Deformations Preserving Gauß Curvature

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1 Introduction

In industrial surface generation, it is important to consider surfaces with minimal areas for two main reasons: these surfaces require less material than non-minimal surfaces, and they are cheaper to manufacture. Based on a prototype, a so-called masterpiece, the final product is created using small deformations to adapt a surface to the desired shape. We present a linear deformation technique preserving the total curvature of the masterpiece. In particular, we derive sufficient conditions for these linear deformations to be total curvature preserving when applied to the masterpiece. It is useful to preserve total curvature of a surface in order to minimise the amount of material needed, and to minimise bending energy [15, 9].

Efimov was the first to introduce partial differential equations as a tool to study infinitesimal bending. He gives an overview of the state of the art of infinitesimal bendings in his textbook [6]. Hagen et al. [10] visualise the momentarial rotation field that is associated with infinitesimal bending. They then use the structure of this rotation field as a tool to analyse the deformations that were generated by this bending. Hahmann et al. [11] investigate numerical aspects of discretising the deformation vector field. Ivanova and Subitov [13] examine infinitesimal bendings of surfaces of revolution and of polyhedra. Meziani [18] studies infinitesimal bending

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of homogeneous surfaces that have a flat point and positive curvature. More recent works on infinitesimal bending for curves and non-parametric surfaces have been published by L. Velimirović et al. They study total mean curvature variation on oriented, boundary-free surfaces [22], and they visualise changes of bent curves as surfaces constructed from different stages of deformation [21]. Eigensatz et al. [8] use curvature as a tool to control surface deformation. They extend this work to allow various user-specified local restrictions on deformation [7]. Other works have addressed perturbations preserving the topological form of polyhedra [1], and deformations preserving ambient isotopy of curves [14, 16].

In this work, rather than studying total curvature changes after bending, or using curvature as a tool to deform surfaces, we employ total curvature as a tool to restrict bending and avoid large changes. We assume a rigid material that can be bent out of shape through exterior deformations but that cannot be stretched in tangent direction through interior deformations, as common in engineering [2, 4].

Section 2 gives an introduction into some fundamentals of differential geometry that our method is based on. In Section 3, we describe and prove our approach, rounded off by two examples in Section 4.

2 Fundamentals of Differential Geometry

We start by defining parametrised surfaces, tangents, derivatives, and Gauß frames. Then, we recall the definitions of the first and second fundamental forms, and finally, we discuss various well-established definitions of curvature. For more definitions, see [5].

A *parametrised C^r surface* is a C^r-differentiable mapping $X : U \to \mathbb{E}^3$ of an open domain $U \subset \mathbb{E}^2$ into the Euclidean space $\mathbb{E}^3$, whose differential $\mathrm{d}X$ is one-to-one for each $q \in U$.

Remark 1.

(a) A change of variables of X is a diffeomorphism $\tau : \tilde{U} \to U$, where τ is an open domain in $\mathbb{E}^2$, such that τ's differential $\mathrm{d}\tau$ always has rank $= 2$, if the determinant of its Jacobian matrix $\det(\tau^*) > 0$ is orientation-preserving.
(b) Relationship: the change of variables defines an equivalence relation on the class of all parametrised surfaces. An equivalence class of parametrised surfaces is called a surface in $\mathbb{E}^3$.
(c) Let us denote in the following $X_u := \frac{\partial X}{\partial u}, X_w := \frac{\partial X}{\partial w}, X_{uv} := \frac{\partial^2 X}{\partial u \partial w}$ or alternatively $X_i, X_j, i, j \in \{u, w\}$. The differential $\mathrm{d}X$ is one-to-one if and only if $\frac{\partial X}{\partial u}$ and $\frac{\partial X}{\partial w}$ are linearly independent.

We can define a tangent plane which is spanned by the tangents of the surface. This tangent plane, in conjunction with the surface normal, defines a local coordinate system on the manifold.

Definition 1.

(a) The tangent plane is a two-dimensional linear subspace $T_{\mathbf{u}}X$ of $\mathbb{E}^3$ generated by $\operatorname{span}\{X_u, X_w\}$, and it is called the *tangent space of* X *at* $\mathbf{u} = (u,w) \in U$.
(b) Elements of $T_{\mathbf{u}}X$ are called *tangent vectors*.
(c) The vector field $N := \frac{[X_u,X_w]}{\|[X_u,X_w]\|}$, where $[.,.]$ is the cross product, is called a *unit normal field*.
(d) The map $N : U \to S^2 \subset \mathbb{E}^3$ is called *Gauß map*, and the moving frame $\{X_u, X_w, N\}$ is called the *Gauß frame* of the surface as displayed in Figure 1.

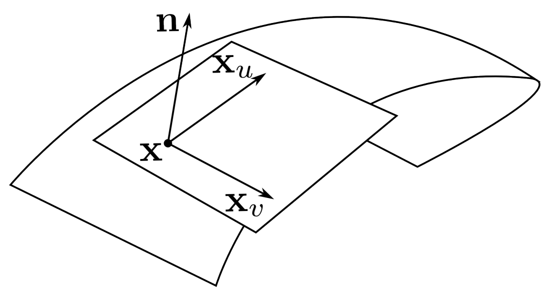


Fig. 1: Gauß frame $\{X_u, X_w, N\}$ for the surface X.

Some properties of the surface can be determined using the first and second fundamental forms. The first fundamental form allows to make measurements on the surface: lengths of curves, angles between tangent vectors, areas of regions, etc. without referring back to the ambient space $\mathbb{E}^3$.

Definition 2. Let $X : U \to \mathbb{E}^3$ be a surface. The bilinear form of $T_{\mathbf{u}}X$ induced by the scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{E}^3$ by restriction is called the *first fundamental form* $I_{\mathbf{u}}$ of the surface.

Remark 2. Properties of the first fundamental form:

(a) The matrix representation of the first fundamental form with respect to the basis $\{X_u, X_w\}$ of $T_{\mathbf{u}}X$ is given by

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle X_u, X_u \rangle & \langle X_u, X_w \rangle \\ \langle X_w, X_u \rangle & \langle X_w, X_w \rangle \end{pmatrix}. \tag{1}$$

(b) Let us denote by

$$g := \det(g_{ij})$$

the determinant of the first fundamental form.
(c) The first fundamental form is symmetric, positive definite, and a geometric invariant.

The second fundamental form allows us to study surface curvature and torsion. One especially interesting consequence of the second fundamental form can be found in the Weingarten equations which will prove useful when considering the main theorem of this paper.

Definition 3. Let $X : U \to \mathbb{E}^3$ be a surface and $\mathbf{u} \in U$.

