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ENDLESS CONTINUABILITY AND CONVOLUTION PRODUCT

by

Yafei Ou & Eric Delabaere

Abstract. — We provide a rigorous analysis for the so-called endlessly continuable germs of holomorphic functions or in other words, the Ecalle’s resurgent functions. We follow and complete an approach due to Pham, based on the notion of discrete filtered set Ω_* and the associated Riemann surface defined as the space of Ω_* -homotopy classes of paths. Our main contribution consists in a complete though simple proof of the stability under convolution product of the space of endlessly continuable germs.

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1. Introduction

Today, the resurgence theory of Ecalle has demonstrated its efficiency in many instances for dealing with divergent series arising from differential and difference equations, PDEs, semiclassical analysis, etc.. see for instance [11, 9, 15, 16, 4, 5, 24] and references therein. In particular, this theory provides the necessary tools for understanding the nonlinear Stokes phenomena in asymptotic analysis and leads up to both theoretical results and even numerical methods, see e.g. [8].

In its simplest form, the Ecalle’s theory generalizes the Borel resummation theory, whose main objects are 1-Gevrey formal series $\tilde{\varphi}(z) = \sum_{n \geq 1} \frac{a_n}{z^n} \in \mathbb{C}[[z^{-1}]]$, that is the formal Borel transform $\hat{\varphi}(\zeta) = \sum_{n \geq 1} \frac{a_n}{(n-1)!} \zeta^n \in \mathbb{C}\{\zeta\}$ defines a germ of holomorphic functions at the origin. The formal series $\tilde{\varphi}$ is said to be resurgent if $\hat{\varphi}$ is “endlessly

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continuable". Roughly speaking, $\widehat{\varphi}$ is endlessly continuable if $\widehat{\varphi}$ can be analytically continued along any path on \mathbb{C} avoiding a set of possibly singular points, each of them being locally isolated. However, this set may be everywhere dense and is so for many important applications, in particular those stemming from high energy physics, string theory and related models [6, 7, 1, 19, 10]

It turns out that the Cauchy product $(\widetilde{\varphi}.\widetilde{\psi})(z)$ of 1-Gevrey formal series becomes the convolution product $\widehat{\varphi} * \widehat{\psi}(\zeta) = \int_0^\zeta \widehat{\varphi}(\eta)\widehat{\psi}(\zeta - \eta) d\eta$ of germs of holomorphic functions by Borel transformation. Since the Ecalle's theory aims at analyzing nonlinear problems, it is an essential demand that the convolution product preserves the notion of endless continuability. This ends up with the definition of various subalgebras of resurgent functions in the so-called Borel ζ -plane and their counterpart in the initial z -plane, as well as alien operators designed for encoding the singularities by microlocalisation and describing the nonlinear Stokes phenomena [12, 13, 2, 24, 5].

As a matter of fact, there exist various definitions of "endless continuability" in the literature. The more general one is that of Ecalle [12, 13]. Another one is due to Pham *et al.* [3, 2], is easier to handle with and is based on the construction of a Riemann surface governed by the datum of a discrete filtered set. This provides the notion of endless Riemann surface. A germ of holomorphic functions that can be analytically continued to such a Riemann surface is by definition endlessly continuable. This is this second approach that we use in this paper.

As already said, the endless continuability and its stability under convolution product are key-properties at the very root of the resurgence theory. Unfortunately, the existing proofs for the stability in its full generality [12, 2] are difficult and subject to controversy. Our main goal in this article is to show in a simple and rigorous way that the convolution product of any two endlessly continuable functions is endlessly continuable. Though inspired by [2], our methods differ from these authors for key-arguments. The method that we present in this paper can be seen as an extension of ideas detailed in [20, 22, 24] for the case of holomorphic functions that can be analytically continued along any path avoiding closed discrete subsets of \mathbb{C} . Consequently, we thought that our results were worth the attention of the specialists in this field of research.

The paper is organized as follows. We introduce the notion of discrete filtered sets and their associated Riemann surfaces that we study properly (Sect. 2). We define endless Riemann surfaces and endlessly continuable germs of holomorphic functions and we make a link with the endlessly continuable functions of Ecalle (Sect. 3). The main result of the paper concerns the stability under convolution product and this is detailed in Sect. 4. We end the paper with some open problems.

2. Discrete filtered set and associated Riemann surface

2.1. Discrete filtered sets. — The following definitions are adapted from [2].

Definition 2.1. — A **discrete filtered set** Ω_\star centred at $\omega \in \mathbb{C}$ is an increasing sequence of finite sets $\Omega_L \subset \mathbb{C}$, $L > 0$, such that :

- for any $L > 0$, Ω_L belongs to the open disc centred at ω with radius L ;
- if $L_1 \leq L_2$ then $\Omega_{L_1} \subseteq \Omega_{L_2}$;
- for $L > 0$ small enough, $\Omega_L = \{\omega\}$.

For $L > 0$, we denote $\Omega_L^\star = \Omega_L \setminus \{\omega\}$. The number $\rho_{\Omega_\star}(\omega) = \sup\{L > 0 \mid \Omega_L^\star = \emptyset\}$ is called the **distance of ω to Ω_\star** .

Definition 2.2. — Let Ω_\star and Ω'_\star be two discrete filtered sets centred at $\omega \in \mathbb{C}$. Their **union** $\Omega_\star \cup \Omega'_\star$ is the discrete filtered set centred at ω defined by : for every $L > 0$, $(\Omega_\star \cup \Omega'_\star)_L = \Omega_L \cup \Omega'_L$. Their **sum** $\Omega_\star + \Omega'_\star$ is the filtered set centred at ω defined by : for every $L > 0$, $(\Omega_\star + \Omega'_\star)_L = \{-\omega + \Omega_L + \Omega'_L\} \cap D(\omega, L)$. Their **fine sum** $\Omega_\star * \Omega'_\star$ is the filtered set centred at ω given by : for every $L > 0$, $(\Omega_\star * \Omega'_\star)_L = \{\zeta = -\omega + \omega_1 + \omega_2 \mid \omega_1 \in \Omega_{L_1}, \omega_2 \in \Omega'_{L_2}, L_1 + L_2 = L\}$.

If Ω_\star is a discrete filtered set, we remark that $\bigcup_{L>0} \Omega_L$ can be dense in \mathbb{C} as it is shown in the following example.

Example 2.3. — Assume that $\omega_1 \in \mathbb{C}^\star$ and define

- for any $L \in]0, |\omega_1|]$, $\Omega_L = \{0\}$,
- for any $n \in \mathbb{N}^\star$ and any $L \in]n|\omega_1|, (n+1)|\omega_1|]$, $\Omega_L = \{0, \pm\omega_1, \dots, \pm n\omega_1\}$.

This define a discrete filtered set $\Omega_{1\star}$ centred at 0.

Assume now that $\omega_1, \omega_2, \omega_3 \in \mathbb{C}^\star$ are rationally independent, that is linearly independent over \mathbb{Z} . We consider the three discrete filtered sets $\Omega_{1\star}, \Omega_{2\star}$ and $\Omega_{3\star}$ centred at 0 defined as above. We note $\Omega_\star = \Omega_{1\star} + \Omega_{2\star} + \Omega_{3\star}$ their sum. Then $\bigcup_{L>0} \Omega_L$ is everywhere dense in \mathbb{C} . The conclusion is the same when $\Omega_\star = \Omega_{1\star} * \Omega_{2\star} * \Omega_{3\star}$ is defined by fine sums.

For a given discrete filtered set Ω_\star centred at ω , its iterated fine sums $\sum_{n\star} \Omega_\star = \underbrace{\Omega_\star * \dots * \Omega_\star}_{n \text{ times}}$ makes a direct system (for the injections $(\sum_{n\star} \Omega_\star)_L \hookrightarrow (\sum_{(n+1)\star} \Omega_\star)_L$, for every $L > 0$). The fine sums enjoys the following property (the proof is left to the reader):

Proposition 2.4. — Let Ω_\star be a discrete filtered set Ω_\star centred at ω . Then the direct limit $\Omega_\star^\infty = \lim_{\rightarrow} \sum_{n\star} \Omega_\star$ is a discrete filtered set at ω .

Definition 2.5. — The discrete filtered set Ω_\star^∞ is called the **saturated** of Ω_\star .

2.2. Reminder about paths. — In what follows, a path λ in a topological space X is any continuous function $\lambda : [a, a+l] \rightarrow X$, where $[a, a+l] \subset \mathbb{R}$ is a (compact) interval possibly reduced to $\{a\}$. We often work with standard paths, that is paths defined on $[0, 1]$. The path $\underline{\lambda} : t \in [0, 1] \mapsto \lambda(a+tl)$ is the standardized path of λ .

For two paths $\lambda_1 : [a, a + l] \rightarrow X$, $\lambda_2 : [b, b + k] \rightarrow X$ so that $\lambda_1(a + l) = \lambda_2(b)$, one defines their product (or concatenation) by

$$\lambda_1 \lambda_2 : t \in [a, a + l + k] \mapsto \begin{cases} \lambda_1(t), & t \in [a, a + l] \\ \lambda_2(t - a - l + b), & t \in [a + l, a + l + k] \end{cases}$$

When the two paths λ_1, λ_2 have same extremities, they are homotopic when there exists a continuous map $H : [0, 1] \times [0, 1] \rightarrow X$ that realizes a homotopy between the standardized paths $\underline{\lambda}_1$ and $\underline{\lambda}_2$. We recall that any path $\lambda : I \rightarrow \mathbb{C}$ can be uniformly approached by C^∞ -paths. When $\lambda : I \rightarrow \mathbb{C}$ is piecewise C^1 , we denote its length by \mathcal{L}_λ .

2.3. Ω_\star -allowed path, Ω_\star -homotopy. — The following definitions are inspired from [2] but for slight modifications ⁽¹⁾.

Definition 2.6. — For Ω_\star a discrete filtered set centred at $\omega \in \mathbb{C}$, one denotes by $\mathfrak{R}_{\Omega_\star}(L)$ the set of paths $\lambda : I \rightarrow \mathbb{C}$ starting from ω and such that :

- λ is C^1 piecewise and its length satisfies $\mathcal{L}_\lambda < L$;
- λ is the constant path or there exists $t_0 \in [0, 1[$ such that $\underline{\lambda}([0, t_0]) = \{\omega\}$ and $\underline{\lambda}([t_0, 1]) \subset D(\omega, L) \setminus \Omega_L$.

A path λ is said to be Ω_\star -allowed if $\lambda \in \mathfrak{R}_{\Omega_\star}(L)$ for some $L > 0$. We denote by $\mathfrak{R}_{\Omega_\star} = \bigcup_{L>0} \mathfrak{R}_{\Omega_\star}(L)$ the set of Ω_\star -allowed paths.

Definition 2.7. — Let Ω_\star be a discrete filtered set centred at $\omega \in \mathbb{C}$. A continuous map $H : (s, t) \in [0, 1]^2 \mapsto H_t(s) \in \mathbb{C}$ is a Ω_\star -homotopy if H has a continuous partial derivative $\frac{\partial H}{\partial s}$ and, for every $t \in [0, 1]$, the path H_t is Ω_\star -allowed.

