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Gradient Schemes for Stokes problem

Jérôme Droniou[*], Robert Eymard[†] and Pierre Feron [‡]

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Abstract

We develop a framework, which encompasses a large family of conforming and nonconforming numerical schemes, for the approximation of the steady state and transient incompressible Stokes equations with homogeneous Dirichlet's boundary conditions. This framework provides general convergence proofs, by error estimates in the case of the steady problem and by compactness arguments in the case of the transient one. Three classical methods (MAC, Taylor–Hood and Crouzeix-Raviart schemes) are shown to belong to this framework, which also inspires the construction of a novel scheme, whose advantage is to be $H_{\rm div}$ conforming and to retain a small number of degrees of freedom.

1 Introduction

The notion of gradient schemes, designed for linear and nonlinear elliptic and parabolic problems in [8, 9, 11, 16, 15], has the benefit of providing common convergence and error estimates results, which hold for a wide variety of numerical methods (finite element methods, nonconforming and mixed finite element methods, hybrid and mixed mimetic finite difference methods...). Checking a minimal set of properties for a given numerical method suffices to prove that it belongs to the gradient schemes framework, and therefore that it is convergent on the aforementioned problems. The aim of this paper is to extend this framework to the steady and transient Stokes problems:

$$\left\{\begin{array}{rll} \eta\overline{u} - \Delta\overline{u} + \nabla\overline{p} & = f - \mathrm{div}(G) & \text{in } \Omega \\ \mathrm{div}\overline{u} & = 0 & \text{in } \Omega \\ \overline{u} & = 0 & \text{on } \partial\Omega \end{array}\right. \tag{1}$$

and

$$\left\{\begin{array}{rll} \partial_t\overline{u} - \Delta\overline{u} + \nabla\overline{p} & = f - \mathrm{div}(G) & \text{in } \Omega\times(0,T) \\ \mathrm{div}\overline{u} & = 0 & \text{in } \Omega\times(0,T) \\ \overline{u} & = 0 & \text{on } \partial\Omega\times(0,T) \\ \overline{u}(\cdot,0) & = \overline{u}_{\rm ini} & \text{a.e on } \Omega, \end{array}\right. \tag{2}$$

where $\overline{u}$ represents the velocity field, $\overline{p}$ is the pressure and the domain Ω is a bounded open set in $\mathbb{R}^d$, $d \geq 1$.

We may cite (without exhaustivity) separate works done for analyzing different methods, that are among the most commonly used in many industrial applications. In [1], a first error estimate is given for the Taylor-Hood approximation of the Stokes problem, further improved in [25]. In [5], the authors analyze the application of the P^1-nonconforming method for the velocity, with piecewise P^0 approximation for

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the pressure, known as the Crouzeix-Raviart scheme, and they provide an error estimate. In [21], the author provides, in a pioneering work, the first convergence proof for the famous Marker-And-Cell (MAC) scheme [18, 22, 26], intensively used in the engineering and fluid numerics frameworks.
In this paper, we exhibit the common principles of all these methods, which are sufficient for proving their convergence. We define the notion of gradient discretization for the Stokes problem, and we show how the classical schemes may be included in this framework: this is done in Section 4.1 for the MAC scheme (see [3] for an extended version of the MAC scheme which also could be included in the present framework), the Taylor–Hood scheme, the Crouzeix-Raviart scheme and a new method which is $H_{\rm div}$ conforming, and hence satisfies an interesting property regarding pressure-independent errors [19].
The paper is organised as follows. In Section 2, we detail the discrete space and operators used in gradient discretisations for the Stokes problem, and we provide the common properties that are sufficient for showing convergence results of the resulting gradient scheme. Then, in Section 3, we give our main results. These results concern the convergence of the general gradient scheme for the steady and transient Stokes problems. The steady results are obtained through error estimates: the first one is given in the most general hypotheses, and the next ones are improved, accounting for more restrictive hypotheses on the data. The transient results are obtained via compactness tools. In all these results, the pressure plays a key role. Section 4 is devoted to the proof that standard schemes are included in the general framework. We first consider the MAC scheme, for which the appropriate tools are those of the discrete functional analysis, such as the one developed in [12] for the study of finite volume schemes for elliptic problems. We then consider the Taylor-Hood scheme, which easily enters into the general framework thanks to its conformity, and we next study the nonconforming Crouzeix-Raviart scheme. This scheme is known to be useful in many industrial applications, due to the approximation of the velocity at the barycenter of the faces, which allows for easy couplings with finite volume schemes for more complex situations. We finally turn to an original scheme, which has a few interesting characteristics. It is an extension of the MAC scheme to simplicial meshes but, contrary to other extensions, it remains formulated in velocity/pressure (no use of the vorticity is introduced). Hence this scheme happens to have the same characteristics as the Crouzeix-Raviart scheme for its domain of application, in addition to the fact that the number of unknowns is reduced. Moreover, in 2D, a simple change of unknowns (use of the scalar discrete streamline potential) allows for the elimination of the pressure and the resolution of a much smaller number of unknowns. Nevertheless, it has the drawback of showing large stencils, and the 2D change of variable does not easily apply in 3D. A short numerical insight is proposed to enlighten the comparison between these two last schemes.
The proofs of the results stated in Section 3 are then detailed in Section 5. One shall notice that these proofs are not straightforward, in particular in the case of the transient problem. Appropriate compactness tools have to be provided, demanding to define suitable discrete norms.

Notations In the following, if F is a vector space we denote by $\boldsymbol{F}$ the space F^d. Thus, $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$ and $\boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega)^d$. The space $E(\Omega)$ is the space of fields $v \in \boldsymbol{H}_0^1(\Omega)$ such that $\mathrm{div}(v) = 0$. $L_0^2(\Omega)$ is the space of functions in $L^2(\Omega)$ with a zero mean value over Ω. Finally, $\boldsymbol{H}_{\rm div}(\Omega)$ is the space of fields $v \in \boldsymbol{L}^2(\Omega)$ such that $\mathrm{div}(v) \in L^2(\Omega)$.

2 Gradient discretisations

Gradient discretisations provide the foundations, in terms of discrete spaces, operators and properties, upon which the gradient scheme framework is designed.

2.1 Space

Definition 2.1 (Gradient discretisation for the steady Stokes problem) *A gradient discretisation D for the incompressible steady Stokes problem, with homogeneous Dirichlet's boundary conditions, is de-*

fined by $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \mathrm{div}_D)$*, where:*

1. $X_{D,0}$ *is a finite-dimensional vector space on* $\mathbb{R}$*.*
2. Y_D *is a finite-dimensional vector space on* $\mathbb{R}$*.*
3. *The linear mapping* $\Pi_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)$ *is the reconstruction of the approximate velocity field.*
4. *The linear mapping* $\chi_D : Y_D \to L^2(\Omega)$ *is the reconstruction of the approximate pressure, and must be chosen such that* $\|\chi_D \cdot\|_{L^2(\Omega)}$ *is a norm on* Y_D*. We then set* $Y_{D,0} = \{q \in Y_D, \int_\Omega \chi_D q \,\mathrm{d}x = 0\}$*.*
5. *The linear mapping* $\nabla_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)^d$ *is the discrete gradient operator. It must be chosen such that* $\|\cdot\|_D := \|\nabla_D \cdot\|_{\boldsymbol{L}^2(\Omega)^d}$ *is a norm on* $X_{D,0}$*.*
6. *The linear mapping* $\mathrm{div}_D : X_{D,0} \to L^2(\Omega)$ *is the discrete divergence operator.*

Remark 2.2 (Boundary conditions) *Gradient discretisations adapted to other boundary conditions than the homogeneous Dirichlet conditions can be easily designed, see [8].*

The *coercivity* of a sequence of gradient discretisations ensure that a discrete Poincaré inequality, a control of the discrete divergence and a discrete Ladyzenskaja-Babuka-Brezzi (LBB) conditions can be establish, all uniform along the sequence of discretisations.

Definition 2.3 (Coercivity) *Let* D *be a discretisation in the sense of Definition 2.1. Let* C_D *and* β_D *be defined by*

$$C_D = \max_{v \in X_{D,0}, \|v\|_D = 1} \|\Pi_D v\|_{\boldsymbol{L}^2(\Omega)} + \max_{v \in X_{D,0}, \|v\|_D = 1} \|\mathrm{div}_D v\|_{L^2(\Omega)}, \tag{3}$$

$$\beta_D = \min\left\{ \max_{v \in X_{D,0}, \|v\|_D = 1} \int_\Omega \chi_D q \ \mathrm{div}_D v \,\mathrm{d}x \ : \ q \in Y_{D,0} \text{ such that } \|\chi_D q\|_{L^2(\Omega)} = 1 \right\}. \tag{4}$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation is said to be* ***coercive*** *if there exist* $C_P \geq 0$ *and* $\beta > 0$ *such that* $C_{D_m} \leq C_P$ *and* $\beta_{D_m} \geq \beta$*, for all* $m \in \mathbb{N}$*.*

The *consistency* of a sequence of gradient discretisations states that the discrete space and operators "fill in" the continuous space as the discretisation is refined.

Definition 2.4 (Consistency) *Let* D *be a gradient discretisation in the sense of Definition 2.1, and let* $S_D : \boldsymbol{H}^1_0(\Omega) \to [0, +\infty)$*, and* $\widetilde{S}_D : L^2_0(\Omega) \to [0, +\infty)$ *be defined by*

$$\forall \varphi \in \boldsymbol{H}^1_0(\Omega)\,, \quad S_D(\varphi) = \min_{v \in X_{D,0}} \left(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\mathrm{div}_D v - \mathrm{div}\varphi\|_{L^2(\Omega)} \right)$$

and

$$\forall \psi \in L^2_0(\Omega)\,, \quad \widetilde{S}_D(\psi) = \min_{z \in Y_{D,0}} \|\chi_D z - \psi\|_{L^2(\Omega)}.$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation is said to be* ***consistent*** *if, for all* $\varphi \in \boldsymbol{H}^1_0(\Omega)$*,* $S_{D_m}(\varphi)$ *tends to* 0 *as* $m \to \infty$ *and, for all* $\psi \in L^2_0(\Omega)$*,* $\widetilde{S}_{D_m}(\psi)$ *tends to* 0 *as* $m \to \infty$*.*

Definition 2.5 (Limit-conformity) *Let* D *be a gradient discretisation in the sense of Definition 2.1 and let* $\overline{W}_D : Z(\Omega) \mapsto [0, +\infty)$*, with* $Z(\Omega) = \{(\varphi, \psi) \in \boldsymbol{L}^2(\Omega)^d \times L^2(\Omega), \mathrm{div}\varphi - \nabla\psi \in \boldsymbol{L}^2(\Omega)\}$*, be defined by*

$$\forall (\varphi, \psi) \in Z(\Omega)\,, \quad \overline{W}_D(\varphi, \psi) = \max_{\substack{v \in X_{D,0} \\ \|v\|_D = 1}} \left(\int_\Omega \left[\nabla_D v : \varphi + \Pi_D v \cdot (\mathrm{div}\varphi - \nabla\psi) - \psi \mathrm{div}_D v \right] \mathrm{d}x \right).$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation is said to be* ***limit-conforming*** *if, for all* $(\varphi, \psi) \in Z(\Omega)$*,* $\overline{W}_{D_m}(\varphi, \psi)$ *tends to* 0 *as* $m \to \infty$*.*

Remark 2.6 *As in [14, 8], the consistency of a coercive sequence of gradient discretisation only needs to be checked on dense subspaces of $\boldsymbol{H}_0^1(\Omega)$ and $L_0^2(\Omega)$. This is also true for limit-conformity, but we discuss this particular case in the appendix (Section 6.1).*

2.2 Space-time

The notion of gradient discretisation for transient problems requires the addition of time steps and an interpolation (not necessarily linear) of the initial condition.

Definition 2.7 (Space-time gradient discretisation) *A space-time gradient discretisation D for the transient Stokes problem, with homogenous Dirichlet boundary conditions, is defined by a family $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \mathrm{div}_D, (t^{(n)})_{n=0,\dots,N}, J_D)$ where:*

- *$D^s = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \mathrm{div}_D)$ a gradient discretisation of Ω in the sense of Definition 2.1,*
- *$J_D : \boldsymbol{L}^2(\Omega) \mapsto X_{D,0}$ an interpolation operator;*
- *$t^{(0)} = 0 < t^{(1)} < \dots < t^{(N)} = T$.*

We define $\delta t^{n+\frac{1}{2}} = t^{(n+1)} - t^{(n)}$ for all $n = 0, \dots, N-1$ and $\delta t_D = \max\limits_{n=0,\dots,N-1} (\delta t^{n+\frac{1}{2}})$.

A sequence of space-time gradient discretisation $(D_m)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming) if its spatial component $(D_m^s)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming).

Definition 2.8 (Space-time consistency) *A sequence $(D_m)_{m\in\mathbb{N}}$ of space-time gradient discretisations in the sense of Definition 2.7 is said* ***consistent*** *if*

1. *$(D_m^s)_{m\in\mathbb{N}}$ is consistent in the sense of Definition 2.4,*
2. *for all $\varphi \in \boldsymbol{L}^2(\Omega)$, $\Pi_{D_m} J_{D_m} \varphi \to \varphi$ in $\boldsymbol{L}^2(\Omega)$,*
3. *$\delta t_{D_m} \to 0$ as $m \to \infty$.*

3 Gradient schemes and main results

3.1 Steady Stokes problem

Our assumptions for the steady Stokes problem (1) are the following:

$$\begin{array}{l} \Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d \ (d \geq 1), \\ f \in \boldsymbol{L}^2(\Omega),\ G \in \boldsymbol{L}^2(\Omega)^d \text{ and } \eta \in [0, +\infty). \end{array} \tag{5}$$

Definition 3.1 (Weak solution to the steady Stokes problem) *Under Hypotheses (5), $(\overline{u}, \overline{p})$ is a weak solution to (1) if*

$$\left\{ \begin{array}{l} \overline{u} \in \boldsymbol{H}_0^1(\Omega),\ \overline{p} \in L_0^2(\Omega), \\ \eta \displaystyle\int_\Omega \overline{u} \cdot \overline{v}\,\mathrm{d}x + \int_\Omega \nabla\overline{u} : \nabla\overline{v}\,\mathrm{d}x - \int_\Omega \overline{p}\mathrm{div}\overline{v}\,\mathrm{d}x = \int_\Omega (f \cdot \overline{v} + G : \nabla\overline{v})\,\mathrm{d}x,\ \forall \overline{v} \in \boldsymbol{H}_0^1(\Omega), \\ \displaystyle\int_\Omega q\mathrm{div}\overline{u}\,\mathrm{d}x = 0\,,\ \forall q \in L_0^2(\Omega), \end{array} \right. \tag{6}$$

where "$\cdot$" is the dot product on $\mathbb{R}^d$, and if $\xi = (\xi_{i,j})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}$ and $\chi = (\chi_{i,j})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}$, $\xi : \chi = \sum_{i,j=1}^d \xi_{i,j}\chi_{i,j}$ is the doubly contracted product on $\mathbb{R}^{d\times d}$.

Remark 3.2 *Under Hypotheses* (5)*, the existence and uniqueness of a weak solution* $(\overline{u},\overline{p})$ *to Problem* (1) *in the sense of Definition 3.1 follows from [24, Ch.I, Theorem 2.1].*

The gradient scheme for the steady Stokes problem is based on a discretisation of the weak formulation (6), in which the continuous spaces and operators are replaced with discrete ones (in (6), we wrote the property "$\mathrm{div}\overline{u}=0$" using test functions to make clearer this parallel between the weak formulation and the gradient scheme). If D is a gradient discretisation in the sense of Definition 2.1, the scheme is given by:

$$\left\{\begin{array}{l} u\in X_{D,0}\,,\ p\in Y_{D,0},\\ \eta\displaystyle\int_\Omega \Pi_D u\cdot\Pi_D v\,\mathrm{d}x+\int_\Omega \nabla_D u:\nabla_D v\ x\\ \qquad\qquad -\displaystyle\int_\Omega \chi_D p\ \mathrm{div}_D v\,\mathrm{d}x=\int_\Omega (f\cdot\Pi_D v+G:\nabla_D v)\ \mathrm{d}x,\ \forall v\in X_{D,0},\\ \displaystyle\int_\Omega \chi_D q\mathrm{div}_D u\,\mathrm{d}x=0,\ \ \forall q\in Y_{D,0}. \end{array}\right. \tag{7}$$

Our main result on the gradient schemes for steady Stokes problem is the following theorem.

Theorem 3.3 (Error estimates the steady Stokes problem) *Under Hypotheses* (5)*, let* $(\overline{u},\overline{p})$ *be the unique solution of the incompressible steady Stokes problem* (1) *in the sense of Definition 3.1. Let* D *be a gradient discretisation in the sense of Definition 2.1 such that* $\beta_D>0$ *(see Definition 2.3). Then there exists a unique* $(u_D,p_D)\in X_{D,0}\times Y_{D,0}$ *solution of the gradient scheme* (7)*, and there exists* $C_e>0$*, non-decreasing w.r.t.* η*,* C_D *and* $\frac{1}{\beta_D}$*, such that*

$$\|\overline{u}-\Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla\overline{u}-\nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d}+\|\overline{p}-\chi_D p\|_{L^2(\Omega)}$$
$$\leq C_e\left(\overline{W}_D(\nabla\overline{u}-G,\overline{p})+S_D(\overline{u})+\widetilde{S}_D(\overline{p})\right). \tag{8}$$

Remark 3.4 *As a consequence, if* $(D_m)_{m\in\mathbb{N}}$ *is a coercive, consistent and limit-conforming sequence of gradient discretisations (see Definitions 2.3, 2.4 and 2.5) and if* (u_m,p_m) *are the solutions to the corresponding gradient schemes, then, as* $m\to\infty$*,* $\Pi_{D_m}u_m\to\overline{u}$ *in* $\boldsymbol{L}^2(\Omega)$*,* $\nabla_{D_m}u_m\to\nabla\overline{u}$ *in* $\boldsymbol{L}^2(\Omega)^d$ *and* $\chi_{D_m}p_m\to\overline{p}$ *in* $L^2(\Omega)$*.*

The constant C_e in the preceding estimate explodes if β_D becomes small. For some gradient schemes, we can obtain an estimate on the velocity which is independent on the constant in the inf-sup condition. For a gradient discretisation D, we define the space of discrete divergence-free functions (in the dual sense), discrete version of the space $E(\Omega)$, by

$$E_D=\left\{v\in X_{D,0}\ :\ \forall q\in Y_{D,0}\,,\ \int_\Omega \chi_D q\,\mathrm{div}_D v\,\mathrm{d}x=0\right\}. \tag{9}$$

Theorem 3.5 (Error estimates on the velocity without inf-sup constant) *Under the assumptions of Theorem 3.3, we suppose that*

$$\begin{array}{c}\forall v\in X_{D,0}\,,\ \ \textit{if}\ \displaystyle\int_\Omega \chi_D q\mathrm{div}_D v\,\mathrm{d}x=0\ \textit{for all}\ q\in Y_{D,0},\ \textit{then}\ \mathrm{div}_D v=0\ \textit{a.e. in}\ \Omega\\ \textit{(that is to say,}\ E_D=\{v\in X_{D,0}\ :\ \mathrm{div}_D v=0\ \textit{a.e.}\}\textit{).}\end{array} \tag{10}$$

Then

$$\|\nabla\overline{u}-\nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d}\leq(\eta C_D+2)S_{D,E_D}(\overline{u})+\overline{W}_D(\nabla\overline{u}-G,\overline{p}) \tag{11}$$

$$\|\overline{u}-\Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)}\leq\left[C_D(\eta C_D+1)+1\right]S_{D,E_D}(\overline{u})+C_D\overline{W}_D(\nabla\overline{u}-G,\overline{p}), \tag{12}$$

where $S_{D,E_D}(\overline{u})=\min_{v\in E_D}\left(\|\Pi_D v-\overline{u}\|_{\boldsymbol{L}^2(\Omega)}+\|\nabla_D v-\nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d}\right)$*.*

Remark 3.6 *Most classical schemes for Stokes problem satisfy* (10) *and have interpolants* $E(\Omega) \to E_D$ *which ensure that* $S_{D,E_D}(\overline{u}) \to 0$ *as the mesh size tend to* 0*. This is for example the case of all schemes presented in Section 4.1.*

Estimates (11) and (12) on the discrete velocity still depend on the continuous pressure $\overline{p}$. This means that even in the case of purely irrotational forces, with the solution to the Stokes equation $(\overline{u}, \overline{p}) = (0, \overline{p})$, the pressure terms can lead to errors on the velocity [19]. This dependency on the pressure can be sometimes be removed.