(a) The linear map $L : T_{\mathbf{u}}X \to T_{\mathbf{u}}X$ defined by $L := -\mathrm{d}N_{\mathbf{u}} \cdot \mathrm{d}X_{\mathbf{u}}$ is called the *Weingarten map*.
(b) The bilinear form $I\!I_{\mathbf{u}}$ defined by $I\!I_{\mathbf{u}}(A,B) := \langle L(A), B\rangle$ for each $A,B \in T_{\mathbf{u}}X$ is called the *second fundamental form* of the surface.

Remark 3. Properties of the second fundamental form:

(a) The matrix representation of $I\!I_{\mathbf{u}}$ with respect to the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $T_{\mathbf{u}}\mathbb{E}^2$ (identified with $\mathbb{E}^2$) and the associated basis $\{X_u, X_w\}$ of $T_{\mathbf{u}}X$ is given by

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} \langle -N_u, X_u\rangle & \langle -N_u, X_w\rangle \\ \langle -N_w, X_u\rangle & \langle -N_w, X_w\rangle \end{pmatrix} = \begin{pmatrix} \langle N, X_{uu}\rangle & \langle N, X_{uw}\rangle \\ \langle N, X_{wu}\rangle & \langle N, X_{ww}\rangle \end{pmatrix}, \tag{2}$$

i.e.

$$h_{ij} := \langle -N_i, X_j\rangle = \langle N, X_{ij}\rangle .$$

We can assume that $h_{12} = h_{21}$ since we are considering C^r-continuous surfaces.
(b) Let us denote by

$$h := \det(h_{ij})$$

the determinant of the second fundamental form.
(c) We call two geometric objects *congruent* to each other iff. there is an isometric transformation (i.e. only translation, rotation, and reflection are employed) from one to the other. Congruences preserve lengths and angles.
(d) The second fundamental form is invariant under congruences of $\mathbb{E}^3$ and orientation-preserving changes of variables.
(e) It can be shown that $\langle N_i, N\rangle = 0;\ i = 0,1$. Thus, N_i can be represented by the local frame of the tangent plane, and the following relation holds

$$N_i = -\sum_{k=0}^{1} h_i^k X_k , \tag{3}$$

where the following equations

$$h_1^1 = \frac{h_{11}g_{22} - h_{12}g_{12}}{g} \qquad h_1^2 = \frac{h_{12}g_{11} - h_{11}g_{12}}{g} \tag{4}$$

$$h_2^1 = \frac{h_{12}g_{22} - h_{22}g_{12}}{g} \qquad h_2^2 = \frac{h_{22}g_{11} - h_{12}g_{12}}{g} \tag{5}$$

are called *Weingarten equations* [5]. Further, N_{ij} can be expressed in terms of the Gauß frame, and it can be shown that the following relation holds

$$X_{ij} = h_{ij}N + \sum_{k=0}^{1} \Gamma_{ij}^k X_k . \tag{6}$$

where $\Gamma_{ij}^{k} = \langle X_k, X_{ij} \rangle$ are called the *Christoffel Symbols*.

Curvature is of great interest in the context of differential geometry. The minimal and maximal curvatures k_1, k_2 at a surface point are the basis for the more interesting definitions of mean curvature and Gauß curvature. In this work, we examine total curvature of surfaces under deformation in normal direction.

Considering surface curves, we get to know the geometric interpretations of the second fundamental form:

Let $A := \lambda^1 X_u + \lambda^2 X_w$ be a tangent vector with $||A|| = 1$. If we intersect the surface with the plane given by N and A, we get an intersection curve y with the following properties:

$$y'(s) = A \text{ and } \mathbf{e}_2 = \pm N,$$

where $\mathbf{e}_2$ is the principal normal vector of the space curve y.

The implicit function theorem implies the existence of this so-called *normal section curve*. To calculate the minimal and maximal curvature of a normal section curve (the so-called *normal section curvature*), we can use the method of *Lagrange multipliers* because we are looking for extreme values of the normal section curvature k_N with the condition $g_{ij}\lambda^i\lambda^j = 1 = ||y'||$.

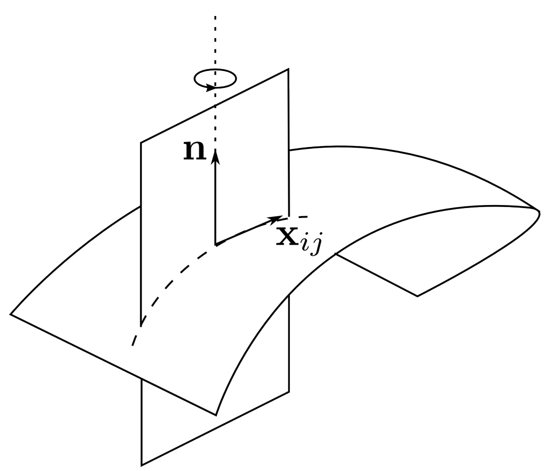


Fig. 2: Construction of normal section curves.

As a result of these considerations, we can define various notions of curvature:

Definition 4. Let $X : U \to \mathbb{E}^3$ be a surface and $A = \lambda^1 X_u + \lambda^2 X_w$ a tangent vector of X at $\mathbf{u}$.

(a) The Weingarten map L is self-adjoint.
(b) The *normal section curvature* $k_N(\lambda^1, \lambda^2)$ can be computed as:

$$k_N(\lambda^1, \lambda^2) = \frac{h_{ij}\lambda^i\lambda^j}{g_{ij}\lambda^i\lambda^j}.$$

Unless the normal section curvature is the same for all directions (umbilical points), there are two perpendicular directions A_1 and A_2 in which k_N attains its absolute maximum and its absolute minimum.

(c) A_1 and A_2 are the *principal directions*.

(d) The corresponding normal section curvatures, k_1 and k_2, are called *principal curvatures* of the surface.

(e) Let $X : U \to \mathbb{E}^3$ be a surface and $y : I \to \mathbb{E}^3$ be a surface curve. We denote by $\hat{y}(t)$ the orthogonal projection of $y(t)$ on the tangent plane $T_{\mathbf{u}}X$ at (an arbitrary) point $P := X(\mathbf{u})$. The *geodesic curvature* k_g of y at P is defined as the curvature of the projected curve $\hat{y}(t)$ at P. A curve $y(t)$ on a surface X is called *geodesic* if its geodesic curvature k_g vanishes identically.

(f) $k_g = \det(\dot{y}, \ddot{y}, N)$, where dots denote derivatives with respect to the arc length of y.