Two Ω_\star -allowed paths λ_0 and λ_1 with same extremities are Ω_\star -homotopic when there exists a Ω_\star -homotopy that realises a homotopy between the standardized paths $\underline{\lambda}_0$ and $\underline{\lambda}_1$.

For λ a Ω_\star -allowed path, we denote by $\text{cl}(\lambda)$ its equivalence class for the relation \sim_{Ω_\star} of Ω_\star -homotopy of paths in $\mathfrak{R}_{\Omega_\star}$ with fixed extremities.

It is quite important to understand what is the Ω_\star -homotopy and we make the following remark that we formulate as a lemma:

Lemma 2.8. — Assume that Ω_\star be a discrete filtered set centred at $\omega \in \mathbb{C}$ and let H be a Ω_\star -homotopy. Then there exists a good ⁽²⁾ open covering $(I_i)_{0 \leq i \leq n}$ of $[0, 1]$ and real positive numbers L_0, L_1, \dots, L_n such that,

- for every $i = 0, \dots, n$ and every $t \in I_i$, H_t belongs to $\mathfrak{R}_{\Omega_\star}(L_i)$.
- for every $i = 0, \dots, n-1$ and for every $t \in I_i \cap I_{i+1}$, $H_t \in \mathfrak{R}_{\Omega_\star}(L_i) \cap \mathfrak{R}_{\Omega_\star}(L_{i+1})$.

Proof. — From the very definition of a discrete filtered set, one can define an increasing sequence of real numbers $0 = l_{-1} < l_0 < l_1 < l_2 < \dots$ with the properties:

- $\Omega_{l_0} = \{\omega\}$ and for any integer $i \geq 1$, $\Omega_{l_{i-1}} \subsetneq \Omega_{l_i}$;

1. These definitions are less general than those of [2] but sufficient in practice as far as we know.

2. By “good”, we mean that the covering has finite elements, that each of these element I_i is a connected interval and that there are no 3-by-3 intersections.

– $\Omega_L = \Omega_{l_i}$ for every $L \in]l_{i-1}, l_i]$, $i \in \mathbb{N}$.

Pick any $t_\star \in [0, 1]$ and assume that $\mathcal{L}_{H_{t_\star}} \in]l_{i-1}, l_i[$ for some $i \in \mathbb{N}$. One has $H_{t_\star} \in \mathfrak{R}_{\Omega_\star}(l_i)$ necessarily since H_{t_\star} is Ω_\star -allowed. We say that $H_t \in \mathfrak{R}_{\Omega_\star}(l_i)$ for any $t \in [0, 1]$ close enough to t_\star . Indeed, we remark that the map $t \in [0, 1] \mapsto \mathcal{L}_{H_t}$ is continuous because of the existence and the continuity of the partial derivative $\frac{\partial H}{\partial s}$. Thus, if $\mathcal{L}_{H_{t_\star}} \in]l_{i-1}, l_i[$, then $\mathcal{L}_{H_t} \in]l_{i-1}, l_i[$ for t close enough to t_\star . Now if $\mathcal{L}_{H_{t_\star}} = l_{i-1}$ ($i \geq 1$), then $\mathcal{L}_{H_t} \in]l_{i-2}, l_i[$ for t close enough to t_\star . However, \mathcal{L}_{H_t} belongs also to $\mathfrak{R}_{\Omega_\star}(l_i)$ for t close enough to t_\star because of the continuity of H (the euclidean distance $d(H_{t_\star}, \Omega_{l_i})$ of the path H_t to the set Ω_{l_i} is > 0 , so does $d(H_t, \Omega_{l_i})$ for t close enough to t_\star). This way one gets an open covering of $[0, 1]$ from which one deduces a finite open covering by compactity. One easily concludes. \square

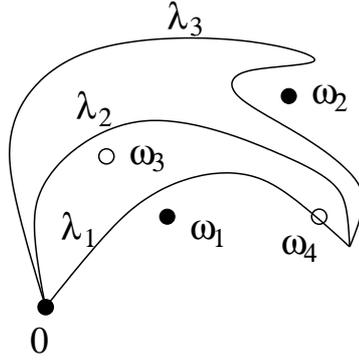


FIGURE 1. We assume that Ω_\star is a discrete filtered set centred at 0. For $0 < L_1 < L_2$, $\Omega_{L_1} = \{0, \omega_1, \omega_2\}$, $\Omega_{L_2} = \Omega_{L_1} \cup \{\omega_3, \omega_4\}$. The paths $\lambda_1, \lambda_2 \in \mathfrak{R}_{\Omega_\star}(L_1)$ are Ω_\star -homotopic, the paths $\lambda_2, \lambda_3 \in \mathfrak{R}_{\Omega_\star}(L_2)$ are Ω_\star -homotopic, thus λ_1 and λ_3 are Ω_\star -homotopic despite the fact that $\lambda_1 \notin \mathfrak{R}_{\Omega_\star}(L_2)$.

From lemma 2.8, observe that if $L_2 > L_1$, a path $\lambda_1 \in \mathfrak{R}_{\Omega_\star}(L_1)$ can be Ω_\star -homotopic to another path $\lambda_2 \in \mathfrak{R}_{\Omega_\star}(L_2)$ and at the same time λ_1 not being homotopic to λ_2 in the usual way, when both are seen as paths in $\mathfrak{R}_{\Omega_\star}(L_2)$. Even, we may have $\lambda_1 \notin \mathfrak{R}_{\Omega_\star}(L_2)$, see Fig. 1.

2.4. Riemann surface associated with a discrete filtered set. —

Definition 2.9. — Let Ω_\star be a discrete filtered set centred at $\dot{\omega} \in \mathbb{C}$. We set:

$$\mathcal{R}_{\Omega_\star} = \{\zeta = \text{cl}(\lambda) \mid \lambda \in \mathfrak{R}_{\Omega_\star}\} \text{ and } \mathfrak{p} : \zeta = \text{cl}(\lambda) \mapsto \dot{\zeta} = \underline{\lambda}(1) \in \mathbb{C}.$$

Remark that $\mathfrak{p}^{-1}(\dot{\omega})$ is reduced to a single point $\omega = \text{cl}(\text{constant path})$. This is why one usually considers $\mathcal{R}_{\Omega_\star}$ as a pointed space (\mathcal{R}, ω) .

Let Ω_\star be a discrete filtered set centred at $\dot{\omega} \in \mathbb{C}$, and set $\omega = \mathfrak{p}^{-1}(\dot{\omega})$. One can endow $\mathcal{R}_{\Omega_\star}$ with a separated topology, a basis $\mathcal{B} = \{\mathcal{U}\}$ of open sets defining

this topology being given as follows⁽³⁾. (We adapt the classical construction of a universal covering [17]). Let us consider a point $\zeta \in \mathcal{R}_{\Omega_\star}$.

- Assume that $\zeta = \omega$. For some $L > 0$ we consider $\mathcal{U} \subset D(\dot{\omega}, L) \setminus \Omega_L^\star$ a star-shaped domain with respect to $\dot{\omega}$. Let $\mathcal{U} \subset \mathcal{R}_{\Omega_\star}$ be the set of all $\xi = \text{cl}(\lambda)$ where $\lambda \in \mathfrak{R}_{\Omega_\star}(L)$ is any path ending at $\xi \in \mathcal{U}$ and whose image is the line segment $[\omega, \xi]$. (For a given ξ , the length of these paths is $|\xi| < L$ and all these paths belong to the same Ω_\star -homotopy class).
- Suppose that $\zeta \neq \omega$. We choose a path $\lambda_1 \in \mathfrak{R}_{\Omega_\star}(L)$ such that $\text{cl}(\lambda_1) = \zeta$. For some $L_2 > 0$ such that $\mathcal{L}_{\lambda_1} + L_2 < L$, we consider $\mathcal{U} \subset D(\dot{\zeta}, L_2) \setminus \Omega_{L_2} \subset D(\dot{\omega}, L) \setminus \Omega_L$ such that \mathcal{U} is a star-shaped domain with respect to $\dot{\zeta} = \lambda_1(1)$. For $\xi \in \mathcal{U}$, consider a path λ_2 starting from $\dot{\zeta}$, ending at ξ and whose image is the line segment $[\zeta, \xi]$. Then the product $\lambda_1 \lambda_2$ belongs to $\mathfrak{R}_{\Omega_\star}(L)$ and we consider its Ω_\star -homotopy class $\xi = \text{cl}(\lambda_1 \lambda_2)$. We note \mathcal{U} the set of such points ξ .

We show that the system $\mathcal{B} = \{\mathcal{U}\}$ made of these sets provides a basis for a topology on $\mathcal{R}_{\Omega_\star}$. Obviously, every element $\xi \in \mathcal{R}_{\Omega_\star}$ belongs to at least one $\mathcal{U} \in \mathcal{B}$. Now assume that $\xi \in \mathcal{U} \cap \mathcal{V}$, $\mathcal{U}, \mathcal{V} \in \mathcal{B}$.

- If $\xi = \omega$, then necessarily \mathcal{U} and \mathcal{V} are two star-shaped domains with respect to $\dot{\omega}$, \mathcal{U} is a subset of $D(\dot{\omega}, L_1) \setminus \Omega_{L_1}^\star$ and \mathcal{V} is a subset of $D(\dot{\omega}, L_2) \setminus \Omega_{L_2}^\star$. Set $L = \max\{L_1, L_2\}$. Then $\mathcal{W} = \mathcal{U} \cap \mathcal{V} \subset D(\dot{\omega}, L) \setminus \Omega_L^\star$ is also a star-shaped domain with respect to $\dot{\omega}$ to which is associated a $\mathcal{W} \in \mathcal{B}$ that satisfies: $\xi \in \mathcal{W} \subset \mathcal{U} \cap \mathcal{V}$.
- Otherwise $\xi \neq \omega$ and $\xi \in \mathcal{U} \cap \mathcal{V}$. There is no loss of generality in assuming also that $\omega \notin \mathcal{U} \cap \mathcal{V}$. Thus :

- for some $L > 0$, $\mathcal{U} \subset D(\dot{\zeta}_1, L_2) \setminus \Omega_{L_2}$ is a star-shaped domain with respect to $\dot{\zeta}_1 = \lambda_1(1)$ where $\lambda_1 \in \mathfrak{R}_{\Omega_\star}(L)$ satisfies $\mathcal{L}_{\lambda_1} + L_2 < L$. Also we have $\xi = \text{cl}(\lambda_1 \lambda_2)$ where λ_2 starts from $\dot{\zeta}_1$, ends at ξ and is such that $\underline{\lambda}_2([0, 1]) = [\dot{\zeta}_1, \xi]$.
- for some $L' > 0$, $\mathcal{V} \subset D(\dot{\zeta}'_1, L'_2) \setminus \Omega_{L'_2}$ is a star-shaped domain with respect to $\dot{\zeta}'_1 = \lambda'_1(1)$ where $\lambda'_1 \in \mathfrak{R}_{\Omega_\star}(L')$ satisfies $\mathcal{L}_{\lambda'_1} + L'_2 < L'$. Also $\xi = \text{cl}(\lambda'_1 \lambda'_2)$ where λ'_2 starts from $\dot{\zeta}'_1$, ends at ξ and is such that $\underline{\lambda}'_2([0, 1]) = [\dot{\zeta}'_1, \xi]$.

