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NEUMANN PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS WITH L^1 DATA

M.F. BETTA, O. GUIBÉ, AND A. MERCALDO

ABSTRACT. In the present paper we prove existence results for solutions to nonlinear elliptic Neumann problems whose prototype is

$$\begin{cases} -\Delta_p u - \operatorname{div}(c(x)|u|^{p-2}u) = f & \text{in } \Omega, \\ (|\nabla u|^{p-2}\nabla u + c(x)|u|^{p-2}u) \cdot \underline{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

when f is just a summable function. Our approach allows also to deduce a stability result for renormalized solutions and an existence result for operator with a zero order term.

MATHEMATICS SUBJECT CLASSIFICATION: MSC 2000 : 35J25

KEY WORDS: Nonlinear elliptic equations, Neumann problems, renormalized solutions, existence results

1. INTRODUCTION

In the present paper we prove existence results for solutions to nonlinear elliptic Neumann problems whose prototype is

$$(1.1) \quad \begin{cases} -\Delta_p u - \operatorname{div}(c(x)|u|^{p-2}u) = f & \text{in } \Omega, \\ (|\nabla u|^{p-2}\nabla u + c(x)|u|^{p-2}u) \cdot \underline{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of \mathbb{R}^N , $N \geq 2$, with Lipschitz boundary, $1 < p \leq N$, \underline{n} is the outer unit normal to $\partial\Omega$, the datum f belongs to $L^1(\Omega)$ and satisfies the compatibility condition $\int_{\Omega} f = 0$. Finally the coefficient $c(x)$ belongs to an appropriate Lebesgue space.

When $c(x) = 0$ and f is an element of the dual space of the Sobolev space $W^{1,p}(\Omega)$, the existence and uniqueness (up to additive constants) of weak solutions to problem (1.1) is consequence of the classical theory of pseudo monotone operators (cfr. [21], [22]). But if f is just an L^1 -function, and not more an element of the dual space of $W^{1,p}(\Omega)$, one has to give a meaning to the notion of solution.

When Dirichlet boundary conditions are prescribed, various definitions of solution to nonlinear elliptic equations with right-hand side in L^1 or measure have been introduced. In [5], [13], [23], [24] different notions of solution are defined even if they turn out to be equivalent, at least when the datum is an L^1 -function. The study of existence or uniqueness for Dirichlet boundary value problems has been the object of several papers. We just recall that the linear case has been studied in [26], while the nonlinear case began to be faced in [8] and [9] and was continued in various contributions, including

[2], [4], [5], [6], [7], [12], [13], [19], [20]; mixed boundary value problems have been also studied (see [4]). In the present paper we refer to the so-called renormalized solutions (see [12], [23], [24]) whose precise definition is recalled in Section 2.

The existence for Neumann boundary value problems with L^1 -data when $c = 0$ has been treated in various contests. In [3], [11], [15], [16] and [25] the existence of a distributional solution which belongs to a suitable Sobolev space and which has null mean value is proved. Nevertheless when p is close to 1, i.e. $p \leq 2 - 1/N$, the distributional solution to problem (1.1) does not belong to a Sobolev space and in general is not a summable function; this implies that its mean value has not meaning. This difficulty is overcome in [14] by considering solutions u which are not in $L^1(\Omega)$, but for which $\Phi(u)$ is in $L^1(\Omega)$, where $\Phi(t) = \int_0^t \frac{ds}{(1+|s|)^\alpha}$ with appropriate $\alpha > 1$. In [1] the case where both the datum f and the domain Ω are not regular is studied and solutions whose median is equal to zero are obtained with a natural process of approximations. We recall that the median of u is defined by

$$(1.2) \quad \text{med}(u) = \sup\{t \in \mathbb{R} : \text{meas}\{u > t\} \geq \text{meas}(\Omega)/2\}.$$

Neumann problems have been studied by a different point of view in [17, 18].

In this paper we face two difficulties: one due to the presence of the lower order term $-\text{div}(c(x)|u|^{p-2}u)$ and the other due to the low integrability properties of the datum f .

Our main result is Theorem 4.1 which asserts the existence of a renormalized solution to (1.1) having $\text{med}(u) = 0$. Its proof, contained in Section 4, is based on an usual procedure of approximation which consists by considering problems of type (1.1) having smooth data which strongly converge to f in L^1 . For such a sequence of problems we prove in Section 3 an existence results for weak solutions which is obtained by using a fixed point arguments. A priori estimates allow to prove that these weak solutions converge in some sense to a function u and a delicate procedure of passage to the limit allows to prove that u is a renormalized solution to (2.1).

In Section 5 we give a stability result and we prove that, under larger assumptions on the summability of f , a renormalized solution to (2.1) is in turn a weak solution to the same problem. At last Section 6 is concerned with Neumann problems with a zero order term; adapting the proof of Theorem 4.1 allows to derive an existence result for this type of operators.

2. ASSUMPTIONS AND DEFINITIONS

Let us consider the following nonlinear elliptic Neumann problem

$$(2.1) \quad \begin{cases} -\text{div}(\mathbf{a}(x, u, \nabla u) + \Phi(x, u)) = f & \text{in } \Omega, \\ (\mathbf{a}(x, u, \nabla u) + \Phi(x, u)) \cdot \underline{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a connected open subset of \mathbb{R}^N , $N \geq 2$, having finite Lebesgue measure and Lipschitz boundary, \underline{n} is the outer unit normal to $\partial\Omega$. We

assume that p is a real number such that $1 < p \leq N$ and

$$\mathbf{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N,$$

$$\Phi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^N$$

are Carathéodory functions. Moreover \mathbf{a} satisfies:

$$(2.2) \quad \mathbf{a}(x, s, \xi) \cdot \xi \geq \alpha |\xi|^p, \quad \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \text{ a.e. in } \Omega$$

where $\alpha > 0$ is a given real number;

$$(2.3) \quad (\mathbf{a}(x, s, \xi) - \mathbf{a}(x, s, \eta)) \cdot (\xi - \eta) \geq 0$$

$\forall s \in \mathbb{R}, \forall \xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$ and a.e. in Ω ;

for any $k > 0$ there exist $a_k > 0$ and b_k belonging to $L^{p'}(\Omega)$ such that

$$(2.4) \quad |\mathbf{a}(x, s, \xi)| \leq a_k |\xi|^{p-1} + b_k(x), \quad \forall |s| < k, \forall \xi \in \mathbb{R}^N, \text{ a.e. in } \Omega.$$

We assume that Φ satisfies the following growth condition

$$(2.5) \quad |\Phi(x, s)| \leq c(x)(1 + |s|^{p-1})$$

$\forall s \in \mathbb{R}$, a.e. in Ω , with $c \in L^{\frac{N}{p-1}}(\Omega)$ if $p < N$ and $c \in L^q(\Omega)$ with $q > N/(N-1)$ if $p = N$.

