l = 1$, only player 2 is allowed to stop at both points. If some player can stop at x_0 then it is the other player who is authorized to stop uniformly in the small interval of the dicretization just after x_0, and the intervals in which players can stop uniformly alternate until the point y_0, and the last interval belongs to the player who is not allowed to stop at y_0.

Third case. $x_0 = y_0$, implying $h_k(x_0) < g_k(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0, x_0 + \eta[$ and $h_l(x_0) < f_l(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0 - \eta, x_0[$ for some $k \in \{1, 2\}$ and $l \in \{1, 2\}$ (if x_0 is 0 or 1, then only one of the inequalities holds). Suppose for example that $h_1(x_0) < g_1(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0, x_0 + \eta[$. We let $D_1 = \{0 = t_0 < ... < t_K\}$ be a discretization on the left of x_0, not including x_0, and empty if $x_0 = 0$; let $D_2 = \{s_0 < ... < s_K = 1\}$ be a discretization on the right of y_0, not including y_0, and empty if $y_0 = 1$. Again, without any loss of generality, assume that the mesh of the discretizations is smaller than $\eta > 0$, so that payoff functions f and g do not change by more than $\frac{\varepsilon}{2}$ if a player moves by no more than η. Consider a strategic approximation where Player 2 is allowed to play in $D_1 \cup D_2 \cup \{x_0\}$ and player 1 to play in $D_1 \cup D_2$ or to mix uniformly in the length $[x_0, s_0]$. Let $y \in [0, 1]$ be some $\varepsilon/2$-best reply of player 1 to some mixed strategy of player 2 which is not in D_1 (if such strategy does not exist, this is finished). If $y < x_0$, moving from y to the highest element in D_1 smaller than y gives an ε-best reply for player 1. If $y > x_0$, moving from y to the smallest element in D_1 larger than y gives an ε-best reply for player 1. Last, if $y = x_0$, playing uniformly in $[x_0, y_0]$ instead of playing x_0 is an ε-best reply for player 1, because of $h_1(x_0) < g_1(x, x_0) + \frac{\varepsilon}{4}$ for $x \in]x_0, x_0 + \eta[$. We treat in a similar way the case of player 2, and the case $k = 2$.

6.7. *Proof of Proposition 4.12*

Case c2: the multiplayer private value setting

Define a weak strategic approximation of the initial game G as follows: for each integer m, a strategy (in M_i) of player i (whatever his type) is some element of the finite set D^m of uniform distributions on $I_m^k = [\frac{k}{m}, \frac{k+1}{m}]$ ($k \in \{0, 1, ..., m-1\}$). By Nash theorem, this finite (Bayesian) game has a mixed Nash equilibrium σ^m. We shall prove that if players $j \neq i$ play according to σ^m_{-i}, each type t_i of player i has some ε-best response (in G') which belongs to D^m. This proves that σ^m is an ε-Nash equilibrium of G' for m large enough.

Consider $\varepsilon > 0$, and suppose m is large enough so that for every $t \in T$, $f_i(t, ., y)$ and $g_i(t, ., y)$ does not move by more than ε in the interval $[\frac{k}{m}, \frac{k+2}{m}]$ ($k = 0, ..., m-2$) uniformly in y. If player i of type t_i chooses a pure strategy $x \in [0, 1]$ and if the realized strategy profile of its opponents is x_{-i}, then its payoff can be written $w_i(t_i, x, \phi(x_{-i}))$, where $w_i(t_i, x, \phi(x_{-i}))$ is almost surely equal to $f_i(t_i, x, \phi(x_{-i}))$ or $g_i(t_i, x, \phi(x_{-i}))$, depending on the position of $\phi(x_{-i})$ with respect to x. This is because the image probability measure of σ^m_{-i} by ϕ has no atoms[26]. It also implies that the expected payoff of player i is a continuous function of x. Consequently, there exists $x^* \in [0, 1]$, a pure best response of player t_i (in the game G'). From x^*, one can construct an ε-best response in D^m as follows: if $x^* \in [0, \frac{1}{m}]$, from assumption (a), replacing x^* by the uniform distribution on I_m^1 is a ε-best response for m large enough. If $\frac{k}{m} < x^* < \frac{k+1}{m}$ for some $k = 1, ..., m-2$ then, either $g_i(t_i, x^*, x^*) \leq f_i(t_i, x^*, x^*)$ and then replace with the uniform strategy on I_m^{k+1}, or $g_i(t_i, x^*, x^*) > f_i(t_i, x^*, x^*)$, and then replace x^* with the uniform distribution on I_m^{k-1}. In both cases, this gives an ε-best response in D^m for m large enough. Last if Assumption b1) is satisfied and not b2) and $x^* \in [1 - \frac{1}{m}, 1]$, then replace x^* with the uniform distribution on I_m^{m-1}. Note that this proof works also when the payoff of type