Now choose $L'' > 0$ such that $\mathcal{L}_{\lambda_1 \lambda_2} + L'' < L$ and $\mathcal{L}_{\lambda'_1 \lambda'_2} + L'' < L'$. Consider $\mathcal{W} \subset \mathcal{U} \cap \mathcal{V} \cap D(\dot{\xi}, L'')$ a star-shaped domain with respect to $\dot{\xi}$. To \mathcal{W} is associated $\mathcal{W} \in \mathcal{B}$ and $\xi \in \mathcal{W} \subset \mathcal{U} \cap \mathcal{V}$.

We show that the topology thus defined by \mathcal{B} is Hausdorff. We consider two points ζ and ζ' in $\mathcal{R}_{\Omega_\star}$. Clearly if $\zeta \neq \zeta'$, then ζ and ζ' have disjoint neighbourhoods. Thus

3. This topology is not detailed in [2].

assume that $\overset{\bullet}{\zeta} = \overset{\bullet}{\zeta}' (\neq \overset{\bullet}{\omega}, \text{ say})$ but $\zeta \neq \zeta'$. Suppose the existence of a neighbourhood $\mathcal{U} \in \mathfrak{B}$ of ζ , a neighbourhood $\mathcal{U}' \in \mathfrak{B}$ of ζ' such that $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$. This means that there exists $\xi \in \mathcal{U} \cap \mathcal{U}'$ that satisfies:

- $\xi = \text{cl}(\lambda_1 \lambda_2)$ with $\zeta = \text{cl}(\lambda_1)$, λ_2 starting from $\overset{\bullet}{\zeta}$, ending at $\overset{\bullet}{\xi}$ with image the line segment $[\overset{\bullet}{\zeta}, \overset{\bullet}{\xi}] \subset \mathcal{U}$;
- $\xi = \text{cl}(\lambda'_1 \lambda'_2)$ with $\zeta' = \text{cl}(\lambda'_1)$, λ'_2 starting from $\overset{\bullet}{\zeta}' = \overset{\bullet}{\zeta}$, endings at $\overset{\bullet}{\xi}$ the line segment $[\overset{\bullet}{\zeta}, \overset{\bullet}{\xi}] \subset \mathcal{U}'$ for its range.

This implies that λ_1 and λ'_1 are in the same class, that is $\zeta = \zeta'$ and we get a contradiction.

The topological space $\mathcal{R}_{\Omega_\star}$ is obviously (arc)connected. Also, by the very construction of the topology, for every $\mathcal{U} \in \mathfrak{B}$, the restriction $\mathfrak{p}|_{\mathcal{U}} : \zeta \mapsto \overset{\bullet}{\zeta} \in \mathcal{U}$ is a homeomorphism. This means that $\mathcal{R}_{\Omega_\star}$ is an étalé space on \mathbb{C} (but of course not a covering space). We have thus shown the following proposition.

Proposition 2.10. — *Let Ω_\star be is a discrete filtered set. The (pointed) space $\mathcal{R}_{\Omega_\star}$ is a topologically (arc)connected separated space. With the projection \mathfrak{p} , the space $\mathcal{R}_{\Omega_\star}$ is an étalé space on \mathbb{C} .*

When pulling back by \mathfrak{p} the complex structure of \mathbb{C} , $\mathcal{R}_{\Omega_\star}$ becomes a Riemann surface. This Riemann surface is the smallest in the following sense:

Lemma 2.11. — *Let $(\mathcal{R}', \mathfrak{p}', \omega')$ be a Riemann surface over \mathbb{C} , with $\mathfrak{p}'(\omega') = \overset{\bullet}{\omega}$. We suppose that any Ω_\star -allowed path can be lifted on \mathcal{R}' from ω' with respect to \mathfrak{p}' . Then $(\mathcal{R}_{\Omega_\star}, \mathfrak{p}, \omega)$ is contained in $(\mathcal{R}', \mathfrak{p}', \omega')$, that is there exists a continuous map $\tau : \mathcal{R}_{\Omega_\star} \rightarrow \mathcal{R}'$ such that $\mathfrak{p}' \circ \tau = \mathfrak{p}$ and $\tau(\omega) = \omega'$.*

Proof. — From the very definition of the Riemann surface $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ associated with a discrete filtered set Ω_\star centred at $\overset{\bullet}{\omega} \in \mathbb{C}$, every Ω_\star -allowed path can be lifted with respect to \mathfrak{p} into a path on $\mathcal{R}_{\Omega_\star}$ with initial point $\omega = \mathfrak{p}^{-1}(\overset{\bullet}{\omega})$. Indeed, assume that $\underline{\lambda} \in \mathfrak{R}_{\Omega_\star}(L)$ and define $\underline{\lambda}_t : s \in [0, 1] \mapsto \underline{\lambda}_t(s) = \underline{\lambda}(ts)$ for $t \in [0, 1]$. Then $\underline{\lambda}_t \in \mathfrak{R}_{\Omega_\star}(L)$ and the mapping $\underline{\lambda} : t \in [0, 1] \mapsto \Lambda(t) = \text{cl}(\underline{\lambda}_t)$ is continuous and is a lifting of $\underline{\lambda}$ from ω . This lifting is unique thanks to the uniqueness of lifting [17]. Pick any Ω_\star -allowed path λ . Its lifting with respect to \mathfrak{p} from ω ends at ζ while its lifting with respect to \mathfrak{p}' from ω' ends at ξ . This gives a mapping $\tau : \zeta \in \mathcal{R}_{\Omega_\star} \mapsto \xi \in \mathcal{R}'$ which is well-defined and injective (uniqueness of lifting), continuous (we work with étalé spaces) and preserves fibers. \square

Definition 2.12. — Let Ω_\star be a discrete filtered set centred at $\overset{\bullet}{\omega}$. Any Riemann surface $(\mathcal{R}', \mathfrak{p}', \omega')$ over \mathbb{C} which is isomorphic to $(\mathcal{R}_{\Omega_\star}, \mathfrak{p}, \omega)$, is called a Riemann surface associated with Ω_\star .

In this definition, isomorphic means the existence of a fiber preserving homeomorphism $\tau : \mathcal{R}_{\Omega_\star} \rightarrow \mathcal{R}'$.

We would like to point out a consequence of the topology considered on these Riemann surfaces. On Fig. 2, we consider a discrete filtered set Ω_\star centred at 0

such that, for $0 < L_1 < L_2$, $\Omega_{L_1} = \{0, \omega_1, \omega_2\}$, $\Omega_{L_2} = \Omega_{L_1} \cup \{\omega_3, \omega_4\}$. We have drawn two paths $\lambda_1, \lambda_2 \in \mathfrak{R}_{\Omega_*}(L_1)$ ending at the same point ζ and Ω_* -homotopic, $\text{cl}(\lambda_1) = \text{cl}(\lambda_2)$. Also we have drawn a path λ_3 starting at ζ and ending at ξ such that the product path $\lambda_2\lambda_3$ belongs to $\mathfrak{R}_{\Omega_*}(L_2)$. We remark that $\lambda_1\lambda_3 \notin \mathfrak{R}_{\Omega_*}(L_2)$. The path λ_1 , *resp.* λ_2 , can be lifted with respect to \mathfrak{p} to a path Λ_1 , *resp.* Λ_2 , on \mathcal{R}_{Ω_*} with initial point $0 = \mathfrak{p}^{-1}(0)$ and common end point $\text{cl}(\lambda_1) = \text{cl}(\lambda_2)$. From that point, λ_3 can be lifted with respect to \mathfrak{p} to a path Λ_3 ending at $\text{cl}(\lambda_2\lambda_3)$. We thus see that the path $\Lambda_1\Lambda_3$ is well defined and $\mathfrak{p}(\Lambda_1\Lambda_3) = \lambda_1\lambda_3$. This means that $\lambda_1\lambda_3$ can be lifted on \mathcal{R}_{Ω_*} with respect to \mathfrak{p} from 0 despite the fact that $\lambda_1\lambda_3$ is not Ω_* -allowed. We say that ω_3 is a “removable Ω_* -point” for $\lambda_1\lambda_3$.

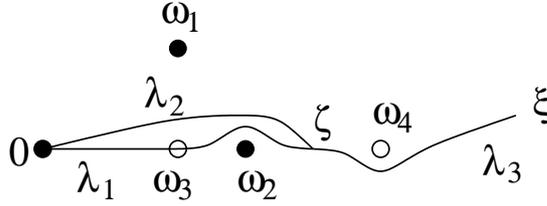


FIGURE 2

Definition 2.13. — Let Ω_* be a discrete filtered set centred at $\dot{\omega}_0 \in \mathbb{C}$ and $(\mathcal{R}_{\Omega_*}, \mathfrak{p})$ its associated Riemann surface. Let Λ be a piecewise \mathcal{C}^1 path on \mathcal{R}_{Ω_*} starting from $\omega_0 = \mathfrak{p}^{-1}(\dot{\omega}_0)$ and $\lambda = \mathfrak{p} \circ \Lambda$. Let $L > \mathcal{L}_\lambda$. If λ meets a point $\omega \in \Omega_L$, then ω is called a **removable Ω_* -point** for λ .

2.5. Distance of a path to Ω_* . — Let Ω_* be a discrete filtered set. We consider a point $\zeta \in \mathcal{R}_{\Omega_*}$. For $r > 0$ small enough and since a disc is a star-shaped domain with respect to its origin, there exists a connected neighbourhood $\mathcal{U} \subset \mathcal{R}_{\Omega_*}$ of ζ so that $\mathfrak{p}|_{\mathcal{U}} : \mathcal{U} \rightarrow D(\zeta, r)$ is a homeomorphism.

Definition 2.14. — For any $\zeta \in \mathcal{R}_{\Omega_*}$ and for $r > 0$ small enough, the **ball** $D(\zeta, r)$ centred at ζ with radius r is the connected neighbourhood of ζ so that $\mathfrak{p}|_{D(\zeta, r)} : D(\zeta, r) \rightarrow D(\dot{\zeta}, r) \subset \mathbb{C}$ is a homeomorphism. The distance $\rho_{\Omega_*}(\zeta)$ of ζ to Ω_* is the supremum of the $r > 0$ such that ζ has a neighbourhood of the form $D(\zeta, r)$.

In other words, $\rho_{\Omega_*}(\zeta)$ is the distance of ζ (for the norm $|\cdot|$) to the boundary of \mathcal{R}_{Ω_*} . If Ω_* is centred at $\dot{\omega} = \mathfrak{p}(\omega)$, then of course $\rho_{\Omega_*}(\dot{\omega}) = \rho_{\Omega_*}(\omega)$ and there is no risk of misunderstanding. Notice that the mapping $\zeta \in \mathcal{R}_{\Omega_*} \mapsto \rho_{\Omega_*}(\zeta)$ is continuous since \mathfrak{p} is continuous. This implies that $t \in [0, 1] \mapsto \rho_{\Omega_*}(\underline{\Lambda}(t))$ is a continuous mapping when Λ on \mathcal{R}_{Ω_*} starting from ω , thus $\inf_{t \in [0, 1]} \rho_{\Omega_*}(\Lambda(t)) > 0$ by compactness.