Theorem 3.7 (Pressure-independent error-estimates on the velocity) *Under the assumptions of Theorem 3.3, we suppose that*

$$\forall v \in E_D\,,\ \forall \psi \in H^1(\Omega)\,,\ \int_\Omega \Pi_D v \cdot \nabla \psi \,\mathrm{d}x = 0. \tag{13}$$

Then, if $\nabla \overline{u} - G \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$ *(which amounts to asking that* $\overline{p} \in H^1(\Omega)$*), we have the following pressure-independent estimates on the velocity:*

$$\|\nabla \overline{u} - \nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d} \le (\eta C_D + 2) S_{D,E_D}(\overline{u}) + W_D(\nabla \overline{u} - G) \tag{14}$$

$$\|\overline{u} - \Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} \le [C_D(\eta C_D + 1) + 1]\, S_{D,E_D}(\overline{u}) + C_D W_D(\nabla \overline{u} - G), \tag{15}$$

where W_D *is given in Definition 6.1.*

Remark 3.8 *In case of purely irrotational forces* $(f, G) = (\nabla V, 0)$*, then the solution to the Stokes problem is* $(\overline{u}, \overline{p}) = (0, V)$ *and Estimates* (14) *and* (15) *show that the velocity is exactly approximated. In other words, for such irrotational forces, the discrete velocity provided by the scheme is zero.*

Remark 3.9 *Assumption* (13) *is obviously satisfied by conforming methods, such as the Taylor–Hood scheme (cf. Section 4.2). It is not satisfied in general by non-conforming methods such as the Crouzeix-Raviart scheme, when* Π_D *is the "classical" reconstruction of function (see Section 4.3). As suggested in [19], a way to solve these schemes' poor mass conservation (arising from the action at the discrete level of purely irrotational forces on the velocity) is to replace* Π_D *with a non-standard reconstruction which satisfies* (13)*. The reconstruction proposed in [19] consists in defining* Π_D *as an interpolation in the lowest order Raviart-Thomas space of functions in the Crouzeix-Raviart space: if* v *is a function in the non-conforming* $\mathbb{P}_1^d$ *space and* $(v_\sigma)_{\sigma\in\mathcal{E}}$ *(*$\mathcal{E}$ *being the set of all faces) are its values at the centers of gravity of the faces* $(x_\sigma)_{\sigma\in\mathcal{E}}$*, then* $\Pi_D v = \Pi_D^{\mathrm{RT}} v$ *is the function in the lowest order Raviart-Thomas space which satisfies, for any simplicial cell* T *and any edge* σ *of* T*,*

$$(\Pi_D v)(x_\sigma) \cdot \mathbf{n}_{T,\sigma} = v(x_\sigma) \cdot \mathbf{n}_{T,\sigma},$$

where $\mathbf{n}_{T,\sigma}$ *is the outer unit normal to* T *on* σ*. Then* (13) *is satisfied. Indeed,* $\Pi_D^{\mathrm{RT}} v$ *is* H_{div} *conforming and satisfies* $\mathrm{div}(\Pi_D^{\mathrm{RT}} v) = \mathrm{div}_D v$*, where* $\mathrm{div}_D v$ *is the broken piecewise constant divergence of* v*. Hence, if* $v \in E_D$ *we have* $\mathrm{div}(\Pi_D^{\mathrm{RT}} v) = 0$ *in* Ω *and* (13) *holds.*

3.2 Transient Stokes problem

We consider the transient Stokes problem (2) under the assumptions

$$\begin{array}{l} \Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d\ (d \ge 1),\ T > 0, \\ \overline{u}_{\mathrm{ini}} \in \boldsymbol{L}^2(\Omega),\ f \in \boldsymbol{L}^2(\Omega \times (0,T)) \text{ and } G \in \boldsymbol{L}^2(\Omega \times (0,T))^d. \end{array} \tag{16}$$

The solution to (2) is initially understood in the following weak sense, in which the pressure is eliminated by the choice of divergence-free test functions. Existence and uniqueness of this solution is proved in [24, Ch.III, Theorem 1.1].

Definition 3.10 *Under Hypothesis* (16)*,* $\overline{u}$ *is a weak solution to* (2) *if*

$$\left\{\begin{array}{l} \overline{u} \in L^2(0,T,E(\Omega)), \\ \displaystyle\int_0^T \int_\Omega -\overline{u}\cdot\partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \overline{u}_{\mathrm{ini}}\cdot\varphi(\cdot,0)\,\mathrm{d}x + \int_0^T\int_\Omega \nabla\overline{u} : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t \\ \qquad \displaystyle + \int_0^T\int_\Omega G : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t,\ \forall\varphi = \theta w \text{ with } \theta\in C_c^\infty([0,T)) \text{ and } w\in E(\Omega). \end{array}\right. \tag{17}$$

It can however be seen, see Section 6.2 in the appendix, that if $u_{\mathrm{ini}} \in E(\Omega)$ and $\overline{u}$ is the solution to (17), then there exists a pressure $\overline{p}$ such that $(\overline{u},\overline{p})$ is a solution to (2) in the following sense.

Proposition 3.11 *Assume Hypotheses* (16) *and* $u_{\mathrm{ini}} \in E(\Omega)$ *and let* $\overline{u}$ *be the solution to* (17)*. Then there exists* $\overline{p}$ *such that* $(\overline{u},\overline{p})$ *satisfies:*

$$\left\{\begin{array}{l} \overline{u}\in L^2(0,T,E(\Omega))\cap C([0,T];\boldsymbol{L}^2(\Omega))\,,\ \partial_t\overline{u}\in L^2(0,T,\boldsymbol{H}^{-1}(\Omega))\,, \\ \overline{p}\in L^2(0,T,L^2_0(\Omega))\,, \\ \displaystyle\int_0^T\int_\Omega \langle\partial_t\overline{u},\varphi\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \nabla\overline{u}:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \overline{p}(x,t)\mathrm{div}\varphi\,\mathrm{d}x\,\mathrm{d}t \\ \qquad \displaystyle = \int_0^T\int_\Omega f\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega G:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\,,\ \forall\varphi\in L^2(0,T;\boldsymbol{H}^1_0(\Omega)) \\ \overline{u}(\cdot,0) = \overline{u}_{\mathrm{ini}} \text{ a.e on } \Omega \end{array}\right. \tag{18}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $\boldsymbol{H}^{-1}(\Omega)$ *and* $\boldsymbol{H}^1_0(\Omega)$*.*

Let D be a space-time gradient discretisation in the sense of Definition (2.7). With the notation

$$\delta_D^{n+\frac12}u_D = \frac{u_D^{(n+1)} - u_D^{(n)}}{\delta t^{n+\frac12}},$$

the implicit gradient scheme for (2) is based on the following approximation of (18):

$$\left\{\begin{array}{l} u_D = (u_D^{(n)})_{n=0,\dots,N}\,,\ p_D = (p_D^{(n)})_{n=1,\dots,N} \text{ such that } u_D^{(0)} = J_D u_{\mathrm{ini}} \text{ and, } \forall n = 0,\dots,N-1: \\ u_D^{(n+1)}\in X_{D,0}\,,\ p_D^{(n+1)}\in Y_{D,0}\,, \\ \displaystyle\int_\Omega \Pi_D\delta_D^{n+\frac12}u_D\cdot\Pi_D v\,\mathrm{d}x + \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v\,\mathrm{d}x - \int_\Omega \chi_D p_D^{(n+1)}\mathrm{div}_D v\,\mathrm{d}x \\ \qquad \displaystyle = \frac{1}{\delta t^{n+\frac12}}\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega f\cdot\Pi_D v\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\delta t^{n+\frac12}}\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega G\cdot\nabla_D v\,\mathrm{d}x\,\mathrm{d}t,\quad \forall v\in X_{D,0}. \\ \displaystyle\int_\Omega \mathrm{div}_D u_D^{(n+1)}\chi_D q\,\mathrm{d}x = 0,\quad \forall q\in Y_{D,0}. \end{array}\right. \tag{19}$$

It is common to use Π_D and ∇_D to denote space-time functions the following way: if $v = (v^n)_{n=0,\dots,N} \in X_{D,0}$, the functions $\Pi_D v : \Omega\times(0,T)\to\mathbb{R}^d$ and $\nabla_D v : \Omega\times(0,T)\to\mathbb{R}^{d\times d}$ are defined by

$$\begin{array}{l} \forall n = 0,\dots,N-1\,,\ \forall t\in(t^{(n)},t^{(n+1)}]\,,\ \forall x\in\Omega\,, \\ \Pi_D v(x,t) = \Pi_D v^{(n+1)}(x)\,,\ \nabla_D v(x,t) = \nabla_D v^{(n+1)}(x) \text{ and } \delta_D v(t) = \delta_D^{n+\frac12}v. \end{array} \tag{20}$$

Our first convergence result deals only with the velocity.

Theorem 3.12 (Convergence of the velocity for the transient Stokes problem)
Under Hypotheses (16)*, let* $\overline{u}$ *the unique weak solution of the incompressible transient Stokes problem* (2) *in the sense of Definition 3.10 and let* $(D_m)_{m\in\mathbb{N}}$ *be a sequence of space-time gradient discretisations in the sense of Definition 2.7, which is space-time consistent, limit-conforming and coercive in the sense of Definitions 2.8, 2.5 and 2.3. Then for any* m *there is a unique solution* (u_{D_m}, p_{D_m}) *to* (19) *with* $D = D_m$ *and, as* $m\to\infty$*,*

- $\Pi_{D_m} u_{D_m}$ *converges to* $\overline{u}$ *in* $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$,
- $\nabla_{D_m} u_{D_m}$ *converges to* $\nabla\overline{u}$ *in* $\boldsymbol{L}^2(\Omega\times(0,T))^d$.

Remark 3.13 *Note that since the functions* $\Pi_{D_m} u_{D_m}$ *are piecewise constant in time, their convergence in* $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ *is actually a uniform-in-time convergence (not "uniform a.e. in time").*

Under slightly more restrictive (but usual) conditions on the initial data, we can also prove a convergence result on the pressure.

Theorem 3.14 (Convergence of the pressure for the transient Stokes problem) *Under the assumptions and notations of Theorem 3.12, we suppose that* $G=0$, $u_{\rm ini}\in E(\Omega)$, $(||J_{D_m}u_{\rm ini}||_{D_m})_{m\in\mathbb{N}}$ *is bounded and, for all* $m\in\mathbb{N}$, $J_{D_m}u_{\rm ini}\in E_{D_m}$ *(where* E_{D_m} *is defined by* (9) *with* $D=D_m^s$*, the spatial gradient discretisation corresponding to* D_m*). Then*

- $\Pi_{D_m} u_{D_m}$ *converges to* $\overline{u}$ *in* $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$,
- $\nabla_{D_m} u_{D_m}$ *converges to* $\nabla\overline{u}$ *in* $\boldsymbol{L}^2(\Omega\times(0,T))^d$,
- $\chi_{D_m} p_{D_m}$ *weakly converges to* $\overline{p}$ *in* $L^2(\Omega\times(0,T))$,

where $(\overline{u},\overline{p})$ *is the weak solution to* (2) *in the sense* (18).

4 Examples of gradient discretisations

4.1 The MAC scheme on rectangular meshes

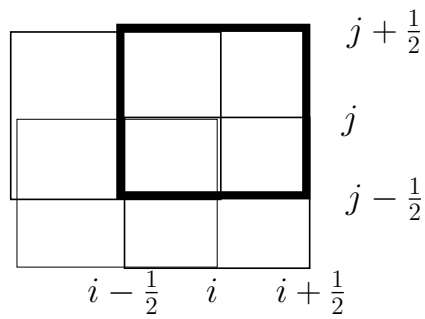


Figure 1: Notations for the meshes

The Marker-And-Cell (MAC) scheme [18, 22, 26] can be easily defined on domains where the boundary is composed of subparts parallel to the axes. Let us assume, for simplicity, that $\Omega=(a,b)\times(c,d)$, for given real values $a<b$, $c<d$. We introduce, for given $N,M\in\mathbb{N}^\star$, the finite real sequences $x_{\frac12}=a<x_{1+\frac12}\ldots<b=x_{N+\frac12}$ and $y_{\frac12}=c<y_{1+\frac12}\ldots<d=y_{M+\frac12}$. We define $x_0=a$, $x_i=\frac12(x_{i-\frac12}+x_{i+\frac12})$ for $i=1,\ldots,N$, $x_{N+1}=b$, $y_0=c$, $y_j=\frac12(y_{j-\frac12}+y_{j+\frac12})$ for $j=1,\ldots,M$, $y_{M+1}=d$. We then define a gradient discretization $D=(X_{D,0},Y_D,\Pi_D,\chi_D,\nabla_D,\mathrm{div}_D)$ by:

1. $X_{D,0}$ is the set of families of reals $u=((u_{i+\frac12,j})_{i=0,\ldots,N,j=0,\ldots,M+1},(u_{i,j+\frac12})_{i=0,\ldots,N+1,j=0,\ldots,M})$ such that $u_{\frac12,j}=u_{N+\frac12,j}=u_{i,\frac12}=u_{i,M+\frac12}=0$ for all i,j,
2. Y_D is the set of all families of reals $p=(p_{i,j})_{i=1,\ldots,N,j=1,\ldots,M}$,

3. $\Pi_D u = (\Pi_D^{(1)} u, \Pi_D^{(2)} u) \in L^2(\Omega)^2$ with $\Pi_D^{(1)} u$ piecewise constant equal to $u_{i+\frac{1}{2},j}$ in $(x_i, x_{i+1}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 0, \ldots, N$ and $j = 1, \ldots, M$, and $\Pi_D^{(2)} u$ piecewise constant equal to $u_{i,j+\frac{1}{2}}$ in $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_j, y_{j+1})$ for $i = 1, \ldots, N$ and $j = 0, \ldots, M$ (this definition accounts for the boundary conditions on the velocity),

4. $\chi_D p$ is piecewise constant equal to $p_{i,j}$ in $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$,

5. $\nabla_D u = (\nabla_D^{(a,b)} u)_{a,b=1,2} \in L^2(\Omega)^4$ with $\nabla_D^{(a,b)} u$ the piecewise constant approximation of the b-th derivative of the a-th component defined by:

   (a) $\nabla_D^{(1,1)} u = \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}$ on $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$,

   (b) $\nabla_D^{(1,2)} u = \frac{u_{i+\frac{1}{2},j+1} - u_{i+\frac{1}{2},j}}{y_{j+1} - y_j}$ on $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ for $i = 0, \ldots, N$ and $j = 0, \ldots, M$,

   (c) $\nabla_D^{(2,1)} u = \frac{u_{i+1,j+\frac{1}{2}} - u_{i,j+\frac{1}{2}}}{x_{i+1} - x_i}$ on $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ for $i = 0, \ldots, N$ and $j = 0, \ldots, M$,

   (d) $\nabla_D^{(2,2)} u = \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}}$ on $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$,

6. $\mathrm{div}_D u = \mathrm{Tr}(\nabla_D u) = \nabla_D^{(1,1)} u + \nabla_D^{(2,2)} u$ (constant in $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$).

We then denote $h_D = \max_{i,j}(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})$ and we have the following result.

Proposition 4.1 (Coercivity, consistency and limit-conformity of the MAC discretization) *Let $D_m = (X_{D_m,0}, Y_{D_m}, \Pi_{D_m}, \chi_{D_m}, \nabla_{D_m}, \mathrm{div}_{D_m})$ be defined as in the beginning of this section, with h_{D_m} tending to 0 as $m \to \infty$. Then D_m is a gradient discretisation in the sense of Definition 2.1 and the family $(D_m)_{m\in\mathbb{N}}$ is coercive, consistent and limit-conforming in the sense of Definitions 2.3, 2.4 and 2.5.*

Proof
We drop the indices m for legibility.
Coercivity: Since the definition of ∇_D corresponds to the discrete gradient of a finite volume scheme on a mesh satisfying the usual orthogonality property, the bound on C_D is a consequence of the discrete Poincaré inequality [12, Lemma 9.1 p 765] (the control of div_D by ∇_D is trivial from its definition). The lower bound on β_D is a consequence of Nečas' result [20], see also [2]. Indeed, we can find $w \in \boldsymbol{H}_0^1(\Omega)$ such that $\mathrm{div} w = \chi_D q$ and $\|w\|_{\boldsymbol{H}_0^1(\Omega)} \le C\|\chi_D q\|_{\boldsymbol{L}^2(\Omega)}$. Considering then $v \in X_{D,0}$ defined by averaging this function w on all edges and applying [12, Lemma 9.4 p 776] provides the result.

Consistency: The consistency for the pressure stems from the fact that, given a family of meshes whose size tend to 0, any L^2 function can be approximated by sequences of piecewise constant functions on the meshes. The consistency for the velocity is equally immediate, since Taylor expansions show that, for a regular φ, the interpolation $u \in X_{D,0}$ defined by $u_{i+\frac{1}{2},j} = \varphi(x_{i+\frac{1}{2}}, y_j)$ and $u_{i,j+\frac{1}{2}} = \varphi(x_i, y_{j+\frac{1}{2}})$ has a reconstruction $\Pi_D u$ and a discrete gradient $\nabla_D u$ close respectively to φ and $\nabla\varphi$ if the mesh size is small.