(g) $H := \mathrm{trace}(L) = \frac{1}{2} \cdot (k_1 + k_2)$ is called the *mean curvature*.

(h) $K := k_1 \cdot k_2 = \det(L) = \frac{\det(II)}{\det(I)}$ is called the *Gauß curvature*.

(i) *Total Gauß curvature*, or short, *total curvature*, is defined as $K_{\mathrm{tot}} = \iint_X K \mathrm{d}X$.

Remark 4 (Geodesics and curvature).

(a) An arc of minimum length on a surface joining two arbitrary points must be an arc of a geodesic.

(b) Assuming the boundary of a surface is given and we have to fit in a surface patch of minimal area, then the minimal curvature of this patch has to vanish, in which case, the mean curvature $H \equiv 0$ will also vanish.

3 Deformations

Let $X(u,w)$ be the masterpiece of an industrial surface. Let us further assume that it is a minimal surface (i.e. $H \equiv 0$), such that it covers a minimal area. This masterpiece should be deformed along its normal direction $N(u,w)$ by applying a deformation function $F(u,w)$ $(F : U \to \mathbb{E})$. Deformations along the normal mean that interior deformations of the surface are not permitted (no inner bending).

We consider linear deformations of the form

$$\tilde{X}(u,w,t) := X(u,w) + t \cdot F(u,w) \cdot N(u,w)\,, \tag{7}$$

for $t \in (-\varepsilon, \varepsilon)\,,\ \tilde{g} = g + o(t^2)\,,$ such that $o(t^2)$ constitutes an infinitesimal change. Let us notice that the more general case of linear deformations

$$\tilde{X}(u,w,t) = X(u,w) + tZ(u,w)\,, \tag{8}$$

where $Z(u,w)$ is a continuous vector field ($Z : U \to \mathbb{E}^3$), is called an infinitesimal bending if $\mathrm{d}s_t^2 = \mathrm{d}s^2 + o(t^2)$, i.e. the difference of the squares of the line elements of these surfaces has at least second order [6, 11, 10].

Let us first prove two properties of minimal surfaces which will be needed to prove Theorem 1.

Lemma 1. *For a minimal surface $X(u,w)$, i.e. a surface with $H \equiv 0$, we get*

(a) $[X_u, N_w] + [N_u, X_w] = 0\,,$
(b) $\langle N, h_{11}N_{ww} + h_{22}N_{uu} - h_{12}N_{uw} - h_{12}N_{wu}\rangle = 0\,,$

where $N = \frac{[X_u, X_w]}{g}$ *for* $g = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} = g_{11}g_{22} - g_{12}^2\,.$

Proof. We will prove part (a) and part (b) of this Lemma separately.

(a) To prove Lemma 1a, we can expand Equation 3 to

$$N_u = \frac{h_{12}g_{12} - h_{11}g_{22}}{g}X_u + \frac{h_{11}g_{12} - h_{12}g_{11}}{g}X_w \tag{9}$$

$$N_w = \frac{h_{22}g_{12} - h_{12}g_{22}}{g}X_u + \frac{h_{12}g_{12} - h_{22}g_{11}}{g}X_w\,, \tag{10}$$

from which we can immediately conclude the assumption:

$$\begin{aligned}
[X_u, N_w] + [N_u, X_w] &= \frac{h_{12}g_{12} - h_{22}g_{11}}{g}[X_u, X_w] + \frac{h_{12}g_{12} - h_{11}g_{22}}{g}[X_u, X_w] \\
&= (h_{12}g_{12} - h_{22}g_{11} + h_{12}g_{12} - h_{11}g_{22}) \cdot \frac{[X_u, X_w]}{g} \\
&= -(h_{11}g_{22} + h_{22}g_{11} - 2h_{12}g_{12}) \cdot N \\
&= -\frac{h_{11}g_{22} + h_{22}g_{11} - 2h_{12}g_{12}}{g} \cdot g \cdot N \\
&= -(k_1 + k_2) \cdot g \cdot N \\
&= -2H \cdot g \cdot N = 0\,.
\end{aligned}$$

(b) To prove Lemma 1b, we first compute the second derivatives of N. Then, using the Weingarten equations, we can conclude the following relations:

$$\begin{aligned}
N_{uu} &= -\frac{\partial h_1^1}{\partial u}X_u - h_1^1 X_{uu} - \frac{\partial h_1^2}{\partial u}X_w - h_1^2 X_{wu} \\
N_{ww} &= -\frac{\partial h_2^1}{\partial w}X_u - h_2^1 X_{uw} - \frac{\partial h_2^2}{\partial w}X_w - h_2^2 X_{ww} \\
N_{uw} &= -\frac{\partial h_1^1}{\partial w}X_u - h_1^1 X_{uw} - \frac{\partial h_1^2}{\partial w}X_w - h_1^2 X_{ww} \\
N_{wu} &= -\frac{\partial h_2^1}{\partial u}X_u - h_2^1 X_{uu} - \frac{\partial h_2^2}{\partial u}X_w - h_2^2 X_{wu}\,.
\end{aligned}$$

Next, we look at the scalar product of the normal vector and its second partial derivatives. From this computation, we receive all basic components needed to express part the formula given in part (b) of this Lemma:

$$
\begin{aligned}
\langle N, N_{uu}\rangle &= -h_1^1\langle N, X_{uu}\rangle - h_1^2\langle N, X_{wu}\rangle && = -h_1^1 h_{11} - h_1^2 h_{12}\\
\langle N, N_{ww}\rangle &= -h_2^1\langle N, X_{uw}\rangle - h_2^2\langle N, X_{ww}\rangle && = -h_2^1 h_{12} - h_2^2 h_{22}\\
\langle N, N_{uw}\rangle &= -h_1^1\langle N, X_{uw}\rangle - h_1^2\langle N, X_{ww}\rangle && = -h_1^1 h_{12} - h_1^2 h_{22}\\
\langle N, N_{wu}\rangle &= -h_2^1\langle N, X_{uu}\rangle - h_2^2\langle N, X_{wu}\rangle && = -h_2^1 h_{11} - h_2^2 h_{12}\,.
\end{aligned}
$$

