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**MEASURE BOUNDARY VALUE PROBLEM FOR SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL HARDY POTENTIALS**

Konstantinos T. Gkikas<sup>1</sup>, Laurent Véron<sup>2</sup>

RÉSUMÉ. Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^2$  domain and  $\mathcal{L}_\kappa = -\Delta - \frac{\kappa}{d^2}$  the Hardy operator where  $d = \text{dist}(\cdot, \partial\Omega)$  and  $0 < \kappa \leq \frac{1}{4}$ . Let  $\alpha_\pm = 1 \pm \sqrt{1 - 4\kappa}$  be the two Hardy exponents,  $\lambda_\kappa$  the first eigenvalue of  $\mathcal{L}_\kappa$  with corresponding positive eigenfunction  $\phi_\kappa$ . If  $g$  is a continuous nondecreasing function satisfying  $\int_1^\infty (g(s) + |g(-s)|) s^{-2\frac{2N-2+\alpha_\pm}{2N-4+\alpha_\pm}} ds < \infty$ , then for any Radon measures  $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$  and  $\mu \in \mathfrak{M}(\partial\Omega)$  there exists a unique weak solution to problem  $P_{\nu,\mu} : \mathcal{L}_\kappa u + g(u) = \nu$  in  $\Omega$ ,  $u = \mu$  on  $\partial\Omega$ . If  $g(r) = |r|^{q-1}u$  ( $q > 1$ ) we prove that, in the supercritical range of  $q$ , a necessary and sufficient condition for solving  $P_{0,\mu}$  with  $\mu > 0$  is that  $\mu$  is absolutely continuous with respect to the capacity associated to the Besov space  $B^{2-\frac{2+\alpha_\pm}{2q}, q'}(\mathbb{R}^{N-1})$ . We also characterize the boundary removable sets in terms of this capacity. In the subcritical range of  $q$  we classify the isolated singularities of positive solutions.

**Problèmes aux limites avec données mesures pour des équations semi linéaires elliptiques avec des potentiels de Hardy critiques**

RÉSUMÉ. Soient  $\Omega \subset \mathbb{R}^N$  un domaine de classe  $C^2$  et  $\mathcal{L}_\kappa = -\Delta - \frac{\kappa}{d^2}$  l'opérateur de Hardy où  $d = \text{dist}(\cdot, \partial\Omega)$  et  $0 < \kappa \leq \frac{1}{4}$ . Soient  $\alpha_\pm = 1 \pm \sqrt{1 - 4\kappa}$  les deux exposants de Hardy,  $\lambda_\kappa$  première valeur propre de  $\mathcal{L}_\kappa$  et  $\phi_\kappa$  la fonction propre positive correspondante. Si  $g$  est une fonction continue croissante vérifiant  $\int_1^\infty (g(s) + |g(-s)|) s^{-2\frac{2N-2+\alpha_\pm}{2N-4+\alpha_\pm}} ds < \infty$ , alors pour toutes mesures de Radon  $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$  et  $\mu \in \mathfrak{M}(\partial\Omega)$  il existe une unique solution faible au problème  $P_{\nu,\mu} : \mathcal{L}_\kappa u + g(u) = \nu$  dans  $\Omega$ ,  $u = \mu$  sur  $\partial\Omega$ . Si  $g(r) = |r|^{q-1}u$  ( $q > 1$ ) nous démontrons qu'une condition nécessaire et suffisante pour résoudre  $P_{0,\mu}$  avec  $\mu > 0$  est que  $\mu$  soit absolument continue par rapport à la capacité associée à l'espace de Besov  $B^{2-\frac{2+\alpha_\pm}{2q}, q'}(\mathbb{R}^{N-1})$ . Nous caractérisons les ensembles éliminables pour les valeurs sur critiques de  $q$ . Dans le cas sous-critique nous classifions les singularités isolées au bord des solutions positives.