Finally we assume that the datum f is a measurable function in a Lebesgue space $L^r(\Omega)$, $1 \leq r \leq +\infty$, which belongs to the dual space of the classical Sobolev space $W^{1,p}(\Omega)$ or is just an L^1 -function. Moreover it satisfies the compatibility condition

$$(2.6) \quad \int_{\Omega} f \, dx = 0.$$

As explained in the Introduction we deal with solutions whose median is equal to zero. Let us recall that if u is a measurable function, we denote the median of u by

$$(2.7) \quad \text{med}(u) = \sup \left\{ t \in \mathbb{R} : \text{meas}\{x \in \Omega : u(x) > t\} > \frac{\text{meas}(\Omega)}{2} \right\}.$$

Let us explicitly observe that if $\text{med}(u) = 0$ then

$$\begin{aligned} \text{meas}\{x \in \Omega : u(x) > 0\} &\leq \frac{\text{meas}(\Omega)}{2}, \\ \text{meas}\{x \in \Omega : u(x) < 0\} &\leq \frac{\text{meas}(\Omega)}{2}. \end{aligned}$$

In this case a Poincaré-Wirtinger inequality holds (see e.g. [27]):

Proposition 2.1. *If $u \in W^{1,p}(\Omega)$, then*

$$(2.8) \quad \|u - \text{med}(u)\|_{L^p(\Omega)} \leq C \|\nabla u\|_{(L^p(\Omega))^N}$$

where C is a constant depending on p, N, Ω .

As pointed out in the Introduction, when the datum f is not an element of the dual space of the classical Sobolev space $W^{1,p}(\Omega)$ or is just an L^1 -function, the classical notion of weak solution does not fit. We will refer to the notion of renormalized solution to (2.1) (see [12, 24] for elliptic equations with Dirichlet boundary conditions) which we give below.

In the whole paper, T_k , $k \geq 0$, denotes the truncation at height k that is $T_k(s) = \min(k, \max(s, -k))$, $\forall s \in \mathbb{R}$.

Definition 2.2. A real function u defined in Ω is a renormalized solution to (2.1) if

(2.9) u is measurable and finite almost everywhere in Ω ,

(2.10) $T_k(u) \in W^{1,p}(\Omega)$, for any $k > 0$,

(2.11)
$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{x \in \Omega; |u(x)| < n\}} \mathbf{a}(x, u, \nabla u) \nabla u \, dx = 0$$

and if for every function h belonging to $W^{1,\infty}(\mathbb{R})$ with compact support and for every $\varphi \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ we have

$$(2.12) \quad \int_{\Omega} h(u) \mathbf{a}(x, u, \nabla u) \nabla \varphi \, dx + \int_{\Omega} h'(u) \mathbf{a}(x, u, \nabla u) \nabla u \varphi \, dx \\ + \int_{\Omega} h(u) \Phi(x, u) \nabla \varphi \, dx + \int_{\Omega} h'(u) \Phi(x, u) \nabla u \varphi \, dx = \int_{\Omega} f \varphi h(u) \, dx.$$

Remark 2.3. A renormalized solution is not an $L^1_{loc}(\Omega)$ -function and therefore it has not a distributional gradient. Condition (2.10) allows to define a generalized gradient of u according to Lemma 2.1 of [5], which asserts the existence of a unique measurable function v defined in Ω such that $\nabla T_k(u) = \chi_{\{|u| < k\}} v$ a.e. in Ω , $\forall k > 0$. This function v is the generalized gradient of u and it is denoted by ∇u .

Equality (2.12) is formally obtained by using in (2.1) the test function $\varphi h(u)$ and by taking into account Neumann boundary conditions. Actually in a standard way one can check that every term in (2.12) is well-defined under the structural assumptions on the elliptic operator.

Remark 2.4. It is worth noting that growth assumption (2.5) on Φ together with (2.9)–(2.11) allow to prove that any renormalized solution u verifies

$$(2.13) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\Omega} |\Phi(x, u)| \times |\nabla T_n(u)| \, dx = 0.$$

Without loss of generality we can assume that $\text{med}(u) = 0$. Growth assumption (2.5) implies that

$$\int_{\Omega} |\Phi(x, u)| \times |\nabla T_n(u)| \, dx \leq \frac{1}{n} \int_{\Omega} c(x) (1 + |T_n(u)|)^{p-1} |\nabla T_n(u)| \, dx.$$

In the case $N > p$, using Hölder inequality we obtain

$$(2.14) \quad \int_{\Omega} c(x)(1 + |T_n(u)|)^{p-1} |\nabla T_n(u)| dx \\ \leq C \|c\|_{L^{N/(p-1)}(\Omega)} (1 + \|T_n(u)\|_{L^{p^*}(\Omega)}^{p-1}) \|\nabla T_n(u)\|_{(L^p(\Omega))^N}.$$

Since $\text{med}(T_n(u)) = 0$, by Poincaré–Wirtinger inequality, i.e. Proposition 2.1, and Sobolev embedding theorem it follows that

$$\int_{\Omega} c(x)(1 + |T_n(u)|)^{p-1} |\nabla T_n(u)| dx \\ \leq C \|c\|_{L^{N/(p-1)}(\Omega)} (1 + \|\nabla T_n(u)\|_{(L^p(\Omega))^N}^{p-1}) \|\nabla T_n(u)\|_{(L^p(\Omega))^N}$$

where $C > 0$ is a generic constant independent of n . Therefore Young inequality leads to

$$(2.15) \quad \frac{1}{n} \int_{\Omega} c(x)(1 + |T_n(u)|)^{p-1} |\nabla T_n(u)| dx \\ \leq \frac{C}{n} \|c\|_{L^{N/(p-1)}(\Omega)} (1 + \|\nabla T_n(u)\|_{(L^p(\Omega))^N}^p)$$

In the case $N = p$ a similar inequality involving $\|c\|_{L^q(\Omega)}$ with $q > N/(N-1)$ occurs.

