

MEASURE BOUNDARY VALUE PROBLEM FOR SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL HARDY POTENTIALS

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RÉSUMÉ. Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain and $\mathcal{L}_\kappa = -\Delta - \frac{\kappa}{d^2}$ the Hardy operator where $d = \text{dist}(\cdot, \partial\Omega)$ and $0 < \kappa \leq \frac{1}{4}$. Let $\alpha_\pm = 1 \pm \sqrt{1 - 4\kappa}$ be the two Hardy exponents, λ_κ the first eigenvalue of \mathcal{L}_κ with corresponding positive eigenfunction ϕ_κ . If g is a continuous nondecreasing function satisfying $\int_1^\infty (g(s) + |g(-s)|) s^{-2\frac{2N-2+\alpha_\pm}{2N-4+\alpha_\pm}} ds < \infty$, then for any Radon measures $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$ and $\mu \in \mathfrak{M}(\partial\Omega)$ there exists a unique weak solution to problem $P_{\nu,\mu} : \mathcal{L}_\kappa u + g(u) = \nu$ in Ω , $u = \mu$ on $\partial\Omega$. If $g(r) = |r|^{q-1}u$ ($q > 1$) we prove that, in the subcritical range of q , a necessary and sufficient condition for solving $P_{0,\mu}$ with $\mu > 0$ is that μ is absolutely continuous with respect to the capacity associated to the Besov space $B^{2-\frac{2+\alpha_\pm}{2q}, q'}(\mathbb{R}^{N-1})$. We also characterize the boundary removable sets in terms of this capacity. In the subcritical range of q we classify the isolated singularities of positive solutions.

Problèmes aux limites avec données mesures pour des équations semi linéaires elliptiques avec des potentiels de Hardy critiques

RÉSUMÉ. Soient $\Omega \subset \mathbb{R}^N$ un domaine de classe C^2 et $\mathcal{L}_\kappa = -\Delta - \frac{\kappa}{d^2}$ l'opérateur de Hardy où $d = \text{dist}(\cdot, \partial\Omega)$ et $0 < \kappa \leq \frac{1}{4}$. Soient $\alpha_\pm = 1 \pm \sqrt{1 - 4\kappa}$ les deux exposants de Hardy, λ_κ première valeur propre de \mathcal{L}_κ et ϕ_κ la fonction propre positive correspondante. Si g est une fonction continue croissante vérifiant $\int_1^\infty (g(s) + |g(-s)|) s^{-2\frac{2N-2+\alpha_\pm}{2N-4+\alpha_\pm}} ds < \infty$, alors pour toutes mesures de Radon $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$ et $\mu \in \mathfrak{M}(\partial\Omega)$ il existe une unique solution faible au problème $P_{\nu,\mu} : \mathcal{L}_\kappa u + g(u) = \nu$ dans Ω , $u = \mu$ sur $\partial\Omega$. Si $g(r) = |r|^{q-1}u$ ($q > 1$) nous démontrons qu'une condition nécessaire et suffisante pour résoudre $P_{0,\mu}$ avec $\mu > 0$ est que μ soit absolument continue par rapport à la capacité associée à l'espace de Besov $B^{2-\frac{2+\alpha_\pm}{2q}, q'}(\mathbb{R}^{N-1})$. Nous caractérisons les ensembles éliminables pour les valeurs sur critiques de q . Dans le cas sous-critique nous donnons une classifications des singularités isolées au bord des solutions positives.