We want to show that $\langle N, h_{11}N_{ww} + h_{22}N_{uu} - h_{12}N_{uw} - h_{12}N_{wu}\rangle = 0$. Taking the above results, combined with Equation 4, we arrive at

$$
\begin{aligned}
&\langle N, h_{11}N_{ww} + h_{22}N_{uu} - h_{12}N_{uw} - h_{12}N_{wu}\rangle\\
&= \langle N, h_{11}N_{ww}\rangle + \langle N, h_{22}N_{uu}\rangle - \langle N, h_{12}N_{uw}\rangle - \langle N, h_{12}N_{wu}\rangle\\
&= -h_{11}h_2^1 h_{12} - h_{11}h_2^2 h_{22} - h_{22}h_1^1 h_{11} - h_{22}h_1^2 h_{12}\\
&\quad + h_{12}h_1^1 h_{12} + h_{12}h_1^2 h_{22} + h_{12}h_2^1 h_{11} + h_{12}h_2^2 h_{12}\\
&= -h_{11}h_{22}(h_1^1 + h_2^2) + (h_{12})^2(h_1^1 + h_2^2)\\
&= (h_{11}h_{22} - (h_{12})^2)(-h_2^2 - h_1^1)\\
&= \frac{h}{g}(h_{12}g_{12} - h_{11}g_{22} + h_{12}g_{12} - h_{22}g_{11})\\
&= \frac{h}{g}(-2Hg)\\
&= -2hH = 0\,.
\end{aligned}
$$

□

We are now interested in shape-preserving modification of the masterpiece. We consider infinitesimal deformations which do not change the Gauß curvature, and therefore preserve the total curvature of the minimal surface.

We restrict ourselves to exterior deformations, i.e. deformations in normal direction. Interior deformations, such as perturbations in the tangent plane, are not permitted. This restriction serves the purpose of exaggerating or reducing features that are present in the masterpiece but refraining from introducing additional perturbations. We can now introduce the main theorem of this paper:

Theorem 1. *A linear deformation*

$$\tilde{X}(u,w,t) = X(u,w) + tF(u,w)N(u,w) \tag{11}$$

of a minimal surface $X(u,w)$ with $t \in (-\varepsilon, \varepsilon)$, and $\tilde{g} = g + o(t^2)$ preserves the Gauß curvature if

$$
\begin{aligned}
\mathrm{D}F := {}& h_{11}F_{ww} + h_{22}F_{uu} - 2h_{12}F_{uw} - F_u(h_{11}\Gamma_{22}^1 - h_{12}\Gamma_{12}^1 + h_{22}\Gamma_{11}^1)\sqrt{g}\\
&+ F_w(h_{11}\Gamma_{22}^2 - h_{12}\Gamma_{12}^2 + h_{22}\Gamma_{11}^2)\sqrt{g}\\
={}& 0\,,
\end{aligned}
$$

and therefore preserves the total curvature $\iint_S K\mathrm{d}s$ of our minimal surface $X(u,w)$.

To examine the impact of linear deformation as given in Equation 11, we need to observe changes in some surface properties. We start with normal vectors, along which we perturb the surface. Normal vectors deform as follows:

$$\tilde{N}(u,w,t) = \frac{1}{||\tilde{N}||}\left\{[X_u, X_w] + t\cdot[F_w X_u - F_u X_w, N]\right\} + \mathrm{o}(t^2)\,.$$

The deformed second fundamental form $\widetilde{II}$, defined as

$$\begin{pmatrix}\tilde{h}_{11} & \tilde{h}_{12}\\ \tilde{h}_{12} & \tilde{h}_{22}\end{pmatrix} = \begin{pmatrix}\langle\tilde{N},\tilde{X}_{uu}\rangle & \langle\tilde{N},\tilde{X}_{uw}\rangle\\ \langle\tilde{N},\tilde{X}_{wu}\rangle & \langle\tilde{N},\tilde{X}_{ww}\rangle\end{pmatrix},$$

can be written as

$$\begin{aligned}
\tilde{h}_{11} &= \langle N + t\cdot[F_w X_u - F_u X_w, N], X_{uu} + tF_{uu}N + 2tF_u N_u + tFN_{uu}\rangle + o(t^2)\\
&= h_{11} + t\cdot\{F_{uu} + \det(F_w X_u - F_u X_w, N, X_{uu}) + F\langle N, N_{uu}\rangle\} + o(t^2)\\
\tilde{h}_{22} &= h_{22} + t\cdot\{F_{ww} + \det(F_w X_u - F_u X_w, N, X_{ww}) + F\langle N, N_{ww}\rangle\} + o(t^2)\\
\tilde{h}_{12} &= h_{12} + t\cdot\{F_{uw} + \det(F_w X_u - F_u X_w, N, X_{uw}) + F\langle N, N_{uw}\rangle\} + o(t^2)\\
\tilde{h}_{21} &= h_{21} + t\cdot\{F_{wu} + \det(F_w X_u - F_u X_w, N, X_{wu}) + F\langle N, N_{wu}\rangle\} + o(t^2)\,.
\end{aligned}$$

We know that $\tilde{K} = \frac{\det(\widetilde{II})}{\det(\tilde{I})}$, and we already have the determinant $\det(\tilde{I}) = g$ of the first fundamental form, which remains identical to $\det(I)$ up to an infinitesimal change under deformation. To compute $\tilde{K}$, we have to compute the determinant of the second fundamental form, $\det(\widetilde{II}) = \tilde{h}_{11}\cdot\tilde{h}_{22} - \tilde{h}_{12}\cdot\tilde{h}_{12}\,.$

$$\begin{aligned}
\det(\widetilde{II}) &= \tilde{h}_{11}\cdot\tilde{h}_{22} - \tilde{h}_{12}\cdot\tilde{h}_{12}\\
&= h_{11}h_{22} - h_{12}h_{12} + o(t^2)\\
&\quad + t\cdot\{h_{11}F_{ww} + h_{11}\det(F_w X_u - F_u X_w, N, X_{ww}) + h_{11}F\langle N, N_{ww}\rangle\}\\
&\quad + t\cdot\{h_{22}F_{uu} + h_{22}\det(F_w X_u - F_u X_w, N, X_{uu}) + h_{22}F\langle N, N_{uu}\rangle\}\\
&\quad - t\cdot\{h_{12}F_{wu} + h_{12}\det(F_w X_u - F_u X_w, N, N_{wu}) + h_{12}F\langle N, N_{wu}\rangle\}\\
&\quad - t\cdot\{h_{12}F_{uw} + h_{12}\det(F_w X_u - F_u X_w, N, N_{uw}) + h_{12}F\langle N, N_{uw}\rangle\}\\
&= h_{11}h_{22} - h_{12}^2 + o(t^2)\\
&\quad + t\{h_{11}F_{ww} + h_{11}\det(F_w X_u - F_u X_w, N, X_{ww})\\
&\qquad + h_{22}F_{uu} + h_{22}\det(F_w X_u - F_u X_w, N, X_{uu})\\
&\qquad - 2h_{12}F_{uw} - 2h_{12}\det(F_w X_u - F_u X_w, N, N_{uw})\}\,.
\end{aligned}$$