**Version française abrégée.** Soit  $\Omega$  un domaine de  $\mathbb{R}^N$  de classe  $C^2$ . On désigne par  $d(x)$  la distance de  $x$  à  $\partial\Omega$  et on définit l'opérateur de Hardy dans  $\Omega$  par

$$(1) \quad \mathcal{L}_\kappa u = -\Delta u - \frac{\kappa}{d^2}u$$

où  $0 < \kappa \leq \frac{1}{4}$  et ses exposants caractéristiques

$$(2) \quad \alpha_+ = 1 + \sqrt{1 - 4\kappa} \quad \alpha_- = 1 - \sqrt{1 - 4\kappa}.$$

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On supposera  $\Omega$  convexe si  $\kappa = \frac{1}{4}$ . Il est bien connu que sous ces conditions  $\mathcal{L}_\kappa$  possède une première valeur propre  $\lambda_\kappa > 0$  définie par

$$(3) \quad \lambda_\Omega := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega d^{-2} u^2 dx}.$$

La première fonction propre positive associée  $\phi_\kappa$  n'appartient à  $H_0^1(\Omega)$  que si  $0 < \kappa < \frac{1}{4}$ , et dans tous les cas elle vérifie  $\phi_\kappa(x) \sim (d(x))^{\alpha_+}$  au voisinage de  $\partial\Omega$ . On dénote par  $G_\kappa$  et  $K_\kappa$  les noyaux de Green et de Poisson de  $\mathcal{L}_\kappa$  dans  $\Omega$  et par  $\omega^{x_0}$  la mesure  $\mathcal{L}_\kappa$ -harmonique dans  $\Omega$  ( $x_0 \in \Omega$ ). Si  $g$  est une fonction continue et croissante sur  $\mathbb{R}$  telle que  $g(0) \geq 0$ , nous étudions tout d'abord le problème  $(P_{\nu,\mu})$  suivant :

$$(4) \quad \begin{aligned} \mathcal{L}_\kappa u + g(u) &= \nu && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega, \end{aligned}$$

où  $\nu, \mu$  sont des mesures de Radon.

**Théorème 1.** *Supposons que  $g$  vérifie*

$$(5) \quad \int_1^\infty (g(s) + |g(-s)|) s^{-2\frac{N-1+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} ds < \infty;$$

alors pour toutes mesures de Radon  $\nu$  et  $\mu$  dans  $\Omega$  et  $\partial\Omega$  respectivement,  $\nu$  vérifiant en outre  $\int_\Omega \phi_\kappa d|\nu| < \infty$ , il existe une unique fonction  $u = u_{\nu,\mu} \in L_{\phi_\kappa}^1(\Omega)$  telle que  $g \circ u \in L_{\phi_\kappa}^1(\Omega)$  vérifiant

$$(6) \quad \begin{aligned} \int_\Omega (u \mathcal{L}_\kappa \zeta + \zeta g \circ u) dx &= \int_\Omega \int_\Omega G_\kappa(x, y) d\nu(y) \zeta(x) dx \\ &+ \int_\Omega \int_{\partial\Omega} K_\kappa(x, y) d\mu(y) \mathcal{L}_\kappa \zeta(x) dx \end{aligned}$$

pour toute  $\zeta \in \mathbf{X}_\kappa(\Omega)$  où

$$(7) \quad \mathbf{X}_\kappa(\Omega) = \{\zeta \in H_{loc}^1(\Omega) : (\phi_\kappa)^{-1} \zeta \in H_0^1(\Omega, \phi_\kappa dx), (\phi_\kappa)^{-1} \mathcal{L}_\kappa \zeta \in L^\infty(\Omega)\}.$$

En outre l'application  $(\nu, \mu) \mapsto u_{\nu,\mu}$  de  $\mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$  dans  $L_{\phi_\kappa}^1(\Omega)$  est croissante et stable pour la convergence faible des mesures.

La démonstration utilise des estimations des noyaux de Green et de Poisson déduits des propriétés de la mesure  $\omega_\kappa$ . Dans le cas où  $g \circ u = |u|^{q-1} u$ , (5) est vérifiée si  $0 < q < q_c := \frac{2N+\alpha_+}{2N+\alpha_+-4}$ . Dans le cas  $q > 1$  nous dénotons par  $C_{2-\frac{2+\alpha_+}{2q}, q'}^{\mathbb{R}^{N-1}}$

la capacité associée à l'espace de Besov  $B^{2-\frac{2+\alpha_+}{2q}, q'}(\mathbb{R}^{N-1})$  et nous démontrons :

**Théorème 2.** *Soit  $q \geq q_c$  et  $\nu \in \mathfrak{M}_+(\partial\Omega)$ . Alors le problème*

$$(8) \quad \begin{aligned} \mathcal{L}_\kappa u + |u|^{q-1} u &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega \end{aligned}$$

admet une unique solution  $u := u_\mu$  si et seulement si pour tout borélien  $E \subset \partial\Omega$ ,

$$(9) \quad C_{2-\frac{2+\alpha_+}{2q}, q'}^{\mathbb{R}^{N-1}}(E) = 0 \implies \mu(E) = 0.$$