[26] To prove that, consider the event $[\phi(x_1, ..., x_{N-1}) = \alpha]$ for some $\alpha \in [0, 1]$. Let $S = \{(x_1, ..., x_{N-1}) \in [0, 1]^{N-1} : \sum_{k=1}^{N-1} x_k = 1\}$ be the $(N-2)$-simplex. The monotonicity assumption guarantees that for every $(x_1, ..., x_{N-1}) \in S$, there is no more than one $y \in \mathbf{R}$ such that $\phi((x_1, ..., x_{N-1}) + y(1, ..., 1)) = \alpha$. For every $x = (x_1, ..., x_{N-1}) \in S$, define $g(x_1, ..., x_{N-1}) = y$ if such y exists, and $g(x_1, ..., x_{N-1}) = 0$ otherwise. Clearly, identifying in a natural way $S \times \mathbf{R}$ to a subset of $\mathbf{R}^{N-1}$, the graph of g contains the event $[\phi(x_1, ..., x_{N-1}) = \alpha]$. But g is measurable, and thus its graph has a 0-Lebesgue-measure in $\mathbf{R}^{N-1}$ (from Fubini theorem). The assertion follows from the fact that σ^m_{-i} is a convex combination of uniform probabilities whose supports are rectangles with nonempty interiors.

t_i depends also on (t_i, t_{-i}) if we add the following assumption: for every player i and every $x^* \in [0,1]$, if $g_i(t, x^*, x^*) \leq f_i(t, x^*, x^*)$ is true for one $\bar{t}_{-i}$ then it is true for every t_{-i}, and similarly for the inequality $g_i(t, x^*, x^*) \geq f_i(t, x^*, x^*)$. Remark that this proof only requires the (strict) Monotonicity of ϕ, and not the other properties.

Case c1: the two-player general value setting

When there are two players, by uninimity $\phi(y) = y$. Now, we mimic the proof and the approximation scheme of the second case of Proposition 3.21 with $x_0 = 0$ and $y_0 = 1$, proving that if σ is some mixed strategy profile of player 2 in the approximated game, then any type t_1 has an ε-best response against σ in the full game that belongs to his set of authorized strategies. That is, take the following discretization of $[0,1]$: $0 = s_0 < t_0 < s_1 < t_1 < ... < t_K < s_{K+1} = 1$. Player 1 is restricted to play uniformly on one of the intervals $[s_k, t_k]$, $k = 0, ..., K$, or to choose $x = 1$. Player 2 is restricted to play uniformly on one of the intervals $[t_k, s_{k+1}]$, $k = 0, ..., K$, or to choose $x = 0$. Observe that the intervals where players are mixing are disjoint and alternate (player 1 can stop uniformly in the first interval, player 2 in the second, player 1 in the third, etc.). This kind of discretizations do not work for 3 player-player games...

Finally, in both cases (two players or private value with N players), by construction, in the weak strategic approximations, ties have zero probability. Consequently, our ε-equilibria are independent on the tie-breaking rule h.

6.8. *Proof of Example 4.17*

Start with the maxmin. We let α be the probability with which player 1 stops at $x = 0$ (so with probability $(1 - \alpha)$ he stops after zero). If $\alpha = 0$, player 2 by stoping at time zero gets 1 (and so player 1 gets -1). If $\alpha > 0$, type A for player 2 can stop uniformly between 0 and some ε where ε is very small so that with high probability, if the game has not been stopped at time zero, he is stopped by player 2 (just after zero). Assume that type B of player 2 stops at time zero. Payoff of player 1 is thus very close to $\alpha(\frac{1}{2} \times 1 + \frac{1}{2} \times -2) + (1 - \alpha) \times -1$. Consequently, the best strategy for player 1 against such behavior by player 2 is to stop at $t = 0$ with probability 1 so that $\max\min \leq -\frac{1}{2}$.

Let us now compute the $\min\max$. Let us restrict player 1 to playing best-replies to the following kind of strategies : (1) to stop at time $t = 0$ or (2) to stop uniformly between 0 and some ε very small, which depends of course on the strategy of player 2. Knowing this behavior, type B must stop at time zero. We let β be the probability that type A stops at time zero. The payoff of player 1 if he stops at 0 (choose option 1) is $\frac{1}{2} \times -2 + \frac{1}{2} \times (\beta \times 3 + (1-\beta) \times 1) = \frac{-1}{2} + \beta$, while if he chooses option 2 his payoff is close to $\frac{1}{2} \times -1 + \frac{1}{2}(\beta \times -1 + (1 - \beta) \times 1) = -\beta$. Thus, the optimal β for type B against this behavior of player 1 must be equalizing and so is $\beta = \frac{1}{4}$. Consequently, $\min\max \geq -\frac{1}{4}$.

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