Version française abrégée. Soit Ω un domaine de \mathbb{R}^N de classe C^2 . On désigne par $d(x)$ la distance de x à $\partial\Omega$ et on définit l'opérateur de Hardy dans Ω par

$$(1) \quad \mathcal{L}_\kappa u = -\Delta u - \frac{\kappa}{d^2} u$$

où $0 < \kappa \leq \frac{1}{4}$ et ses exposants caractéristiques

$$(2) \quad \alpha_+ = 1 + \sqrt{1 - 4\kappa} \quad \alpha_- = 1 - \sqrt{1 - 4\kappa}.$$

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On supposera Ω convexe si $\kappa = \frac{1}{4}$. Il est bien connu que sous ces conditions \mathcal{L}_κ possède une première valeur propre $\lambda_\kappa > 0$ définie par

$$(3) \quad \lambda_\kappa := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega d^{-2} u^2 dx}.$$

La première fonction propre positive associée ϕ_κ n'appartient à $H_0^1(\Omega)$ que si $0 < \kappa < \frac{1}{4}$, et dans tous les cas elle vérifie $\phi_\kappa(x) \sim (d(x))^{\alpha_+}$ au voisinage de $\partial\Omega$. On dénote par G_κ et K_κ les noyaux de Green et de Poisson de \mathcal{L}_κ dans Ω et par ω^{x_0} la mesure harmonique dans Ω ($x_0 \in \Omega$). Si g est une fonction continue et croissante sur \mathbb{R} telle que $g(0) \geq 0$, nous étudions tout d'abord le problème $(P_{\nu,\mu})$ suivant

$$(4) \quad \begin{aligned} \mathcal{L}_\kappa u + g(u) &= \nu && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega, \end{aligned}$$

où ν, μ sont des mesures de Radon.

Théorème 1. *Supposons que g vérifie*

$$(5) \quad \int_1^\infty (g(s) + |g(-s)|) s^{-2\frac{N-1+\alpha_+}{N-2+\frac{\alpha_+}{2}}} ds < \infty;$$

alors pour toutes mesures de Radon ν et μ dans Ω et $\partial\Omega$ respectivement, ν vérifiant en outre $\int_\Omega \phi_\kappa d|\nu| < \infty$, il existe une unique fonction $u = u_{\nu,\mu} \in L_{\phi_\kappa}^1(\Omega)$ telle que $g \circ u \in L_{\phi_\kappa}^1(\Omega)$ vérifiant

$$(6) \quad \int_\Omega (u \mathcal{L}_\kappa \zeta + \zeta g \circ u) dx = \int_\Omega G_\kappa(x, y) d\nu(y) + \int_{\partial\Omega} K_\kappa(x, y) d\mu(y)$$

pour toute $\zeta \in \mathbf{X}_\kappa(\Omega)$ où

$$(7) \quad \mathbf{X}_\kappa(\Omega) = \{\zeta \in H_{loc}^1(\Omega) : (\phi_\kappa)^{-1} \in H_0^1(\Omega, \phi_\kappa dx), (\phi_\kappa)^{-1} \mathcal{L}_\kappa \zeta \in L^\infty(\Omega)\}.$$

En outre l'application $(\nu, \mu) \mapsto u_{\nu,\mu}$ de $\mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$ dans $L_{\phi_\kappa}^1(\Omega)$ est croissante et stable pour la convergence faible des mesures.

La démonstration utilise des estimations des noyaux de Green et de Poisson obtenus à partir des propriétés de la mesure harmonique. Dans le cas où $g \circ u = |u|^{q-1} u$ l'inégalité (5) est vérifiée si $0 < q < q_c := \frac{2N+\alpha_+}{2N+\alpha_+-4}$. Dans le cas $q > 1$ nous dénotons par $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}$ la capacité associée à l'espace de Besov $B^{2-\frac{2+\alpha_+}{2q'}, q'}(\mathbb{R}^{N-1})$ et nous avons le résultat suivant :

Théorème 2. *Soit $q \geq q_c$ et $\nu \in \mathfrak{M}_+(\partial\Omega)$. Alors le problème*

$$(8) \quad \begin{aligned} \mathcal{L}_\kappa u + |u|^{q-1} u &= 0 && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega \end{aligned}$$

admet une unique solution $u := u_\mu$ si et seulement si pour tout borélien $E \subset \partial\Omega$,

$$(9) \quad C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}(E) = 0 \implies \mu(E) = 0.$$