Nous caractérisons aussi les sous ensembles du bord éliminables pour l'équation

$$(10) \quad \mathcal{L}_\kappa u + |u|^{q-1} u = 0 \quad \text{in } \Omega.$$

Définissons

$$(11) \quad W(x) = \begin{cases} (d(x))^{\frac{\alpha_-}{2}} & \text{if } 0 < \kappa < \frac{1}{4} \\ \sqrt{d(x)} \ln |d(x)| & \text{if } \kappa = \frac{1}{4}. \end{cases}$$

**Théorème 3.** *Soit  $q > 1$  et  $K \subset \partial\Omega$  un sous-ensemble compact. Toute solution  $u \in C(\overline{\Omega} \setminus \{K\})$  de (10) qui vérifie*

$$(12) \quad \lim_{x \rightarrow y} \frac{u(x)}{W(x)} = 0 \quad \forall y \in \partial\Omega \setminus \{K\},$$

*est identiquement nulle dans  $\Omega$  si et seulement si  $C_{2-\frac{2+\alpha_+}{2q}, q'}^{\mathbb{R}^{N-1}}(K) = 0$ .*

Nous montrons que si  $q > 1$ , toute solution positive de (10) dans  $\Omega$  admet une trace au bord représentée par une mesure de Borel régulière. En supposant que  $0 \in \partial\Omega$  et  $1 < q < q_c$ , nous étudions aussi le comportement au voisinage de 0 des solutions positives de (10) qui vérifient (12) avec  $K = \{0\}$ .

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Let  $\Omega$  be a bounded  $C^2$  domain in  $\mathbb{R}^N$ ,  $N \geq 3$  and  $d(x) = \text{dist}(x, \Omega)$ . We define  $\lambda_\Omega$  by (3). It is well known that  $\lambda_\Omega \in (0, \frac{1}{4}]$ . Also we define the Hardy operator  $\mathcal{L}_\kappa$  in  $\Omega$  by (1) with  $0 < \kappa < \lambda_\Omega$  if  $\lambda_\Omega < \frac{1}{4}$  or  $0 < \kappa \leq \frac{1}{4}$  if  $\lambda_\Omega = \frac{1}{4}$  and the characteristic exponents by (2). We assume that  $\Omega$  is convex if  $\kappa = \frac{1}{4}$ . It is well known that  $\mathcal{L}_\kappa$  possesses a first eigenvalue  $\lambda_\kappa > 0$  defined by

$$(13) \quad \lambda_\kappa := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx - \kappa \int_\Omega d^{-2} u^2 dx}{\int_\Omega u^2 dx}.$$

The first positive eigenfunction  $\phi_\kappa > 0$  may or may not belong to  $H_0^1(\Omega)$  according  $0 < \kappa < \frac{1}{4}$  or  $\kappa = \frac{1}{4}$ , and  $\phi_\kappa(x) \sim (d(x))^{\frac{\alpha_+}{2}}$ ,  $|\nabla \phi_\kappa(x)| \sim (d(x))^{\frac{\alpha_+}{2}-1}$  as  $d(x) \rightarrow 0$ . Let  $G_\kappa(x, y)$  (resp.  $K_\kappa(x, y)$ ) be the Green (resp. Poisson) kernel of  $\mathcal{L}_\kappa$ , then

$$(14) \quad G_\kappa(x, y) \sim \min \left\{ \frac{1}{|x-y|^{N-2}}, \frac{(d(x))^{\frac{\alpha_+}{2}}}{|x-y|^{N-2+\alpha_+}} \frac{(d(y))^{\frac{\alpha_+}{2}}}{|x-y|^{N-2+\alpha_+}} \right\} \quad \forall (x, y) \in \Omega \times \Omega, x \neq y,$$

$$(15) \quad K_\kappa(x, y) \sim \frac{(d(x))^{\frac{\alpha_+}{2}}}{|x-y|^{N-2+\alpha_+}} \quad \forall (x, y) \in \Omega \times \partial\Omega.$$

The corresponding Green and Poisson operators are denoted by  $\mathbb{G}_\kappa[\cdot]$  and  $\mathbb{K}_\kappa[\cdot]$ . We first consider the boundary value problem (4) where  $g$  is a continuous nondecreasing function such that  $g(0) \geq 0$  and  $\nu$  and  $\mu$  are Radon measures in  $\Omega$  and  $\partial\Omega$  respectively. We say that  $g$  is a subcritical nonlinearity if it satisfies (5).