Limit-conformity: We will use Definition 6.1 (and Remark 6.2). We start by taking $\varphi \in \mathbf{C}^\infty(\overline{\Omega})^2$ and we show that $W_D(\varphi) \to 0$ as $h_D \to 0$. The study is simplified by considering each component of the

gradient separately. For $u \in X_{D,0}$ we have

$$\int_\Omega \nabla_D^{(1,1)} u\, \varphi^{(1,1)}\, \mathrm{d}x\, \mathrm{d}y = \sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} \int_{(x_{i-\frac12},x_{i+\frac12})\times(y_{j-\frac12},y_{j+\frac12})} \frac{(u_{i+\frac12,j}-u_{i-\frac12,j})\varphi^{(1,1)}(x,y)}{x_{i+\frac12}-x_{i-\frac12}}\, \mathrm{d}x\, \mathrm{d}y$$
$$= \sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} (u_{i+\frac12,j}-u_{i-\frac12,j})\varphi_{i,j}(y_{j+\frac12}-y_{j-\frac12})$$

where $\varphi_{i,j}$ is the average of $\varphi^{(1,1)}$ on $(x_{i-\frac12},x_{i+\frac12})\times(y_{j-\frac12},y_{j+\frac12})$. Using a discrete integration by parts and the Dirichlet boundary conditions, we may write

$$\sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} (u_{i+\frac12,j}-u_{i-\frac12,j})\varphi_{i,j}(y_{j+\frac12}-y_{j-\frac12}) = -\sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} u_{i+\frac12,j}\,\frac{\varphi_{i+1,j}-\varphi_{i,j}}{x_{i+\frac12}-x_{i-\frac12}}(y_{j+\frac12}-y_{j-\frac12})(x_{i+\frac12}-x_{i-\frac12})$$
$$\le -\int_\Omega \Pi_D^{(1)} u\, \partial_x\varphi^{(1,1)}\, \mathrm{d}x\, \mathrm{d}y + Ch_D\|u\|_D$$

where C only depends on φ and C_D. We used the fact that $\varphi_{i,j} = \varphi^{(1,1)}(x_i,y_j)+O((x_{i+\frac12}-x_{i-\frac12})(y_{j+\frac12}-y_{j-\frac12}))$, which implies $\frac{\varphi_{i+1,j}-\varphi_{i,j}}{x_{i+\frac12}-x_{i-\frac12}} = \partial_x\varphi^{(1,1)}(x_i,y_j)+\mathcal{O}(y_{j+\frac12}-y_{j-\frac12}) = \partial_x\varphi^{(1,1)}+\mathcal{O}(h_D)$ on $(x_i,x_{i+1})\times(y_{j-\frac12},y_{j+\frac12})$. Combining the last inequation with (**??**), we finally get

$$\int_\Omega (\nabla_D^{(1,1)} u\, \varphi^{(1,1)} + \Pi_D^{(1)} u\, \partial^{(1)}\varphi^{(1,1)})\, \mathrm{d}x\, \mathrm{d}y \le Ch_D||u||_D.$$

Using the same idea for each component of ∇_D, we can conclude that $W_D(\varphi)\to 0$ as $m\to\infty$.
We now have to prove that $\widetilde{W}_D(\psi)\to 0$ for all $\psi\in C^\infty(\overline{\Omega})$. Since $\mathrm{div}_D = \nabla_D^{(1,1)}+\nabla_D^{(2,2)}$, defining $\varphi\in \mathbf{C}^\infty(\overline{\Omega})^2$ by $\varphi^{(1,1)}=\varphi^{(2,2)}=\psi$ and $\varphi^{(1,2)}=\varphi^{(2,1)}=0$ we see that $\widetilde{W}_D(\psi)=W_D(\varphi)$, and the previously established limit-conformity of the discrete gradient therefore shows that the discrete divergence is also limit-conforming. □

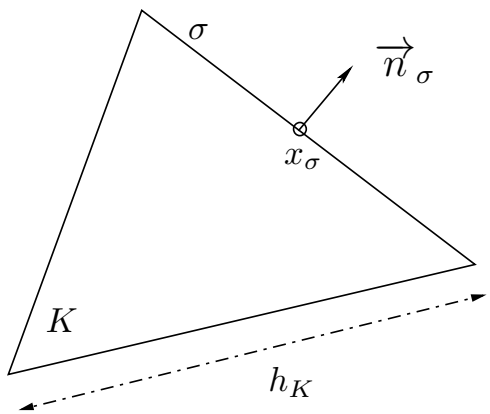
4.2 Conforming Taylor–Hood scheme

In this section, we assume that Ω is an open bounded domain with a polygonal boundary. We consider simplicial meshes $\mathcal{T}$, i.e. meshes made of triangles in 2D or tetrahedra in 3D – see Figure 2 for some general notations.
We let $h_\mathcal{T} = \max_{K\in\mathcal{T}} h_K$ be the maximum diameter of the simplices in $\mathcal{T}$. We define $\theta_\mathcal{T} = \inf\{\frac{\xi_K}{h_K}, K\in\mathcal{T}\}$, where ξ_K is the diameter of the largest ball included in K, and we say that $(\mathcal{T}_m)_{m\in\mathbb{N}}$ is an admissible sequence of triangulations if $(\theta_{\mathcal{T}_m})_{m\in\mathbb{N}}$ remains bounded. This implies in particular $h_K|\sigma|\le C|K|$, for any $K\in\mathcal{T}_m$ and any edge (face in 3D) σ of K, with C not depending on m. Finally, we define $\mathcal{E}$ the set of the edges/faces of $\mathcal{T}$, $\mathcal{E}_{\mathrm{int}} = \mathcal{E}\setminus\partial\Omega$ and $\mathcal{E}_K$ the set of edges of K.
The Taylor–Hood scheme [23] on a simplicial mesh $\mathcal{T}$ can be seen as the gradient scheme corresponding to the gradient discretisation $D = (X_{D,0}, Y_D, \Pi_D, \chi_D, \nabla_D, \mathrm{div}_D)$ defined by:

1. $X_{D,0}$ and Y_D are the vector spaces of the degrees of freedom for the velocity and the pressure in the Taylor–Hood element (that d-dimension vectors at the internal vertices and edge mid-points for $X_{D,0}$, and scalar values at the cell centres for $Y_{D,0}$),

2. Π_D and χ_D are respectively obtained through the $\mathbb{P}^2$ and $\mathbb{P}^1$ finite element basis functions,

3. ∇_D and div_D are the conforming operators $\nabla_D = \nabla\circ\Pi_D$ and $\mathrm{div}_D = \mathrm{div}\circ\Pi_D$.



K is an element of $\mathcal{T}$,
$|K|$ is the d-dimension measure of K,
σ is an edge/face of K,
$|\sigma|$ is the $(d-1)$-dimension measure of σ,
x_σ is the center of gravity of σ,
$\mathbf{n}_{K,\sigma}$ is the unit normal to σ, pointing outward K,
h_K is the diameter of K.

Figure 2: Notations in a triangle.

Proposition 4.2 (Coercivity, consistency and limit-conformity) *Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be an admissible sequence of triangulations of Ω. We assume that every mesh element has at least d edges in Ω and that $h_{\mathcal{T}_m} \to 0$ as $m \to \infty$. Let $D_m = (X_{D_m,0}, Y_{D_m}, \Pi_{D_m}, \chi_{D_m}, \nabla_{D_m}, \mathrm{div}_{D_m})$ corresponding to the conforming Taylor–Hood scheme for $\mathcal{T}_m$. Then D_m is a gradient discretisation in the sense of Definition 2.1 and the family $(D_m)_{m\in\mathbb{N}}$ is coercive, consistent and limit-conforming in the sense of Definitions 2.3, 2.4 and 2.5.*

Proof
Coercivity: Since $\Pi_{D_m}(X_{D_m,0})$ is the set of continuous, piecewise $\mathbb{P}^2$ functions, it is a subset of $H_0^1(\Omega)$ and the Poincaré inequality $||\Pi_{D_m}u||_{\boldsymbol{L}^2(\Omega)} \le \mathrm{diam}(\Omega)||\nabla(\Pi_{D_m}u)||_{\boldsymbol{L}^2(\Omega)^d}$ is therefore satisfied. Applying then [10, Lemma 4.24] to estimate β_{D_m}, we obtain the coercivity of $(D_m)_{m\in\mathbb{N}}$.

Consistency: The consistency is proved in [4, Theorem 3.1.6] in the general case of finite element $\mathbb{P}^k$, thus we just apply this result with $k = 2$ for the discrete velocity and $k = 1$ for the discrete pressure.

Limit-conformity: Because of the definition of ∇_{D_m} and div_{D_m}, $\overline{W}_{D_m}$ is identically null. □

4.3 The Crouzeix-Raviart scheme

We still consider a simplicial mesh $\mathcal{T}$. The Crouzeix-Raviart scheme [5] can be seen as a gradient scheme with the gradient discretisation defined by:

1. $X_{D,0}$ is the vector space of all families of vectors of $\mathbb{R}^d$ at the center of all internal faces of the mesh,
2. Y_D is the vector space of all families of values in the simplices,
3. The linear mapping $\Pi_D^{\mathrm{CR}} : X_{D,0} \to \boldsymbol{L}^2(\Omega)$ is the nonconforming piecewise affine reconstruction of each component of the velocity,
4. The linear mapping $\chi_D : Y_D \to L^2(\Omega)$ is the piecewise constant reconstruction in the simplices,
5. The linear mapping $\nabla_D^{\mathrm{CR}} : X_{D,0} \to \boldsymbol{L}^2(\Omega)^d$ is the so-called "broken gradient" of the velocity, defined as the piecewise constant field of the velocity's gradients in each simplex,
6. The linear mapping $\mathrm{div}_D^{\mathrm{CR}} : X_{D,0} \to L^2(\Omega)$ is the discrete divergence operator, with piecewise constant values in the cells equal to the balance of the normal velocities over the cell's faces.

Proposition 4.3 (Coercivity, consistency and limit-conformity) *Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be an admissible sequence of triangulations of Ω. We define $D_m = (X_{D_m,0}, Y_{D_m}, \Pi^{\rm CR}_{D_m}, \chi_{D_m}, \nabla^{\rm CR}_{D_m}, {\rm div}^{\rm CR}_{D_m})$ as above for $\mathcal{T} = \mathcal{T}_m$, and we assume that $h_{\mathcal{T}_m} \to 0$ as $m \to \infty$. Then D_m is a gradient discretisation in the sense of Definition 2.1 and the family $(D_m)_{m\in\mathbb{N}}$ is coercive, consistent and limit-conforming in the sense of Definitions 2.3, 2.4 and 2.5.*

Proof
Coercivity: Direct consequence of [10, Lemmas 4.30 and 4.31].
Consistency: The consistency for the operators related to the velocity is shown in [17, Theorem 2.1]. The consistency for the interpolation of the pressure is straightforward since, as $h_{\mathcal{T}_m} \to 0$, any function can be approximated in $L^2(\Omega)$ by piecewise constant functions on $\mathcal{T}_m$.

Limit-conformity: We use Proposition 6.4 in the appendix to establish the limit-conformity through W_{D_m} and $\widetilde{W}_{D_m}$ instead of $\overline{W}_{D_m}$. Note that, since smooth functions are dense in $\boldsymbol{H}_{\rm div}(\Omega)^d$ and in $H^1(\Omega)$, we only need to study the convergence of W_{D_m} and $\widetilde{W}_{D_m}$ on smooth functions (see e.g.[8]).

Let us handle W_{D_m} first. To simplify the notations, we drop the index m. We also only consider one component of the discrete velocity $v \in X_{D,0}$ and we therefore treat $\Pi_D v$ as a scalar function and $\nabla_D v$ as a function with values in $\mathbb{R}^d$. Let $\varphi \in \boldsymbol{C}^\infty(\overline{\Omega})$ and let $v \in X_{D,0}$. We have

$$\begin{aligned}\int_\Omega (\nabla^{\rm CR}_D v \cdot \varphi + \Pi^{\rm CR}_D v {\rm div}\varphi)\,{\rm d}x &= \sum_{K\in T_m} \int_K (\nabla^{\rm CR}_D v \cdot \varphi + \Pi^{\rm CR}_D v {\rm div}\varphi)\,{\rm d}x \\ &= \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \varphi \cdot [\Pi^{\rm CR}_D v\ \mathbf{n}]_\sigma\,{\rm d}\gamma(x)\end{aligned}$$

where $[\Pi^{\rm CR}_D v\ \mathbf{n}]_\sigma = \Pi^{\rm CR}_K v\,\mathbf{n}_{K,\sigma} + \Pi^{\rm CR}_L v\ \mathbf{n}_{L,\sigma}$, with K, L the cells on each side of σ and $\Pi^{\rm CR}_K v = (\Pi^{\rm CR}_D v)_{|K}$. Noticing that, for $x \in \sigma$, $\Pi^{\rm CR}_K v(x) = \Pi^{\rm CR}_K v(x_\sigma) + \nabla^{\rm CR}_K v \cdot (x - x_\sigma)$ (with $\nabla^{\rm CR}_K v = (\nabla^{\rm CR}_D v)_{|K}$) and that $\Pi^{\rm CR}_K v(x_\sigma) = \Pi^{\rm CR}_L v(x_\sigma)$, we may write, with obvious notations,

$$\int_\Omega (\nabla^{\rm CR}_D v \cdot \varphi + \Pi^{\rm CR}_D v {\rm div}\varphi)\,{\rm d}x = \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \varphi \cdot [\mathbf{n} \otimes \nabla^{\rm CR}_D v]_\sigma (x - x_\sigma)\,{\rm d}\gamma(x). \tag{21}$$

The smoothness of φ gives, for any $x \in \sigma$, $|\varphi(x) - \varphi(x_\sigma)| \le ||\nabla\varphi||_\infty h_K$. Moreover, since $[\mathbf{n} \otimes \nabla^{\rm CR}_D v]_\sigma$ is constant over σ and x_σ is the center of gravity of σ,

$$\int_\sigma \varphi(x_\sigma) \cdot [\mathbf{n} \otimes \nabla^{\rm CR}_D v]_\sigma (x - x_\sigma)\,{\rm d}\gamma(x) = \varphi(x_\sigma) \cdot [\mathbf{n} \otimes \nabla^{\rm CR}_D v]_\sigma \int_\sigma (x - x_\sigma)\,{\rm d}\gamma(x) = 0.$$

Introducing $\varphi(x_\sigma)$ into (21) and using $h_K|\sigma| \le C|K|$ for all $K \in \mathcal{T}_m$ and all $\sigma \in \mathcal{E}_K$ (consequence of the admissibility of $(\mathcal{T}_m)_{m\in\mathbb{N}}$), we infer that

$$\begin{aligned}\left|\int_\Omega (\nabla^{\rm CR}_D v \cdot \varphi + \Pi^{\rm CR}_D v {\rm div}\varphi)\,{\rm d}x\right| &\le ||\nabla\varphi||_\infty \sum_{\sigma\in\mathcal{E}_{\rm int}} |\sigma|(h_K^2|\nabla_K v| + h_L^2|\nabla_L v|) \\ &\le C||\nabla\varphi||_\infty h_{\mathcal{T}_m} \sum_{K\in\mathcal{T}_m} |\nabla_K v| \sum_{\sigma\in\mathcal{E}_K} |K| \\ &= (d+1)C||\nabla\varphi||_\infty h_{\mathcal{T}_m} ||\nabla_D v||_{L^1(\Omega)^d}.\end{aligned}$$

This shows that $W_{D_m}(\varphi) \to 0$ as $m \to \infty$. We now turn to $\widetilde{W}_{D_m}$. If $\psi \in C^\infty(\overline{\Omega})$ and $v \in X_{D,0}$, we have

$$\int_\Omega (\Pi^{\rm CR}_D v \cdot \nabla\psi + \psi {\rm div}^{\rm CR}_D v)\,{\rm d}x = \sum_{K\in T_m} \left(\int_K ({\rm div}^{\rm CR}_D v - {\rm div}(\Pi^{\rm CR}_D v))\psi\,{\rm d}x + \sum_{\sigma\in\mathcal{E}_K} \int_\sigma \psi[\Pi^{\rm CR}_D v \cdot \mathbf{n}]_\sigma\,{\rm d}\gamma(x)\right).$$

By definition of the piecewise affine reconstruction $\Pi_D^{\rm CR} v$ and of $\mathrm{div}_D^{\rm CR}$, we have

$$\mathrm{div}(\Pi_D^{\rm CR} v) = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| v(x_\sigma) \cdot \mathbf{n}_{K,\sigma} = \mathrm{div}_D^{\rm CR} v \quad \text{ in } K.$$

Hence, $\int_\Omega (\Pi_D^{\rm CR} v \cdot \nabla \psi + \psi \mathrm{div}_D^{\rm CR} v)\,\mathrm{d}x = \sum_{\sigma \in \mathcal{E}_{\rm int}} \int_\sigma \psi [\Pi_D^{\rm CR} v \cdot \mathbf{n}]_\sigma \,\mathrm{d}\gamma(x)$ and, following similar computations as for W_{D_m}, we conclude that $\widetilde{W}_{D_m}(\psi) \to 0$ as $m \to \infty$. $\square$

4.4 An $\boldsymbol{H}_{\rm div}$ conforming scheme

We now turn to a scheme which, in some way, is a synthesis of the MAC scheme and of the Crouzeix-Raviart scheme. On a simplicial mesh, we only keep as velocity unknowns the normal velocities at the barycenters of the faces, instead of all the components. Then the velocity is reconstructed using the Raviart-Thomas $H_{\rm div}$ conforming. The interest of this scheme is multiple.

1. Since it is $H_{\rm div}$ conforming, the velocity field is continuously divergence free, which prevents the method from introducing error terms in the velocity when the source terms are gradients.
2. The number of unknowns is largely reduced compared to the Crouzeix-Raviart scheme.
3. One can change unknowns in the 2D case, using the discrete streamline potential unknown, to eliminate the pressure. This is also possible in the Navier-Stokes framework and it leads to systems of equations that are easier to solve.

The mesh $\mathcal{T}$ is still 2D or 3D simplicial. For each edge (face in 3D) σ, we select one unit normal $\mathbf{n}_\sigma$ to σ and we make the following assumption.

Assumption 4.4 *[Construction of R_σ] We assume that for any $\sigma \in \mathcal{E}_{\rm int}$ between two cells K and L, there exists a set $\mathcal{E}_\sigma$ of 6 edges (in 2D) or 12 faces (in 3D) such that*

1. *$\sigma \in \mathcal{E}_\sigma$,*
2. *the sets $\mathcal{E}_K \setminus \mathcal{E}_\sigma$ and $\mathcal{E}_L \setminus \mathcal{E}_\sigma$ contain at most one element,*
3. *for any real family $(u_{\sigma'})_{\sigma' \in \mathcal{E}_\sigma}$, there exists one and only one affine function $\psi_\sigma : \Omega \to \mathbb{R}^d$ such that $\psi_\sigma(x_{\sigma'}) \cdot \mathbf{n}_{\sigma'} = u_{\sigma'}$.*

Note that a linear function ψ_σ can always be written $\psi_\sigma(x) = \mathbf{v}_\sigma + A_\sigma(x - x_\sigma)$ with $\mathbf{v}_\sigma \in \mathbb{R}^d$ and $A_\sigma \in \mathbb{R}^{d \times d}$, and Assumption 4.4 means that the linear system

$$\psi_\sigma(x_{\sigma'}) \cdot \mathbf{n}_\sigma = (\mathbf{v}_\sigma + A_\sigma(x_{\sigma'} - x_\sigma)) \cdot \mathbf{n}_{\sigma'} = u_{\sigma'} \text{ for all } \sigma' \in \mathcal{E}_\sigma$$

is uniquely solvable for $(\mathbf{v}_\sigma, A_\sigma)$. We then define the operators $R_\sigma u$ and $D_\sigma u$ by

$$R_\sigma u = \mathbf{v}_\sigma = \psi_\sigma(x_\sigma) \quad \text{ and } \quad D_\sigma u = A_\sigma.$$

Let $a_\sigma > 0$ be the minimal real value such that, for all real family $(u_{\sigma'}, \sigma' \in \mathcal{E}_\sigma)$,

$$|R_\sigma u| \le a_\sigma \sum_{\sigma' \in \mathcal{E}_\sigma} |u_{\sigma'}|\,, \quad |D_\sigma u| \le \frac{a_\sigma}{\mathrm{diam}(\sigma)} \sum_{\sigma' \in \mathcal{E}_\sigma} |u_{\sigma'}| \quad \text{ and } \quad \forall \sigma' \in \mathcal{E}_\sigma\,,\ \mathrm{dist}(x_\sigma, \sigma') \le a_\sigma \mathrm{diam}(\sigma).$$

We define $\mathcal{R}_\mathcal{T} = \max\{a_\sigma, \sigma \in \mathcal{E}_{\rm int}\}$.