Due to the coercivity of the operator \mathbf{a} and to (2.11) we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\Omega} |\nabla T_n(u)|^p dx = 0.$$

By (2.14) and (2.15) we conclude that (2.13) holds.

3. A BASIC EXISTENCE RESULT FOR WEAK SOLUTIONS

In this section we assume more restrictive conditions on the right-hand side f , on Φ and on the operator \mathbf{a} in order to prove the existence of a weak solution u to problem (2.1), that is

$$u \in W^{1,p}(\Omega), \\ \int_{\Omega} \mathbf{a}(x, u, \nabla u) \nabla v dx + \int_{\Omega} \Phi(x, u) \nabla v dx = \int_{\Omega} f v dx$$

for any $v \in W^{1,p}(\Omega)$.

We assume

$$(3.1) \quad f \in L^r(\Omega) \cap (W^{1,p}(\Omega))'$$

$$(3.2) \quad |\Phi(x, s)| \leq c(x) \quad \forall s \in \mathbb{R}, \text{ a.e. in } \Omega$$

with $c \in L^\infty(\Omega)$. Moreover the operator \mathbf{a} satisfies

$$(3.3) \quad (\mathbf{a}(x, s, \xi) - \mathbf{a}(x, s, \eta)) \cdot (\xi - \eta) > 0$$

$\forall s \in \mathbb{R}, \forall \xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$ and a.e. in Ω ;

$$(3.4) \quad |\mathbf{a}(x, s, \xi)| \leq a_0(|\xi|^{p-1} + |s|^{p-1}) + a_1(x) \quad \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \text{ a.e. in } \Omega,$$

with $a_0 > 0$, $a_1 \in L^{p'}(\Omega)$.

Theorem 3.1. *Assume that (2.2), (3.1)–(3.4) and (2.6) hold. There exists at least one weak solution u to problem (2.1) having $\text{med}(u) = 0$.*

Proof. The proof relies on a fixed point argument.

Let $v \in L^p(\Omega)$. Due to (2.2), (3.3) and (3.4), $(x, \xi) \in \Omega \times \mathbb{R}^N \mapsto \mathbf{a}(x, v(x), \xi)$ is a strictly monotone operator and verifies

$$|\mathbf{a}(x, v(x), \xi)| \leq a_0(|\xi|^{p-1} + |v(x)|^{p-1}) + a_1(x) \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. in } \Omega.$$

Since $\Phi(x, v(x)) \in (L^\infty(\Omega))^N$, classical arguments (see e.g. [21], [22]) allow to deduce that there exists a unique u such that

$$(3.5) \quad u \in W^{1,p}(\Omega), \quad \text{med}(u) = 0$$

and

$$(3.6) \quad \int_{\Omega} \mathbf{a}(x, v, \nabla u) \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx - \int_{\Omega} \Phi(x, v) \nabla \varphi \, dx, \quad \forall \varphi \in W^{1,p}(\Omega).$$

It follows that we can consider the functional $\Gamma : L^p(\Omega) \rightarrow L^p(\Omega)$ defined by

$$\Gamma(v) = u, \quad \forall v \in L^p(\Omega),$$

where u is the unique element of $W^{1,p}(\Omega)$ verifying (3.5) and (3.6). We now prove that Γ is a continuous and compact operator.

Let us begin by proving that Γ is continuous. Let $v_n \in L^p(\Omega)$ such that $v_n \rightarrow v$ in $L^p(\Omega)$. Up to a subsequence (still denoted by v_n) $v_n \rightarrow v$ a.e. in Ω . Let $u_n = \Gamma(v_n)$ belonging to $W^{1,p}(\Omega)$ such that $\text{med}(u_n) = 0$ and such that (3.6) holds with v_n in place of v .

Choosing $\varphi = u_n$ as test function in (3.6) and using (2.2) we obtain that

$$\alpha \int_{\Omega} |\nabla u_n|^p \, dx \leq \int_{\Omega} |f u_n| \, dx + \int_{\Omega} |\Phi(x, v_n) \nabla u_n| \, dx.$$

Since $\text{med}(u_n) = 0$, from Poincaré-Wirtinger inequality (2.8), (3.1) and (3.2) Young inequality and Sobolev embedding theorem lead to

$$(3.7) \quad \int_{\Omega} |\nabla u_n|^p \, dx \leq M$$

where $M > 0$ is a constant independent of n . Using again (2.8), it follows that u_n is bounded in $W^{1,p}(\Omega)$.

As a consequence and in view of (3.4), there exists a subsequence (still denoted by u_n), a measurable function u and a field σ belonging to $(L^{p'}(\Omega))^N$ such that

$$(3.8) \quad u_n \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),$$

$$(3.9) \quad u_n \rightarrow u \quad \text{strongly in } L^p(\Omega),$$

$$(3.10) \quad u_n \rightarrow u \quad \text{a.e. in } \Omega,$$

$$(3.11) \quad \mathbf{a}(x, v_n, \nabla u_n) \rightharpoonup \sigma \quad \text{weakly in } (L^{p'}(\Omega))^N.$$

Since $\text{med}(u_n) = 0$ for any n and since $u \in W^{1,p}(\Omega)$ the point-wise convergence of u_n to u implies that $\text{med}(u) = 0$.

To get the continuity of Γ it remains to prove that $u = \Gamma(v)$ that is u satisfies (3.6). Using (3.6) with v_n in place of v and the test function $u_n - u$ we have

$$(3.12) \quad \int_{\Omega} \mathbf{a}(x, v_n, \nabla u_n)(\nabla u_n - \nabla u) dx = \int_{\Omega} f(u_n - u) dx \\ - \int_{\Omega} \Phi(x, v_n)(\nabla u_n - \nabla u) dx.$$

The point-wise convergence of v_n and assumption (3.2) imply that $\Phi(x, v_n)$ converges to $\Phi(x, v)$ almost everywhere in Ω and in L^∞ weak-* as n goes to infinity. Therefore from (3.8) and (3.9), passing to the limit in the right-hand side of (3.12), we obtain

$$(3.13) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} \mathbf{a}(x, v_n, \nabla u_n)(\nabla u_n - \nabla u) dx = 0.$$

Let us recall the classical arguments, so-called Minty arguments, (see [21], [22]) which allow to identify σ with $\mathbf{a}(x, v, \nabla u)$. Let ϕ belonging to $(L^\infty(\Omega))^N$. Due to assumption (3.4) and the convergence of v_n the Lebesgue theorem shows that for any $t \in \mathbb{R}$

$$\mathbf{a}(x, v_n, \nabla u + t\phi) \rightarrow \mathbf{a}(x, v, \nabla u + t\phi) \text{ strongly in } (L^{p'}(\Omega))^N.$$