Definition 2.15. — Let Ω_* be a discrete filtered set centred at $\dot{\omega} = \mathfrak{p}(\omega) \in \mathbb{C}$. For any path λ issued from $\dot{\omega}$ that can be lifted to \mathcal{R}_{Ω_*} with respect to \mathfrak{p} from ω into the path Λ , one defines the distance $d(\lambda, \Omega_*)$ of λ to Ω_* by $d(\lambda, \Omega_*) = \inf_{t \in [0, 1]} \rho_{\Omega_*}(\Lambda(t)) > 0$.

2.6. Some properties of the Riemann surface $\mathcal{R}_{\Omega_\star}$. —

Proposition 2.16. — *The Riemann surface $\mathcal{R}_{\Omega_\star}$ associated with the discrete filtered set Ω_\star is simply connected.*

Thus $\mathcal{R}_{\Omega_\star}$ is conformally equivalent to the open unit disc as a consequence of the uniformization theorem.

Proof. — Let Ω_\star be a discrete filtered set Ω_\star centred at $\dot{\omega} \in \mathbb{C}$ and let $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ be its associated pointed Riemann surface, $\omega = \mathfrak{p}^{-1}(\dot{\omega})$. Pick a non-constant closed curve Λ on $\mathcal{R}_{\Omega_\star}$. We want to show that Λ is null-homotopic. Since $\mathcal{R}_{\Omega_\star}$ is arcconnected, one can suppose that Λ starts and ends at ω and there is no loss of generality in assuming that Λ does not meet ω apart from its extremities. Also, up to making a slight deformation of Λ in its homotopy class, one can assume that $\lambda = \mathfrak{p} \circ \Lambda$ is \mathcal{C}^1 , with length $\mathcal{L}_\lambda < L$ for some $L > 0$ and that $\lambda|_{]0,1[}$ avoids Ω_L . One can write Λ under the form ⁽⁴⁾ $\Lambda = \Lambda_1 \Lambda_2^{-1}$ where both Λ_1, Λ_2 are paths starting from ω and ending at a point $\zeta \in \mathcal{R}_{\Omega_\star}$ with $\zeta \neq \omega$. We set $\lambda_1 = \mathfrak{p} \circ \Lambda_1$ and $\lambda_2 = \mathfrak{p} \circ \Lambda_2$. Both λ_1, λ_2 are Ω_\star -allowed paths, precisely their belong to $\mathfrak{R}_{\Omega_\star}(L)$. The path λ_1 , resp. λ_2 , can be lifted with respect to \mathfrak{p} from ω and, by uniqueness of lifting, corresponds to Λ_1 , resp. Λ_2 . This implies that λ_1 and λ_2 are Ω_\star -homotopic with $\zeta = \text{cl}(\lambda_1) = \text{cl}(\lambda_2)$. The Ω_\star -homotopy between λ_1 and λ_2 can be lifted with respect to \mathfrak{p} and this provides a homotopy between Λ_1 and Λ_2 . Therefore, $\Lambda = \Lambda_1 \Lambda_2^{-1}$ is null-homotopic. \square

Proposition 2.17. — *Let $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ be the Riemann surface associated with a discrete filtered set $\Omega_\star = \Omega_\star(\omega)$ centred at $\dot{\omega}$. Then, for every $\zeta \in \mathcal{R}_{\Omega_\star}$, there exists a discrete filtered set $\Omega_\star(\zeta)$ centred at $\dot{\zeta} = \mathfrak{p}(\zeta)$ such that every $\Omega_\star(\zeta)$ -allowed path starting from ζ can be lifted on $\mathcal{R}_{\Omega_\star}$ from ζ with respect to \mathfrak{p} .*

Proof. — We consider the Riemann surface $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ associated with a discrete filtered set Ω_\star centred at $\dot{\omega} \in \mathbb{C}$ and set $\omega = \mathfrak{p}^{-1}(\dot{\omega})$. Pick a point $\zeta \in \mathcal{R}_{\Omega_\star}$ with $\zeta \neq \omega$ and assume that $\zeta = \text{cl}(\lambda_0)$, $\lambda_0 \in \mathfrak{R}_{\Omega_{L_0}}$ for some $L_0 > 0$. We consider a path λ (\mathcal{C}^1 piecewise) starting from $\dot{\zeta}$ and of length \mathcal{L}_λ . If $\mathcal{L}_\lambda < \rho_{\Omega_\star}(\zeta)$, then λ can be lifted from ζ with respect to \mathfrak{p} : this is just a consequence of the topology considered on $\mathcal{R}_{\Omega_\star}$.

Assume now that λ satisfies the properties : $\lambda(0) = \dot{\zeta}$, $\lambda(]0,1]) \subset \mathbb{C} \setminus \Omega_{L_0+L}$ and $\mathcal{L}_\lambda < L$ for some $L > 0$. For $\varepsilon > 0$ small enough, one can construct a Ω_\star -homotopy $H : t \in [0,1] \mapsto H_t \in \mathfrak{R}_{\Omega_\star}$ such that

- $H_0 = \lambda_0$;
- for every $t \in [0, \varepsilon]$, $H_t \in \mathfrak{R}_{\Omega_\star}(L_0)$ and $H_t(1) = \lambda(t)$;
- H_ε belongs to $\mathfrak{R}_{\Omega_\star}(L_0) \cap \mathfrak{R}_{\Omega_\star}(L_0 + L)$;
- for every $t \in [\varepsilon, 1]$, $H_t \in \mathfrak{R}_{\Omega_\star}(L_0 + L)$ and $H_t(1) = \lambda(t)$.

4. Λ_2^{-1} is the inverse path, $\underline{\Lambda}_2^{-1}(s) = \underline{\Lambda}_2^{-1}(1 - s)$.

Indeed, for $t \in [0, \varepsilon]$, H_t realizes a small deformation of $\underline{\lambda}_0$ so as to avoid the points of Ω_{L_0+L} while for $t \in [\varepsilon, 1]$, H_t is for instance the standardized product of H_ε with $\underline{\lambda}|_{[\varepsilon, t]}$. This Ω_\star -homotopy can be lifted with respect to \mathfrak{p} into a homotopy $\mathcal{H} : t \in [0, 1] \mapsto \mathcal{H}_t$ where, for every $t \in [0, 1]$, $\mathcal{H}_t : [0, 1] \rightarrow \mathcal{R}_{\Omega_\star}$ is a path starting at ω . Therefore, the path $\Lambda : t \in [0, 1] \mapsto \mathcal{H}_t(1) \in \mathcal{R}_{\Omega_\star}$ is a lifting of $\underline{\lambda}$ from ζ . This has the following consequences. There exists a discrete filtered set $\Omega_\star(\zeta)$ centred at ζ ,

- for $L > 0$ small enough, $\Omega_L(\zeta) = \{\zeta\}$,
- for $L > 0$ large enough, $\Omega_L(\zeta) = (\Omega_{L_0+L} \cap D(\zeta, L)) \cup \{\zeta\}$

such that every $\Omega_\star(\zeta)$ -allowed path can be lifted on $\mathcal{R}_{\Omega_\star}$ with respect to \mathfrak{p} from ζ . \square

2.7. Seen and glimpsed points. — We denote by $\mathbb{S}^1 \subset \mathbb{C}$ the circle of directions about 0 of half-lines on \mathbb{C} . We usually identify \mathbb{S}^1 with $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 2.18. — Let $I \subset \mathbb{S}^1$ be an open arc, $L > 0$ and $\omega \in \mathbb{C}$. We denote by $\mathfrak{s}_\omega^L(I)$ the following open sector adherent to ω :

$$\mathfrak{s}_\omega^L(I) = \{\zeta = \omega + \xi e^{i\theta} \in \mathbb{C} \mid \theta \in I, 0 < \xi < L\}.$$

Assume that Ω_\star is a discrete filtered set centred at $\omega \in \mathbb{C}$, $\theta \in \mathbb{S}^1$ is a given direction and $L > 0$. Since Ω_L is a finite set, observe that $\Omega_L \cap \mathfrak{s}_\omega^L(I) = \Omega_L \cap]\omega, \omega + e^{i\theta}L[$ when $I =]-\alpha + \theta, \theta + \alpha[$ with $\alpha > 0$ chosen small enough.

Definition 2.19. — Let Ω_\star be a discrete filtered set centred at $\omega \in \mathbb{C}$, $\theta \in \mathbb{S}^1$ and $L > 0$. We denote $\Omega_L^\star(\theta) = \Omega_L \cap]\omega, \omega + e^{i\theta}L[$. One says that $\alpha \in]0, \frac{\pi}{2}[$ is a $\Omega_\star(\theta, L)$ -angle if $\Omega_L \cap \mathfrak{s}_\omega^L(I) = \Omega_L^\star(\theta)$ with $I =]-\alpha + \theta, \theta + \alpha[$.

Definition 2.20. — Let Ω_\star be a discrete filtered set centred at $\omega \in \mathbb{C}$, $\theta \in \mathbb{S}^1$ and $L > 0$. We denote by $\mathfrak{R}(\Omega_\star, \theta, L)$ the set of piecewise \mathcal{C}^1 paths λ that satisfy the conditions:

- $\underline{\lambda}(0) = \omega$ and $\mathcal{L}_\lambda < L$;
- for every $t \in [0, 1]$, the right and left derivatives $\underline{\lambda}'(t)$ do not vanish;
- there exists a $\Omega_\star(\theta, L)$ -angle $\alpha \in]0, \frac{\pi}{2}[$ such that for every $t \in [0, 1]$, $\arg \lambda'(t) \in]-\alpha + \theta, \theta + \alpha[$.

Remark that, apart from its origin, a path $\lambda \in \mathfrak{R}(\Omega_\star, \theta, L)$ stays in an open sector of the form $\mathfrak{s}_\omega^L(I)$, $I =]-\alpha + \theta, \theta + \alpha[$, with a α a $\Omega_\star(\theta, L)$ -angle. Moreover λ always moves forward in that sector.

Proposition 2.21. — Let Ω_\star be a discrete filtered set centred at $\omega_0 \in \mathbb{C}$ and $\theta \in \mathbb{S}^1$ a direction. There exists a uniquely defined discrete and closed set $\text{GLIMP}_{\Omega_\star}^\star(\theta) \subset \mathbb{C}$ that satisfies the following conditions for any $L > 0$:

- $\text{GLIMP}_{\Omega_\star}^\star(\theta, L) \subset \Omega_L^\star(\theta)$, where $\text{GLIMP}_{\Omega_\star}^\star(\theta, L) = \text{GLIMP}_{\Omega_\star}^\star(\theta) \cap D(0, L)$;
- any path belonging to $\mathfrak{R}(\Omega_\star, \theta, L)$ that circumvents to the right or the left the set $\text{GLIMP}_{\Omega_\star}^\star(\theta, L)$, can be lifted on the Riemann surface $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ with respect to \mathfrak{p} from $\mathfrak{p}^{-1}(\omega_0)$.
- when at least one point is removed from $\text{GLIMP}_{\Omega_\star}^\star(\theta)$, then the above property is no more satisfied.