Nous caractérisons aussi les sous ensembles du bord éliminables pour

$$(10) \quad \mathcal{L}_\kappa u + |u|^{q-1} u = 0 \quad \text{in } \Omega.$$

Pour cela nous posons

$$(11) \quad W(x) = \begin{cases} (d(x))^{\frac{\alpha_-}{2}} & \text{if } 0 < \kappa < \frac{1}{4} \\ \sqrt{d(x)} \ln |d(x)| & \text{if } \kappa = \frac{1}{4}. \end{cases}$$

Théorème 3. *Soit $q > 1$ et $K \subset \partial\Omega$ un sous-ensemble compact. Toute solution $u \in C(\overline{\Omega} \setminus \{K\})$ de (10) qui vérifie*

$$(12) \quad \lim_{x \rightarrow y} \frac{u(x)}{W(x)} = 0 \quad \forall y \in \partial\Omega \setminus \{K\},$$

est identiquement nulle dans Ω si et seulement si $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}(K) = 0$.

Nous montrons que si $q > 1$, toute solution positive de (10) dans Ω admet une trace au bord représentée par une mesure de Borel régulière. En supposant que $0 \in \partial\Omega$ et $1 < q < q_c$, nous étudions aussi le comportement au voisinage de 0 des solutions positives de (10) qui vérifient (12) avec $K = \{0\}$.

Let Ω be a bounded C^2 domain in \mathbb{R}^N and $d(x) = \text{dist}(x, \Omega)$. We define the Hardy operator \mathcal{L}_κ in Ω by (1) with $0 < \kappa \leq \frac{1}{4}$ and the characteristic exponents by (2). We assume that Ω is convex if $\kappa = \frac{1}{4}$. It is well known that \mathcal{L}_κ possesses a first eigenvalue $\lambda_\kappa > 0$ defined by (3) and that the first positive eigenfunction $\phi_\kappa > 0$ may or may not belong to $H_0^1(\Omega)$ according $0 < \kappa < \frac{1}{4}$ or $\kappa = \frac{1}{4}$, and it satisfies $\phi_\kappa(x) \sim (d(x))^{\frac{\alpha_+}{2}}$, $|\nabla \phi_\kappa(x)| \sim (d(x))^{\frac{\alpha_+}{2}-1}$ as $d(x) \rightarrow 0$. The Green and the Poisson kernels are denoted by $G_\kappa(x, y)$ and $K_\kappa(x, y)$, and they satisfy

$$(13) \quad G_\kappa(x, y) \sim \min \left\{ \frac{1}{|x-y|^{N-2}}, \frac{(d(x))^{\frac{\alpha_+}{2}} (d(y))^{\frac{\alpha_+}{2}}}{|x-y|^{N-2+\alpha_+}} \right\} \quad \forall (x, y) \in \Omega \times \Omega, x \neq y,$$

$$(14) \quad K_\kappa(x, y) \sim \frac{(d(x))^{\frac{\alpha_+}{2}}}{|x-y|^{N-2+\alpha_+}} \quad \forall (x, y) \in \Omega \times \partial\Omega.$$

The corresponding Green and Poisson operators are denoted by $\mathbb{G}_\kappa[\cdot]$ and $\mathbb{K}_\kappa[\cdot]$. We first consider the boundary value problem (4) where g is a continuous nondecreasing function such that $g(0) \geq 0$ and ν and μ are Radon measures in Ω and $\partial\Omega$ respectively. We say that g is a subcritical nonlinearity if it satisfies (5).