As we know from Lemma 1b, $\langle N, h_{11}N_{ww} + h_{22}N_{uu} - h_{12}N_{uw} - h_{12}N_{wu}\rangle = 0$ holds. Assuming $F_{uw} = F_{wu}$, we can conclude that

$$\begin{aligned}\tilde{K} &= \frac{\tilde{h}_{11}\tilde{h}_{22}-\tilde{h}_{12}^2}{g} + o(t^2)\\ &= K + t\cdot\{h_{11}F_{ww} + h_{22}F_{uu} - 2h_{12}F_{uw}\\ &\qquad + h_{11}(F_w\Gamma_{22}^2 - F_u\Gamma_{22}^1)\det(X_u,N,X_w)\\ &\qquad + h_{22}(F_w\Gamma_{11}^2 - F_u\Gamma_{11}^1)\det(X_u,N,X_w)\\ &\qquad + 2h_{12}(F_w\Gamma_{12}^2 - F_u\Gamma_{12}^1)\det(X_u,N,X_w)\}\,.\end{aligned}$$

Since $\det(X_u,N,X_w) = -\det(N,X_u,X_w) = -\langle N,[X_u,X_w]\rangle = -\sqrt{g}$, the Gauß curvature changes als follows under deformation:

$$\begin{aligned}\tilde{K} = K + t\cdot\{&h_{11}F_{ww} + h_{22}F_{uu} - 2h_{12}F_{uw} + h_{11}(F_w\Gamma_{22}^2 - F_u\Gamma_{22}^1)\sqrt{g}\\ &+ h_{22}(F_w\Gamma_{11}^2 - F_u\Gamma_{11}^1)\sqrt{g} - 2h_{12}(F_w\Gamma_{12}^2 - F_u\Gamma_{12}^1)\sqrt{g}\}\,.\end{aligned}$$

□

This concludes the proof of the main theorem of this paper.

4 Examples

In the following examples, we consider linear deformations, assuming bilinear distribution function $F(u,w) = au + bw + c$ which has the derivatives $F_u = a$, $F_w = b$, and $F_{uw} = F_{wu} = 0$.

Example 1 (Helicoid). We deform a helicoid X, which is a minimal surface of the form

$$X(u,w) = \begin{pmatrix} u\cos w \\ u\sin w \\ d\cdot w \end{pmatrix},$$

with $d = \frac{d_0}{2\pi}$, where d_0 is the number of windings.
This gives us the following derivatives and normal vector:

$$X_u = \begin{pmatrix}\cos w\\ \sin w\\ 0\end{pmatrix} \quad X_w = \begin{pmatrix}-u\sin w\\ u\cos w\\ d\end{pmatrix} \quad X_{uw} = \begin{pmatrix}-\sin w\\ \cos w\\ 0\end{pmatrix}$$

$$N = \frac{[X_u,X_w]}{||[X_u,X_w]||} = \frac{1}{\sqrt{u^2+d^2}}\begin{pmatrix} d\sin w\\ -d\cos w\\ u\end{pmatrix}.$$

Next, we compute the elements g_{ij} of the first fundamental form, and the elements h_{ij} of the second fundamental form.

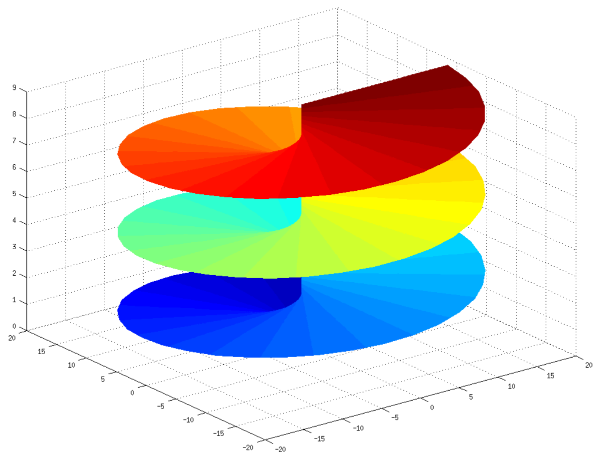


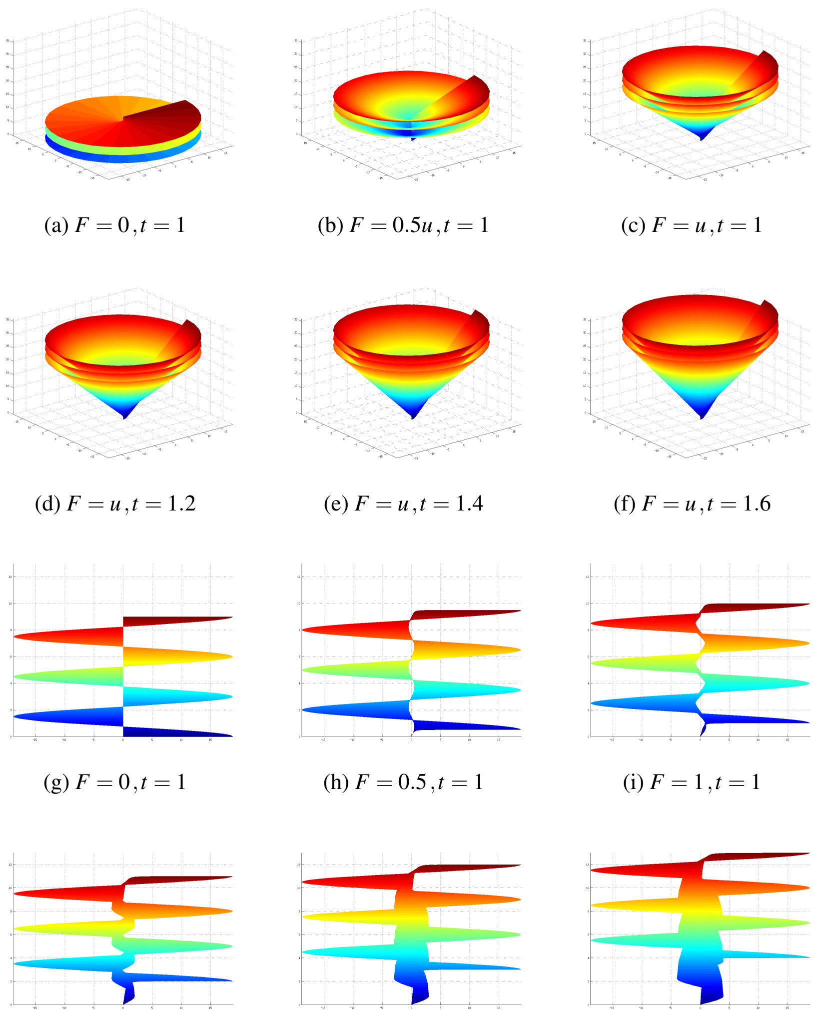
Fig. 3: A helicoid with $u, w \in [0, 6\pi], d = \frac{3}{2\pi}$.