**Theorem 1.** *Assume that  $g$  is a subcritical nonlinearity. Then for all  $(\nu, \mu) \in \mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$  there exists a unique function  $u = u_{\nu, \mu} \in L_{\phi_\kappa}^1(\Omega)$  such that  $g \circ u \in L_{\phi_\kappa}^1(\Omega)$  verifying (6) for all  $\zeta$  in the space of test functions  $\mathbf{X}_\kappa(\Omega)$  defined by (7). Furthermore the mapping  $(\nu, \mu) \mapsto u_{\nu, \mu}$  from  $\mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$  into  $L_{\phi_\kappa}^1(\Omega)$  is nondecreasing and stable for the weak convergence of measures.*

When  $g(u) = |u|^{q-1}u$  with  $q > 0$ , the inequality (6) means

$$(16) \quad 0 < q < q_c := \frac{2N + \alpha_+}{2N + \alpha_+ - 4}.$$

When  $q \geq q_c$  not all the measures  $\mu$  are eligible for solving (8). We denote by  $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}$  the capacity associated to the Besov space  $B^{2-\frac{2+\alpha_+}{2q'}, q'}(\mathbb{R}^{N-1})$ .

**Theorem 2.** *Let  $q > 1$  and  $\nu \in \mathfrak{M}_+(\partial\Omega)$ . Then problem (8) admits a solution if and only if  $\mu$  is absolutely continuous with respect to  $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}$ , i.e. for any Borel set  $E \subset \partial\Omega$ , implication (9) holds.*

We also characterize the boundary removable sets for (10).

**Theorem 3.** *Let  $q > 1$  and  $K \subset \partial\Omega$  be compact. Any  $u \in C(\overline{\Omega} \setminus \{K\})$  solution of (10) which verifies (12) is identically zero in  $\Omega$  if and only if  $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}(K) = 0$ .*

When  $1 < q < q_c$  only the empty set has zero capacity. There exist singular solutions of (10) with an isolated singularity on the boundary, either solutions  $u_{k\delta_a}$  of (8) with  $\mu = k\delta_a$  for  $k > 0$  and  $a \in \partial\Omega$  or solutions  $u_a = \lim_{k \rightarrow \infty} u_{k\delta_a}$ . This very singular solution is described by considering the following problem on the half upper-sphere  $S_+^{N-1} = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x| = 1, x_N > 0\}$

$$(17) \quad \begin{aligned} -\Delta' \omega - \ell_{N,q,\kappa} \omega - \frac{\kappa}{(\mathbf{e}_N \cdot \sigma)^2} \omega + |\omega|^{q-1} \omega &= 0 & \text{in } S_+^{N-1} \\ \omega &= 0 & \text{in } \partial S_+^{N-1} \end{aligned}$$

where  $\Delta'$  is the Laplace-Beltrami operator on  $S^{N-1}$ ,  $(\mathbf{e}_1, \dots, \mathbf{e}_N)$  is the canonic basis in  $\mathbb{R}^N$ ,  $\sigma = \frac{x}{|x|}$  and

$$\ell_{N,q} = \left( \frac{2}{q-1} \right) \left( \frac{2q}{q-1} - N \right).$$

The spherical Hardy operator  $\omega \mapsto \mathcal{L}'_\kappa := -\Delta' \omega - \frac{\kappa}{(\mathbf{e}_N \cdot \sigma)^2} \omega$  on  $S_+^{N-1}$  admits a first eigenvalue  $\mu_\kappa$  defined by

$$(18) \quad \mu_{\kappa,1} = \inf_{\psi \in H_0^1(S_+^{N-1}) \setminus \{0\}} \frac{\int_{S_+^{N-1}} (|\nabla' \psi|^2 - \kappa (\mathbf{e}_N \cdot \sigma)^{-2} \omega^2) dS}{\int_{\Omega} (\mathbf{e}_N \cdot \sigma)^{-2} \psi^2 dS}.$$