Theorem 1. *Assume that g is a subcritical nonlinearity. Then for all $(\nu, \mu) \in \mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$ there exists a unique function $u = u_{\nu, \mu} \in L_{\phi_\kappa}^1(\Omega)$ such that $g \circ u \in L_{\phi_\kappa}^1(\Omega)$ verifying (6) for all ζ in the space of test functions $\mathbf{X}_\kappa(\Omega)$ defined by (7). Furthermore the mapping $(\nu, \mu) \mapsto u_{\nu, \mu}$ from $\mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$ into $L_{\phi_\kappa}^1(\Omega)$ is nondecreasing and stable for the weak convergence of measures.*

When $g(u) = |u|^{q-1}u$ with $q > 0$, the inequality (6) means

$$(15) \quad 0 < q < q_c := \frac{2N + \alpha_+}{2N + \alpha_+ - 4}.$$

When $q \geq q_c$ not all the measures μ are eligible for solving (8). We denote by $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}$ the capacity associated to the Besov space $W^{2-\frac{2+\alpha_+}{2q'}, q'}(\mathbb{R}^{N-1})$.

Theorem 2. *Let $q > 1$ and $\nu \in \mathfrak{M}_+(\partial\Omega)$. Then problem (8) admits a solution if and only if μ is absolutely continuous with respect to $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}$, i.e. for any Borel set $E \subset \partial\Omega$, implication (9) holds.*

We also characterize the boundary removable sets for (10).

Theorem 3. *Let $q > 1$ and $K \subset \partial\Omega$ is compact. Any $u \in C(\overline{\Omega} \setminus \{K\})$ solution of (10) which verifies (12) is identically zero in Ω if and only if $C_{2-\frac{2+\alpha_+}{2q'}, q'}^{\mathbb{R}^{N-1}}(K) = 0$.*

When $1 < q < q_c$ only the empty set has zero capacity. There exist singular solutions of (10) with an isolated singularity on the boundary either solutions $u_{k\delta_a}$ of (8) with $\mu = k\delta_a$ for $k > 0$ and $a \in \partial\Omega$ or solutions $u_a = \lim_{k \rightarrow \infty} u_{k\delta_a}$. This very singular solution is described by considering the following problem on the half sphere $S_+^{N-1} = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x| = 1, x_N = 1\}$

$$(16) \quad \begin{aligned} -\Delta' \omega - \ell_{N,q,\kappa} \omega - \frac{\kappa}{(\mathbf{e}_N \cdot \sigma)^2} \omega + |\omega|^{q-1} \omega &= 0 & \text{in } S_+^{N-1} \\ \omega &= 0 & \text{in } \partial S_+^{N-1} \end{aligned}$$

where Δ' is the Laplace-Beltrami operator on S^{N-1} , $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ is the canonic basis in \mathbb{R}^N , $\sigma = \frac{x}{|x|}$ and

$$\ell_{N,q} = \left(\frac{2}{q-1} \right) \left(\frac{2q}{q-1} - N \right).$$

The spherical Hardy operator $\omega \mapsto \mathcal{L}'_\kappa := -\Delta' \omega - \frac{\kappa}{(\mathbf{e}_N \cdot \sigma)^2} \omega$ on S_+^{N-1} admits a first eigenvalue μ_κ defined by

$$(17) \quad \mu_{\kappa,1} = \inf_{\psi \in H_0^1(S_+^{N-1}) \setminus \{0\}} \frac{\int_{S_+^{N-1}} (|\nabla' \psi|^2 - \kappa (\mathbf{e}_N \cdot \sigma)^{-2} \omega^2) dS}{\int_{\Omega} (\mathbf{e}_N \cdot \sigma)^{-2} \psi^2 dS}.$$

We prove that $\mu_{\kappa,1} = \frac{\alpha_+}{2} (N + \frac{\alpha_+}{2} - 2)$ with corresponding positive eigenfunction $\rho_\kappa = (\mathbf{e}_N \cdot \sigma)^{\frac{\alpha_+}{2}}$. There exists a second eigenvalue $\mu_{\kappa,2} = \mu_{\kappa,1} + N + \alpha_+ - 1$ with $N-1$ independent eigenfunctions $\rho_{\kappa,j} = (\mathbf{e}_N \cdot \sigma)^{\frac{\alpha_+}{2}} \mathbf{e}_j \cdot \sigma$ for $j = 1, \dots, N-1$. We denote by \mathcal{E}_κ the set of functions ω such that $\rho_\kappa^{-1} \omega \in L_{\rho_\kappa^{q+1}}^{q+1}(S_+^{N-1}) \cap H_0^1(S_+^{N-1}, \rho_\kappa^2 dS)$ which satisfy (16), and by \mathcal{E}_κ^+ the set of positive solutions.