By (3.11) and (3.13), it follows that for any $t \in \mathbb{R}$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} [\mathbf{a}(x, v_n, \nabla u_n) - \mathbf{a}(x, v_n, \nabla u + t\phi)](\nabla u_n - \nabla u - t\phi) dx \\ = \int_{\Omega} [\sigma - \mathbf{a}(x, v, \nabla u + t\phi)]t\phi dx.$$

Using the monotone character (3.3) of \mathbf{a} we obtain that for any $t \neq 0$

$$\text{sign}(t) \int_{\Omega} [\sigma - \mathbf{a}(x, v, \nabla u + t\phi)]\phi dx \geq 0.$$

Since $\mathbf{a}(x, v, \nabla u + t\phi)$ converges strongly to $\mathbf{a}(x, v, \nabla u)$ in $(L^{p'}(\Omega))^N$ as t goes to zero, letting $t \rightarrow 0$ in the above inequality leads to

$$\int_{\Omega} [\sigma - \mathbf{a}(x, v, \nabla u)]\phi dx = 0$$

for any ϕ belonging to $(L^\infty(\Omega))^N$. We easily conclude that

$$(3.14) \quad \sigma = \mathbf{a}(x, v, \nabla u).$$

By using (3.11) and (3.14) we can pass to the limit as $n \rightarrow +\infty$ in (3.6) with v_n in place of v and we get

$$\int_{\Omega} \mathbf{a}(x, v, \nabla u) \nabla \varphi dx = \int_{\Omega} f \varphi dx - \int_{\Omega} \Phi(x, v) \nabla \varphi dx, \quad \forall \varphi \in W^{1,p}(\Omega).$$

Since there exists a unique weak solution to (3.6) with median equal to zero we obtain that the whole sequence u_n converges to u in $L^p(\Omega)$ and $u = \Gamma(v)$. It follows that Γ is continuous.

Compactness of Γ immediatly follows. Indeed, thank to the assumptions, for any $v \in L^p(\Omega)$, we have

$$\int_{\Omega} |\nabla u|^p dx \leq C,$$

where C is a constant depending on α , a_0 , a_1 , $\|c\|_{L^\infty(\Omega)}$, Ω , N , p and f . Then, using Poincaré-Wirtinger inequality and Rellich theorem, $u = \Gamma(v)$ belongs to a compact set of $L^p(\Omega)$. By choosing a ball of $L^p(\Omega)$, $B_{L^p}(0, r)$ such that

$$\Gamma(B_{L^p}(0, r)) \subset B_{L^p}(0, r),$$

Leray-Schauder fixed point theorem ensures the existence of at least one fixed point. \square

4. EXISTENCE RESULT FOR RENORMALIZED SOLUTIONS

In this section we prove our main result which gives the existence of a renormalized solution to problem (2.1).

Theorem 4.1. *Assume (2.2)–(2.6). If the datum f belongs to $L^1(\Omega)$, then there exists at least one renormalized solution u to problem (2.1) having $\text{med}(u) = 0$.*

Proof. The proof is divided into 7 steps. In a standard way we begin by introducing a sequence of approximate problems whose data are smooth enough and converge in some sense to the datum f . Then we prove that the weak solutions u_ε to the approximate problems and their gradients ∇u_ε satisfy a priori estimates; such estimates allow to prove that u_ε and ∇u_ε converge to a function u and its gradient ∇u respectively. The final step consists in proving that u is a renormalized solution to (2.1) by showing that it is possible to pass to the limit in the approximate problems.

Step 1. Approximate problems.

For $\varepsilon > 0$, let us define

$$\mathbf{a}_\varepsilon(x, s, \xi) = \mathbf{a}(x, T_{\frac{1}{\varepsilon}}(s), \xi) + \varepsilon|\xi|^{p-2}\xi,$$

$$\Phi_\varepsilon(x, s) = T_{\frac{1}{\varepsilon}}(\Phi(x, s))$$

and $f_\varepsilon \in L^{p'}(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} f_\varepsilon dx &= 0, \\ f_\varepsilon &\rightarrow f \quad \text{strongly in } L^1(\Omega), \\ \|f_\varepsilon\|_{L^1(\Omega)} &\leq \|f\|_{L^1(\Omega)}, \quad \forall \varepsilon > 0. \end{aligned}$$

Let us denote by u_ε one weak solution belonging to $W^{1,p}(\Omega)$ such that

$$\text{med}(u_\varepsilon) = 0$$

and

$$(4.1) \quad \int_{\Omega} \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \nabla \varphi dx + \int_{\Omega} \Phi_\varepsilon(x, u_\varepsilon) \nabla \varphi dx = \int_{\Omega} f_\varepsilon \varphi dx,$$

for every $\varphi \in W^{1,p}(\Omega)$. The existence of such a function u_ε follows from Theorem 3.1.

Step 2. A priori estimates

Using $\varphi = T_k(u_\varepsilon)$ for $k > 0$, as test function in (4.1) we have

$$\begin{aligned} \int_{\Omega} \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) dx + \int_{\Omega} \Phi_\varepsilon(x, T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) dx \\ = \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) dx. \end{aligned}$$

which implies, by (2.2) and (2.5),

$$\alpha \int_{\Omega} |\nabla T_k(u_\varepsilon)|^p dx \leq \int_{\Omega} c(x)(1 + |T_k(u_\varepsilon)|^{p-1}) |\nabla T_k(u_\varepsilon)| dx + k \|f\|_{L^1(\Omega)}.$$

By Young inequality we get

$$(4.2) \quad \int_{\Omega} |\nabla T_k(u_\varepsilon)|^p dx \leq M(k + k^p)$$

for a suitable positive constant M which depends on the data, but does not depend on k and ε .

We deduce that, for every $k > 0$,

$$T_k(u_\varepsilon) \text{ is bounded in } W^{1,p}(\Omega).$$

Moreover taking into account (2.4) and (4.2), we obtain that for any $k > 0$

$$\mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \text{ is bounded in } (L^{p'}(\Omega))^N$$

uniformly with respect to ε . Therefore there exists a measurable function $u : \Omega \rightarrow \overline{\mathbb{R}}$ and for any $k > 0$ there exists a function σ_k belonging to $(L^{p'}(\Omega))^N$ such that, up to a subsequence still indexed by ε ,

$$(4.3) \quad u_\varepsilon \rightarrow u \text{ a.e. in } \Omega,$$

$$(4.4) \quad T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } W^{1,p}(\Omega),$$

$$(4.5) \quad \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \rightharpoonup \sigma_k \text{ weakly in } (L^{p'}(\Omega))^N \quad \forall k > 0.$$

Step 3. The function u is finite a.e. in Ω and $\text{med}(u) = 0$.