Proof. — We show proposition 2.21 by constructing $\text{GLIMP}_{\Omega_\star}^*(\theta)$.

If $\bigcup_{L>0} \Omega_L^*(\theta) = \emptyset$, then $\text{GLIMP}_{\Omega_\star}^*(\theta) = \emptyset$. Otherwise, from the very definition of Ω_\star , one can define an increasing sequence $0 = L_{-1} < L_0 < L_1 < L_2 < \dots$ such that

- $\Omega_{L_0}^*(\theta) = \emptyset$,
- for every $i \in \mathbb{N}^*$, $\Omega_{L_i}^*(\theta) = \Omega_{L_{i-1}}^*(\theta) \cup \{\omega_{i_1}, \dots, \omega_{i_i}\}$ with $\{\omega_{i_1}, \dots, \omega_{i_i}\}$ a finite subset of $] \omega_0, \omega_0 + e^{i\theta} L_{i-1}]$;
- $\Omega_L^*(\theta) = \Omega_{L_i}^*(\theta)$ for every $L \in]L_{i-1}, L_i]$ and every $i \in \mathbb{N}$.

We construct $\text{GLIMP}_{\Omega_\star}^*(\theta)$ by induction on $i \in \mathbb{N}$.

Case $i = 0$. Since $\Omega_{L_0}^*(\theta) = \emptyset$, then for every $L \leq L_0$, every $\lambda \in \mathfrak{R}(\Omega_\star, L, \theta)$ is Ω_\star -allowed and thus can be lifted on $\mathcal{R}_{\Omega_\star}$ with respect to \mathfrak{p} from $\mathfrak{p}^{-1}(\omega)$. Therefore, we set $\text{GLIMP}_{\Omega_\star}^*(\theta, L) = \emptyset$ for $L \leq L_0$.

Case $i = 1$. From the above property, if $\omega \in \Omega_{L_1}^*(\theta)$ satisfied $|\omega| < L_0$, then it is a removable Ω_\star -point for every path $\lambda \in \mathfrak{R}(\Omega_\star, \theta, L)$ with $L \leq L_1$. Let us take $L \in]L_0, L_1]$, so that $\Omega_L^*(\theta) = \Omega_{L_1}^*(\theta)$:

- either there is $\omega \in \Omega_{L_1}^*(\theta)$ of the form $\omega = L_0 e^{i\theta}$. In this case the condition $\text{GLIMP}_{\Omega_\star}^*(\theta, L) = \text{GLIMP}_{\Omega_\star}^*(\theta, L_0) \cup \{\omega\}$ is both needed and sufficient so as to ensure that for every $L \leq L_1$, any path $\lambda \in \mathfrak{R}(\Omega_\star, \theta, L)$ that circumvents $\text{GLIMP}_{\Omega_\star}^*(\theta, L)$ to the right or the left can be lifted on $\mathcal{R}_{\Omega_\star}$;
- or we set $\text{GLIMP}_{\Omega_\star}^*(\theta, L) = \text{GLIMP}_{\Omega_\star}^*(\theta, L_0) = \emptyset$.

Induction. Pick some $i \in \mathbb{N}^*$ and suppose that the following properties are valid for every integer $j \in [1, i]$:

- for every $L \in]L_{j-1}, L_j]$, $\text{GLIMP}_{\Omega_\star}^*(\theta, L) = \text{GLIMP}_{\Omega_\star}^*(\theta, L_j) \subset \Omega_{L_j}^*(\theta)$;
- $\text{GLIMP}_{\Omega_\star}^*(\theta, L_j) \cap] \omega_0, \omega_0 + e^{i\theta} L_{j-1}] = \text{GLIMP}_{\Omega_\star}^*(\theta, L_{j-1})$;
- for every $L \leq L_i$, any path $\lambda \in \mathfrak{R}(\Omega_\star, L, \theta)$ that circumvents $\text{GLIMP}_{\Omega_\star}^*(\theta, L)$ to the right or the left, can be lifted on $\mathcal{R}_{\Omega_\star}$ with respect to \mathfrak{p} from $\mathfrak{p}^{-1}(\omega_0)$.

From these properties, every $\omega \in \Omega_{L_{i+1}}^*(\theta) \setminus \text{GLIMP}_{\Omega_\star}^*(\theta, L_i)$ such that $|\omega| < L_i$ is a removable Ω_\star -point for every path $\lambda \in \mathfrak{R}(\Omega_\star, \theta, L)$ with $L \leq L_{i+1}$. From the fact that $\Omega_{L,\theta}^* = \Omega_{L_{i+1},\theta}^*$ for every $L \in]L_i, L_{i+1}]$:

- either there is $\omega \in \Omega_{L_{i+1}}^*(\theta)$ of the form $\omega = L_i e^{i\theta}$. In that case one sets $\text{GLIMP}_{\Omega_\star}^*(\theta, L) = \text{GLIMP}_{\Omega_\star}^*(\theta, L_i) \cup \{\omega\}$ for $L \in]L_i, L_{i+1}]$ and this provides a necessary and sufficient condition to ensure that any path $\lambda \in \mathfrak{R}(\Omega_\star, \theta, L)$ that circumvents $\text{GLIMP}_{\Omega_\star}^*(\theta, L)$ to the right or the left, can be lifted on $\mathcal{R}_{\Omega_\star}$, for every $L \leq L_{i+1}$;
- or we simply set $\text{GLIMP}_{\Omega_\star}^*(\theta, L) = \text{GLIMP}_{\Omega_\star}^*(\theta, L_i)$ for $L \in]L_i, L_{i+1}]$.

This ends the proof. \square

Definition 2.22. — Let Ω_\star be a discrete filtered set centred at $\omega_0 \in \mathbb{C}$, $\theta \in \mathbb{S}^1$. The discrete and closed set $\text{GLIMP}_{\Omega_\star}^*(\theta) = \{\omega_i \in] \omega_0, \omega_0 + e^{i\theta} \infty [, \omega_0 \prec \omega_1 \prec \omega_2 \dots \}$ given by proposition 2.21 is the set of **glimpsed** Ω_\star -points in the direction

5. The symbol \prec stands for the total order on $[\omega_0, \omega_0 + e^{i\theta} \infty [$ induced by $r \in [0, \infty] \mapsto \omega_0 + r e^{i\theta} \in [\omega_0, \omega_0 + e^{i\theta} \infty [$.

θ . The glimpsed point ω_1 is the **seen** Ω_\star -point in the direction θ . The **completed set of glimpsed** Ω_\star -points $\text{GLIMP}_{\Omega_\star}(\theta)$ in the direction θ is defined by $\text{GLIMP}_{\Omega_\star}(\theta) = \text{GLIMP}_{\Omega_\star}^*(\theta) \cup \{\omega_0\}$.

Remark 2.23. — The notion of glimpsed point can be defined in a simpler way but the presentation we have made here is fitted to the methods that we develop in the paper.

3. Endless continuability

3.1. Endless Riemann surface. —

Definition 3.1 (Endless Riemann surface). — A Riemann surface $(\mathcal{R}, \mathfrak{p})$, given as an étalé space on \mathbb{C} , is said to be **endless** if for every $\zeta \in \mathcal{R}$, there exists a discrete filtered set $\Omega_\star(\zeta)$ centred at $\zeta = \mathfrak{p}(\zeta)$ so that every $\Omega_\star(\zeta)$ -allowed path can be lifted on \mathcal{R} with respect to \mathfrak{p} from ζ .

Example 3.2. — Let Ω be a closed discrete subset of \mathbb{C} . Then the universal covering $\widetilde{\mathbb{C} \setminus \Omega}$ of $\mathbb{C} \setminus \Omega$ is an endless Riemann surface.

The following result is a direct consequence of proposition 2.17.

Proposition 3.3. — Let Ω_\star be a discrete filtered set. Then the associated Riemann surface $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ is endless.

3.2. Endless continuability. —

Definition 3.4 (Endless continuability). — A germ of holomorphic functions $\widehat{\varphi} \in \mathcal{O}_\omega$ at $\omega \in \mathbb{C}$ is endlessly continuable on \mathbb{C} if $\widehat{\varphi}$ can be analytically continued to an endless Riemann surface. One denotes by $\widehat{\mathcal{R}}_{\omega, \text{endl}}$ the space of germ of holomorphic functions at ω that are endlessly continuable on \mathbb{C} . When $\omega = 0$ we use the abridged notation $\widehat{\mathcal{R}}_{\text{endl}} = \widehat{\mathcal{R}}_{0, \text{endl}}$.

Proposition 3.5. — A germ of holomorphic functions $\widehat{\varphi} \in \mathcal{O}_\omega$ at $\omega \in \mathbb{C}$ is endlessly continuable on \mathbb{C} if and only if there exists a discrete filtered set Ω_\star centred at ω such that $\widehat{\varphi}$ can be analytically continued along any Ω_\star -allowed path.

Proof. — We suppose that $\widehat{\varphi} \in \mathcal{O}_\omega$ is endlessly continuable, thus $\widehat{\varphi}$ can be analytically continued to an endless Riemann surface $(\mathcal{R}, \mathfrak{p})$. This means that there exist a neighbourhood $\mathcal{U} \subset \mathbb{C}$ of $\omega = \mathfrak{p}(\omega)$ and a neighbourhood $\mathcal{U} \subset \mathcal{R}$ of ω such that the restriction $\mathfrak{p}|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{U}$ is a homeomorphism, and there is a function Φ holomorphic on \mathcal{R} so that $\phi = \Phi \circ \mathfrak{p}|_{\mathcal{U}}^{-1}$ represents the germ $\widehat{\varphi}$. By the very definition of an endless Riemann surface, one can find a discrete filtered set Ω_\star centred at ω so that every Ω_\star -allowed path λ can be lifted with respect to \mathfrak{p} into a path Λ starting at ω . Since Φ can be analytically continued along Λ , one gets that $\widehat{\varphi}$ can be analytically continued along λ as an upshot.

We now suppose that $\hat{\varphi} \in \mathcal{O}_{\dot{\omega}}$ can be analytically continued along any Ω_\star -allowed path, where Ω_\star is a discrete filtered set Ω_\star centred at $\dot{\omega}$. By proposition 2.17, the Riemann surface $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ associated with this discrete filtered set is endless. To $\hat{\varphi}$ is associated a germ of holomorphic functions Φ at ω , $\dot{\omega} = \mathfrak{p}(\omega)$ that can be analytically continued along the path Λ starting from ω and deduced from any Ω_\star -allowed path λ . Since $\mathcal{R}_{\Omega_\star}$ is simply connected, this implies that Φ can be analytically continued to a function holomorphic on $\mathcal{R}_{\Omega_\star}$. \square

Definition 3.6. — If Ω_\star is a discrete filtered set centred at ω , one denotes by $\hat{\mathcal{R}}_{\Omega_\star}$ the space of germs of holomorphic functions at ω that can be analytically continued to the Riemann surface $\mathcal{R}_{\Omega_\star}$.