We prove that  $\mu_{\kappa,1} = \frac{\alpha_+}{2} (N + \frac{\alpha_+}{2} - 2)$  with corresponding positive eigenfunction  $\rho_\kappa = (\mathbf{e}_N \cdot \sigma)^{\frac{\alpha_+}{2}}$ . There exists a second eigenvalue  $\mu_{\kappa,2} = \mu_{\kappa,1} + N + \alpha_+ - 1$  with  $N-1$  independent eigenfunctions  $\rho_{\kappa,j} = (\mathbf{e}_N \cdot \sigma)^{\frac{\alpha_+}{2}} \mathbf{e}_j \cdot \sigma$  for  $j = 1, \dots, N-1$ . We denote by  $\mathcal{E}_\kappa$  the set of functions  $\omega$  such that  $\rho_\kappa^{-1} \omega \in L_{\rho_\kappa}^{q+1}(S_+^{N-1}) \cap H_0^1(S_+^{N-1}, \rho_\kappa^2 dS)$  which satisfy (17), and by  $\mathcal{E}_\kappa^+$  the set of positive solutions.

**Theorem 4.** *I- If  $q \geq q_c$ ,  $\mathcal{E}_\kappa = \{\emptyset\}$ .*

*II- If  $1 < q < q_c$ ,  $\mathcal{E}_\kappa^+ = \{0, \omega_\kappa\}$  where  $\omega_\kappa$  is the unique positive solution of (17).*

*III- If  $q_e \leq q < q_c$ ,  $\mathcal{E}_\kappa = \{0, \omega_\kappa, -\omega_\kappa\}$  where  $q_e := \frac{2N + 2 + \alpha_+}{2N - 2 + \alpha_+}$ .*

This allows us to describe the isolated boundary singularities of positive solutions of (10). Assume  $0 \in \partial\Omega$  and  $\mathbf{e}_N$  is the outward normal unit vector to  $\partial\Omega$  at 0.

**Theorem 5.** Assume ,  $1 < q < q_c$  and  $u \in C(\overline{\Omega} \setminus \{0\})$  is a positive solution of (10) which verifies(12) with  $K = \{0\}$ . Then

(i) either there exists  $k \geq 0$  such that  $u = u_{k\delta_0}$  and  $\lim_{|x| \rightarrow 0} |x|^{N + \frac{\alpha_+}{2} - 2} u(x) = c_N k (\mathbf{e}_N \cdot \frac{x}{|x|})^{\frac{\alpha_+}{2}}$ ,

(ii) or  $\lim_{|x| \rightarrow 0} |x|^{\frac{2}{q-1}} u(x) = \omega_\kappa(\frac{x}{|x|})$ .

The above two convergence hold locally uniformly on  $S_+^{N-1}$ .

We can also define a boundary trace of any positive solution  $u$  of (10). For  $\delta > 0$  small enough, we denote by  $\omega_{\Omega'_\delta}^{x_0}$  the harmonic measure relative to the operator  $\mathcal{L}_\kappa$  in  $\Omega'_\delta = \{x \in \Omega : d(x) > \delta\}$  where  $x_0 \in \Omega$  (with  $d(x_0) \geq \delta_1 > \delta$ ) and set  $\Sigma_\delta = \partial\Omega'_\delta$ .

**Theorem 6.** Assume  $q > 1$  and  $u \in C(\overline{\Omega} \setminus \{0\})$  is a positive solution of (10) in  $\Omega$ . Then for any  $y \in \partial\Omega$ , the following dichotomy occurs :

(i) Either there exist an open subset  $U \subset \mathbb{R}^N$  containing  $y$  and a positive Radon measure  $\lambda_U$  on  $\partial\Omega \cap U$  such that

$$(19) \quad \lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} Z(x) u(x) d\omega_{\Omega'_\delta}^{x_0} = \int_{\partial\Omega \cap U} Z d\lambda_U \quad \forall Z \in C_0(U).$$

(ii) Or for any open subset  $U \subset \mathbb{R}^N$  containing  $y$ , there holds

$$(20) \quad \lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} u(x) d\omega_{\Omega'_\delta}^{x_0} = \infty.$$

The set  $\mathcal{R}_u$  of  $y$  such that (i) holds is relatively open in  $\partial\Omega$  and it carries a positive Radon measure  $\mu_u$  such that (19) occurs with  $U$  replaced by  $\mathcal{R}_u$  and  $\lambda_U$  by  $\mu_u$ ; its complement  $\mathcal{S}_u$  in  $\partial\Omega$  has the property that (20) occurs for any open subset  $U$  such that  $U \cap \mathcal{S}_u \neq \{\emptyset\}$ .