Since $\text{med}(u_\varepsilon) = 0$, Poincaré-Wirtinger inequality allows us to use a log-type estimate (see [4, 10, 15, 16] for similar non coercive problems). We consider the function

$$\Psi_p(r) = \int_0^r \frac{1}{(1+|s|)^p} ds, \quad \forall r \in \mathbb{R}.$$

We observe that $\text{med}(\Psi_p(u_\varepsilon)) = \text{med}(u_\varepsilon) = 0$. Using $\Psi_p(u_\varepsilon)$ as test function in (4.1), we get

$$\int_{\Omega} \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \frac{\nabla u_\varepsilon}{(1 + |u_\varepsilon|)^p} dx + \int_{\Omega} \Phi_\varepsilon(x, u_\varepsilon) \frac{\nabla u_\varepsilon}{(1 + |u_\varepsilon|)^p} dx = \int_{\Omega} f_\varepsilon \Psi_p(u_\varepsilon) dx.$$

By ellipticity condition (2.2), growth condition (2.5) and since $\|\Psi_p(u_\varepsilon)\|_{L^\infty(\Omega)} \leq \frac{1}{p-1}$, we get

$$\begin{aligned} \alpha \int_{\Omega} \frac{|\nabla u_\varepsilon|^p}{(1 + |u_\varepsilon|)^p} dx &\leq \int_{\Omega} c(x)(1 + |u_\varepsilon|^{p-1}) \frac{|\nabla u_\varepsilon|}{(1 + |u_\varepsilon|)^p} dx + \frac{1}{p-1} \|f\|_{L^1(\Omega)} \\ &\leq C \int_{\Omega} c(x) \frac{|\nabla u_\varepsilon|}{(1 + |u_\varepsilon|)} dx + \frac{1}{p-1} \|f\|_{L^1(\Omega)}, \end{aligned}$$

where C is a generic and positive constant independent of ε . By Young inequality we deduce

$$(4.6) \quad \int_{\Omega} \frac{|\nabla u_\varepsilon|^p}{(1 + |u_\varepsilon|)^p} dx \leq Cp' \|c\|_{L^{p'}(\Omega)}^{p'} + \frac{p'}{p-1} \|f\|_{L^1(\Omega)}.$$

Let us define

$$\Psi_1(u_\varepsilon) = \int_0^{u_\varepsilon} \frac{1}{(1 + |s|)} ds = \text{sign}(u_\varepsilon) \ln(1 + |u_\varepsilon|).$$

By (4.6) we have

$$\|\nabla \Psi_1(u_\varepsilon)\|_{(L^p(\Omega))^N} \leq C$$

and since $\text{med}(\Psi_1(u_\varepsilon)) = 0$, Poincaré-Wirtinger inequality leads to

$$\|\Psi_1(u_\varepsilon)\|_{L^p(\Omega)} \leq C.$$

According to the definition of Ψ_1 we obtain that

$$(4.7) \quad \sup_{\varepsilon > 0} \text{meas}(\{x \in \Omega; |u_\varepsilon(x)| > A\}) \leq \frac{C}{\ln(1 + A)}$$

and this implies that u is finite almost everywhere in Ω .

Since $\text{med}(u_\varepsilon) = 0$ for any $\varepsilon > 0$ we also have, for any $k > 0$, $\text{med}(T_k(u_\varepsilon)) = 0$, for any $\varepsilon > 0$. Due to the point-wise convergence of u_ε and to the fact that $T_k(u) \in W^{1,p}(\Omega)$ we obtain that $\text{med}(T_k(u)) = 0$ for any $k > 0$. It follows that $\text{med}(u) = 0$.

Step 4. We prove

$$(4.8) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_n(u_\varepsilon) dx = 0.$$

Using the test function $\frac{1}{n} T_n(u_\varepsilon)$ in (4.1) we have

$$\begin{aligned} \frac{1}{n} \int_{\Omega} \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_n(u_\varepsilon) dx + \frac{1}{n} \int_{\Omega} \Phi_\varepsilon(x, T_n(u_\varepsilon)) \nabla T_n(u_\varepsilon) dx \\ = \frac{1}{n} \int_{\Omega} f_\varepsilon T_n(u_\varepsilon) dx, \end{aligned}$$

which yields that

$$(4.9) \quad \frac{1}{n} \int_{\Omega} \mathbf{a}_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla T_n(u_{\varepsilon}) dx \leq \frac{1}{n} \int_{\Omega} |f_{\varepsilon}| \times |T_n(u_{\varepsilon})| dx \\ + \frac{1}{n} \int_{\Omega} c(x)(1 + |T_n(u_{\varepsilon})|^{p-1}) |\nabla T_n(u_{\varepsilon})| dx.$$

Due to (4.3) the sequence $T_n(u_{\varepsilon})$ converges to $T_n(u)$ as ε goes to zero in $L^{\infty}(\Omega)$ weak-*. Since f_{ε} strongly converges to f in $L^1(\Omega)$ it follows that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} |f_{\varepsilon}| \times |T_n(u_{\varepsilon})| dx = \frac{1}{n} \int_{\Omega} |f| \times |T_n(u)| dx.$$

Recalling that u is finite almost everywhere in Ω , the sequence $T_n(u)/n$ converges to 0 as n goes to infinity in $L^{\infty}(\Omega)$ weak-*. Therefore we deduce that

$$(4.10) \quad \lim_{n \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} |f_{\varepsilon}| \times |T_n(u_{\varepsilon})| dx = 0.$$

If R is a positive real number which will be chosen later, let us define for any $\varepsilon > 0$ the set $E_{\varepsilon, R} = \{x \in \Omega : |u_{\varepsilon}(x)| > R\}$. We have for any $n > R$

$$(4.11) \quad \frac{1}{n} \int_{\Omega} c(x)(1 + |T_n(u_{\varepsilon})|^{p-1}) |\nabla T_n(u_{\varepsilon})| dx \\ \leq \frac{1}{n} \int_{\Omega \setminus E_{\varepsilon, R}} c(x)(1 + |T_R(u_{\varepsilon})|^{p-1}) |\nabla T_R(u_{\varepsilon})| dx \\ + \frac{1}{n} \int_{E_{\varepsilon, R}} c(x)(1 + |T_n(u_{\varepsilon})|^{p-1}) |\nabla T_n(u_{\varepsilon})| dx.$$