Theorem 4. *I- If $q \geq q_c$, $\mathcal{E}_\kappa = \{\emptyset\}$.*

II- If $1 < q < q_c$, $\mathcal{E}_\kappa^+ = \{0, \omega_\kappa\}$ where ω_κ is the unique positive solution of (16).

III- If $q_e \leq q < q_c$, $\mathcal{E}_\kappa = \{0, \omega_\kappa, -\omega_\kappa\}$ where $q_e := \frac{2N+2+\alpha_+}{2N-2+\alpha_+}$.

This result allows to describe the isolated boundary singularities of positive solutions of (10). We assume that $0 \in \partial\Omega$ and the outward normal unit vector to $\partial\Omega$ at 0 is \mathbf{e}_N .

Theorem 5. *Assume, $1 < q < q_c$ and $u \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (10) which verifies (12) with $K = \{0\}$. Then*

(i) either there exists $k \geq 0$ such that $u = u_{k\delta_0}$ and $\lim_{|x| \rightarrow 0} |x|^{N+\frac{\alpha_+}{2}-2} u(x) = c_N k (\mathbf{e}_N \cdot \frac{x}{|x|})^{\frac{\alpha_+}{2}}$,

(ii) or $\lim_{|x| \rightarrow 0} |x|^{\frac{2}{q-1}} u(x) = \omega_\kappa\left(\frac{x}{|x|}\right)$.

The above two convergence hold locally uniformly on S_+^{N-1} .

We can also define a boundary trace of any positive solution u of (10). For $\delta > 0$ small enough, we denote by $\omega_{\Omega'_\delta}^{x_0}$ the harmonic measure relative to the operator \mathcal{L}_κ in $\Omega'_\delta = \{x \in \Omega : d(x) > \delta\}$ where $x_0 \in \Omega$ (with $d(x_0) \geq \delta_1 > \delta$) and set $\Sigma_\delta = \partial\Omega'_\delta$.

Theorem 6. *Assume $q > 1$ and $u \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (10) in Ω . Then for any $y \in \partial\Omega$, the following dichotomy occurs :*

(i) *Either there exists an open subset $U \subset \mathbb{R}^N$ containing y and a positive Radon measure λ_U on $\partial\Omega \cap U$ such that*

$$(18) \quad \lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} Z(x)u(x)d\omega_{\Omega'_\delta}^{x_0} = \int_{\partial\Omega \cap U} Zd\lambda_U \quad \forall Z \in C_0(U).$$

(ii) *Or for any open subset $U \subset \mathbb{R}^N$ containing y , there holds*

$$(19) \quad \lim_{\delta \rightarrow 0} \int_{\Sigma_\delta \cap U} u(x)d\omega_{\Omega'_\delta}^{x_0} = \infty.$$

The set \mathcal{R}_u of x_0 such that (i) holds is relatively open in $\partial\Omega$ and it carries a positive Radon measure μ_u such that (18) occurs with U replaced by \mathcal{R}_u and λ_U by μ_u ; its complement \mathcal{S}_u in $\partial\Omega$ has the property that (19) occurs for any open subset U such that $U \cap \mathcal{S}_u \neq \{\emptyset\}$.