3.3. Seen and glimpsed points. — We have introduced the notion of glimpsed Ω_\star -points (definition 2.22) associated with a discrete filtered set Ω_\star centred at ω_0 . If $\hat{\varphi}$ is an endless continuable germ at ω_0 that belongs to $\hat{\mathcal{R}}_{\Omega_\star}$ then, by the very definition of $\text{GLIMP}_{\Omega_\star}^*(\theta)$, $\hat{\varphi}$ can be analytically continued along any path that closely follows the half-line $[\omega_0, \omega_0 + e^{i\theta}\infty[$ in the forward direction, while circumventing to the right or to the left each point from the set $\text{GLIMP}_{\Omega_\star}^*(\theta)$. However, this set is not always the smaller one and one easily gets the following proposition.

Proposition 3.7. — Let $\hat{\varphi} \in \hat{\mathcal{R}}_{\omega, \text{endl}}$ be an endlessly continuable germ of holomorphic functions at $\omega \in \mathbb{C}$ and let $\theta \in \mathbb{S}^1$ be a direction. There exists a uniquely defined discrete and closed set $\text{GLIMP}_{\hat{\varphi}}^*(\theta) = \{\omega_i \in]\omega_0, \omega_0 + e^{i\theta}\infty[, \omega_0 \prec \omega_1 \prec \omega_2 \dots\}$ such that:

- $\hat{\varphi}$ can be analytically continued along any path that closely follows the half-line $[\omega_0, \omega_0 + e^{i\theta}\infty[$ in the forward direction, while circumventing (to the right or to the left) each point of the set $\text{GLIMP}_{\hat{\varphi}}^*(\theta)$.
- this property is no more valid if at least one point is removed from $\text{GLIMP}_{\hat{\varphi}}^*(\theta)$.

If Ω_\star a discrete filtered set centred at ω_0 , then $\text{GLIMP}_{\hat{\varphi}}^*(\theta) \subseteq \text{GLIMP}_{\Omega_\star}^*(\theta)$ for any $\hat{\varphi}$ belonging to $\hat{\mathcal{R}}_{\Omega_\star}$.

Definition 3.8. — The elements of $\text{GLIMP}_{\hat{\varphi}}^*(\theta)$ are called the **glimpsed** singular points in the direction $\theta \in \mathbb{S}^1$ for the endlessly continuable germ $\hat{\varphi} \in \hat{\mathcal{R}}_{\omega, \text{endl}}$. Specifically, ω_1 is the **seen** singular point in the direction θ for $\hat{\varphi}$.

3.4. Continuability without cut. — We complete this Sect. with a brief comparison to Ecalle's endless continuability.

Definition 3.9 (Riemann surface without cut [12]). — Let $(\mathcal{R}, \mathfrak{p})$ be a Riemann surface given as an étalé space on \mathbb{C} . This surface is said to be **without cut** if for every $\omega \in \mathcal{R}$, there exists a closed and discrete set $\text{sing}(\omega) \subset \mathbb{C}$ that satisfies the following properties. Introducing $\dot{\omega} = \mathfrak{p}(\omega)$:

1. if the line segment $[\dot{\omega}, \dot{\omega}'] \subset \mathbb{C}$ does not meet $\text{sing}(\omega)$, then $[\dot{\omega}, \dot{\omega}']$ can be lifted homeomorphically on \mathcal{R} with respect to \mathfrak{p} from ω .

2. for every line segment $[\dot{\omega}, \dot{\omega}'] \subset \mathbb{C}$ that meets the points $\dot{\omega}_1, \dots, \dot{\omega}_r$ of $\text{sing}(\omega)$, there exists an open rectangle W neighbourhood of $[\dot{\omega}, \dot{\omega}']$ such that each of the 2^r simply connected open sets W_j deduced from W by making lateral cuts at $\dot{\omega}_1, \dots, \dot{\omega}_r$ (see Fig. 3) can be lifted homeomorphically on \mathcal{R} with respect to \mathfrak{p} to an open set $\mathcal{W}_j \subset \mathcal{R}$ containing ω .
3. if at least one point is removed from $\text{sing}(\omega)$, then properties 1-2 are no more satisfied.

One says that the point $\dot{\omega}_1 \in \text{sing}(\omega)$ is **seen** from ω if $[\dot{\omega}, \dot{\omega}_1] \cap \text{sing}(\omega) = \{\dot{\omega}_1\}$. Otherwise the points $\dot{\omega}_j \in \text{sing}(\omega)$ are **glimpsed** from ω .

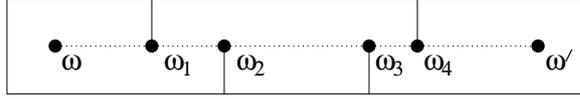


FIGURE 3

We note that in Definition 3.9, condition 3 is added so as to define the seen and glimpsed singular points.

Definition 3.10 (Continuability without cut). — A germ of holomorphic functions $\hat{\varphi} \in \mathcal{O}_0$ at $0 \in \mathbb{C}$ is said to be **analytically continuable without cut** on \mathbb{C} if its Riemann surface is without cut.

There is the following relationship between endless continuability in the sense of definition 3.1 and continuability without cut.

Proposition 3.11. — *Let $(\mathcal{R}, \mathfrak{p})$ be a Riemann surface. If \mathcal{R} is endless, then \mathcal{R} is without cut.*

Proof. — We assume that the Riemann surface $(\mathcal{R}, \mathfrak{p})$ is endless. We consider a point $\omega \in \mathcal{R}$, $\dot{\omega} = \mathfrak{p}(\omega)$: there exists a discrete filtered set Ω_\star centred at $\dot{\omega}$ such that every Ω_\star -allowed path $\lambda \in \mathfrak{R}_{\Omega_\star}^\star$ can be lifted on \mathcal{R} from ω with respect to \mathfrak{p} . We consider the line segment $[\dot{\omega}, \dot{\omega}'] \subset \mathbb{C}$ and $L > l > 0$ where $l = |\dot{\omega}' - \dot{\omega}|$.

1. Assume that the line segment $[\dot{\omega}, \dot{\omega}']$ does not meet Ω_L apart from $\dot{\omega}$. Then the path $\lambda : t \in [0, 1] \mapsto \dot{\omega} + t(\dot{\omega}' - \dot{\omega})$ belongs to $\mathfrak{R}_{\Omega_\star}(L)$ and thus can be lifted by \mathfrak{p} from ω .
2. Assume that the line segment $[\dot{\omega}, \dot{\omega}']$ meets the points $\dot{\omega}, \dot{\omega}_1, \dots, \dot{\omega}_r$ of Ω_L . Consider an open rectangle W centred on (and thus neighbourhood of) $[\dot{\omega}, \dot{\omega}']$, of length $l+2l'$ and width $2l'$ where $l' > 0$ satisfies $l+2l' > L$. For l' small enough one has $W \cap \Omega_L = \{\dot{\omega}, \dot{\omega}_1, \dots, \dot{\omega}_r\}$ and for every $\zeta \in W \setminus \{\dot{\omega}_1, \dots, \dot{\omega}_r\}$, there exists a path $\lambda \in \mathfrak{R}_{\Omega_\star}(L)^\star$ such that $\lambda([0, 1]) \subset W \setminus \{\dot{\omega}_1, \dots, \dot{\omega}_r\}$, $\lambda(0) = \dot{\omega}$ and $\lambda(1) = \zeta$.

Now assume that W_j is one of the 2^r simply connected open sets W_j deduced from W by making lateral cuts at $\dot{\omega}_1, \dots, \dot{\omega}_r$ (see Fig. 3). Then for every $\dot{\zeta} \in W_j$, there exists a path $\lambda \in \mathfrak{R}_{\Omega_\star}(L)$ such that $\lambda([0, 1]) \subset W_j$, $\lambda(0) = \dot{\omega}$ and $\lambda(1) = \dot{\zeta}$. This path can be lifted with respect to \mathfrak{p} into a path starting from ω and ending at a point ζ such that $\mathfrak{p}(\zeta) = \dot{\zeta}$. We note \mathcal{W}_j the set of these points ζ .

By its very definition, \mathcal{W}_j is an open arcconnected subset of \mathcal{R} such that $\mathfrak{p}(\mathcal{W}_j) = W_j$. Moreover $\mathfrak{p}|_{\mathcal{W}_j}$ is injective. Indeed, if one considers two paths $\lambda_0, \lambda_1 \in \mathfrak{R}_{\Omega_\star}(L)$ such that $\lambda_1([0, 1]) \subset W_j$ and $\lambda_2([0, 1]) \subset W_j$ and ending at the same point $\dot{\zeta} \in W_j$, one can easily constructs a Ω_\star -homotopy $\Gamma : t \in [0, 1] \mapsto \Gamma_t \in \mathfrak{R}_{\Omega_\star}(L)$ between $\underline{\lambda}_0$ and $\underline{\lambda}_1$, because W_j is simply connected. Finally, since \mathfrak{p} is a local homeomorphism, $\mathfrak{p}|_{\mathcal{W}_j}$ is a homeomorphism between \mathcal{W}_j and W_j .

□

4. Endless continuability and convolution product

For two germs $\widehat{\varphi}, \widehat{\psi} \in \mathcal{O}_\omega$ of holomorphic functions at $\omega \in \mathbb{C}$, their convolution product $\widehat{\varphi} * \widehat{\psi} \in \mathcal{O}_\omega$ is the germ of holomorphic functions at ω defined by the integral,

$$(1) \quad \widehat{\varphi} * \widehat{\psi}(\zeta) = \int_\omega^\zeta \widehat{\varphi}(\eta) \widehat{\psi}(\zeta + \omega - \eta) d\eta,$$

for ζ close enough to ω .

4.1. Endless continuability, stability under convolution product. — We state the main result of the paper.

Theorem 4.1. — *Let $\widehat{\varphi}, \widehat{\psi} \in \widehat{\mathcal{R}}_{\omega, \text{endl}}$ be two endlessly continuable germs of holomorphic functions at ω . Then their convolution product $\widehat{\varphi} * \widehat{\psi}$ is endlessly continuable as well, $\widehat{\varphi} * \widehat{\psi} \in \widehat{\mathcal{R}}_{\omega, \text{endl}}$. More precisely, let Ω_\star and Ω'_\star be two discrete filtered sets centred at ω . If $\widehat{\varphi} \in \widehat{\mathcal{R}}_{\Omega_\star}$ and $\widehat{\psi} \in \widehat{\mathcal{R}}_{\Omega'_\star}$, then their convolution product $\widehat{\varphi} * \widehat{\psi}$ belongs to $\widehat{\mathcal{R}}_{\Omega_\star * \Omega'_\star}$ where $\Omega_\star * \Omega'_\star$ is the fine sum of the two discrete filtered sets.*

This theorem has an obvious but interesting corollary.

Corollary 4.2. — *Let Ω_\star be a discrete filtered sets centred at ω and $\widehat{\varphi} \in \widehat{\mathcal{R}}_{\Omega_\star}$. Then the iterated convolution products $\widehat{\varphi}^{*n}$ belong to $\widehat{\mathcal{R}}_{\Omega_\star^\infty}$ with Ω_\star^∞ the saturated of Ω_\star .*

Theorem 4.1 is given in [2] and proved there up to sometimes alluded-to key-points arguments. The rest of this section is devoted to showing this result rigorously. Our method differs from that of [2].