Hölder inequality yields that

$$\frac{1}{n} \int_{\Omega \setminus E_{\varepsilon, R}} c(x)(1 + |T_R(u_{\varepsilon})|^{p-1}) |\nabla T_R(u_{\varepsilon})| dx \leq \frac{1 + R^{p-1}}{n} \int_{\Omega} c(x) |\nabla T_R(u_{\varepsilon})| dx \\ \leq \frac{1 + R^{p-1}}{n} \|c\|_{L^{p'}(\Omega)} \|\nabla T_R(u_{\varepsilon})\|_{(L^p(\Omega))^N}$$

and since $T_R(u_{\varepsilon})$ is bounded in $W^{1,p}(\Omega)$ uniformly with respect to ε we obtain

$$(4.12) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\{|u_{\varepsilon}| \leq R\}} c(x)(1 + |T_R(u_{\varepsilon})|^{p-1}) |\nabla T_R(u_{\varepsilon})| dx = 0.$$

To control the second term of the right-hand side of (4.11) we distinguish the case $p < N$ and $p = N$. If $p < N$ we have

$$\frac{p-1}{N} + \frac{(N-p)(p-1)}{Np} + \frac{1}{p} = 1$$

so that Hölder inequality gives

$$\begin{aligned} \frac{1}{n} \int_{E_{\varepsilon,R}} c(x)(1 + |T_n(u_\varepsilon)|^{p-1}) |\nabla T_n(u_\varepsilon)| dx &\leq \frac{1}{n} \|c\|_{L^{N/(p-1)}(E_{\varepsilon,R})} \\ &\times \left(\text{meas}(\Omega)^{Np/((N-p)(p-1))} + \|T_n(u_\varepsilon)\|_{L^{pN/(N-p)}(\Omega)} \right) \|\nabla T_n(u_\varepsilon)\|_{(L^p(\Omega))^N}. \end{aligned}$$

Recalling that $\text{med}(T_n(u_\varepsilon)) = 0$ Poincaré-Wirtinger inequality and Sobolev embedding theorem lead to

$$(4.13) \quad \begin{aligned} \frac{1}{n} \int_{E_{\varepsilon,R}} c(x)(1 + |T_n(u_\varepsilon)|^{p-1}) |\nabla T_n(u_\varepsilon)| dx \\ \leq \frac{C}{n} \|c\|_{L^{N/(p-1)}(E_{\varepsilon,R})} \left(1 + \|\nabla T_n(u_\varepsilon)\|_{(L^p(\Omega))^N}^p \right) \end{aligned}$$

where $C > 0$ is a constant independent of n and ε . If $p = N$, since c belongs to $L^q(\Omega)$ with $q > \frac{N}{N-1}$ similar arguments lead to

$$(4.14) \quad \begin{aligned} \frac{1}{n} \int_{E_{\varepsilon,R}} c(x)(1 + |T_n(u_\varepsilon)|^{p-1}) |\nabla T_n(u_\varepsilon)| dx \\ \leq \frac{C}{n} \|c\|_{L^q(E_{\varepsilon,R})} \left(1 + \|\nabla T_n(u_\varepsilon)\|_{(L^p(\Omega))^N}^p \right) \end{aligned}$$

where $C > 0$ is a constant independent of n and ε .

In view of (4.7) and the equi-integrability of c in $L^q(\Omega)$ (with $q = N/(p-1)$ if $p < N$ and $q > N/(N-1)$ if $p = N$) let $R > 0$ such that for any $\varepsilon > 0$

$$(4.15) \quad C \|c\|_{L^q(E_{\varepsilon,R})} < \frac{\alpha}{2},$$

where α denotes the ellipticity constant in (2.2). Using the ellipticity condition (2.2) together with (4.9)–(4.15) leads to

$$\frac{1}{n} \int_{\Omega} \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_n(u_\varepsilon) dx \leq \frac{C}{n} \|c\|_{L^q(\Omega)} + \omega(\varepsilon, n)$$

with $q = N/(p-1)$ if $p < N$ and $q > N/(N-1)$ if $p = N$ and where $\omega(\varepsilon, n)$ is such that $\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \omega(\varepsilon, n) = 0$.

It follows that (4.8) holds.

Step 5. We prove that for any $k > 0$

$$(4.16) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (\mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - \mathbf{a}(x, T_k(u), \nabla T_k(u))) \\ \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx = 0. \end{aligned}$$

Let h_n defined by

$$(4.17) \quad h_n(s) = \begin{cases} 0 & \text{if } |s| > 2n, \\ \frac{2n - |s|}{n} & \text{if } n < |s| \leq 2n, \\ 1 & \text{if } |s| \leq n. \end{cases}$$

Using the admissible test function $h_n(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ to (4.1) we have

$$(4.18) \quad \int_{\Omega} h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx \\ = A_{k,n,\varepsilon} + B_{k,n,\varepsilon} + C_{k,n,\varepsilon} + D_{k,n,\varepsilon} + E_{k,n,\varepsilon}$$

with

$$A_{k,n,\varepsilon} = \int_{\Omega} h_n(u_\varepsilon) f_\varepsilon(T_k(u_\varepsilon) - T_k(u)) dx, \\ B_{k,n,\varepsilon} = - \int_{\Omega} h_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx, \\ C_{k,n,\varepsilon} = - \int_{\Omega} h'_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon (T_k(u_\varepsilon) - T_k(u)) dx, \\ D_{k,n,\varepsilon} = - \int_{\Omega} h'_n(u_\varepsilon) \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon (T_k(u_\varepsilon) - T_k(u)) dx, \\ E_{k,n,\varepsilon} = -\varepsilon \int_{\Omega} h_n(u_\varepsilon) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx.$$

We now pass to the limit in (4.18) first as ε goes to zero and then as n goes to infinity.