Then an $\boldsymbol{H}_{\rm div}$ conforming scheme can be defined as a gradient scheme through the following gradient discretisation:

1. $X_{D,0}$ is the vector space on $\mathbb{R}$ of all families of real values approximating the normal component (with respect to the chosen orientations $(\mathbf{n}_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$) of the velocity at the center of gravity x_σ of all internal faces $\sigma \in \mathcal{E}_{\rm int}$ of the mesh (the normal components of the velocity are 0 for all external faces).

2. Y_D is the vector space on $\mathbb{R}$ of all families of constant values in the simplices.

3. The linear mapping $\Pi_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)$ is the Raviart-Thomas reconstruction of the velocity, obtained from the normal components (and taking into account the orientation of $\mathbf{n}_\sigma$). We then have that $\Pi_D u \in \boldsymbol{H}_{\rm div}(\Omega)$ with zero normal trace at the boundary. Note that this reconstruction is not conforming in $\boldsymbol{H}^1_0(\Omega)$.

4. The linear mapping $\chi_D : Y_D \to L^2(\Omega)$ is the piecewise constant reconstruction in the simplices.

5. The linear mapping $\nabla_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)^d$ is the so-called "broken gradient" (Crouzeix-Raviart) of the velocity, defined in the following way: if $u \in X_{D,0}$, we reconstruct in each simplex K an affine velocity equal to $R_\sigma u$ at all points x_σ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm int}$, and equal to 0 at all x_σ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$, and $\nabla_D u$ is equal in K to the gradient of this (Crouzeix-Raviart) reconstruction.

6. The linear mapping $\mathrm{div}_D : X_{D,0} \to L^2(\Omega)$ is the discrete divergence operator defined by $\mathrm{div}_D u = \mathrm{div}(\Pi_D u)$, that is to say $\mathrm{div}_D u$ is the piecewise constant function given on each cell by the balance of the outer normal components of u on the edges/faces of the cell: $(\mathrm{div}_D u)_{|K} = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma|\delta_{K,\sigma} u_\sigma$ where $\delta_{K,\sigma} = 1$ if $\mathbf{n}_\sigma$ points outside K, $\delta_{K,\sigma} = -1$ otherwise, and $u_\sigma = 0$ if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$.

We prove in the following proposition that this gradient discretisation satisfies all the required properties.

Proposition 4.5 (Coercivity, consistency and limit-conformity) *Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be an admissible sequence of triangulations of Ω, such that Assumption 4.4 holds for all $\mathcal{T}_m$. We define the family $D_m = (X_{D_m,0}, Y_{D_m}, \Pi_{D_m}, \chi_{D_m}, \nabla_{D_m}, \mathrm{div}_{D_m})$ as in the introduction of this section, with $\mathcal{T}_m$ instead of $\mathcal{T}$. We assume that $h_{\mathcal{T}_m} \to 0$ as $m \to \infty$ and that $(\mathcal{R}_{\mathcal{T}_m})_{m\in\mathbb{N}}$ remains bounded. Then D_m is a gradient discretisation in the sense of Definition 2.1 and the family $(D_m)_{m\in\mathbb{N}}$ is coercive, consistent and limit-conforming in the sense of Definitions 2.3, 2.4 and 2.5.*

Proof
Coercivity
As is usual, in the following proof the notation C (without index) is used to designate various positive constants which do not depend on the mesh or the selected functions.

Step 1: estimate on C_{D_m}.
Let $v \in X_{D_m,0}$ and let v_σ be the value of $\Pi_{D_m} v$ at x_σ for all $\sigma \in \mathcal{E}$. We let $Rv = (R_\sigma v)_{\sigma\in\mathcal{E}_{\rm int}}$ and we denote by $\Pi^{\rm CR}_{D_m} Rv$ the Crouzeix-Raviart reconstruction from the edge values of Rv.
We drop the index m to simplify the notations. Let $K \in \mathcal{T}$. If we set $Z = (R_\sigma v)_{\sigma\in\mathcal{E}_K}$ then, with the notations of Lemma 4.6 below, $\Pi^{\rm CR} Z = \Pi^{\rm CR}_D Rv$, $\nabla(\Pi^{\rm CR} Z) = \nabla_D v$ and, since $R_\sigma v \cdot \mathbf{n}_{K,\sigma} = \delta_{K,\sigma} v_\sigma$, $\Pi^{\rm RT}(Z\cdot\mathbf{n}) = \Pi_D v$. Hence, taking the square of the estimate in Lemma 4.6, integrating over K and summing on K, we find

$$||\Pi^{\rm CR}_D Rv - \Pi_D v||_{\boldsymbol{L}^2(\Omega)} \le C h_{\mathcal{T}} ||\nabla_D v||_{\boldsymbol{L}^2(\Omega)^d}, \tag{22}$$

which implies

$$||\Pi_D v||_{\boldsymbol{L}^2(\Omega)} \le ||\Pi^{\rm CR}_D Rv||_{\boldsymbol{L}^2(\Omega)} + C||\nabla_D v||_{\boldsymbol{L}^2(\Omega)^d}. \tag{23}$$

We also notice that, on $K \in \mathcal{T}$,

$$\mathrm{div}^{\rm CR}_D(Rv) = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma| R_\sigma v \cdot \mathbf{n}_{K,\sigma} = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma|\delta_{K,\sigma} v_\sigma = \mathrm{div}_D v. \tag{24}$$

We now recall that, from Proposition 4.3,

$$\|\Pi_D^{\rm CR} Rv\|_{\boldsymbol{L}^2(\Omega)} + ||{\rm div}_D^{\rm CR} Rv||_{L^2(\Omega)} \le C\|\nabla_D^{\rm CR} Rv\|_{\boldsymbol{L}^2(\Omega)^d} = C\|\nabla_D v\|_{\boldsymbol{L}^2(\Omega)^d}.$$

Combined with (23) and (24), this concludes the proof that $(C_{D_m})_{m\in\mathbb{N}}$ is bounded.

Step 2: estimate on β_{D_m}, i.e. discrete inf-sup (or LBB) condition.
Once more, we drop the index m. Let $q \in Y_{D,0}$ be given. Thanks to Nečas' result [20, 2], we can find $w \in \boldsymbol{H}_0^1(\Omega)$ such that ${\rm div} w = \chi_D q$ and $\|w\|_{\boldsymbol{H}_0^1(\Omega)} \le C\|\chi_D q\|_{\boldsymbol{L}^2(\Omega)}$. We then consider $v \in X_{D,0}$ such that $v_\sigma = \frac{1}{|\sigma|}\int_\sigma w\cdot \mathbf{n}_\sigma$, which gives ${\rm div}_D v = \chi_D q$ and thus $\int_\Omega \chi_D q {\rm div}_D v \,{\rm d}x = ||\chi_D q||^2_{L^2(\Omega)}$. A lower bound on β_D ensues if we can establish that

$$||\nabla_D v||_{\boldsymbol{L}^2(\Omega)^d} \le C||\chi_D q||_{L^2(\Omega)}. \tag{25}$$

We set $v_K = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} \delta_{K,\sigma} v_\sigma (x_\sigma - x_K)|\sigma|$, where x_K is the centre of gravity of the simplex K. We denote by ψ_σ the basis function of the Crouzeix-Raviart space associated with $\sigma \in \mathcal{E}$. On K we have $\sum_{\sigma\in\mathcal{E}_K} \nabla\psi_\sigma = 0$ and $|\nabla\psi_\sigma| \le C/h_K$, and thus

$$\begin{aligned}\int_\Omega |\nabla_D v|^2 \,{\rm d}x &= \sum_{K\in\mathcal{T}} \int_K |\nabla_D v|^2 \,{\rm d}x = \sum_{K\in\mathcal{T}} \int_K \left|\sum_{\sigma\in\mathcal{E}_K} R_\sigma v \otimes \nabla\psi_\sigma\right|^2 {\rm d}x \\ &= \sum_{K\in\mathcal{T}} \int_K \left|\sum_{\sigma\in\mathcal{E}_K} (R_\sigma v - v_K) \otimes \nabla\psi_\sigma\right|^2 {\rm d}x \le C^2(d+1)\sum_{K\in\mathcal{T}} \frac{|K|}{h_K^2} \sum_{\sigma\in\mathcal{E}_K} |R_\sigma v - v_K|^2. \end{aligned} \tag{26}$$

We then write

$$\begin{aligned}|R_\sigma v - v_K| &= \left| R_\sigma v - \frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K} \delta_{K,\sigma'} v_{\sigma'}(x_{\sigma'} - x_K)|\sigma'|\right| \\ &= \left| R_\sigma v - \frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K\cap\mathcal{E}_\sigma} \delta_{K,\sigma'} v_{\sigma'}(x_{\sigma'} - x_K)|\sigma'| - \frac{1}{|K|}\sum_{\sigma''\in\mathcal{E}_K\setminus\mathcal{E}_\sigma} \delta_{K,\sigma''} v_{\sigma''}(x_{\sigma''} - x_K)|\sigma''|\right|.\end{aligned}$$

Let us first assume that there exists $\sigma'' \in \mathcal{E}_K \setminus \mathcal{E}_\sigma$ (thanks to Assumption 4.4, there is at most one such σ''). We deduce from $({\rm div}_D v)_{|K} = q_K$ that $\delta_{K,\sigma''} v_{\sigma''}|\sigma''| = q_K|K| - \sum_{\sigma'\in\mathcal{E}_K\cap\mathcal{E}_\sigma} \delta_{K,\sigma'} v_{\sigma'}|\sigma'|$, which leads to

$$|R_\sigma v - v_K| \le |q_K| h_K + \left| R_\sigma v - \frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K\cap\mathcal{E}_\sigma} \delta_{K,\sigma'} v_{\sigma'}(x_{\sigma'} - x_{\sigma''})|\sigma'|\right|. \tag{27}$$

This estimate is also valid if $\mathcal{E}_K\setminus\mathcal{E}_\sigma = \emptyset$ (just fix any $\sigma'' \in \mathcal{E}_K$ in the reasoning above, and notice that the term involving $\sigma' = \sigma''$ in the last sum vanishes). For all $\sigma' \in \mathcal{E}_K \cap \mathcal{E}_\sigma$ we have

$$v_{\sigma'} = (R_\sigma v + D_\sigma v(x_{\sigma'} - x_\sigma))\cdot \mathbf{n}_{\sigma'}$$

and, using [6, Lemma 6.1],

$$R_\sigma v = \frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K} R_\sigma v\cdot \mathbf{n}_{K,\sigma'}(x_{\sigma'} - x_{\sigma''})|\sigma'| = \frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K\cap\mathcal{E}_\sigma} R_\sigma v\cdot \mathbf{n}_{\sigma'}\delta_{K,\sigma'}(x_{\sigma'} - x_{\sigma''})|\sigma'|.$$

Equation (27) thus gives

$$|R_\sigma v - v_K| \le |q_K| h_K + \left|\frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K\cap\mathcal{E}_\sigma} D_\sigma v(x_{\sigma'} - x_\sigma)\cdot \mathbf{n}_{\sigma'}(x_{\sigma'} - x_{\sigma''})|\sigma'|\right|.$$

Along an admissible sequence of triangulations we have, for all $\sigma' \in \mathcal{E}_K$, $|\sigma'|h_K \le C|K|$, which leads to $|R_\sigma v - v_K| \le |q_K|h_K + Cd|D_\sigma v|h_K$. Plugging this estimate into (26), we infer

$$\begin{aligned}\int_\Omega |\nabla_D v|^2 \,\mathrm{d}x &\le C \sum_{K\in\mathcal{T}} \frac{|K|}{h_K^2} \sum_{\sigma\in\mathcal{E}_K} (2q_K^2 h_K^2 + 2C^2 d^2 |D_\sigma v|^2 h_K^2) \\ &\le C||\chi_D q||^2_{L^2(\Omega)} + C \sum_{K\in\mathcal{T}} |K| \sum_{\sigma\in\mathcal{E}_K} |D_\sigma v|^2. \end{aligned} \tag{28}$$

For $\sigma \in \mathcal{E}_K$, we define the family $V(\sigma) = (V(\sigma)_{\sigma'})_{\sigma'\in\mathcal{E}_\sigma}$ by $V(\sigma)_{\sigma'} = \frac{1}{|\sigma|}\int_\sigma w \cdot \mathbf{n}_{\sigma'}$. Then $D_\sigma V(\sigma) = 0$ (and $R_\sigma V(\sigma) = \frac{1}{|\sigma|}\int_\sigma w$), and therefore, by definition of a_σ,

$$\begin{aligned}|D_\sigma v|^2 = |D_\sigma(v - V(\sigma))|^2 &\le C \left(\frac{a_\sigma}{\mathrm{diam}(\sigma)} \sum_{\sigma'\in\mathcal{E}_\sigma} \left| v_{\sigma'} - \frac{1}{|\sigma|}\int_\sigma w\cdot\mathbf{n}_{\sigma'} \right| \right)^2 \\ &\le C(d+d^2)\frac{a_\sigma^2}{\mathrm{diam}(\sigma)^2} \sum_{\sigma'\in\mathcal{E}_\sigma} \left| \frac{1}{|\sigma'|}\int_{\sigma'} w\cdot\mathbf{n}_{\sigma'} - \frac{1}{|\sigma|}\int_\sigma w\cdot\mathbf{n}_{\sigma'} \right|^2 \end{aligned} \tag{29}$$

(recall that $\mathrm{Card}(\mathcal{E}_\sigma) = d+d^2$). Using [12, Equation (9.38) p 777] or [6, Lemma 6.3] and the admissibility of the sequence of triangulations (which implies $|\sigma| \le C|\sigma'|$ for all $\sigma, \sigma' \in \mathcal{E}_K$), we have

$$\begin{aligned}\left| \frac{1}{|\sigma'|}\int_{\sigma'} w\cdot\mathbf{n}_{\sigma'} - \frac{1}{|\sigma|}\int_\sigma w\cdot\mathbf{n}_{\sigma'} \right|^2 &\le 2\left| \frac{1}{|\sigma'|}\int_{\sigma'} w - \frac{1}{|K|}\int_K w\,\mathrm{d}x \right|^2 + 2\left| \frac{1}{|\sigma|}\int_\sigma w - \frac{1}{|K|}\int_K w\,\mathrm{d}x \right|^2 \\ &\le \frac{Ch_K}{|\sigma|}\int_K |\nabla w|^2\,\mathrm{d}x. \end{aligned}$$

Combined with (29) and (28), this leads to

$$\int_\Omega |\nabla_D v|^2\,\mathrm{d}x \le C||\chi_D q||^2_{L^2(\Omega)} + C\mathcal{R}_\mathcal{T}^2 \sum_{K\in\mathcal{T}} \int_K |\nabla w|^2\,\mathrm{d}x \sum_{\sigma\in\mathcal{E}_K} \frac{h_K|K|}{\mathrm{diam}(\sigma)^2|\sigma|}.$$

The admissibility of the sequence of triangulations ensures that $\frac{h_K|K|}{\mathrm{diam}(\sigma)^2|\sigma|} \le C$, which allows us to conclude, by choice of w, that

$$\int_\Omega |\nabla_D v|^2\,\mathrm{d}x \le C||\chi_D q||^2_{L^2(\Omega)} + C\mathcal{R}_\mathcal{T}^2||\nabla w||_{\boldsymbol{L}^2(\Omega)} \le C(1+\mathcal{R}_\mathcal{T}^2)||\chi_D q||^2_{L^2(\Omega)}.$$

Estimate (25) is therefore established and the proof of the coercivity of $(D_m)_{m\in\mathbb{N}}$ is complete.

Consistency: the consistency for the pressure is straightforward, as for the Crouzeix-Raviart method (see the proof of Proposition 4.3). To establish the consistency for the velocity, we use Lemma 4.7 below. Following Remark 2.6, we just have to prove that $S_{D_m}(\varphi) \to 0$, as $m \to \infty$, for all $\varphi \in \mathbf{C}_c^\infty(\Omega)$. We will prove that, for such a function and defining $v_m = \mathcal{I}_{D_m}\varphi \in X_{D_m,0}$ as in Lemma 4.7, we have $\Pi_{D_m} v_m \to \varphi$ in $\boldsymbol{L}^2(\Omega)$, $\nabla_{D_m} v_m \to \nabla\varphi$ in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} v_m \to \mathrm{div}\varphi$ in $L^2(\Omega)$ as $m \to \infty$ (which implies $S_{D_m}(\varphi) \to 0$).

By construction, on K we have $\mathrm{div}_{D_m} v_m = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \varphi\cdot\mathbf{n}_{K,\sigma}\,\mathrm{d}s = \frac{1}{|K|}\int_K \mathrm{div}\varphi$. In other words, $\mathrm{div}_{D_m} v_m$ is the L^2 projection of $\mathrm{div}\varphi$ on the set of piecewise constant functions on $\mathcal{T}_m$. Hence the convergence $\mathrm{div}_{D_m} v_m \to \mathrm{div}\varphi$ in $L^2(\Omega)$ as $m \to \infty$.