$$
\begin{aligned}
g_{11} &= \langle X_u, X_u \rangle = \cos^2 w + \sin^2 w + 0 && = 1 \\
g_{12} &= \langle X_u, X_w \rangle = -u \sin w \cos w + u \sin w \cos w + 0 && = 0 \\
g_{22} &= \langle X_w, X_w \rangle = u^2 \sin^2 w + u^2 \cos^2 w + d^2 && = u^2 + d^2 \\
h_{11} &= \langle N, X_{uu} \rangle = \frac{d \sin w \cos w - d \cos w \sin w}{\sqrt{u^2 + d^2}} && = 0 \\
h_{12} &= \langle N, X_{uw} \rangle = \frac{-d \sin^2 w - d \cos^2 w}{\sqrt{u^2 + d^2}} && = \frac{-d}{\sqrt{u^2 + d^2}} \\
h_{22} &= \langle N, X_{ww} \rangle = \frac{-du \cos w \sin w + du \cos w \sin w}{\sqrt{u^2 + d^2}} && = 0\,.
\end{aligned}
$$

If we compute DF for our surface X and our deformation function F, and set $DF = 0$ (Theorem 1), we end up with

$$
\begin{aligned}
\mathrm{D}F &= h_{11}F_{ww} + h_{22}F_{uu} - 2h_{12}F_{uw} - (h_{11}\Gamma^1_{22} - h_{12}\Gamma^1_{12} + h_{22}\Gamma^1_{11}) \cdot F_u\sqrt{g} \\
&\quad + (h_{11}\Gamma^2_{22} - h_{12}\Gamma^2_{12} + h_{22}\Gamma^2_{11}) \cdot F_w\sqrt{g} \\
&= 0 \cdot F_{ww} + 0 \cdot F_{uu} - 2\frac{-d}{\sqrt{u^2+d^2}} \cdot F_{uw} \\
&\quad - \left(0 \cdot \Gamma^1_{22} - \frac{-d}{\sqrt{u^2+d^2}}\Gamma^1_{12} + 0 \cdot \Gamma^1_{11}\right) \cdot F_u\sqrt{u^2+d^2} \\
&\quad + \left(0 \cdot \Gamma^2_{22} - \frac{-d}{\sqrt{u^2+d^2}}\Gamma^2_{12} + 0 \cdot \Gamma^2_{11}\right) \cdot F_w\sqrt{u^2+d^2} \\
&= \frac{2d}{\sqrt{u^2+d^2}} \cdot F_{uw} + d \cdot 0 \cdot F_u + d \cdot u \cdot F_w \\
&= \frac{2d}{\sqrt{u^2+d^2}} \cdot 0 + du \cdot F_w \\
&= du \cdot F_w \\
&= du \cdot b \overset{!}{=} 0 \\
&\Leftrightarrow b = 0\,,
\end{aligned}
$$

since $\Gamma^1_{12} = \langle X_{uw}, X_u\rangle = 0$ and $\Gamma^2_{12} = \langle X_{uw}, X_w\rangle = u$.
Therefore, to make the deformation (total) curvature-preserving, the bilinear distribution function has to be simplified to the form $au + c$.
The resulting deformation is shown in Figure 4. The influence of the linear coefficient $a \in \{0, 0.5, 1\}$ is given in the first row: in the beginning, the surface is a helicoid, but with increasing a, it deforms to a funnel. This effect is amplified by the scaling parameter $t \in \{1, 1.25, 1.5\}$, since both are multipliers for the normal, as seen in the second row. The influence of the constant coefficient $c \in \{0, 0.5, 1\}$ is given in the third row: in the beginning, the helicoid's centre curve is a straight line, but with increasing c, it deforms into a helix, dragging along the adjacent portions of the surface. The last row demonstrates the effect of $t \in \{1, 2, 3\}$ on this additive portion: the almost vertical surface parts are stretched from little more than a line to long sheets hanging down.
In real-world examples, parameters have to be chosen carefully (and small) to avoid such drastic deformations. We used extremely large parameters for this example to convey a general impression of the nature of change.
Our method is targeted at infinitesimal deformations. For the sake of illustration, we have chosen extremely large parameters for the deformations in Figure 4. More realistically, one has to choose a much smaller t since we assume $o(t^2)$ to be negligible in our proof. Thus arises $t < 1$ as a necessary reqirement.
With an initial $K_{\mathrm{tot}} = 0.0011$, we consider a change of $\Delta K_{\mathrm{tot}} = 1$ a sufficiently small change. The discretised helicoid consists of 10201 points, so this results an average change of 0.000098 in Gauss curvature per point. This threshold is first reached for $t = 0.015$ with $F = 1$, and it is last reached for $t = 0.086$ with $F = u$ in our example.
In Figures 5 and Figures 6, we use the same deformation function parameters as in Figure 4. Both Figures illustrate how ΔK_{tot} changes with increasing t. In Figure 5,



(a) $F = 0, t = 1$ (b) $F = 0.5u, t = 1$ (c) $F = u, t = 1$

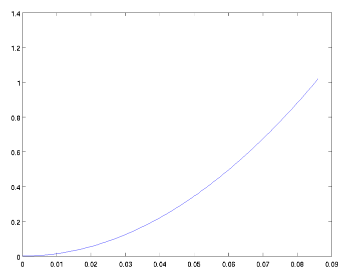
(d) $F = u, t = 1.2$ (e) $F = u, t = 1.4$ (f) $F = u, t = 1.6$

(g) $F = 0, t = 1$ (h) $F = 0.5, t = 1$ (i) $F = 1, t = 1$

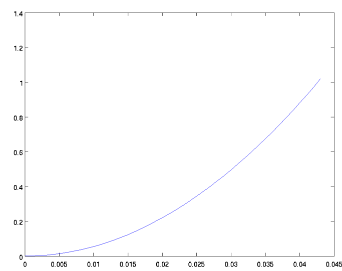
(j) $F = 1, t = 2$ (k) $F = 1, t = 3$ (l) $F = 1, t = 4$

Fig. 4: The impact of deformation on the helicoid, shown separately for a, c, t. In the first and third row, only $F = au + c$ is varied. The first row shows a varying coefficient a with a fixed coefficient $c = 0$, while the third row shows a varying coefficient c with a fixed coefficient $a = 0$. For the second and fourth row, we keep F fixed, while varying the scaling parameter t in order to demonstrate the influence of scaling on the linear and constant coefficients. Figures (4a) and (4g) display the same surface from different perspectives.