Due to the point-wise convergence of u_ε the sequence $T_k(u_\varepsilon) - T_k(u)$ converges to zero almost everywhere in Ω and in $L^\infty(\Omega)$ weak* as ε goes to zero. Since f_ε converges to f strongly in $L^1(\Omega)$ we obtain that

$$\lim_{\varepsilon \rightarrow 0} A_{k,n,\varepsilon} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h_n(u_\varepsilon) f_\varepsilon(T_k(u_\varepsilon) - T_k(u)) dx = 0.$$

For $\varepsilon < 1/n$ we have $h_n(s)\Phi_\varepsilon(x, s) = h_n(s)\Phi(x, s)$ for any $s \in \mathbb{R}$ and a.e. in Ω . Using the point-wise convergence of u_ε $h_n(u_\varepsilon)\Phi_\varepsilon(x, u_\varepsilon)$ converges to $h_n(u)\Phi(x, u)$ a.e. in Ω as ε goes to zero while by (2.5) we have $h_n(u_\varepsilon)|\Phi_\varepsilon(x, u_\varepsilon)| \leq (1 + (2n)^{p-1})c(x)$. It follows that $h_n(u_\varepsilon)\Phi_\varepsilon(x, u_\varepsilon)$ converges to $h_n(u)\Phi(x, u)$ strongly in $(L^q(\Omega))^N$ with $q = N/(p-1)$ if $N > p$ and $q > N/(N-1)$ if $N = p$. Due to (4.4) we deduce that

$$\lim_{\varepsilon \rightarrow 0} B_{k,n,\varepsilon} = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx = 0.$$

With arguments already used we also have for any $n \geq 1/\varepsilon$

$$\lim_{\varepsilon \rightarrow 0} C_{k,n,\varepsilon} = - \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h'_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon (T_k(u_\varepsilon) - T_k(u)) dx = 0.$$

Since

$$|D_{k,n,\varepsilon}| \leq \frac{2k}{n} \int_{\{|u_\varepsilon| \leq 2n\}} \mathbf{a}_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon dx$$

and due to (4.8) we obtain that

$$\lim_{n \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} D_{k,n,\varepsilon} = 0.$$

The identification $h_n(u_\varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon = h_n(u_\varepsilon)|\nabla T_{2n}(u_\varepsilon)|^{p-2}\nabla T_{2n}(u_\varepsilon)$ a.e. in Ω and estimate (4.2) imply that

$$h_n(u_\varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon(\nabla T_k(u_\varepsilon) - \nabla T_k(u))$$

is bounded in $L^1(\Omega)$ uniformly with respect to ε . It follows that

$$\varepsilon h_n(u_\varepsilon)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon(\nabla T_k(u_\varepsilon) - \nabla T_k(u))$$

converges to 0 strongly in $L^1(\Omega)$ so that

$$\lim_{\varepsilon \rightarrow 0} E_{k,n,\varepsilon} = 0.$$

As a consequence we obtain that for any $k > 0$

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx = 0.$$

Recalling that for any $n > k$, we have

$$h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) = \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) \quad \text{a.e. in } \Omega.$$

It follows that

$$(4.19) \quad \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) dx \\ \leq \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u) dx.$$

According to the definition of h_n we have

$$h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) = h_n(u_\varepsilon) \mathbf{a}(x, T_{2n}(u_\varepsilon), \nabla T_{2n}(u_\varepsilon)) \quad \text{a.e. in } \Omega$$

so that (4.3) and (4.5) give

$$(4.20) \quad \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u) dx = \int_{\Omega} h_n(u) \sigma_{2n} \nabla T_k(u) dx.$$

If $n > k$ we have

$$\mathbf{a}(x, T_n(u_\varepsilon), \nabla T_n(u_\varepsilon)) \chi_{\{|u_\varepsilon| < k\}} = \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \chi_{\{|u_\varepsilon| < k\}}$$

almost everywhere in Ω . From (4.3) and (4.5) it follows that

$$\sigma_n \chi_{\{|u| < k\}} = \sigma_k \chi_{\{|u| < k\}} \quad \text{a.e. in } \Omega \setminus \{|u| = k\}$$

and then we obtain for any $n > k$

$$\sigma_n \nabla T_k(u) = \sigma_k \nabla T_k(u) \quad \text{a.e. in } \Omega.$$

Therefore (4.19) and (4.20) allow to conclude that

$$(4.21) \quad \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) dx \leq \int_{\Omega} \sigma_k \nabla T_k(u) dx.$$

We are now in a position to prove (4.16). Indeed the monotone character of \mathbf{a} implies that for any $\varepsilon > 0$

$$(4.22) \quad 0 \leq \int_{\Omega} (\mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u))) \cdot (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx.$$

Moreover, using the point-wise convergence of $T_k(u_\varepsilon)$ and assumption (2.4), the function $\mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u))$ converges to $\mathbf{a}(x, T_k(u), \nabla T_k(u))$ strongly in $(L^{p'}(\Omega))^N$. Writing

$$\begin{aligned} & \int_{\Omega} (\mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u))) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx \\ &= \int_{\Omega} \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx \\ & \quad - \int_{\Omega} \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u)) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) dx, \end{aligned}$$

using (4.21) and (4.22) allow to conclude that (4.16) holds for any $k > 0$.

Step 6. We prove in this step that for any $k > 0$

$$(4.23) \quad \mathbf{a}(x, T_k(u), \nabla T_k(u)) = \sigma_k$$

$$(4.24) \quad \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) \rightharpoonup \mathbf{a}(x, T_k(u), \nabla T_k(u)) \nabla T_k(u)$$

weakly in $L^1(\Omega)$ as ε goes to zero.

From (4.22) we have for any $k > 0$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) dx = \int_{\Omega} \sigma_k \nabla T_k(u) dx.$$

The monotone character of \mathbf{a} and the usual Minty argument imply (4.23).

From (4.16) we get

$$(\mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u))) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) \rightarrow 0$$

strongly in $L^1(\Omega)$ as ε goes to zero. Using (4.4) and recalling that the sequence $\mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u))$ converges to $\mathbf{a}(x, T_k(u), \nabla T_k(u))$ strongly in $(L^{p'}(\Omega))^N$ the monotone character of \mathbf{a} leads to (4.24).

Step 7. We are now in a position to pass to the limit in the approximated problem.