Up to making a translation, one can suppose that $\omega = 0$ and this is what we do in the sequel. For $\widehat{\varphi} \in \widehat{\mathcal{R}}_{\Omega_\star}$ and $\widehat{\psi} \in \widehat{\mathcal{R}}_{\Omega'_\star}$, notice that the convolution product (1) provides a holomorphic function on $D(0, \rho_{\Omega_\star}(0)) \cap D(0, \rho_{\Omega'_\star}(0))$ since $\widehat{\varphi}$, resp. $\widehat{\psi}$ can be represented by a holomorphic function on $D(0, \rho_{\Omega_\star}(0))$, resp. $D(0, \rho_{\Omega'_\star}(0))$.

4.2. $(\Omega_\star * \Omega'_\star)$ -homotopy. — The following definition generalizes a definition from [22, 24].

Definition 4.3. — Let $\Omega_\star, \Omega'_\star$ be two discrete filtered sets centred at 0, $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$, $(\mathcal{R}_{\Omega'_\star}, \mathfrak{p}')$ their associated Riemann surfaces. Let $H : (s, t) \in [0, 1]^2 \mapsto H(s, t) = H_t(s) \in \mathbb{C}$ be a continuous map and $H^\star : (s, t) \in [0, 1]^2 \mapsto H^\star(s, t) = H_t^\star(s) \in \mathbb{C}$ the continuous map deduced from H through the identity⁽⁶⁾ $H_t^\star(s) = H_t(1) - H_t^{-1}(s)$. One says that H is a $(\Omega_\star * \Omega'_\star)$ -**homotopy** if the following conditions are satisfied for every $t \in [0, 1]$:

- $H_t(0) = 0$;
- H_t can be lifted with respect to \mathfrak{p} on $\mathcal{R}_{\Omega_\star}$ from $0 = \mathfrak{p}^{-1}(0)$;
- H_t^\star can be lifted with respect to \mathfrak{p}' on $\mathcal{R}_{\Omega'_\star}$ from $0 = \mathfrak{p}'^{-1}(0)$;

The path H_0 is the initial path, H_1 is the final and the path $t \in [0, 1] \mapsto H_t(1)$ is the endpoint path of H .

4.3. Useful lemmas. — We start with a technical lemma.

Lemma 4.4. — *Let H be a $(\Omega_\star * \Omega'_\star)$ -homotopy. Then $\inf_{t \in [0, 1]} d(H_t, \Omega_\star) > 0$ and $\inf_{t \in [0, 1]} d(H_t^\star, \Omega'_\star) > 0$.*

Proof. — Let $(\mathcal{R}_{\Omega_\star}, \mathfrak{p})$ be the Riemann surface associated with Ω_\star . Since H_t can be lifted with respect to \mathfrak{p} on $\mathcal{R}_{\Omega_\star}$ from 0 for every $t \in [0, 1]$ and using the homotopy lifting theorem, the $(\Omega_\star * \Omega'_\star)$ -homotopy H can be lifted with respect to \mathfrak{p} into a (unique) homotopy $\mathcal{H} : (s, t) \in [0, 1]^2 \mapsto \mathcal{H}(s, t) = \mathcal{H}_t(s)$ such that $\mathcal{H}_t(0) = 0$ for every $t \in [0, 1]$. Since the mapping $\zeta \in \mathcal{R}_{\Omega_\star} \mapsto d(\zeta, \Omega_\star)$ is continuous, one concludes that $\inf_{(s, t) \in [0, 1]^2} d(\mathcal{H}(s, t), \Omega_\star) > 0$ by compactness. Thus $\inf_{t \in [0, 1]} d(H_t, \Omega_\star) > 0$. The same reasoning holds for $\inf_{t \in [0, 1]} d(H_t^\star, \Omega'_\star)$. \square

Lemma 4.5. — *Let $\Omega_\star, \Omega'_\star$ be two discrete filtered sets centred at 0 and γ be a piecewise \mathcal{C}^1 path such that $|\underline{\gamma}(0)| < \min\{\rho_{\Omega_\star}(0), \rho_{\Omega'_\star}(0)\}$. We suppose the existence of a $(\Omega_\star * \Omega'_\star)$ -homotopy H whose endpoint path is $\underline{\gamma}$ and such that $H_0([0, 1]) \subset D(0, \rho_{\Omega_\star}(0))$ and $H_0^\star([0, 1]) \subset D(0, \rho_{\Omega'_\star}(0))$. Then, for any $\widehat{\varphi} \in \widehat{\mathcal{R}}_{\Omega_\star}$ and any $\widehat{\psi} \in \widehat{\mathcal{R}}_{\Omega'_\star}$, their convolution product $\widehat{\varphi} * \widehat{\psi}$ can be analytically continued along γ .*

Proof. — Just adapt the proof of a similar lemma given in [22, 24] when $\Omega_\star, \Omega'_\star$ are closed discrete subsets of \mathbb{C} , with the help of lemma 4.4. \square

We now state the main lemma of this Sect.

Lemma 4.6 (key-lemma). — *Let $\Omega_\star, \Omega'_\star$ be two discrete filtered sets centred at 0. Let λ_0 and γ be two paths subject to the following conditions:*

- λ_0 satisfies $\underline{\lambda}_0 : s \in [0, 1] \mapsto \underline{\lambda}_0(s) = s\underline{\gamma}(0)$;
- $\underline{\gamma}(0)$ satisfies $|\underline{\gamma}(0)| < \min\{\rho_{\Omega_\star}(0), \rho_{\Omega'_\star}(0)\}$;
- the product path $\lambda_0\gamma$ is $(\Omega_\star * \Omega'_\star)$ -allowed.

6. Remember that $H_t^{-1}(s) = H_t(1 - s)$.

Then there exists a $(\Omega_\star * \Omega'_\star)$ -homotopy H with endpoint path $\underline{\gamma}$ and initial path $H_0 = \underline{\Delta}_0$.

Proof. — Part of our arguments comes from [21, 5]. We also use a construction made in [22, 24] for the case where $\Omega_\star, \Omega'_\star$ are closed discrete subsets of \mathbb{C} , and that simplifies the proof. The later is new up to our knowledge.

We first assume that the product path $\lambda_0\gamma$ is $(\Omega_\star + \Omega'_\star)$ -allowed. Therefore, there exists $L > 0$ such that $\lambda_0\gamma \in \mathfrak{R}_{\Omega_\star + \Omega'_\star}(L)$. In particular, $\mathcal{L}_{\lambda_0\gamma} < L$ and γ avoids the set $(\Omega_\star + \Omega'_\star)_L$,

$$(\Omega_\star + \Omega'_\star)_L = \{\zeta = \omega + \omega' \mid \omega \in \Omega_L, \omega' \in \Omega'_L \text{ and } |\zeta| < L\}.$$

Making a slight deformation of γ (an homotopy in $\mathbb{C} \setminus (\Omega_\star + \Omega'_\star)_L$ with fixed extremities), we can assume that γ is \mathcal{C}^1 . (There is no loss of generality with this assumption).

We pick two functions $\eta_{\Omega_L} : \mathbb{C} \rightarrow \mathbb{R}^+$ and $\eta_{\Omega'_L} : \mathbb{C} \rightarrow \mathbb{R}^+$, both continuous and locally Lipschitz, which furthermore satisfy:

$$\{\zeta \in \mathbb{C} \mid \eta_{\Omega_L}(\zeta) = 0\} = \Omega_L, \quad \{\zeta \in \mathbb{C} \mid \eta_{\Omega'_L}(\zeta) = 0\} = \Omega'_L.$$

(For instance, $\eta_{\Omega_L}(\zeta) = d(\zeta, \Omega_L)$ where d is the euclidean distance). Remark that the mapping $\chi : (\zeta, t) \in \mathbb{C} \times [0, 1] \mapsto \eta_{\Omega_L}(\zeta) + \eta_{\Omega'_L}(\underline{\gamma}(t) - \zeta) \in \mathbb{R}^+$ never vanishes : $\chi(\zeta, t) = 0$ means $\zeta = \omega$ and $\gamma(t) - \zeta = \omega'$ for some $\omega \in \Omega_L$ and $\omega' \in \Omega'_L$, and this implies $\gamma(t) = \omega + \omega'$ which contradicts the hypotheses made on $\lambda_0\gamma$. This implies that the following non-autonomous vector field,

$$X : (\zeta, t) \in \mathbb{C} \times [0, 1] \mapsto X(\zeta, t) = \frac{\eta_{\Omega_L}(\zeta)}{\eta_{\Omega_L}(\zeta) + \eta_{\Omega'_L}(\underline{\gamma}(t) - \zeta)} \underline{\gamma}'(t)$$

is well-defined, continuous, everywhere locally Lipschitz with respect to ζ and bounded, $|X(\zeta, t)| \leq |\underline{\gamma}'(t)| \leq \max_{[0,1]} |\underline{\gamma}'|$. Therefore, its associated flow $g_X : (t_0, t, \zeta) \in [0, 1]^2 \times \mathbb{C} \mapsto g_X^{t_0, t}(\zeta) \in \mathbb{C}$ is \mathcal{C}^1 and globally defined as a consequence of the Cauchy-Lipschitz theorem and the Grönwall lemma.

We start with $H_0 = \underline{\Delta}_0$ and for every $t \in [0, 1]$, we consider the deformation H_t of H_0 along the flow X , precisely we set $H_t = g_X^{0, t}(H_0)$. We get a mapping $H : (s, t) \in [0, 1]^2 \mapsto H(s, t) = H_t(s)$ with the following properties for every $t \in [0, 1]$ (check them or see [22, 24]):

- H is of class \mathcal{C}^1 ;
- $H_t(0) = 0$ and $H_t([0, 1]) \in \mathbb{C} \setminus \Omega_L$;
- $H_0 = \underline{\Delta}_0$ and the endpoint path $t \in [0, 1] \mapsto H_t(1)$ coincides with the path $\underline{\gamma}$.

Let us now consider the family of paths $H^s : t \in [0, 1] \mapsto H^s(t) = H(s, t)$, for $s \in [0, 1]$. These paths satisfy the following properties. For every $s \in [0, 1]$:

- H^s is of \mathcal{C}^1 -class, $H^s(0) = \underline{\Delta}_0(s) = H_0(s)$;
- $\frac{dH^s(t)}{dt} = X(H^s(t), t)$, thus $\left| \frac{dH^s(t)}{dt} \right| \leq |\underline{\gamma}'(t)|$ and this implies that $\mathcal{L}_{H^s} \leq \mathcal{L}_\gamma$;
- $H^0 \equiv 0$ and $H^s([0, 1]) \subset \mathbb{C} \setminus \Omega_L$ for $s \neq 0$;

The product of paths $F^s = H_0|_{[0, s]} H^s$ is well-defined and has the following properties, for any $s \in]0, 1]$:

1. \underline{F}^s is piecewise \mathcal{C}^1 ;
2. $\underline{F}^0 \equiv 0$ otherwise for $s > 0$, $\underline{F}^s(0) = 0$ and $\underline{F}^s(]0, 1]) \subset \mathbb{C} \setminus \Omega_L$;
3. $\mathcal{L}_{F^s} = \mathcal{L}_{H_0|_{[0,s]}} + \mathcal{L}_{H^s}$, hence $\mathcal{L}_{F^s} \leq \mathcal{L}_{\lambda_0\gamma} < L$;

Therefore for any $s \in]0, 1]$, F^s belongs to $\mathfrak{R}_{\Omega_\star}(L)$, thus is Ω_\star -allowed and can be lifted with respect to \mathfrak{p} from 0. This implies that H_s can be lifted from $\text{cl}(\Delta_0|_{[0,s]})$ with respect to \mathfrak{p} and this eventually provides a lifting \mathcal{H} of the mapping H . One concludes that for every $t \in [0, 1]$, the path H_t has a (unique) lifting \mathcal{H}_t with respect to \mathfrak{p} from 0.