To study the convergence of the reconstructed function and gradients, we recall the following formula, a vector-value version of [13, Equation (2.16)] (see also [8, Lemma 9.6]): for any linear mapping $\boldsymbol{\xi} : \mathbb{R}^d \mapsto \mathbb{R}^d$, the constant gradient $\nabla\boldsymbol{\xi} \in \mathbb{R}^{d\times d}$ satisfies

$$\nabla\boldsymbol{\xi} = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma|\boldsymbol{\xi}(x_\sigma)\otimes\mathbf{n}_{K,\sigma} \tag{30}$$

(in applying [13, Equation (2.16)] we use the fact that $\sum_{\sigma\in\mathcal{E}_K}|\sigma|\mathbf{n}_{K,\sigma}=0$). Let us take $\overline{x}\in K$ and let us consider the linear mapping $\overline{\varphi}(x)=\varphi(\overline{x})+\nabla\varphi(\overline{x})(x-\overline{x})$. Then by smoothness of φ we have, for any $\sigma\in\mathcal{E}_K$, $|\overline{\varphi}(x_\sigma)-\varphi(x_\sigma)|\le Ch_K^2||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)}$. Combined with (35), this gives

$$|\Pi_{D_m}^{\rm CR}Rv_m(x_\sigma)-\overline{\varphi}(x_\sigma)|\le Ch_K^2||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)}\quad\text{ for all }\sigma\in\mathcal{E}_K. \tag{31}$$

We then consider the linear function $\boldsymbol{\xi}=(\Pi_{D_m}^{\rm CR}Rv_m)_{|K}-\overline{\varphi}$. We have just established that $|\boldsymbol{\xi}(x_\sigma)|\le Ch_K^2||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)}$ for all $\sigma\in\mathcal{E}_K$, and (30) therefore gives

$$|(\nabla_{D_m}v_m)_{|K}-\nabla\varphi(\overline{x})|=|\nabla\boldsymbol{\xi}|\le\frac{Ch_K^2}{|K|}||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)}\sum_{\sigma\in\mathcal{E}_K}|\sigma|\le Ch_{\mathcal{T}_m}||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)} \tag{32}$$

(we used the admissibility of $(\mathcal{T}_m)_{m\in\mathbb{N}}$ to write $h_K|\sigma|\le C|K|$ for all $\sigma\in\mathcal{E}_K$). Since the preceding estimate is valid for any $\overline{x}\in K$, this means that $||\nabla_{D_m}v_m-\nabla\varphi||_{\boldsymbol{L}^\infty(\Omega)^d}\le Ch_{\mathcal{T}}||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)}$, which proves that $\nabla_{D_m}v_m\to\nabla\varphi$ in $\boldsymbol{L}^2(\Omega)^d$ as $m\to\infty$.
The convergence of the Raviart-Thomas reconstruction $\Pi_{D_m}v_m$ is well-known, but it can also be easily inferred from the previous results. For any $x\in K$ and any $\sigma\in\mathcal{E}_K$, we have

$$|\overline{\varphi}(x_\sigma)-\varphi(x)|\le Ch_K||\varphi||_{\mathbf{W}^{1,\infty}(\Omega)}$$

and

$$|\Pi_{D_m}^{\rm CR}Rv_m(x_\sigma)-\Pi_{D_m}^{\rm CR}Rv_m(x)|\le Ch_K|\nabla(\Pi_{D_m}^{\rm CR}Rv_m)_{|K}|\le Ch_K||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)}$$

(we used (32) and $\nabla(\Pi_{D_m}^{\rm CR}Rv_m)=\nabla_{D_m}v_m$). Injecting these estimates into (31) we infer that

$$||\Pi_{D_m}^{\rm CR}Rv_m-\varphi||_{\boldsymbol{L}^\infty(\Omega)}\le Ch_{\mathcal{T}_m}||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)}$$

and the convergence in $\boldsymbol{L}^2(\Omega)$ of $\Pi_{D_m}v_m$ to φ therefore follows from (22) and from the fact that $(\nabla_{D_m}v_m)_{m\in\mathbb{N}}$ remains bounded in $\boldsymbol{L}^2(\Omega)^d$ (it converges in this space to $\nabla\varphi$).
Limit-conformity: we use the equivalent characterisation given in Proposition 6.4. We start by noticing that $\widetilde{W}_{D_m}=0$ since $\mathrm{div}_{D_m}v=\mathrm{div}(\Pi_{D_m}v)$ for any $v\in X_{D_m,0}$. To estimate W_{D_m} we write, for $\varphi\in\boldsymbol{H}_{\rm div}(\Omega)^d$ and $v\in X_{D_m,0}\setminus\{0\}$,

$$\int_\Omega(\nabla_{D_m}v:\varphi+\Pi_{D_m}v\cdot\mathrm{div}\varphi)\,\mathrm{d}x=\int_\Omega(\nabla_{D_m}^{\rm CR}Rv:\varphi+\Pi_{D_m}^{\rm CR}Rv\cdot\mathrm{div}\varphi)\,\mathrm{d}x+\int_\Omega(\Pi_{D_m}v-\Pi_{D_m}^{\rm CR}Rv)\cdot\mathrm{div}\varphi\,\mathrm{d}x.$$

Invoking (22), we infer

$$\begin{aligned}0\le W_{D_m}(\varphi)&\le\max_{v\in X_{D_m,0}}\frac{\displaystyle\int_\Omega(\nabla_{D_m}^{\rm CR}Rv:\varphi+\Pi_{D_m}^{\rm CR}Rv\cdot\mathrm{div}\varphi)\,\mathrm{d}x}{||\nabla_{D_m}v||_{\boldsymbol{L}^2(\Omega)^d}}+\frac{||\Pi_{D_m}v-\Pi_{D_m}^{\rm CR}Rv||_{\boldsymbol{L}^2(\Omega)}}{||\nabla_{D_m}v||_{\boldsymbol{L}^2(\Omega)^d}}||\mathrm{div}\varphi||_{L^2(\Omega)}\\&\le\max_{v\in X_{D_m,0}}\frac{\displaystyle\int_\Omega(\nabla_{D_m}^{\rm CR}Rv:\varphi+\Pi_{D_m}^{\rm CR}Rv\cdot\mathrm{div}\varphi)\,\mathrm{d}x}{||\nabla_{D_m}^{\rm CR}Rv||_{\boldsymbol{L}^2(\Omega)^d}}+Ch_{\mathcal{T}_m}||\mathrm{div}\varphi||_{L^2(\Omega)}.\end{aligned}$$

The first term in this last right-hand side is precisely the function W_{D_m} corresponding to the Crouzeix-Raviart method. By Proposition 4.3, we now that this term tends to 0 as $m\to\infty$, and we conclude that $W_{D_m}(\varphi)\to0$ as required. □

Lemma 4.6 *Let K be a simplex in $\mathbb{R}^d$ and let $Z=(Z_\sigma)_{\sigma\in\mathcal{E}_K}$ be a family of vectors in $\mathbb{R}^d$. We denote by $\Pi^{\rm CR}Z$ is the Crouzeix-Raviart $\mathbb{P}_1$ reconstruction in K from the values Z, and by $\Pi^{\rm RT}(Z\cdot\mathbf{n})$ the Raviart-Thomas $\mathbb{P}_1$ reconstruction in K from the fluxes $(Z_\sigma\cdot\mathbf{n}_{K,\sigma})_{\sigma\in\mathcal{E}_K}$. Then there exists C only depending on*

the shape regularity of K (i.e. on an upper bound of the ratio between the diameter of K and the diameter of the largest ball included in K) such that, for all $x \in K$,

$$|\Pi^{\mathrm{CR}} Z(x) - \Pi^{\mathrm{RT}}(Z \cdot \mathbf{n})(x)| \leq C h_K |\nabla(\Pi^{\mathrm{CR}} Z)|$$

where $\nabla(\Pi^{\mathrm{CR}} Z)$ is the constant gradient of $\Pi^{\mathrm{CR}} Z$ in K.

Proof By definition, $(\Pi^{\mathrm{CR}} Z)(x_\sigma) = Z_\sigma$ for all $\sigma \in \mathcal{E}_K$ and, for $x \in K$,

$$\Pi^{\mathrm{RT}}(Z \cdot \mathbf{n})(x) = \sum_{\sigma \in \mathcal{E}_K} Z_\sigma \cdot \mathbf{n}_{K,\sigma} \frac{x - S_\sigma}{d(S_\sigma, \sigma)} = \sum_{\sigma \in \mathcal{E}_K} (\Pi^{\mathrm{CR}} Z(x_\sigma)) \cdot \mathbf{n}_{K,\sigma} \frac{x - S_\sigma}{d(S_\sigma, \sigma)},$$

where S_σ is the vertex of K opposite to σ. Since $\Pi^{\mathrm{CR}} Z$ is linear, we also have $\Pi^{\mathrm{CR}} Z(x_\sigma) = \Pi^{\mathrm{CR}} Z(x) + \nabla(\Pi^{\mathrm{CR}} Z)(x_\sigma - x)$. Hence,

$$\Pi^{\mathrm{RT}}(Z \cdot \mathbf{n})(x) = \sum_{\sigma \in \mathcal{E}_K} (\Pi^{\mathrm{CR}} Z(x)) \cdot \mathbf{n}_{K,\sigma} \frac{x - S_\sigma}{d(S_\sigma, \sigma)} + \sum_{\sigma \in \mathcal{E}_K} \nabla(\Pi^{\mathrm{CR}} Z)(x_\sigma - x) \cdot \mathbf{n}_{K,\sigma} \frac{x - S_\sigma}{d(S_\sigma, \sigma)}. \tag{33}$$

We then notice that, for all $\xi \in \mathbb{R}^d$ and all $x \in K$,

$$\sum_{\sigma \in \mathcal{E}_K} \xi \cdot \mathbf{n}_{K,\sigma} \frac{x - S_\sigma}{d(S_\sigma, \sigma)} = \xi. \tag{34}$$

Indeed, the left-hand side of this relation is the Raviart-Thomas reconstruction from the fluxes $(\xi \cdot \mathbf{n}_{K,\sigma})_{\sigma \in \mathcal{E}_K}$. But the constant function ξ belongs to the Raviart-Thomas space, and has the same fluxes. Hence the equality in (34).
We use $\xi = \Pi^{\mathrm{CR}} Z(x)$ in (34) and we plug the result in (33). Since $\frac{|x - S_\sigma|}{d(S_\sigma, \sigma)} \leq C$ with C only depending on the shape regularity of K, we infer

$$\begin{aligned} |\Pi^{\mathrm{RT}}(Z \cdot \mathbf{n})(x) - \Pi^{\mathrm{CR}} Z(x)| &= \left| \sum_{\sigma \in \mathcal{E}_K} \nabla(\Pi^{\mathrm{CR}} Z)(x_\sigma - x) \cdot \mathbf{n}_{K,\sigma} \frac{x - S_\sigma}{d(S_\sigma, \sigma)} \right| \\ &\leq C(d+1) |\nabla(\Pi^{\mathrm{CR}} Z)| h_K, \end{aligned}$$

and the proof is complete. □

Lemma 4.7 *For all $\varphi \in \boldsymbol{H}_0^1(\Omega)$, we define $\mathcal{I}_D \varphi \in X_{D,0}$ by $\mathcal{I}_D \varphi = (\frac{1}{|\sigma|} \int_\sigma \varphi(x) \cdot \mathbf{n}_\sigma \, \mathrm{d}s(x))_{\sigma \in \mathcal{E}_{\mathrm{int}}}$. Then, for all $\varphi \in \boldsymbol{C}_c^\infty(\Omega)$, there exists $C > 0$ only depending on $\mathcal{R}_\mathcal{T}$ such that*

$$\forall \varphi \in \boldsymbol{C}_c^\infty(\Omega)\,,\ \forall \sigma \in \mathcal{E}_{\mathrm{int}}\,,\ |R_\sigma \mathcal{I}_D \varphi - \varphi(x_\sigma)| \leq C \|\varphi\|_{\mathbf{W}^{2,\infty}(\Omega)} \mathrm{diam}(\sigma)^2. \tag{35}$$

Proof
All constants C in this proof only depend (in a non-decreasing way) on $\mathcal{R}_\mathcal{T}$. We first notice that, for all $\sigma' \in \mathcal{E}_\sigma$, since $x_{\sigma'}$ is the center of gravity of σ we have

$$|(\mathcal{I}_D \varphi)_{\sigma'} - \varphi(x_{\sigma'}) \cdot \mathbf{n}_{\sigma'}| \leq C ||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)} \mathrm{diam}(\sigma)^2. \tag{36}$$

Take $\sigma \in \mathcal{E}_{\mathrm{int}}$ and define the linear mapping $\overline{\varphi}_\sigma(x) = \varphi(x_\sigma) + D\varphi(x_\sigma)(x - x_\sigma)$, and the family $u = (u_{\sigma'})_{\sigma' \in \mathcal{E}_\sigma}$ by $u_{\sigma'} = (\mathcal{I}_D \varphi)_{\sigma'} - \overline{\varphi}_\sigma(x_{\sigma'}) \cdot \mathbf{n}_{\sigma'}$. The regularity of φ and the definition of $\mathcal{R}_\mathcal{T}$ ensure that, for all $\sigma' \in \mathcal{E}_\sigma$, $|\overline{\varphi}_\sigma(x_{\sigma'}) - \varphi(x_{\sigma'})| \leq C ||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)} \mathrm{diam}(\sigma)^2$. Combined with (36), this leads to $|u_{\sigma'}| \leq C ||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)} \mathrm{diam}(\sigma)^2$ and therefore to $|R_\sigma u| \leq C ||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)} d a_\sigma \mathrm{diam}(\sigma)^2 \leq C ||\varphi||_{\mathbf{W}^{2,\infty}(\Omega)} \mathrm{diam}(\sigma)^2$. The proof is complete by noticing that, by definition of $\overline{\varphi}_\sigma$, $R_\sigma u = R_\sigma \mathcal{I}_D \varphi - \varphi(x_\sigma)$. □

Short numerical glance

Although providing numerical results on complex situations is the main aim of this work, we give here a brief comparison between the schemes presented in Sections 4.3 and 4.4. Table 1 presents a series of characteristics of these two schemes, and we provide a simple computation of the lid-driven cavity. The mesh used is depicted in Figure 3. The results are similar, with both methods (see Figure 4). Note that more significant differences – in favour of the H_{div} conforming scheme – are expected for the Navier-Stokes equations (a work in progress), and that the computing times of both schemes have to be compared in more realistic situations in 3D.

	Crouzeix-Raviart	H_{div} conforming
number of d.o.f.	#faces $\times\, d$ + #simplices	#faces + #simplices
implementation	simple	complex
stencil	limited, efficient methods for linear systems	extended, linear systems more difficult
2D resolution	standard	facilitated by streamline potential unknown
continuous divergence free	no	yes

Table 1: Comparison between the Crouzeix-Raviart scheme and the new H_{div} conforming scheme

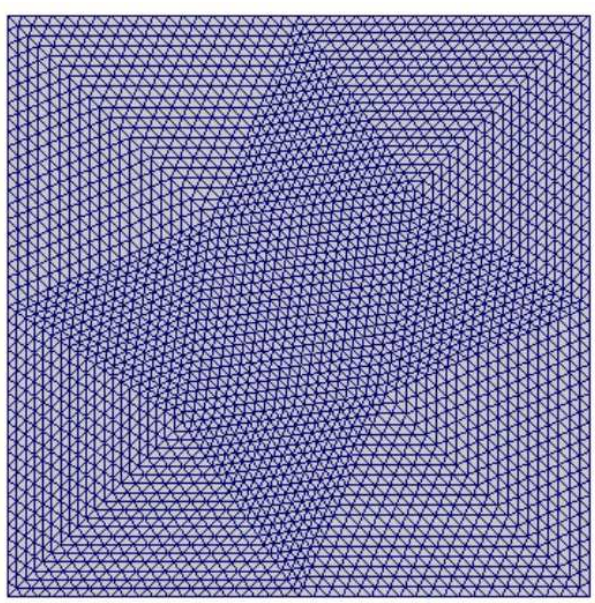


Figure 3: Mesh for the lid driven cavity test case.

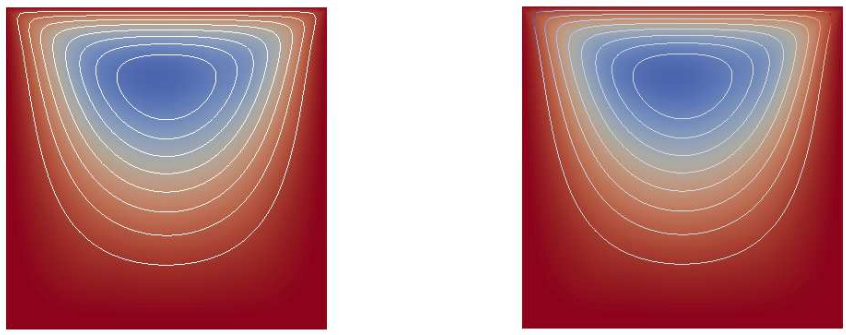


Figure 4: Lid driven cavity, streamline potential. Left: Crouzeix-Raviart, right: $\boldsymbol{H}_{\mathrm{div}}$ conforming .

5 Proof of the convergence results

5.1 Steady problem

Proof of Theorem 3.3
Once established, Estimate (8) shows that if the right-hand of the linear system (7) on (u_D, p_D) is zero (i.e. $f = 0$, $G = 0$, which implies $\overline{u} = 0$ and $\overline{p} = 0$), then the solution (u_D, p_D) is also zero. Hence, this square system is invertible, which ensures the existence and uniqueness of its solution for any right-hand side. We now have to show Estimate (8). Under the hypotheses of the theorem, since $\mathrm{div}(\nabla\overline{u}) - \nabla\overline{p} = -f + \mathrm{div}(G) + \eta\overline{u}$ in the distribution sense, we get $\mathrm{div}(\nabla\overline{u} - G) - \nabla\overline{p} = \eta\overline{u} - f \in \boldsymbol{L}^2(\Omega)$. Using this relation in $\overline{W}_D(\nabla\overline{u} - G, \overline{p})$ we write

$$\overline{W}_D(\nabla\overline{u} - G, \overline{p}) = \max_{\substack{v \in X_{D,0} \\ \|v\|_D = 1}} \int_\Omega [\nabla_D v : (\nabla\overline{u} - G) + \Pi_D v \cdot (\eta\overline{u} - f) - \overline{p}\mathrm{div}_D v]\,\mathrm{d}x.$$

Invoking the gradient scheme (7) to replace $\displaystyle\int_\Omega -(\Pi_D v \cdot f + \nabla_D v : G)\,\mathrm{d}x$, we can write, for any $v \in X_{D,0}$,

$$\int_\Omega \big(\eta(\overline{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla\overline{u} - \nabla_D u_D) : \nabla_D v + (\chi_D p_D - \overline{p})\ \mathrm{div}_D v\big)\,\mathrm{d}x \le \overline{W}_D(\nabla\overline{u} - G, \overline{p})\|v\|_D. \quad (37)$$

Let us introduce $I_D : \boldsymbol{H}_0^1(\Omega) \mapsto X_{D,0}$ and $\widetilde{I}_D : L_0^2(\Omega) \mapsto Y_{D,0}$ defined by

$$\begin{aligned} I_D\varphi &= \operatorname*{argmin}_{v \in X_{D,0}} \left(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\mathrm{div}_D v - \mathrm{div}\varphi\|_{L^2(\Omega)}\right), \\ \widetilde{I}_D\psi &= \operatorname*{argmin}_{z \in Y_{D,0}} \|\chi_D z - \psi\|_{L^2(\Omega)}. \end{aligned} \quad (38)$$

We also define $\varepsilon_D(\overline{u}, \overline{p}) = \overline{W}_D(\nabla\overline{u} - G, \overline{p}) + S_D(\overline{u}) + \widetilde{S}_D(\overline{p})$. We may then write

$$\begin{aligned} &\int_\Omega \big(\eta(\Pi_D I_D\overline{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla_D I_D\overline{u} - \nabla_D u_D) : \nabla_D v\big)\,\mathrm{d}x \\ &\qquad + \int_\Omega (\chi_D p_D - \chi_D \widetilde{I}_D\overline{p})\ \mathrm{div}_D v\,\mathrm{d}x \le (1 + (1+\eta)C_D)\varepsilon_D(\overline{u}, \overline{p})\|v\|_D. \quad (39) \end{aligned}$$