Abridged proof of Theorem 1. Let $(\nu, \mu) \in \mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$. For $\lambda > 0$ we set

$$(20) \quad \mathcal{E}_\lambda(\nu) = \{x \in \Omega : \mathbb{G}_\kappa[|\nu|](x) > \lambda\}, \quad \mathcal{E}_\lambda(\nu) = \int_{\mathcal{E}_\lambda(\nu)} \phi_\kappa dx,$$

and

$$(21) \quad \mathcal{F}_\lambda(\mu) = \{x \in \Omega : \mathbb{K}_\kappa[|\mu|](x) > \lambda\}, \quad \mathcal{F}_\lambda(\mu) = \int_{\mathcal{E}_\lambda(\nu)} dx,$$

and prove

$$(22) \quad \mathcal{E}_\lambda(\nu) + \mathcal{F}_\lambda(\mu) \leq c \left(\frac{\|\nu\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}}{\lambda} \right)^{\frac{2N+\alpha_+}{2N+\alpha_+-4}}.$$

If g satisfies (5) and $\{(\nu_n, \mu_n)\}$ is a sequence of smooth functions which converges in the weak-star topology of measures to (ν, μ) , then the corresponding solutions $\{u_{\nu_n, \mu_n}\}$ of problem P_{ν_n, μ_n} defined in ((4)) converges to some u and $\{g \circ u_{\nu_n, \mu_n}\}$ converges to $g \circ u$ in $L^1_{\phi_\kappa}$ by Vitali convergence theorem. This implies that $u = u_{\nu, \mu}$. Uniqueness holds by adapting Brezis estimates and using monotonicity.

Abridged proof of Theorem 2. Using estimate (14) and the harmonic lifting in Besov spaces introduced in [8, Sect. 3] we prove that for any $\mu \in \mathfrak{M}(\partial\Omega)$ there holds

$$(23) \quad \frac{1}{c} \|\mu\|_{B^{-2+\frac{2+\alpha_+}{2q'}, q}}^q \leq \int_{\Omega} (\mathbb{K}_\kappa[|\mu|])^q \phi_\kappa dx \leq c \|\mu\|_{B^{-2+\frac{2+\alpha_+}{2q'}, q}}^q$$

for some $c = c(\Omega, \kappa, q) > 0$. This implies that we can solve (8) with such a Radon measure. If $\mu \in B^{-2+\frac{2+\alpha_+}{2q'}, q}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$, it is absolutely continuous with respect to the capacity $C_{2-\frac{2+\alpha_+}{2q'}, q}^{\mathbb{R}^{N-1}}$. Finally, if $\mu \in \mathfrak{M}_+(\partial\Omega)$ is absolutely continuous with respect to the capacity $C_{2-\frac{2+\alpha_+}{2q'}, q}^{\mathbb{R}^{N-1}}$, there exists an increasing sequence $\{\mu_n\} \subset$

$B^{-2+\frac{2+\alpha_+}{2q^+},q}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$ which converges to μ . This implies that u_{μ_n} converges to u_μ in $L_{\phi_\kappa}^q(\Omega)$.

Conversely, if $\mu \in \mathfrak{M}_+(\partial\Omega)$ is such that there exists a solution u_μ to (8), we use a variant of the *optimal lifting* $R[\cdot]$ defined in [6, Sect. 1] to prove that for any $\eta \in C^2(\partial\Omega)$ such that $0 \leq \eta \leq 1$ there holds

$$(24) \quad \int_{\partial\Omega} \eta d\mu \leq c \int_{\Omega} u^q \zeta dx + c \left(\int_{\Omega} u^q \zeta dx \right)^{\frac{1}{q}} \left(\int_{\Omega} \phi_\kappa dx + \|\eta\|_{B^{2-\frac{2+\alpha_+}{2q^+},q}}^{q'} \right)^{\frac{1}{q'}}.$$

Here $\zeta = \phi_\kappa(R[\eta])^{q'}$ and $R : C^2(\partial\Omega) \mapsto C^2(\overline{\Omega})$ is a linear mapping which satisfies $0 \leq \eta \leq 1 \implies 0 \leq R[\eta] \leq 1$ and $R[\eta]|_{\partial\Omega} = \eta$. If $K \subset \partial\Omega$ is a compact set with zero $C_{2-\frac{2+\alpha_+}{2q^+},q}^{\mathbb{R}^{N-1}}$ -capacity, there exists a sequence $\{\eta_n\} \subset C^2(\partial\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ on K and $\|\eta_n\|_{B^{2-\frac{2+\alpha_+}{2q^+},q}}^{q'} \rightarrow 0$. This implies $\phi_\kappa(R[\eta_n])^{q'} \rightarrow 0$ and finally $\mu(K) = 0$.