Look at the mapping $H^\star : (s, t) \in [0, 1]^2 \mapsto H^\star(s, t) = H_t^\star(s)$ deduced from H by $H_t^\star(s) = H_t(1) - H_t^{-1}(s)$. It is easy to see that the family of paths $H_t^\star = g_{X^\star}^{0,t}(H_0^\star)$ is obtained by deformation of H_0^\star , where $g_{X^\star} : (t_0, t, \zeta) \in [0, 1]^2 \times \mathbb{C} \mapsto g_{X^\star}^{t_0,t}(\zeta)$ is the flow associated with the non-autonomous vector field,

$$X^\star : (\zeta, t) \in \mathbb{C} \times [0, 1] \mapsto X(\zeta, t) = \frac{\eta_{\Omega'_L}(\zeta)}{\eta_{\Omega'_L}(\zeta) + \eta_{\Omega_L}(\underline{\gamma}(t) - \zeta)} \underline{\gamma}'(t).$$

The above reasoning can be applied as it stands for H^\star and H_t^\star has a (unique) lifting \mathcal{H}_t^\star with respect to \mathfrak{p}' from 0.

We set $H^{\star s}(t) = H^\star(s, t)$, thus $H^{\star(1-s)}(t) = \gamma(t) - H^s(t)$. From the identities $\frac{dH^s(t)}{dt} = X(H^s(t), t)$ and $\frac{dH^{\star(1-s)}(t)}{dt} = X_\star(H^{\star(1-s)}(t), t)$, one

easily gets : $\left| \frac{dH^s(t)}{dt} \right| + \left| \frac{dH^{\star(1-s)}(t)}{dt} \right| = |\underline{\gamma}'(t)|$. The path $H^{\star(1-s)}$ starts

from the point $H^{\star(1-s)}(0) = \Delta_0(1-s) = H_0(1-s)$, thus the product of paths $F^{\star(1-s)} = H_0|_{[0,1-s]} H^{\star(1-s)}$ is well-defined and

$$\mathcal{L}_{F^s} + \mathcal{L}_{F^{\star(1-s)}} = \mathcal{L}_{\lambda_0\gamma} < L.$$

The upshot is that for any $s \in]0, 1[$, F^s belongs to $\mathfrak{R}_{\Omega_\star}(L_1)$ and $F^{\star(1-s)}$ belongs to $\mathfrak{R}_{\Omega'_\star}(L_2)$ with $L_1 + L_2 \leq L$. Therefore, only the points of the form $(\Omega_\star * \Omega'_\star)_L$ actually matter for γ to get the homotopies H and H^\star . This property allows to extend the above construction when the product path $\lambda_0\gamma$ is $(\Omega_\star * \Omega'_\star)$ -allowed. Indeed, denote by $K_L \subset \mathbb{C} \times [0, 1]$ the subset made of the $(\zeta, t) \in \mathbb{C} \times [0, 1]$ such that $\zeta = \omega$, $\gamma(t) - \zeta = \omega'$, $\omega + \omega' \in (\Omega_\star + \Omega'_\star)_L \setminus (\Omega_\star * \Omega'_\star)_L$. It is sufficient to remark that the restriction $X|$ of X to $\mathbb{C} \times [0, 1] \setminus K_L$ is still continuous, locally Lipschitz and bounded, and the above arguments show that the deformation $H_t = g_{X|}^{0,t}(H_0)$ of H_0 along the flow $X|$ can be defined as well, and similarly for H_t^\star . This ends the proof of the lemma. \square

4.4. The proof of theorem 4.1. — Theorem 4.1 is a straightforward consequence of lemma 4.5 and lemma 4.6.

4.5. Convolution product and glimpsed points. — The ideas developed in the proof of theorem 4.1 can be easily adapted to get the following informations on glimpsed points where, to simplify, we only consider discrete filtered sets centred at 0.

Proposition 4.7. — *Let Ω_* , Ω'_* be two discrete filtered sets centred at 0 and $\text{GLIMP}_{\Omega_*}(\theta)$, $\text{GLIMP}_{\Omega'_*}(\theta)$ their respective completed sets of glimpsed points, for a given direction $\theta \in \mathbb{S}^1$. For any two endlessly continuable germs $\widehat{\varphi} \in \widehat{\mathcal{R}}_{\Omega_*}$ and $\widehat{\psi} \in \widehat{\mathcal{R}}_{\Omega'_*}$, the set $\text{GLIMP}_{\widehat{\varphi}*\widehat{\psi}}^*(\theta)$ of glimpsed singular points in the direction $\theta \in \mathbb{S}^1$ for the convolution product $\widehat{\varphi} * \widehat{\psi}$, satisfies the condition : $\text{GLIMP}_{\widehat{\varphi}*\widehat{\psi}}^*(\theta) \subseteq \{\text{GLIMP}_{\Omega_*}(\theta) + \text{GLIMP}_{\Omega'_*}(\theta)\} \setminus \{0\}$.*

Proof. — It is sufficient to consider product paths $\lambda_0\gamma$ of the following form:

- λ_0 satisfies $\underline{\lambda}_0 : s \in [0, 1] \mapsto \underline{\lambda}_0(s) = s\underline{\gamma}(0)$ and $\underline{\gamma}(0)$ satisfies the conditions : $\underline{\gamma}(0) \in]0, e^{i\theta}\infty[$ and $|\underline{\gamma}(0)| < \min\{\rho_{\Omega_*}(0), \rho_{\Omega'_*}(0)\}$;
- γ avoids the set $\{\text{GLIMP}_{\Omega_*}(\theta) + \text{GLIMP}_{\Omega'_*}(\theta)\} \setminus \{0\}$;
- $\lambda_0\gamma$ belongs to $\mathfrak{R}(\Omega_*, \theta, L) \cap \mathfrak{R}(\Omega'_*, \theta, L)$ for some $L > 0$;
- γ is of class \mathcal{C}^1 , its derivative γ' do not vanish and there exists $\alpha \in]0, \pi/2[$ small enough such that for every $t \in [0, 1]$, $\arg \gamma'(t) \in]-\alpha + \theta, \theta + \alpha[$.

We go back to the proof of the key-lemma 4.6 where we replace Ω_L by $\text{GLIMP}_{\Omega_*}(\theta)$ and Ω'_L by $\text{GLIMP}_{\Omega'_*}(\theta)$. We follow the construction of the mapping H . It is easy to see that for every $t \in [0, 1]$, $\arg \frac{dH^s(t)}{dt} \in]-\alpha + \theta, \theta + \alpha[$. Defining F^s like in the proof of lemma 4.6, the upshot is that F^s belongs to $\mathfrak{R}(\Omega_*, \theta, L)$ and avoids $\text{GLIMP}_{\Omega_*}(\theta)$, for any $s \in]0, 1]$. This implies that the mapping H has a (unique) lifting \mathcal{H} with respect to \mathfrak{p} with $\mathcal{H}_t(0) = 0$ for every $t \in [0, 1]$. The same result occurs for the mapping H^* . One concludes with lemma 4.5. \square

One can draw the following consequences from both theorem 4.1 and proposition 4.7, where we use classical notations in resurgence theory for which we refer to [24, 5] :

Corollary 4.8. — *The space of endlessly continuable functions $\widehat{\mathcal{R}}_{\text{endl}}$ makes a differential convolution algebra (without unit) on which the alien operators act. In particular, if Ω_* , Ω'_* are two discrete filtered sets centred at 0, then for any $\widehat{\varphi} \in \widehat{\mathcal{R}}_{\Omega_*}$, $\widehat{\psi} \in \widehat{\mathcal{R}}_{\Omega'_*}$ and any $\omega \in \mathbb{C}^1$, the alien operator Δ_ω^+ acts on $\overset{\nabla}{\varphi} * \overset{\nabla}{\psi}$ with $\overset{\nabla}{\varphi} = \overset{\nabla}{\varphi}$, $\overset{\nabla}{\psi} = \overset{\nabla}{\psi}$ and the following identity holds:*

$$(2) \quad \Delta_\omega^+(\overset{\nabla}{\varphi} * \overset{\nabla}{\psi}) = (\Delta_\omega^+ \overset{\nabla}{\varphi}) * \overset{\nabla}{\psi} + \sum_{\overset{\bullet}{\omega}_1 + \overset{\bullet}{\omega}_2 = \overset{\bullet}{\omega}} (\Delta_{\omega_1}^+ \overset{\nabla}{\varphi}) * (\Delta_{\omega_2}^+ \overset{\nabla}{\psi}) + \overset{\nabla}{\varphi} * (\Delta_\omega^+ \overset{\nabla}{\psi}).$$

In (2), the sum runs over all $\overset{\bullet}{\omega}_1 \in \text{GLIMP}_{\Omega_*}^*(\overset{\bullet}{\theta})$, $\overset{\bullet}{\omega}_2 \in \text{GLIMP}_{\Omega'_*}^*(\overset{\bullet}{\theta})$ with $\overset{\bullet}{\theta} = \dot{\pi}(\theta)$ and $\theta = \arg(\omega) \in \mathbb{S}^1$.

5. Conclusion

This article contributes to the resurgence theory in showing rigorously the stability under convolution product of endlessly continuable functions, thus adds a piece to the very foundation of this theory. We mention that the notion of endless continuability used in this paper is less general than this in [2] and *a fortiori* the endless

continuability of Ecalle. We do not know whether our method could be applied to these more general frames or not, however we know no application where such a generality is needed.

Since theorem 4.1 brings in fine sums of discrete filtered sets, series like $\sum_n a_n \widehat{\varphi}^{*n}$, $a_n \in \mathbb{C}$ can be defined on the endless Riemann surface $\mathcal{R}_{\Omega_\star^\infty}$ provided the uniform convergence of the series of any compact set of $\mathcal{R}_{\Omega_\star^\infty}$. Such a result is given in [2] but for mistakes that have been corrected by Sauzin [23] for the case where Ω_\star stands for a closed discrete subset of \mathbb{C} . Considering the natural link between our method and [23], it is likely that Sauzin's work can be generalized to endlessly continuable functions.

Finally, and like mentioned in [21, 23], extensions of theorem 4.1 for the so-called weighted products [13, 14] would be welcome so as to contribute to the knowledge on the exact WKB analysis or coequational resurgence [11, 6, 7, 9, 15, 16], see also [18]. We hope to make some advances toward that direction in a near future.

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