*Abridged proof of Theorem 1.* Let  $(\nu, \mu) \in \mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$ . For  $\lambda > 0$  we set

$$(21) \quad E_\lambda(\nu) = \{x \in \Omega : \mathbb{G}_\kappa[|\nu|](x) > \lambda\}, \quad \mathcal{E}_\lambda(\nu) = \int_{E_\lambda(\nu)} \phi_\kappa dx,$$

and

$$(22) \quad F_\lambda(\nu) = \{x \in \Omega : \mathbb{K}_\kappa[|\mu|](x) > \lambda\}, \quad \mathcal{F}_\lambda(\mu) = \int_{E_\lambda(\nu)} dx,$$

and prove

$$(23) \quad \mathcal{E}_\lambda(\nu) + \mathcal{F}_\lambda(\mu) \leq c \left( \frac{\|\nu\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}}{\lambda} \right)^{\frac{2N + \alpha_+}{2N + \alpha_+ - 4}}.$$

If  $g$  satisfies (5) and  $\{(\nu_n, \mu_n)\}$  is a sequence of smooth functions which converges in the weak-star topology of measures to  $(\nu, \mu)$ , then the corresponding solutions  $\{u_{\nu_n, \mu_n}\}$  of problem  $P_{\nu_n, \mu_n}$  defined in (4) converges to some  $u$  and  $\{g \circ u_{\nu_n, \mu_n}\}$  converges to  $g \circ u$  in  $L^1_{\phi_\kappa}$  by Vitali convergence theorem. This implies  $u = u_{\nu, \mu}$ . Uniqueness holds by adapting Brezis estimates and using monotonicity.

*Abridged proof of Theorem 2.* Using estimate (15) and the harmonic lifting in Besov spaces introduced in [9, Sect. 3] we prove that for any  $\mu \in \mathfrak{M}(\partial\Omega)$  there holds

$$(24) \quad \frac{1}{c} \|\mu\|_{B^{-2 + \frac{2 + \alpha_+}{2q'}, q}}^q \leq \int_{\Omega} (\mathbb{K}_\kappa[|\mu|])^q \phi_\kappa dx \leq c \|\mu\|_{B^{-2 + \frac{2 + \alpha_+}{2q'}, q}}^q,$$

for some  $c = c(\Omega, \kappa, q) > 0$ . If the above quantity is finite, we can solve (8) with such a  $\mu$ . If  $\mu \in B^{-2+\frac{2+\alpha_+}{2q'}, q}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$ , it is absolutely continuous with respect to the capacity  $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}$ . Finally, if  $\mu \in \mathfrak{M}_+(\partial\Omega)$  is absolutely continuous with respect to the capacity  $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}$ , there exists an increasing sequence  $\{\mu_n\} \subset B^{-2+\frac{2+\alpha_+}{2q'}, q}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$  which converges to  $\mu$ . This implies that  $u_{\mu_n}$  converges to  $u_\mu$  in  $L_{\phi_\kappa}^q(\Omega)$ .

Conversely, if  $\mu \in \mathfrak{M}_+(\partial\Omega)$  is such that there exists a solution  $u_\mu$  to (8), we use a variant of the *optimal lifting*  $R[\cdot]$  defined in [7, Sect. 1] to prove that for any  $\eta \in C^2(\partial\Omega)$  such that  $0 \leq \eta \leq 1$  there holds

$$(25) \quad \int_{\partial\Omega} \eta d\mu \leq c \int_{\Omega} u^q \zeta dx + c \left( \int_{\Omega} u^q \zeta dx \right)^{\frac{1}{q}} \left( \int_{\Omega} \phi_\kappa dx + \|\eta\|_{B^{2-\frac{2+\alpha_+}{2q'}, q'}}^{q'} \right)^{\frac{1}{q'}}.$$