Thanks to Definition 2.3, let us now take $v \in X_{D,0}$ such that $\|v\|_D = 1$ and

$$\int_\Omega \chi_D(p_D - \widetilde{I}_D\overline{p}) \operatorname{div}_D v \,\mathrm{d}x \geq \beta_D \|\chi_D(p_D - \widetilde{I}_D\overline{p})\|_{L^2(\Omega)}.$$

We then get, from (39),

$$\|\chi_D(p_D - \widetilde{I}_D\overline{p})\|_{L^2(\Omega)} \leq \frac{1 + (1+\eta)C_D}{\beta_D}\varepsilon_D(\overline{u},\overline{p}) + \frac{1+\eta C_D}{\beta_D}\|I_D\overline{u} - u_D\|_D. \tag{40}$$

Choosing $v = I_D\overline{u} - u_D$ in (39) and using $\int_\Omega \operatorname{div}_D u_D \ \chi_D q = 0$ for all $q \in Y_{D,0}$, we can write

$$\|I_D\overline{u} - u_D\|_D^2 + \int_\Omega \chi_D(p_D - \widetilde{I}_D\overline{p}) \operatorname{div}_D I_D\overline{u} \,\mathrm{d}x \leq (1 + (1+\eta)C_D)\varepsilon_D(\overline{u},\overline{p})\|I_D\overline{u} - u_D\|_D,$$

which implies, since $\operatorname{div}\overline{u} = 0$,

$$\|I_D\overline{u} - u_D\|_D^2 \leq (1 + (1+\eta)C_D)\varepsilon_D(\overline{u},\overline{p})\|I_D\overline{u} - u_D\|_D + S_D(\overline{u})\|\chi_D(p_D - \widetilde{I}_D\overline{p})\|_{L^2(\Omega)}.$$

Thanks to (40) and to the Young inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, the above estimate yields the existence of C_1, non-decreasing w.r.t. $1/\beta_D$, C_D and η, such that $\|I_D\overline{u} - u_D\|_D \leq C_1\varepsilon_D(\overline{u},\overline{p})$. The conclusion follows from the definitions of $I_D\overline{u}$, $\widetilde{I}_D\overline{p}$ and C_D, the triangle inequality and (40). □

Proof of Theorem 3.5 The proof follows the same idea as the proof of Theorem 3.3, but considering only functions $v \in E_D$. For such functions, owing to (10), Equation (37) can be written

$$\int_\Omega \big(\eta(\overline{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla\overline{u} - \nabla_D u_D) : \nabla_D v\big) \,\mathrm{d}x \leq \overline{W}_D(\nabla\overline{u} - G, \overline{p})\|v\|_D. \tag{41}$$

We introduce $I_{D,E_D} : E(\Omega) \mapsto E_D$ defined by

$$I_{D,E_D}\varphi = \operatorname*{argmin}_{v \in E_D} \big(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d}\big)$$

and from (41) we deduce, by definition of S_{D,E_D},

$$\int_\Omega \big(\eta(\Pi_D I_{D,E_D}\overline{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla_D I_{D,E_D}\overline{u} - \nabla_D u_D) : \nabla_D v\big) \,\mathrm{d}x$$
$$\leq \big[(\eta C_D + 1)S_{D,E_D}(\overline{u}) + \overline{W}_D(\nabla\overline{u} - G, \overline{p})\big] \|v\|_D.$$

Choosing $v = I_{D,E_D}\overline{u} - u_D \in E_D$ leads to

$$\|\nabla_D I_{D,E_D}\overline{u} - \nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d} \leq (\eta C_D + 1)S_{D,E_D}(\overline{u}) + \overline{W}_D(\nabla\overline{u} - G, \overline{p}) \tag{42}$$

and the proof of (11) is complete since $\|\nabla_D I_{D,E_D}\overline{u} - \nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d} \leq S_{D,E_D}(\overline{u})$. Estimate (12) follows from the definition of C_D, (42) and $\|\Pi_D I_{D,E_D}\overline{u} - \overline{u}\|_{\boldsymbol{L}^2(\Omega)^d} \leq S_{D,E_D}(\overline{u})$. □

Proof of Theorem 3.7
The proof starts from the definition of W_D (Definition 6.1). Since $\nabla\overline{u} - G \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$ and $\operatorname{div}(\nabla\overline{u} - G) = \nabla\overline{p} + \eta\overline{u} - f$, we can write, for any $v \in E_D$,

$$\int_\Omega \big(\nabla_D v : (\nabla\overline{u} - G) + \Pi_D v \cdot (\nabla\overline{p} + \eta\overline{u} - f)\big) \,\mathrm{d}x \leq W_D(\nabla\overline{u} - G)\|v\|_D.$$

Owing to Assumption (13) and since $v \in E_D$, we can remove the term $\nabla\overline{p}$. Using the gradient scheme (7) to replace the terms involving f and G, we deduce

$$\int_\Omega (\nabla_D v : (\nabla\overline{u} - \nabla_D u) + \eta\Pi_D v \cdot (\overline{u} - \Pi_D u)) \,\mathrm{d}x \leq W_D(\nabla\overline{u} - G)\|v\|_D.$$

Hence, (41) is satisfied with $W_D(\nabla\overline{u} - G)$ instead of $\overline{W}_D(\nabla\overline{u} - G, \overline{p})$, and the conclusion follows as in the proof of Theorem 3.5. □

5.2 Transient problem

The existence and uniqueness of the solution to the gradient scheme for the transient Stokes problem is a straightforward consequence of the study of the gradient scheme for the steady problem.

Lemma 5.1 (Existence and Uniqueness of the discrete solution) *Under Hypothesis* (16)*, let D be a space-time discretisation in the sense of Definition 2.7. Then there exists a unique solution (u_D, p_D) to the gradient scheme* (19)*.*

Proof We remark that the equation on $(u^{(n+1)}, p^{(n+1)})$ in (19) is the gradient discretisation (7) of the steady Stokes problem, with $\eta = \delta t^{n+\frac{1}{2}}$ and a right-hand side depending on $u^{(n)}$. Existence and uniqueness of the solution therefore follows from Theorem 3.3. □

Let us now establish some *a priori* estimates on the solution to the scheme.

Lemma 5.2 (Estimates) *Under Hypotheses* (16)*, let D be a space-time discretisation in the sense of Definition 2.7 and let (u_D, p_D) be the solution to Scheme* (19)*. Then, for all $m = 0, \ldots, N$,*

$$\int_0^{t^{(m)}} \int_\Omega |\nabla_D u_D|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega \left(\left|\Pi_D u_D^{(m)}\right|^2 - \left|\Pi_D u_D^{(0)}\right|^2 \right) \mathrm{d}x \leq \\ \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_D u_D \,\mathrm{d}x\,\mathrm{d}t + \int_0^{t^{(m)}} \int_\Omega G : \nabla_D u_D \,\mathrm{d}x\,\mathrm{d}t. \quad (43)$$

Moreover, if $C_2 > 0$ is such that $C_2 \geq \|\Pi_D u^{(0)}\|_{L^2(\Omega)}$, then there exist $C_3 \geq 0$ only depending on Ω, d, C_2, f, G and C_D such that

$$\|\Pi_D u_D\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))} + \|\nabla_D u_D\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \leq C_3. \quad (44)$$

Proof Putting $v = \delta t^{n+\frac{1}{2}} u_D^{(n+1)}$ and $q = p_D^{(n+1)}$ in (19) we get

$$\int_\Omega \left(\Pi_D u_D^{(n+1)} - \Pi_D u_D^{(n)}\right) \cdot \Pi_D u_D^{(n+1)} \,\mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_D u_D^{(n+1)}|^2 \,\mathrm{d}x\,\mathrm{d}t = \\ \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D u_D^{(n+1)} \,\mathrm{d}x\,\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D u_D^{(n+1)} \,\mathrm{d}x\,\mathrm{d}t.$$

Using the inequality $(a-b)\cdot a \geq \frac{1}{2}(|a|^2 - |b|^2)$ (valid for any $a, b \in \mathbb{R}^d$) on the first term, it comes:

$$\frac{1}{2}\int_\Omega \left[\left|\Pi_D u_D^{(n+1)}\right|^2 - \left|\Pi_D u_D^{(n)}\right|^2 \right] \mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_D u_D^{(n+1)}|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \\ \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D u_D^{(n+1)} \,\mathrm{d}x\,\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D u_D^{(n+1)} \,\mathrm{d}x\,\mathrm{d}t.$$

We take $m \in \{0, \ldots, N\}$ and sum the obtained equation over $n = 0, \ldots, m-1$. This gives

$$\frac{1}{2}\|\Pi_D u_D^{(m)}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\|\Pi_D u^{(0)}\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^{t^{(m)}} \|\nabla_D u_D\|^2_{\boldsymbol{L}^2(\Omega)^d} \,\mathrm{d}t \leq \\ \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_D u_D \,\mathrm{d}x\,\mathrm{d}t + \int_0^{t^{(m)}} \int_\Omega G : \nabla_D u_D \,\mathrm{d}x\,\mathrm{d}t$$

and the proof of (43) is complete. Estimate (44) is a straightforward consequence of the definition of C_D and Young's inequality applied to (43) with $m = N$. □

Definition 5.3 *The semi-norm* $|\cdot|_{*,D}$ *is defined on* $X_{D,0}$ *by*

$$|w|_{*,D} = \sup\left\{\int_\Omega \Pi_D w \cdot \Pi_D v \,\mathrm{d}x \ : v \in E_D, \|v\|_D = 1\right\},$$

where we recall that E_D *is defined by* (9).

Lemma 5.4 (Estimates on $|\delta_D u_D|_{*,D}$**)** *Under Hypotheses* (16)*, let* D *be a space-time discretisation in the sense of Definition 2.7, and let* (u_D, p_D) *be the solution to Scheme* (19)*. We take* $C_2 \geq \|\Pi_D u^{(0)}\|_{L^2(\Omega)}$*. Then there exist* $C_4 \geq 0$ *only depending on* Ω*,* d*,* C_2*,* f*,* G *and* C_D *such that*

$$\int_0^T |\delta_D u_D|_{*,D}^2 \,\mathrm{d}t \leq C_4. \tag{45}$$

Proof Taking a generic $v \in E_D$ in Scheme (19) and using the definition of $|\cdot|_{*,D}$ gives an estimate on $|\delta_D^{n+\frac{1}{2}} u_D|_{*,D}$ in terms of $||\Pi_D u_D^{(n+1)}||_{\boldsymbol{L}^2(\Omega)}$ and $||\nabla_D u_D^{(n+1)}||_{\boldsymbol{L}^2(\Omega)^d}$. Squaring this estimate, multiplying by $\delta t^{n+\frac{1}{2}}$ and summing over n gives the desired estimate, thanks to (44). □

We can now prove the first convergence result of gradient schemes for the transient Stokes problem.

Proof of Theorem 3.12
Step 1: existence of a weak limit of a subsequence of approximations.
Estimate (44) gives the existence of $\overline{u} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $\zeta \in \boldsymbol{L}^2(\Omega \times (0,T))^d$ such that, up to a subsequence (still indexed by m), $\Pi_{D_m} u_{D_m} \to \overline{u}$ weakly-$*$ in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $\nabla_{D_m} u_{D_m} \to \zeta$ in $\boldsymbol{L}^2(\Omega \times (0,T))^d$. Taking $\varphi \in C^\infty(\overline{\Omega})^d$ and $\theta \in C_c^\infty(0,T)$, we then see that, for all $n = 0, \ldots, N-1$, and all $t \in (t^{(n)}, t^{(n+1)})$,

$$\int_\Omega \nabla_{D_m} u_m^{(n+1)} : \varphi\,\theta + \Pi_{D_m} u_m^{(n+1)} \cdot \mathrm{div}\varphi\,\theta \,\mathrm{d}x \leq \overline{W}_{D_m}(\varphi, 0)\,\theta\,||\nabla_{D_m} u_{D_m}||_{\boldsymbol{L}^2(\Omega)^d}.$$

Integrating this over $t \in (t^{(n)}, t^{(n+1)})$, summing on $n = 0, \ldots, N-1$ and using Estimate (44), we find C_5 not depending on m such that

$$\int_0^T \int_\Omega \nabla_{D_m} u_{D_m} : \varphi\,\theta + \Pi_{D_m} u_{D_m} \cdot \mathrm{div}\varphi\,\theta \,\mathrm{d}x\,\mathrm{d}t \leq C_5 \overline{W}_{D_m}(\varphi, 0).$$

We can then pass to the supremum limit as $m \to \infty$ and apply the resulting inequality to $\pm\varphi$ to see that

$$\int_0^T \int_\Omega \zeta : (\varphi\,\theta) + \overline{u} \cdot \mathrm{div}(\varphi\,\theta) \,\mathrm{d}x\,\mathrm{d}t = 0.$$

This relation first shows, with $\varphi \in C_c^\infty(\Omega)^d$, that $\zeta = \nabla\overline{u}$, and therefore that $\overline{u} \in L^2(0,T;\boldsymbol{H}^1(\Omega))$. Taking then φ which does not vanish on $\partial\Omega$, we also infer that the trace of $\overline{u}$ on $\partial\Omega$ is zero and therefore that $\overline{u} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega))$.
Let us now prove that $\overline{u}$ is divergence-free. From Estimate (44) and the coercivity of the sequence of gradient discretisations, we see that $(\mathrm{div}_{D_m} u_{D_m})_{m\in\mathbb{N}}$ (with the same space-time notations as in (20)) is bounded in $L^2(\Omega \times (0,T))$ and therefore converges weakly in this space, up to a subsequence, to some λ. Taking $\psi \in H^1(\Omega)$ with zero mean value and $\theta \in C^\infty(0,T)$, we have, for any $n = 0, \ldots, N-1$ and any $t \in (t^{(n)}, t^{(n+1)})$,

$$\int_\Omega \Pi_{D_m} u_{D_m}^{(n+1)} \cdot \nabla\psi\,\theta + \mathrm{div}_{D_m} u_{D_m}^{(n+1)} \psi\,\theta \,\mathrm{d}x \leq \overline{W}_{D_m}(0, \psi)\,\theta ||\nabla_{D_m} u_{D_m}||_{\boldsymbol{L}^2(\Omega)^d}.$$

As above, we integrate this over t and sum over n to find

$$\int_0^T \int_\Omega \Pi_{D_m} u \cdot \nabla\psi\, \theta + \mathrm{div}_{D_m} u_{D_m}\, \psi\, \theta \,\mathrm{d}x\,\mathrm{d}t \leq C_6 \overline{W}_{D_m}(0,\psi)$$

with C_6 not depending on m. Using the last equation in the gradient scheme (19), we can introduce $\chi_{D_m}\widetilde{I}_{D_m}\psi$ in the second term of the left-hand side (where $\widetilde{I}_{D_m}$ is defined as in (38)) and we get

$$\int_0^T \int_\Omega \Pi_{D_m} u \cdot \nabla\psi\, \theta \,\mathrm{d}x\,\mathrm{d}t \leq ||\mathrm{div}_{D_m} u_{D_m}||_{L^2(\Omega\times(0,T))} ||\chi_{D_m}\widetilde{I}_{D_m}\psi - \psi||_{L^2_0(\Omega)} ||\theta||_{L^2(0,T)} + C_6 \overline{W}_{D_m}(0,\psi).$$

Passing to the supremum limit $m \to \infty$, thanks to the limit-conformity and the consistency of the gradient discretisations, and applying the resulting inequality to $\pm\psi$, we deduce that $\int_0^T \int_\Omega \overline{u}\cdot\nabla\psi\theta \,\mathrm{d}x\,\mathrm{d}t = 0$. This relation is true for any $\psi \in H^1(\Omega)$ with zero mean value, and hence also for any function in $H^1(\Omega)$. The proof that $\mathrm{div}\overline{u} = 0$ is therefore complete.

Step 2: $\overline{u}$ is the solution to (17).
To simplify notations, we drop the indices m. Let $\theta \in C_c^\infty([0,T))$ and let $w \in E(\Omega)$. As $(w,0)$ is the solution of the incompressible steady stokes problem (Problem (1)) with $f = \eta w$ and $G = \nabla w$, we can find $w_D \in X_{D,0}$ such that $\int_\Omega \chi_D q \mathrm{div}_D w_D = 0$ for all $q \in Y_{D,0}$, $\Pi_D w_D \to w$ in $\boldsymbol{L}^2(\Omega)$ and $\nabla_D w_D \to \nabla w$ in $\boldsymbol{L}^2(\Omega)^d$ (Theorem 3.3 and Remark 3.4). We take $v = \delta t^{(n+\frac{1}{2})}\theta(t^{(n)})w_D$ as test function in Scheme (19) and we sum the resulting equation on $n = 0, ..., N-1$ to get $T_1 + T_2 + T_3 = T_4$ with

$$\begin{aligned}
T_1 &= \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}}\theta(t^{(n)}) \int_\Omega \Pi_D \delta^{(n+\frac{1}{2})} u_D \cdot \Pi_D w_D \,\mathrm{d}x,\\
T_2 &= \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}}\theta(t^{(n)}) \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D w_D \,\mathrm{d}x,\\
T_3 &= -\sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}}\theta(t^{(n)}) \int_\Omega \chi_D p_D^{(n+1)} \mathrm{div}_D w_D \,\mathrm{d}x,\\
T_4 &= \sum_{n=0}^{N-1} \theta(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D w_D \,\mathrm{d}x\,\mathrm{d}t + \sum_{n=0}^{N-1} \theta(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D w_D \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

First, we remark that $T_3 = 0$ since $\int_\Omega \chi_D q \mathrm{div}_D w_D = 0$ for all $q \in Y_{D,0}$. Using discrete integration by parts and writing $\theta(t^{(n+1)}) - \theta(t^{(n)}) = \int_{t^{(n)}}^{t^{(n+1)}} \theta'$, we find

$$T_1 = -\int_0^T \int_\Omega \theta' \Pi_D u_D \cdot \Pi_D w_D \,\mathrm{d}x\,\mathrm{d}t - \theta(0) \int_\Omega \Pi_D u_D^{(0)} \cdot \Pi_D w_D \,\mathrm{d}x.$$

Recall that $u_D^{(0)} = J_D u_{\mathrm{ini}}$, so that the space-time consistency (Definition 2.8) gives $\Pi_D u_D^{(0)} \to u_{\mathrm{ini}}$ in $\boldsymbol{L}^2(\Omega)$ as $m \to \infty$. Thus, by strong convergence in $\boldsymbol{L}^2(\Omega)$ of $\Pi_D w_D$ to w,

$$T_1 \to -\int_0^T \int_\Omega \theta' \overline{u} \cdot w \,\mathrm{d}x\,\mathrm{d}t - \theta(0) \int_\Omega \overline{u}_{\mathrm{ini}} \cdot w \,\mathrm{d}x.$$

It also comes easily, using the regularity of θ and Estimate (44), that

$$T_2 \to \int_0^T \theta \int_\Omega \nabla\overline{u} : \nabla w \,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad T_4 \to \int_0^T \theta \int_\Omega f \cdot w \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \theta \int_\Omega G : \nabla w \,\mathrm{d}x\,\mathrm{d}t.$$