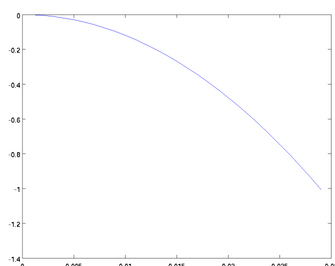
we show the change until the threshold of $\Delta K_{\text{tot}} = 1$ is reached. In Figure 6, we continue deforming until $t = 1.6$, the maximum deformation used for the upper half of Figure 4, to demonstrate the instabilties occuring for large t. In these cases, the prototype of a model has to be adapted before applying further infinitesimal bendings.
For this particular example, the signs of a and c do not affect ΔK_{tot} when varied individually since the helicoid is symmetric and applying the deformation with opposite sign results in a similar deformation in opposite direction.



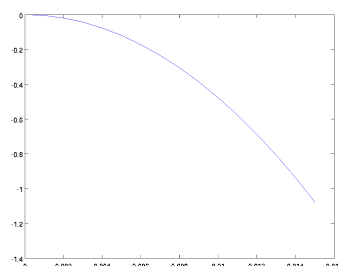
(a) $F = 0.5u$.



(b) $F = 1u$.



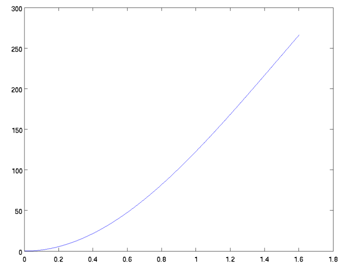
(c) $F = 0.5$.



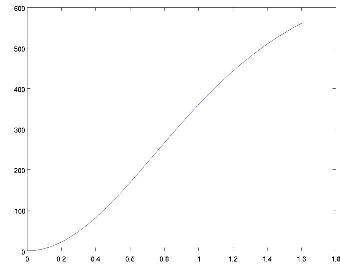
(d) $F = 1$.

Fig. 5: Plot of ΔK_{tot} (vertical) over increasing t (horizontal) up to $|\Delta K_{\text{tot}}| = 1$.

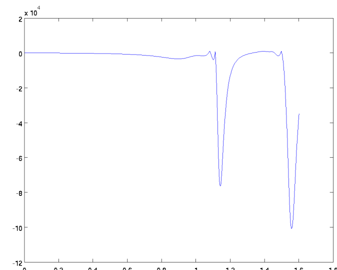
Example 2 (Fandisk). Large industrial surface models are typically composed of smaller parts. E.g. consider a turbine: it is composed of fan blades, fandisks, and many other components. It would not necessarily make sense to deform the entire model at once, but it is relatively easy to modify a single part like a fan blade or a fandisk.
In this example, we present deformations on Hoppe's fandisk model [12]. We have recreated the part marked in the rendering of the original model (Figure 7a) from Bézier surface patches (Figure 7b). As most real-world examples, this model has hard edges. We preserve them as surface patch boundaries between adjacent patches.



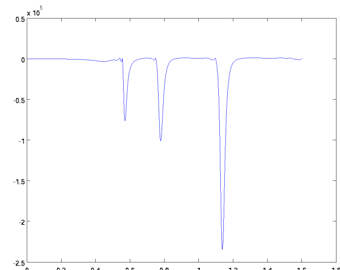
(a) $F = 0.5u$.



(b) $F = 1u$.



(c) $F = 0.5$.



(d) $F = 1$.

Fig. 6: Plot of ΔK_{tot} (vertical) over increasing t (horizontal) up to $t = 1.6$.

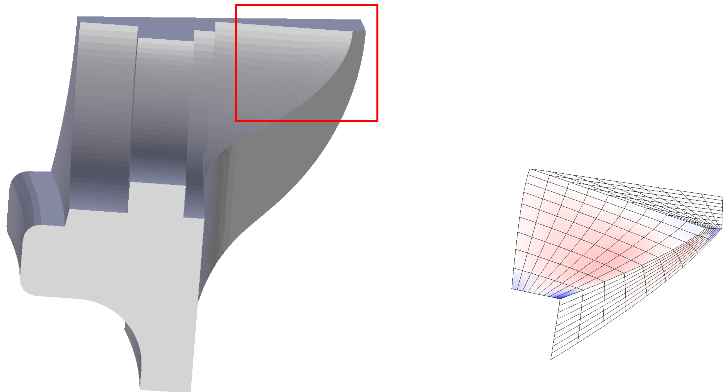
To deform the entire model rather than a single patch at a time, we take the average of adjacent surface normals to perturb edges. Note that this can only be done on oriented manifolds.
We now take the technique we developed for minimal surfaces and adapt it to general surfaces. Our goal remains to keep the surface area as minimal as possible so the material cost remains as minimal as possible.
Now, we deform all surface patches with

$$\tilde{X}(u,w,t) = X(u,w) + t \cdot F(u,w) \cdot N(u,w),$$

where

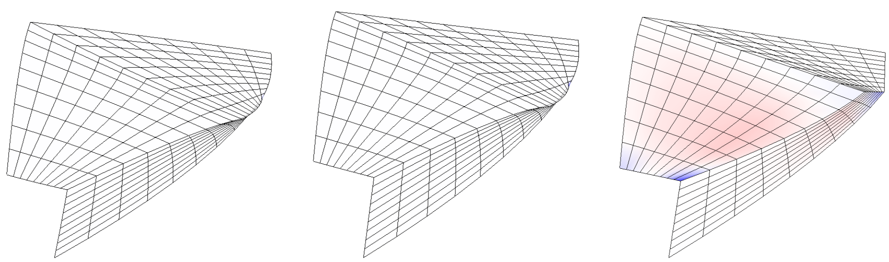
$$F(u,w) = au + bw + c$$

is our deformation function.
In Figure 8, we demonstrate the effect of isolated changes of a, b, c on the deformation. Figure 9 illustrates some deformations with combined parameter changes.
The colour map in Figures 7b, 8, and 9 depends on the Gauß curvature at each point. Blue areas are minima of Gauß curvature, red areas are maxima of Gauß curvature relative to the rest of the model. White areas are close to the median Gauß curvature.