Here  $\zeta = \phi_\kappa(R[\eta])^{q'}$  and  $R : C^2(\partial\Omega) \mapsto C^2(\overline{\Omega})$  is a linear mapping which satisfies  $0 \leq \eta \leq 1 \implies 0 \leq R[\eta] \leq 1$  and  $R[\eta]|_{\partial\Omega} = \eta$ . If  $K \subset \partial\Omega$  is a compact set with zero  $C_{2-\frac{2+\alpha_+}{2q'}, q}^{\mathbb{R}^{N-1}}$ -capacity, there exists a sequence  $\{\eta_n\} \subset C^2(\partial\Omega)$  such that  $0 \leq \eta_n \leq 1$ ,  $\eta_n = 1$  on  $K$  and  $\|\eta_n\|_{B^{2-\frac{2+\alpha_+}{2q'}, q'}}^{q'} \rightarrow 0$ . This implies  $\phi_\kappa(R[\eta_n])^{q'} \rightarrow 0$  and finally  $\mu(K) = 0$ .

*Abridged proof of Theorem 3.* If  $K \subset \partial\Omega$  is compact with  $C_{2-\frac{2+\alpha_+}{2q'}, q}^{\mathbb{R}^{N-1}}(K) > 0$ , its capacity measure  $\mu_K$  belongs to  $B^{-2+\frac{2+\alpha_+}{2q'}, q}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$ . Thus  $u_{\mu_K}$  exists and  $K$  is not removable. Conversely by using again optimal lifting, and test functions of the form  $\phi_\kappa(R[1-\eta])^{2q'}$  where  $0 \leq \eta \leq 1$  and  $\eta = 1$  in a neighborhood of  $K$ , we prove first that that  $u \in L_{\phi_\kappa}^q(\Omega)$  and finally that  $u = 0$ .

*Abridged proof of Theorems 4-5.* Existence is obtained in minimizing the functional  $\mathcal{J}_\kappa$  defined over  $L_{\rho_\kappa}^{q+1}(S_+^{N-1}) \cap H_0^1(S_+^{N-1}, \rho_\kappa^2 dS)$  by

$$(26) \quad \mathcal{J}_\kappa(w) := \int_{S_+^{N-1}} \left( |\nabla' w|^2 - (\ell_{N,q} - \mu_{\kappa,1})w^2 + \frac{2}{q+1} \rho_\kappa^{q-1} |w|^{q+1} \right) \rho_\kappa^2 dS.$$

A non-trivial minimizer  $w$  exists if  $\ell_{N,q} > \mu_{\kappa,1}$  (defined by (18)), i.e.  $1 < q < q_c$ , and  $\omega = \rho_\kappa w$  satisfies (17). Nonexistence in standard since  $\mu_{\kappa,1} < \ell_{N,q}$  if and only if  $1 < q < q_c$ . For uniqueness we assume that  $\omega_j$  ( $j = 1, 2$ ) are positive solutions of (17) and we set  $w_j = \frac{\omega_j}{\rho_\kappa}$ . Then

$$-\operatorname{div}'(\rho_\kappa^2 \nabla' w_j) + (\mu_{\kappa,1} - \ell_{N,q}) \rho_\kappa^2 w_j + \rho_\kappa^{q+1} w_j^q = 0 \quad \text{on } S_+^{N-1}$$

Since  $w_j \sim \rho_\kappa^{\frac{\alpha_+}{2}}$  and  $|\nabla' w_j| \sim \rho_\kappa^{\frac{\alpha_+}{2}-1}$  near  $\partial S_+^{N-1}$ , we use Green formula and get

$$\int_{S_+^{N-1}} \left( \left( \frac{\nabla' w_1}{w_1} - \frac{\nabla' w_2}{w_2} \right) \cdot \nabla' (w_1^2 - w_2^2) + \rho_\kappa^{q-1} (w_1^{q-1} - w_2^{q-1})(w_1^2 - w_2^2) \right) \rho_\kappa^2 dS = 0,$$

thus  $w_1 = w_2$ . For statement III we first prove, by the method used in [11, Th 3.1], that any solution  $\omega$  depends only on the azimuthal angle  $\theta \in [0, \frac{\pi}{2}]$ . Then we show that the corresponding ODE verified by  $\omega$  admits only constant sign solutions. For

Theorem 5, we construct a barrier function as in [8, Appendix] and obtain

$$(27) \quad u(x) \leq c|x|^{-\frac{2}{q-1} + \frac{\alpha_+}{2}} (d(x))^{\frac{\alpha_+}{2}} \quad \forall x \in \Omega.$$

With this estimate we adapt the scaling method developed in [10, Sect. 3.3] to obtain the classification result.

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