Passing to the limit in $T_1+T_2+T_3=T_4$ concludes the proof that $\overline{u}$ satisfies (17).
Step 3: convergence in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$.
The proof of the uniform-in-time convergence follows the same idea as in [7]. Using Lemma 5.5 below and the generalised Ascoli-Arzela theorem [7, Theorem 6.2], we see that for any $\varphi\in E(\Omega)$ the sequence $(\langle\Pi_{D_m}u_{D_m}(\cdot),\varphi\rangle_{\boldsymbol{L}^2})_{m\in\mathbb{N}}$ is relatively compact in $L^\infty(0,T)$. Since this sequence already converges weakly in $L^2(0,T)$ towards $\langle\overline{u}(\cdot),\varphi\rangle_{\boldsymbol{L}^2}$ (because $\Pi_{D_m}u_{D_m}\to\overline{u}$ weakly in $\boldsymbol{L}^2(\Omega\times(0,T))$), we deduce that $\langle\Pi_{D_m}u_{D_m}(\cdot),\varphi\rangle_{\boldsymbol{L}^2}\to\langle\overline{u}(\cdot),\varphi\rangle_{\boldsymbol{L}^2}$ uniformly with respect to t as $m\to\infty$.
Let $\psi\in\boldsymbol{L}^2(\Omega)$ and $\varphi\in E(\Omega)$. By Estimate (44), for any $t\in[0,T]$,

$$|\langle\Pi_{D_m}u_{D_m}(t),\psi\rangle_{\boldsymbol{L}^2}-\langle\overline{u}(t),\psi\rangle_{\boldsymbol{L}^2}|\le|\langle\Pi_{D_m}u_{D_m}(t),\varphi\rangle_{\boldsymbol{L}^2}-\langle\overline{u}(t),\varphi\rangle_{\boldsymbol{L}^2}|+C||\varphi-\psi||_{\boldsymbol{L}^2(\Omega)}$$

where C does not depend on m, t, φ or ψ. Assuming that ψ can be approximated in $\boldsymbol{L}^2(\Omega)$ by functions in $E(\Omega)$ (see [24, Ch. I, Theorem 1.4] for a characterisation of such functions ψ), then the preceding estimate and the uniform-in-time convergence of $(\langle\Pi_{D_m}u_{D_m}(\cdot),\varphi\rangle_{\boldsymbol{L}^2})_{m\in\mathbb{N}}$ show that $\langle\Pi_{D_m}u_{D_m}(\cdot),\psi\rangle_{\boldsymbol{L}^2}\to\langle\overline{u}(\cdot),\psi\rangle_{\boldsymbol{L}^2}$ uniformly-in-time as $m\to\infty$. It is known (see Proposition 6.5 in the appendix) that, for any $T_0\in[0,T]$, $\overline{u}(T_0)$ can be approximated in $\boldsymbol{L}^2(\Omega)$ by functions in $E(\Omega)$; hence, we can apply the preceding result to $\psi=\overline{u}(T_0)$. This allows us to see that, for any $(s_m)_{m\in\mathbb{N}}$ converging to T_0,

$$\|\overline{u}(T_0)\|^2_{\boldsymbol{L}^2(\Omega)}=\lim_{m\to\infty}\langle\overline{u}(s_m),\overline{u}(T_0)\rangle_{\boldsymbol{L}^2}=\lim_{m\to\infty}\langle\Pi_{D_m}u_{D_m}(s_m),\overline{u}(T_0)\rangle_{\boldsymbol{L}^2}\tag{46}$$

(we used the continuity of $\overline{u}:[0,T]\mapsto\boldsymbol{L}^2(\Omega)$, see (55)). Thus, it comes

$$\|\overline{u}(T_0)\|_{\boldsymbol{L}^2(\Omega)}\le\liminf_{m\to\infty}\|\Pi_{D_m}u_{D_m}(s_m)\|_{\boldsymbol{L}^2(\Omega)}.\tag{47}$$

Let $k(m)$ such that $s_m\in(t^{(k(m)-1)},t^{(k(m))}]$, where $(t^l)_l$ are the time steps of the discretisation. Definition (20) gives $\Pi_{D_m}u_{D_m}(s_m)=\Pi_{D_m}u^{(k(m))}_{D_m}$. The discrete energy estimate (43) therefore leads to

$$\begin{aligned}\frac12\int_\Omega|\Pi_{D_m}u_{D_m}(s_m)|^2\,\mathrm{d}x\le\frac12\int_\Omega|\Pi_{D_m}u^{(0)}_{D_m}|^2\,\mathrm{d}x-\int_0^{t^{(k(m))}}\int_\Omega|\nabla_{D_m}u_{D_m}|^2\,\mathrm{d}x\,\mathrm{d}t\\+\int_0^{t^{(k(m))}}\int_\Omega f\cdot\Pi_{D_m}u_{D_m}\,\mathrm{d}x\,\mathrm{d}t+\int_0^{t^{(k(m))}}\int_\Omega G:\nabla_{D_m}u_{D_m}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{48}$$

We notice, by weak convergence in $\boldsymbol{L}^2(\Omega\times(0,T))^d$ of $\nabla_{D_m}u_{D_m}$ to $\nabla\overline{u}$ and strong convergence in $\boldsymbol{L}^2(\Omega\times(0,T))^d$ of $\mathbf{1}_{[0,t^{(k(m))}]}\nabla\overline{u}$ toward $\mathbf{1}_{[0,T]}\nabla\overline{u}$ (notice that $t^{(k(m))}\to T_0$), where $\mathbf{1}_A$ is the characteristic function of A,

$$\begin{aligned}\int_0^{T_0}\int_\Omega|\nabla\overline{u}|^2\,\mathrm{d}x\,\mathrm{d}t&=\int_0^T\int_\Omega\mathbf{1}_{[0,T_0]}\nabla\overline{u}:\nabla\overline{u}\,\mathrm{d}x\,\mathrm{d}t\\&=\lim_{m\to\infty}\int_0^T\int_\Omega\mathbf{1}_{[0,t^{(k(m))}]}\nabla\overline{u}:\nabla_{D_m}u_{D_m}\,\mathrm{d}x\,\mathrm{d}t\\&\le\liminf_{m\to\infty}\left(\left|\left|\mathbf{1}_{[0,t^{(k(m))}]}\nabla\overline{u}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}\left|\left|\mathbf{1}_{[0,t^{(k(m))}]}\nabla_{D_m}u_{D_m}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}\right)\\&=\left|\left|\mathbf{1}_{[0,T_0]}\nabla\overline{u}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}\liminf_{m\to\infty}\left|\left|\mathbf{1}_{[0,t^{(k(m))}]}\nabla_{D_m}u_{D_m}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}.\end{aligned}$$

Hence,

$$\int_0^{T_0}\int_\Omega|\nabla\overline{u}|^2\,\mathrm{d}x\,\mathrm{d}t\le\liminf_{m\to\infty}\int_0^{t^{(k(m))}}|\nabla_{D_m}u_{D_m}|^2\,\mathrm{d}x\,\mathrm{d}t$$

and we can pass to the limit superior in (48), using the weak convergences of $\Pi_{D_m} u_{D_m}$ and $\nabla_{D_m} u_{D_m}$:

$$\limsup_{m\to\infty} \frac{1}{2}\|\Pi_{D_m} u_{D_m}(s_m)\|^2_{\boldsymbol{L}^2(\Omega)} \le \frac{1}{2}\int_\Omega |\overline{u}_{\rm ini}|^2 \,{\rm d}x - \int_0^{T_0}\int_\Omega |\nabla \overline{u}|^2 \,{\rm d}x\,{\rm d}t + \int_0^{T_0}\int_\Omega f\cdot\overline{u}\,{\rm d}x\,{\rm d}t + \int_0^{T_0}\int_\Omega G:\nabla\overline{u}\,{\rm d}x\,{\rm d}t. \quad (49)$$

Plugging $\varphi = \overline{u}\mathbf{1}_{[0,T_0]}$ in Problem (55) and integrating by parts, we obtain the continuous energy estimate

$$\frac{1}{2}\|\overline{u}(T_0)\|^2_{\boldsymbol{L}^2(\Omega)} = \frac{1}{2}\int_\Omega |\overline{u}_{\rm ini}|^2 \,{\rm d}x - \int_0^{T_0}\int_\Omega |\nabla \overline{u}|^2 \,{\rm d}x\,{\rm d}t + \int_0^{T_0}\int_\Omega f\cdot\overline{u}\,{\rm d}x\,{\rm d}t + \int_0^{T_0}\int_\Omega G:\nabla\overline{u}\,{\rm d}x\,{\rm d}t. \quad (50)$$

Combined with (49), this leads to $\limsup_{m\to\infty}\|\Pi_{D_m} u_{D_m}(s_m)\|^2_{\boldsymbol{L}^2(\Omega)} \le \|\overline{u}(T_0)|^2_{\boldsymbol{L}^2(\Omega)}$. Using (47), we deduce $\lim_{m\to\infty}\|\Pi_{D_m} u_{D_m}(s_m)\|^2_{\boldsymbol{L}^2(\Omega)} = \|\overline{u}(T_0)\|^2_{\boldsymbol{L}^2(\Omega)}$. Recalling (46), this allows us to conclude that $||\Pi_{D_m} u_{D_m}(s_m) - \overline{u}(T_0)||^2_{\boldsymbol{L}^2(\Omega)} \to 0$ (just develop the square). Since $\overline{u} : [0,T] \mapsto \boldsymbol{L}^2(\Omega)$ is continuous, we can apply [7, Lemma 5.1] and finally get that $\Pi_{D_m} u_{D_m} \to \overline{u}$ in $\boldsymbol{L}^2(\Omega)$ uniformly-in-time.

Step 4: strong convergence of $\nabla_{D_m} u_{D_m}$.
We write the discrete energy estimate (43) with $t^{(m)} = T$ and use the convergence in $\boldsymbol{L}^2(\Omega)^d$ of $\Pi_{D_m} u_{D_m}(T)$ to $\overline{u}(T)$ to see that

$$\limsup_{m\to\infty} ||\nabla_{D_m} u_{D_m}||^2_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \le \frac{1}{2}\int_\Omega (|\overline{u}(T)|^2 - |\overline{u}_{\rm ini}|^2)\,{\rm d}x - \int_0^{T}\int_\Omega |\nabla \overline{u}|^2 \,{\rm d}x\,{\rm d}t + \int_0^{T}\int_\Omega f\cdot\overline{u}\,{\rm d}x\,{\rm d}t + \int_0^{T_0}\int_\Omega G:\nabla\overline{u}\,{\rm d}x\,{\rm d}t.$$

The energy estimate (50) with $T_0 = T$ then shows that

$$\limsup_{m\to\infty} ||\nabla_{D_m} u_{D_m}||^2_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \le ||\nabla\overline{u}||^2_{\boldsymbol{L}^2(\Omega\times(0,T))^d},$$

which allows us to conclude that the weak convergence of $\nabla_{D_m} u_{D_m}$ to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega\times(0,T))^d$ is in fact strong. □

The following lemma was the initial key to obtain the uniform-in-time convergence result in the previous proof (Step 3).

Lemma 5.5 *Under the assumptions and notations of Theorem 3.12, for all $\varphi \in E(\Omega)$ the sequence of functions $t \mapsto \langle \Pi_{D_m} u_{D_m}(t), \varphi\rangle_{\boldsymbol{L}^2}$ satisfies the following quasi-equi-continuity property: there exist C_7, not depending on m, and a sequence of real numbers $(\omega_{\varphi,D_m})_{m\in\mathbb{N}}$ converging to 0 such that, for all $t, s \in [0,T]$,*

$$|\langle \Pi_{D_m} u_{D_m}(t) - \Pi_{D_m} u_{D_m}(s), \varphi\rangle_{\boldsymbol{L}^2}| \le C_7|t-s|^{\frac{1}{2}} + \omega_{\varphi,D_m}.$$

Proof As in the proof of Theorem 3.12, we drop the index m. Let $\varphi \in E(\Omega)$ and, as in Step 2 of the proof of Theorem 3.12, consider the solution φ_D to the steady gradient scheme (7) with $f = \eta\varphi$ and $G = \nabla\varphi$; then $\Pi_D\varphi \to \varphi$ in $\boldsymbol{L}^2(\Omega)$ and $\nabla_D\varphi_D \to \nabla\varphi$ in $\boldsymbol{L}^2(\Omega)^d$. Since $\varphi_D \in E_D$, the definition 5.3 of $|\cdot|_{*,D}$ gives

$$|\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D\varphi_D\rangle_{\boldsymbol{L}^2}| = \left|\sum_{\substack{n \text{ s.t.}\\ s\le t^{(n)}\le t}} \delta t^{(n+\frac{1}{2})}\int_\Omega \Pi_D\delta_D^{n+\frac{1}{2}} u_D\cdot\Pi_D\varphi_D\,{\rm d}x\right| \le \sum_{\substack{n \text{ s.t.}\\ s\le t^{(n)}\le t}} \delta t^{(n+\frac{1}{2})}|\delta_D^{n+\frac{1}{2}} u_D|_{*,D}\|\varphi_D\|_D.$$

Since $\nabla_D \varphi_D \to \nabla\varphi$ in $\boldsymbol{L}^2(\Omega)^d$, $\|\varphi_D\|_D$ is bounded and so, using Cauchy-Schwarz inequality, we may write

$$|\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D \varphi_D\rangle_{\boldsymbol{L}^2}| \le C \left(\int_0^T |\delta_D u_D|_{*D}^2 \,\mathrm{d}t\right)^{\frac{1}{2}} (|t-s| + \delta t)^{\frac{1}{2}}$$

with C not depending on D. Finally, thanks to Lemma 5.4, we infer

$$|\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D \varphi_D\rangle_{\boldsymbol{L}^2}| \le C\left(|t-s| + \delta t\right)^{\frac{1}{2}}.$$

We then write, using Estimate (44),

$$\begin{aligned}|\langle \Pi_D u_D(t) - \Pi_D u_D(s), \varphi\rangle_{\boldsymbol{L}^2}| &\le |\langle \Pi_D u_D(t) - \Pi_D u_D(s), \varphi - \Pi_D\varphi_D\rangle_{\boldsymbol{L}^2}| \\ &\quad + |\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D\varphi_D\rangle_{\boldsymbol{L}^2}| \\ &\le C\|\varphi - \Pi_D\varphi_D\|_{\boldsymbol{L}^2(\Omega)} + C\left(|t-s| + \delta t\right)^{\frac{1}{2}} \\ &\le C_7|t-s|^{\frac{1}{2}} + C\delta t^{\frac{1}{2}} + C\|\varphi - \Pi_D\varphi_D\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

The limit-conformity of the sequence of gradient discretisations ensures that $\omega_{\varphi,D} := C\delta t^{\frac{1}{2}} + C\|\varphi - \Pi_D\varphi_D\|_{\boldsymbol{L}^2(\Omega)}$ tends to 0, and the proof is complete. □

Let us now turn to the proof of the convergence of the pressure (Theorem 3.14).

Lemma 5.6 (Estimates on discrete time-derivative of velocity) *Under the assumptions of Theorem 3.14, let (u_D, p_D) be the solution to Scheme (19). Let $R \ge C_D + \|J_D u_{\mathrm{ini}}\|_D$. Then there exists $C_8 \ge 0$ only depending on Ω, d, R, f and G such that*

$$\|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega\times(0,T))} \le C_8. \tag{51}$$

Proof Put $v = \delta_D^{n+\frac{1}{2}} u_D$ and $q = p_D^{(n+1)}$ in Scheme (19). Since $u_D^{(0)} = J_D\overline{u}_{\mathrm{ini}} \in E_D$ we have $v \in E_D$ even if $n = 0$ and therefore

$$\|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega)}^2 + \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D \delta_D^{n+\frac{1}{2}} u_D \,\mathrm{d}x = \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D \delta_D^{n+\frac{1}{2}} u_D \,\mathrm{d}x\,\mathrm{d}t \tag{52}$$

(recall that $G = 0$ here). On the other hand, using $a : (a-b) = \frac{1}{2}(|a|^2 - |b|^2) + \frac{1}{2}|a-b|^2 \ge \frac{1}{2}(|a|^2 - |b|^2)$ for any tensors a, b,

$$\int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D \delta_D^{n+\frac{1}{2}} u_D \,\mathrm{d}x \ge \frac{1}{2\delta t^{n+\frac{1}{2}}} \int_\Omega \left[|\nabla_D u_D^{(n+1)}|^2 - |\nabla_D u_D^{(n)}|^2\right] \mathrm{d}x.$$

Plugging this into (52) and multiplying by $\delta t^{n+\frac{1}{2}}$, it comes

$$\int_{t^{(n)}}^{t^{(n+1)}} \|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega)}^2 \,\mathrm{d}t + \frac{1}{2}\int_\Omega \left[|\nabla_D u_D^{(n+1)}|^2 - |\nabla_D u_D^{(n)}|^2\right] \mathrm{d}x \le \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D \delta_D^{n+\frac{1}{2}} u_D \,\mathrm{d}x\,\mathrm{d}t.$$

We sum on $n = 0, \ldots, m-1$ for a given $m = 1, \ldots, N$ to get

$$\int_0^{t^{(m)}} \|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega)}^2 \,\mathrm{d}t \le \frac{1}{2}\int_\Omega |\nabla_D u_D^{(0)}|^2 \,\mathrm{d}x + \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_D \delta_D^{n+\frac{1}{2}} u_D \,\mathrm{d}x\,\mathrm{d}t.$$

The Cauchy-Schwarz' and Young's inequalities conclude the proof. □

Lemma 5.7 (Estimates on discrete pressure) *Under the assumptions of Theorem 3.14, let (u_D, p_D) be the solution to Scheme (19). Let $R \geq C_D + \|J_D u_{\rm ini}\|_D + \beta_D^{-1}$. Then there exists $C_9 \geq 0$ only depending on Ω, d, R, f and G such that*

$$\|\chi_D p_D\|_{L^2(\Omega\times(0,T))} \leq C_9 \tag{53}$$

Proof Let $v \in X_{D,0}$ such that $\|v\|_D = 1$ and $\beta_D \|\chi_D p_D^{(n+1)}\|_{L^2(\Omega)} \leq \int_\Omega \chi_D p_D^{(n+1)} \mathrm{div}_D v \,\mathrm{d}x$ (see the definition of β_D in Definition 2.3). Plugging v in Scheme (19), we obtain

$$\begin{aligned}
&\beta_D \|\chi_D p_D^{(n+1)}\|_{L^2(\Omega)} \\
&\qquad \leq \left| \int_\Omega \Pi_D \delta_D^{n+\frac{1}{2}} u_D \cdot \Pi_D v \,\mathrm{d}x + \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v \,\mathrm{d}x - \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D v \,\mathrm{d}x \,\mathrm{d}t \right|.
\end{aligned}$$

Using Cauchy-Schwarz and the discrete Poincaré inequalities, we deduce

$$\beta_D \|\chi_D p_D^{(n+1)}\|_{L^2(\Omega)} \leq C_D \|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega)} + \|u_D^{(n+1)}\|_D + \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} C_D \|f\|_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t.$$