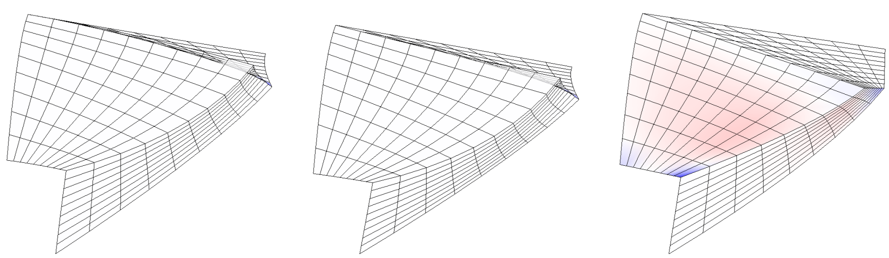


(a) Model shown in Blender. (b) Recreated as surface patches.

Fig. 7: Fandisk model by Hoppe [12] and a portion of it recreated from Bézier surface patches.

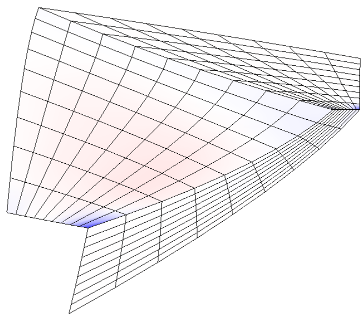


(a) $F(u,w) = u$. $(a > 0)$ (b) $F(u,w) = w$. $(b > 0)$ (c) $F(u,w) = 1$. $(c > 0)$

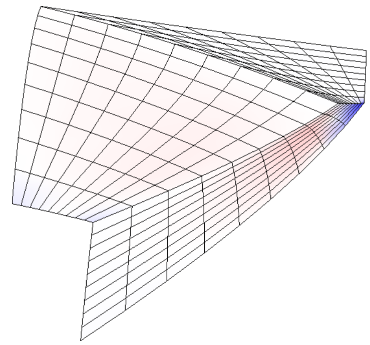


(d) $F(u,w) = -u$. $(a < 0)$ (e) $F(u,w) = -w$. $(b < 0)$ (f) $F(u,w) = -1$. $(c < 0)$

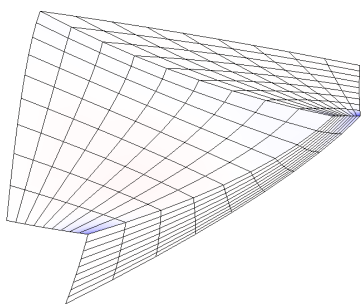
Fig. 8: Fandisk model under deformation with $t = 0.1$.



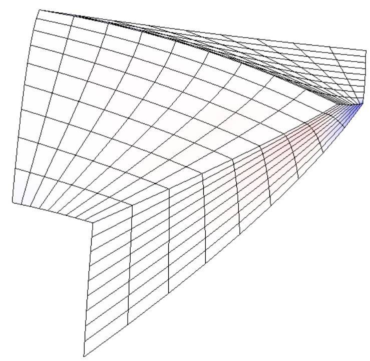
(a) $F(u,w) = u - w$



(b) $F(u,w) = -u + w$



(c) $F(u,w) = 2u - 2w$



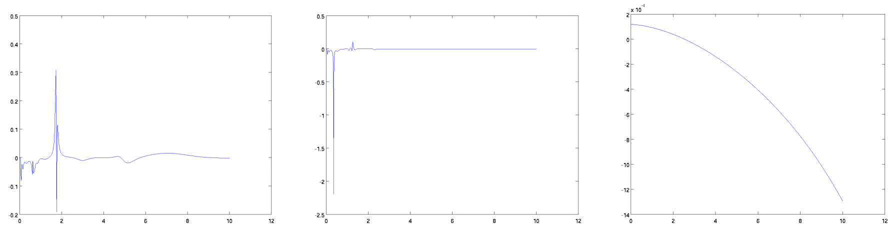
(d) $F(u,w) = -2u + 2w$

Fig. 9: Isolated change of one parameter at a time with $t = 0.1$.

In Figure 10, we show changes in ΔK_{tot} over a deformation with $t \in [0,10]$. For relatively small values of t, the deformation-induced change is stable. However, as the deformation grows, instabilities begin to occur for t approximately between 0.5 and 2.0. For extremely large values of t, the deformation is stable again, however the deformed surface no longer looks similar to the original one.

5 Conclusion

The goal of this work is to deform the masterpiece in a meaningful way, i.e. enhance or decrease features; this can be done by perturbing in normal direction. Since there is a direct connection between total curvature and bending energy, the restriction of total curvature serves to restrict the bending energy. This maintains a surface area that is as minimal as possible and therefore reduces material cost.



(a) $F = u$. (b) $F = w$. (c) $F = 1$.

Fig. 10: Change in ΔK_{tot} for $t \in [0, 10]$.

We have presented a method to perturb surfaces without altering their total curvature, thereby keeping their bending energy low. This results in surfaces with a small surface are which can be manufactured at lower cost than surfaces which have a higher total curvature and higher bending energy. The surface and deformation function given in Example 1 have nice analytic descriptions, so it is possible to make all computations manually. For the surface in Example 2, this is not possible since we have to define normals on hard edges.

We can deform a surface along normal direction, both outward ($F(u,w) > 0$) to increase features, and inward ($F(u,w) < 0$) to decrease features. While the examples in Fig. 4 only present the results for deformation with a positive F, the results look very similar (but upside down) for a negative F.

In real-world examples, the surface description is a lot more complicated, making it more difficult to comprehend what exactly happens to the surface during deformation. A lot of such complex models possess sharp edges on which tangents and normals are not clearly defined. In these cases, they have to be estimated from the neighbourhood of an edge.

Our method is subjected to the same numerical limitations as partial differential equations. It is proven for objects with an analytic description, however, they are applicable to a meshes at the sacrifice of accuracy. In our first example, we computed normals and tangents analytically, but the actual deformation is performed on a discretised version of the model. Given an arbitrary mesh, our approach is limited by the availability of tangents and normals. Solutions to this are presented by [17, 3, 19, 20]. If the surface has a boundary, we are, again, limited by the availability of tangents and normals. However, given this information, the deformation procedure does not discriminate between boundary points and interior points. For a given surface patch, one can assume the normal and tangent on the boundary to be identical to its neighbourhood. Under infinitesimal deformations, the genus of a model will be preserved but if a deformation is very large, deformations can introduce self-intersections.

It is possible to introduce a flow on a given surface. One of the important and complicated challenges we want to address in the future is to apply our deformations in a way that they not only preserve the index sum of all singularities of a vector field defined on this surface, but also leaves the indices of each singularity unchanged.

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