Abridged proof of Theorem 3. If $K \subset \partial\Omega$ is compact with $C_{2-\frac{2+\alpha_+}{2q^+},q}^{\mathbb{R}^{N-1}}(K) > 0$, its capacity measure μ_K belongs to $B^{-2+\frac{2+\alpha_+}{2q^+},q}(\partial\Omega) \cap \mathfrak{M}_+(\partial\Omega)$. Thus u_{μ_K} exists and K is not removable. Conversely by using again optimal lifting, and test functions of the form $\phi_\kappa(R[1-\eta])^{2q'}$ where $0 \leq \eta \leq 1$ and $\eta = 1$ in a neighborhood of K , we prove first that $u \in L_{\phi_\kappa}^q(\Omega)$ and finally that $u = 0$.

Abridged proof of Theorems 4-5. Existence is obtained in minimizing \mathcal{J}_κ defined over $L_{\rho_\kappa^{q+1}}^{q+1}(S_+^{N-1}) \cap H_0^1(S_+^{N-1}, \rho_\kappa^2 dS)$ by

$$(25) \quad \mathcal{J}_\kappa(w) := \int_{S_+^{N-1}} \left(|\nabla' w|^2 - (\ell_{N,q} - \mu_{\kappa,1})w^2 + \frac{2}{q+1} \rho_\kappa^{q-1} |w|^{q+1} \right) \rho_\kappa^2 dS.$$

A non-trivial minimizer exists if $\ell_{N,q} > \mu_{\kappa,1}$ (defined by (17)), i.e. $1 < q < q_c$, and $\omega = \rho_\kappa w$ satisfies (16). Nonexistence in standard since $\mu_{\kappa,1} < \ell_{N,q}$ if and only if $1 < q < q_c$. For uniqueness we assume that ω_j ($j = 1, 2$) are positive solutions of (16) and we set $w_j = \frac{\omega_j}{\rho_\kappa}$. Then

$$-\operatorname{div}' \cdot (\rho_\kappa^2 \nabla' w_j) + (\mu_{\kappa,1} - \ell_{N,q}) \rho_\kappa^2 w_j + \rho_\kappa^{q+1} w_j^q = 0 \quad \text{on } S_+^{N-1}$$

Near ∂S_+^{N-1} we have $w_j \sim \rho_\kappa^{\frac{\alpha_+}{2}}$ and $|\nabla' w_j| \sim \rho_\kappa^{\frac{\alpha_+}{2}-1}$. Then integration by parts is justified and

$$\int_{S_+^{N-1}} \left(\left(\frac{\nabla' w_1}{w_1} - \frac{\nabla' w_2}{w_2} \right) \cdot \nabla' (w_1^2 - w_2^2) + \rho_\kappa^{q-1} (w_1^{q-1} - w_2^{q-1}) (w_1^2 - w_2^2) \right) \rho_\kappa^2 dS = 0.$$

The two terms of the integral are nonnegative, thus $w_1 = w_2$. For statement III we first prove, by the method used in [10, Th 3.1], that any solution ω depends only on the azimuthal angle $\theta \in [0, \frac{\pi}{2}]$. Then we show that the corresponding ODE verified by ω admits only the three mentioned solutions. For Theorem 5, we first construct a barrier function as in [7, Appendix] which yields to the following estimate

$$(26) \quad u(x) \leq c|x|^{-\frac{2}{q-1} + \frac{\alpha_+}{2}} (d(x))^{\frac{\alpha_+}{2}} \quad \forall x \in \Omega.$$

With this estimate we adapt the scaling method developed in [9, Sect. 3.3] to obtain the classification result.

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