We take the square of this estimate, multiply the result by $\delta t^{n+\frac{1}{2}}$ and sum over n. Estimate (44) and (51) then show that (53) holds. □

Proof of Theorem 3.14
We first apply Theorem 3.12 to get the strong convergence of $\Pi_{D_m} u_{D_m}$ to $\overline{u}$ in $\boldsymbol{L}^2(\Omega \times (0,T))$ and of $\nabla_{D_m} u_{D_m}$ to $\nabla \overline{u}$ in $\boldsymbol{L}^2(\Omega \times (0,T))^d$, where $\overline{u} \in L^2(0,T;E(\Omega))$. Thanks to Estimate (53), we can find a function $\overline{p} \in L^2(0,T,L_0^2(\Omega))$ such that, up to a subsequence, $\chi_{D_m} p_{D_m}$ weakly converges to $\overline{p}$ in $L^2(\Omega \times (0,T))$. We then take $\theta \in C_c^\infty([0,T))$, $w \in \boldsymbol{H}_0^1(\Omega)$ and use $v = \delta t^{(n+\frac{1}{2})} \theta(t^{(n)}) I_{D_m} w$ as a test function in Scheme (19), were I_D is defined by (38). Since $\Pi_{D_m} I_{D_m} w \to w$ in $\boldsymbol{L}^2(\Omega)$, $\nabla_{D_m} I_{D_m} w \to \nabla w$ in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} I_{D_m} w \to \mathrm{div}(w)$ in $L^2(\Omega)$ as $m \to \infty$, we can pass to the limit in all terms $T_1, \ldots, T_4$. Note that T_3 is no longer equal to 0, but it converges thanks to the weak convergence of $\chi_{D_m} p_{D_m}$ to $\overline{p}$. We then see that $(\overline{u}, \overline{p})$ satisfy

$$\left\{ \begin{array}{l}
\overline{u} \in L^2(0,T,E(\Omega))\,,\ \overline{p} \in L^2(0,T;L_0^2(\Omega)), \\
\displaystyle\int_0^T \int_\Omega -\overline{u} \cdot \partial_t \varphi \,\mathrm{d}x \,\mathrm{d}t + \int_\Omega \overline{u}_{\rm ini} \cdot \varphi(\cdot,0) \,\mathrm{d}x + \int_0^T \int_\Omega \nabla \overline{u} : \nabla \varphi \,\mathrm{d}x \,\mathrm{d}t \\
\qquad \displaystyle - \int_0^T \int_\Omega \overline{p} \mathrm{div} \varphi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega f \cdot \varphi \,\mathrm{d}x \,\mathrm{d}t,\ \forall \varphi = \theta w \text{ with } \theta \in C_c^\infty([0,T))\,,\ w \in \boldsymbol{H}_0^1(\Omega).
\end{array} \right. \tag{54}$$

The density of tensorial functions in $L^2(0,T;\boldsymbol{H}_0^1(\Omega))$ ensures that this relation is actually satisfied for any $\varphi \in L^2(0,T;\boldsymbol{H}_0^1(\Omega))$. It therefore shows that $\partial_t u \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$. In combination with the fact that $\overline{u} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega))$, this classical implies $u \in C([0,T];\boldsymbol{L}^2(\Omega))$. This regularity of $\overline{u}$ then allows to perform an integration by parts in order to see that (54) is equivalent to (18). □

6 Appendix

6.1 An alternative definition of limit-conformity

The definition 2.5 of the limit-conformity, based on $\overline{W}_D$, is the natural one for the Stokes problem. It is however different in spirit from the definition used in other analyses of gradient schemes, which would rather correspond to splitting the velocity and the pressure as follows.

Definition 6.1 (Limit-conformity for gradient schemes) *Let D be a gradient discretisation in the sense of Definition 2.1, and let $W_D : \boldsymbol{H}_{\rm div}(\Omega) \to [0,+\infty)$ and $\widetilde{W}_D$: $H^1(\Omega) \to [0,+\infty)$ be respectively defined by*

$$\forall \varphi \in \boldsymbol{H}_{\rm div}(\Omega)^d,\ W_D(\varphi) = \max_{v \in X_{D,0}, \|v\|_D = 1} \left(\int_\Omega (\nabla_D v : \varphi + \Pi_D v \cdot {\rm div}\varphi)\, {\rm d}x \right),$$
$$\forall \psi \in H^1(\Omega),\ \widetilde{W}_D(\psi) = \max_{v \in X_{D,0}, \|v\|_D = 1} \left(\int_\Omega (\Pi_D v \cdot \nabla \psi + \psi\ {\rm div}_D v)\, {\rm d}x \right).$$

*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation is said to be **limit-conforming** if, for all $\varphi \in \boldsymbol{H}_{\rm div}(\Omega)$, $W_{D_m}(\varphi)$ tends to 0 as $m \to \infty$ and if, for all $\psi \in H^1(\Omega)$, $\widetilde{W}_{D_m}(\psi)$ tends to 0 as $m \to \infty$.*

Remark 6.2 *For coercive sequences of gradient discretisations, this limit-conformity can be established by only checking that W_{D_m} and $\widetilde{W}_{D_m}$ tend to 0 on dense subsets of $\boldsymbol{H}_{\rm div}(\Omega)^d$ and $H^1(\Omega)$ respectively.*

The convergence of the scheme (7) could be proved by compactness arguments using Definition 6.1, without any other hypothesis other than (5). However, to establish error estimate using Definition 6.1 we would need $\overline{p} \in H^1(\Omega)$ and $\nabla \overline{u} \in \boldsymbol{H}_{\rm div}(\Omega)$. This assumption is more restricting than $(\nabla \overline{u}, \overline{p}) \in Z(\Omega)$. We can nevertheless prove that Definition 2.5 and Definition 6.1 are equivalent, under the non-restrictive additional hypothesis that Ω is locally star-shaped. We first establish a density result, and then use it to prove the equivalence of the definitions of limit-conformity.

Lemma 6.3 *Let Ω be a locally star-shaped open bounded domain in $\mathbb{R}^d$. Then the set $\boldsymbol{C}^\infty(\overline{\Omega})^d \times C^\infty(\overline{\Omega})$ is dense in $Z(\Omega)$ (see (2.5)) for the topology of $Z(\Omega)$ defined by $\|(\varphi,\psi)\|_{Z(\Omega)} = \|\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\psi\|_{L^2(\Omega)} + \|{\rm div}\varphi - \nabla\psi\|_{\boldsymbol{L}^2(\Omega)}$*

Proof The proof follows the ideas of [24, Ch.1, Theorem 1.1, (iii)]. If U is an open set of $\mathbb{R}^d$, we define $L : \boldsymbol{L}^2(U) \times L^2(U) \mapsto \mathcal{D}'(U)^d$ by $L(f,g) = {\rm div} f - \nabla g$. We have $Z(U) = \{(f,g) \in \boldsymbol{L}^2(U)^d \times L^2(U) : L(f,g) \in \boldsymbol{L}^2(U)\}$ and $||(f,g)||_{Z(U)} = ||f||_{\boldsymbol{L}^2(U)^d} + ||g||_{L^2(U)} + ||L(f,g)||_{\boldsymbol{L}^2(U)}$.

Step 1: Stability of $Z(U)$ by localisation.
Let $\theta \in C^\infty(\overline{U})$ and $(f,g) \in Z(U)$. Then $L(\theta f, \theta g) = {\rm div}(\theta f) - \nabla(\theta g) = \theta L(f,g) + f\nabla\theta + g\nabla\theta \in \boldsymbol{L}^2(U)$ and therefore $(\theta f, \theta g) \in Z(U)$ (f is considered here as a matrix, and $f\nabla\theta$ is therefore a matrix-vector product, whereas $g\nabla\theta$ is a scalar–vector product). We also see from this formula that $\theta \mapsto (\theta f, \theta g)$ is continuous $Z(U) \to Z(U)$.

Step 2: Stability of $Z(\cdot)$ by dilatation.
Let $\lambda > 1$ and define, for $h \in L^2(U)$, $h_\lambda \in L^2(\lambda U)$ by $h_\lambda(x) = h(\frac{x}{\lambda})$ Let $(f,g) \in Z(U)$. We have $L(f_\lambda, g_\lambda) = {\rm div} f_\lambda - \nabla g_\lambda = \frac{1}{\lambda} L(f,g)(\frac{\cdot}{\lambda}) \in \boldsymbol{L}^2(\lambda U)$.
It is also well-known that, for any $h \in L^2(U)$, $(h_\lambda)_{|U} \to h$ in $L^2(U)$ as $\lambda \to 1$. Hence, the previous formula for $L(f_\lambda, g_\lambda)$ shows that $(f_{\lambda|_U}, g_{\lambda|_U}) \to (f,g)$ in $Z(U)$ as $\lambda \to 1$.

Step 3: If U is star-shaped, approximation of elements in $Z(U)$ by elements in $Z(\mathbb{R}^d)$.
Let $\lambda > 1$. Since U is star-shaped, we have that $U \subset\subset \lambda U$ and therefore we can find $\gamma \in C_c^\infty(\lambda U)$ such that $\gamma = 1$ on a neighborhood of $\overline{U}$. Then, by combining Steps 1 and 2, we see that $(\gamma f_\lambda, \gamma g_\lambda) \in Z(\lambda U)$ Since γ has a compact support in λU, extending these functions to 0 outside λU gives $(\gamma f_\lambda, \gamma g_\lambda) \in Z(\mathbb{R}^d)$. Moreover, if λ is close to 1, combining the previous step and the fact that $\gamma \equiv 1$ on U (which shows, by Step 1, that $L(\gamma f_\lambda, \gamma g_\lambda) = L(f_\lambda, g_\lambda)$ on U) we see that $(\gamma f_\lambda, \gamma g_\lambda)_{|U}$ is close to (f,g) in $Z(U)$.

Step 4: Approximation by smooth functions in $Z(\mathbb{R}^d)$.

Let $(f,g) \in Z(\mathbb{R}^d)$. let $\rho \in C_c^\infty(\mathbb{R}^d)$ be such that $\rho \geq 0$ and $\int_{\mathbb{R}^d} \rho = 1$. For $n \in \mathbb{N}$, let ρ_n denote the function $x \mapsto n^d \rho(nx)$. Then $(f * \rho_n, g * \rho_n) \in \boldsymbol{C}^\infty(\mathbb{R}^d)^d \times C^\infty(\mathbb{R}^d)$ and $\|f * \rho_n - f\|_{\boldsymbol{L}^2(\mathbb{R}^d)^d} + \|g * \rho_n - g\|_{\boldsymbol{L}^2(\mathbb{R}^d)} \to 0$ as $n \to \infty$. Moreover,

$$L(f * \rho_n, g * \rho_n) = \mathrm{div}(f * \rho_n) - \nabla(g * \rho_n) = \mathrm{div}(f) * \rho_n - \nabla g * \rho_n = L(f,g) * \rho_n.$$

Hence, $L(f * \rho_n, g * \rho_n) \to L(f,g)$ in $\boldsymbol{L}^2(\mathbb{R}^d)$ and therefore $(f * \rho_n, g * \rho_n) \to (f,g)$ in $Z(\mathbb{R}^d)$ as $n \to \infty$.

Step 5: Conclusion.
Now, let $(\varphi, \psi) \in Z(\Omega)$ and $(\mathcal{O}_j)_{j \in J}$ be an open cover of $\overline{\Omega}$ such that, for all $j \in J$, $\Omega \cap \mathcal{O}_j$ is star-shaped. Let us consider a partition of unity $(\theta_j)_{j \in J}$ subordinated to this cover: $\theta_j \in C_c^\infty(\mathcal{O}_j)$ and $1 = \sum_{j \in J} \theta_j$ on Ω. We have $\varphi = \sum_{j \in J} \theta_j \varphi$ and $\psi = \sum_{j \in J} \theta_j \psi$. Let us define, for $n \in \mathbb{N}$ and $\lambda > 1$,

$$\varphi_{n,\lambda} = \sum_{j \in J} (\theta_j \varphi)_\lambda * \rho_n \quad \text{and} \quad \psi_{n,\lambda} = \sum_{j \in J} (\theta_j \psi)_\lambda * \rho_n.$$

The four previous steps show that $(\varphi_n, \psi_n) \to (\varphi, \psi)$ in $Z(\Omega)$, and the proof is complete since $(\varphi_n, \psi_n) \in \boldsymbol{C}^\infty(\overline{\Omega})^d \times C^\infty(\overline{\Omega})$. $\square$

Proposition 6.4 *Let $(D_m)_{m \in \mathbb{N}}$ be a sequence of gradient discretisation in the sense of Definition 2.1 such that $(D_m)_{m \in N}$ is coercive. We assume that Ω is a locally star-shaped open bounded domain of $\mathbb{R}^d$. Then the two following propositions are equivalent*

1. *$(D_m)_{m \in \mathbb{N}}$ is limit-conforming in the sense of Definition 6.1,*
2. *$\forall (\varphi, \psi) \in Z(\Omega)$, $\overline{W}_{D_m}(\varphi, \psi) \to 0$ as $m \to \infty$.*

Proof Let $(\varphi, \psi) \in Z(\Omega)$, the definition of $\overline{W}_{D_m}$ gives:

$$\overline{W}_{D_m}(\varphi, \psi) = \max_{\substack{v \in X_{D_m,0} \\ \|v\|_{D_m} = 1}} \left(\int_\Omega [\nabla_{D_m} v : \varphi + \Pi_{D_m} v \cdot (\mathrm{div}\varphi - \nabla\psi) - \psi \mathrm{div}_{D_m} v] \, \mathrm{d}x \right)$$

Using the previous lemma, we can find $(\overline{\varphi}, \overline{\psi}) \in \boldsymbol{C}^\infty(\overline{\Omega})^d \times C^\infty(\overline{\Omega})$ and $\varepsilon > 0$ such that $\|(\varphi - \overline{\varphi}, \psi - \overline{\psi})\|_{Z(\Omega)} \leq \varepsilon$. Thanks to the coercivity of the sequence of gradient discretisations, we may write

$$\overline{W}_{D_m}(\varphi, \psi) \leq \overline{W}_{D_m}(\overline{\varphi}, \overline{\psi}) + \varepsilon(1 + 2C_P).$$

Moreover, $\overline{\varphi}$ and $\overline{\psi}$ are now enough regular to split $\mathrm{div}\overline{\varphi} - \nabla\overline{\psi}$ in the definition of $\overline{W}_{D_m}(\varphi, \psi)$, which gives

$$\overline{W}_{D_m}(\varphi, \psi) \leq W_{D_m}(\overline{\varphi}) + \widetilde{W}_{D_m}(\overline{\psi}) + \mathcal{O}(\varepsilon).$$

Since the sequence $(D_m)_{m \in \mathbb{N}}$ is limit-conforming in the sense of Definition 6.1, we have $W_{D_m}(\varphi) \to 0$ and $\widetilde{W}_{D_m}(\psi) \to 0$ as $m \to \infty$. Passing to the limit, this shows that $\overline{W}_{D_m}(\varphi, \psi) \to 0$ as $m \to \infty$.
The converse (Item 2 of the proposition implies Item 1) is trivial since for all $\varphi \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$ we have $(\varphi, 0) \in Z(\Omega)$ with $\overline{W}_{D_m}(\varphi, 0) = W_{D_m}(\varphi)$, and for all $\psi \in H^1(\Omega)$ we have $(0, \psi) \in Z(\Omega)$ with $\overline{W}_{D_m}(0, \psi) = \widetilde{W}_{D_m}(\psi)$. $\square$

6.2 About the solution to the transient Stokes problem

The theoretical study of the transient Stokes problem, and in particular of the notion of pressure in this model, is significantly more complex than for the steady problem. The initial sense of weak solution for (2), that is the one given by Definition 3.10, only involves the velocity unknown as the pressure has

been eliminated by the choice of divergence-free test functions. The interest of this formulation is that it provides an existence and uniqueness result based on classical variational arguments [24, Ch.III, Theorem 1.1].
An equivalent and useful formulation of (17) is the following:

$$\left\{\begin{array}{l} \overline{u} \in L^2(0,T;E(\Omega)) \cap C([0,T];\boldsymbol{L}^2(\Omega))\,,\ \partial_t\overline{u} \in L^2(0,T;E(\Omega)')\,, \\ \displaystyle\int_0^T \langle \partial_t\overline{u},\varphi\rangle\,\mathrm{d}t + \int_0^T\int_\Omega \nabla\overline{u} : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ \qquad \displaystyle= \int_0^T\int_\Omega f\cdot\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega G : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\,, \quad \forall\varphi \in L^2(0,T;E(\Omega))\,, \\ \overline{u}(\cdot,0) = \overline{u}_{\mathrm{ini}} \text{ in } \boldsymbol{L}^2(\Omega)\,, \end{array}\right. \tag{55}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $E(\Omega)'$ and $E(\Omega)$. See for example the discussion at the end of the proof of Theorem 3.14 on the equivalence between (54) and (18).

The pressure can be recovered, in a very weak sense. Proposition 1.1 in [24, Ch.III] establishes the existence of a distribution $\overline{p}$ on $\Omega\times(0,T)$ such that, if $\overline{u}$ is the solution to (55) then $(\overline{u},\overline{p})$ satisfies the PDEs in (2) in the sense of distributions. Additional regularity results on $\overline{p}$ can be obtained if we assume that $u_{\mathrm{ini}} \in E(\Omega)$.

Proposition 6.5 (Regularity result) *Let us assume Hypothesis* (16)*, and that* $\overline{u}_{\mathrm{ini}} \in E(\Omega)$*. We denote by* H *the closure of* $\{\varphi \in C_c^\infty(\Omega)^d : \mathrm{div}(\varphi) = 0\}$ *in* $\boldsymbol{L}^2(\Omega)$*. Then the weak solution* $(\overline{u},\overline{p})$ *to* (2) *in the sense of Definition 3.10 (and* $\overline{p}$ *as in the discussion above) satisfies:* $\overline{u} \in L^2(0,T,E(\Omega)) \cap C([0,T];H)$*,* $\partial_t\overline{u} \in L^2(0,T,E(\Omega)')$ *and* $\overline{p} \in L^2(0,T,L^2_0(\Omega))$*.*

Proof This is essentially contained in the proof of [24, Ch.III, Theorem 1.1 and Proposition 1.2]. Proposition 1.2 in this reference is proved under more regularity assumption on Ω and the right-hand side (the term G must not be present and f must be divergence-free, in particular), but this is just useful to recover higher regularity on the solution (H^2 on $\overline{u}$ and H^1 on $\overline{p}$). Under our assumptions, the proof of [24, Ch.III, Proposition 1.2] gives Proposition 6.5. □

These additional regularity results on $(\overline{u},\overline{p})$ make the proof of Proposition 3.11 obvious. Indeed, testing $\partial_t\overline{u} - \Delta\overline{u} + \nabla p = f - \mathrm{div}(G)$ (satisfied in the sense of distributions, see above) against $\varphi \in C_c^\infty(0,T)\times\Omega)$ and using the regularity of $(\overline{u},\overline{p})$, we see that the equation in (18) holds for any smooth φ with compact support. The general case is deduced by density of these functions in $L^2(0,T;\boldsymbol{H}^1_0(\Omega))$.

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