Let h be a function in $W^{1,\infty}(\mathbb{R})$ with compact support, contained in the interval $[-k, k]$, $k > 0$ and let $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Using $\varphi h(u_\varepsilon)$ as a test

function in the approximated problem we have

$$\begin{aligned}
(4.25) \quad & \int_{\Omega} h(u_{\varepsilon}) \mathbf{a}_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla \varphi dx + \int_{\Omega} h'(u_{\varepsilon}) \mathbf{a}_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla u_{\varepsilon} \varphi dx \\
& + \int_{\Omega} h(u_{\varepsilon}) \Phi_{\varepsilon}(x, u_{\varepsilon}) \nabla \varphi dx + \int_{\Omega} h'(u_{\varepsilon}) \Phi_{\varepsilon}(x, u_{\varepsilon}) \nabla u_{\varepsilon} \varphi dx \\
& = \int_{\Omega} f_{\varepsilon} \varphi h(u_{\varepsilon}) dx.
\end{aligned}$$

We want to pass to the limit in this equality. Since $\text{supp } h$ is contained in the interval $[-k, k]$, by the strong converge of f_{ε} to f and (4.3) we immediatly obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} f_{\varepsilon} \varphi h(u_{\varepsilon}) dx = \int_{\Omega} f \varphi h(u) dx.$$

Moreover by growth condition (2.5) and (4.3), using Lebesgue convergence theorem we deduce that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} h(u_{\varepsilon}) \Phi_{\varepsilon}(x, u_{\varepsilon}) \nabla \varphi dx = \int_{\Omega} h(u) \Phi(x, u) \nabla \varphi dx.$$

Analogously from (4.4) we obtain

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega} h'(u_{\varepsilon}) \Phi_{\varepsilon}(x, u_{\varepsilon}) \nabla u_{\varepsilon} \varphi dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h'(u_{\varepsilon}) \Phi(x, T_k(u_{\varepsilon})) \nabla T_k(u_{\varepsilon}) \varphi dx \\
&= \int_{\Omega} h'(u) \Phi(x, u) \nabla T_k(u) \varphi dx.
\end{aligned}$$

In view of the definition of \mathbf{a}_{ε} and since $\varepsilon |\nabla T_k(u_{\varepsilon})|^{p-2} \nabla T_k(u_{\varepsilon})$ converges to zero strongly in $(L^{p'}(\Omega))^N$ as ε goes to zero, (4.5) and (4.23) imply that

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega} h(u_{\varepsilon}) \mathbf{a}_{\varepsilon}(x, u_{\varepsilon}, \nabla T_k(u_{\varepsilon})) \nabla \varphi dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h(u_{\varepsilon}) \mathbf{a}(x, T_k(u_{\varepsilon}), \nabla T_k(u_{\varepsilon})) \nabla \varphi dx \\
&= \int_{\Omega} h(u) \mathbf{a}(x, u, \nabla T_k(u)) \nabla \varphi dx.
\end{aligned}$$

From (4.24) we get

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega} h'(u_{\varepsilon}) \mathbf{a}_{\varepsilon}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla u_{\varepsilon} \varphi dx \\
&= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h'(u_{\varepsilon}) \mathbf{a}(x, T_k(u_{\varepsilon}), \nabla T_k(u_{\varepsilon})) \nabla T_k(u_{\varepsilon}) \varphi dx \\
&= \int_{\Omega} h'(u) \mathbf{a}(x, u, \nabla T_k(u)) \nabla T_k(u) \varphi dx.
\end{aligned}$$

Therefore by passing to the limit in (4.25) we obtain condition (2.12) in the definition of renormalized solution. The decay of the truncated energy (2.11) is a consequence of (4.8) and (4.24). Since u is finite almost everywhere in

Ω and since $T_k(u) \in W^{1,p}(\Omega)$ for any $k > 0$ we can conclude that u is a renormalized solution to (2.1) and that $\text{med}(u) = 0$. \square

5. STABILITY RESULT AND FURTHER REMARKS

This section is devoted to state a stability result and to prove that if the right-hand side f is regular enough, under additional assumptions on \mathbf{a} , then any renormalized solution is also a weak solution.

For $\varepsilon > 0$ let f_ε belonging to $L^1(\Omega)$ and $\Phi_\varepsilon : \Omega \times \mathbb{R} \mapsto \mathbb{R}^N$ a Carathéodory function. Assume that there exists $c \in L^q(\Omega)$ with $q = N/(p-1)$ if $p < N$ and $q > N/(N-1)$ if $p = N$ such that for any $\varepsilon > 0$

$$(5.1) \quad |\Phi_\varepsilon(x, s)| \leq c(x)(|s|^{p-1} + 1)$$

for almost everywhere in Ω and every $s \in \mathbb{R}$. For any $\varepsilon > 0$ let u_ε be a renormalized solution (having null median) to the problem

$$(5.2) \quad \begin{cases} -\text{div}(\mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) + \Phi_\varepsilon(x, u_\varepsilon)) = f_\varepsilon & \text{in } \Omega, \\ (\mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) + \Phi_\varepsilon(x, u_\varepsilon)) \cdot \underline{n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where \mathbf{a} verifies (2.2)–(2.4).

Moreover assume that

$$(5.3) \quad \int_{\Omega} f_\varepsilon dx = 0, \quad f_\varepsilon \rightarrow f \text{ strongly in } L^1(\Omega)$$

and for almost every x in Ω

$$(5.4) \quad \begin{cases} \Phi_\varepsilon(x, s_\varepsilon) \rightarrow \Phi(x, s) \\ \text{for every sequence } s_\varepsilon \in \mathbb{R} \text{ such that } s_\varepsilon \rightarrow s \end{cases}$$

where Φ is a Carathéodory function verifying (as a consequence of (5.1)) the growth condition (2.5)).

Theorem 5.1. *Under the assumptions (5.1), (5.2), (5.3), (5.4), up to a subsequence (still indexed by ε) u_ε converges to u as ε goes to zero where u is a renormalized solution to (2.1) with null median. More precisely we have*

$$(5.5) \quad u_\varepsilon \rightarrow u \text{ a.e in } \Omega,$$

$$(5.6) \quad \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) \rightharpoonup \mathbf{a}(x, T_k(u), \nabla T_k(u)) \nabla T_k(u)$$

weakly $L^1(\Omega)$.

Sketch of proof. We mainly follow the arguments developed in the proof of Theorem 4.1. As usual, the crucial point is to obtain *a priori* estimates, i.e.

$$(5.7) \quad T_k(u_\varepsilon) \text{ bounded in } W^{1,p}(\Omega),$$

$$(5.8) \quad \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \text{ bounded in } (L^{p'}(\Omega))^N \text{ for any } k > 0$$

and

$$(5.9) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_n(u_\varepsilon) dx = 0.$$

Even if $T_k(u_\varepsilon)$ is not an admissible test function in the renormalized formulation (see Definition 2.2) it is well known that it can be achieved through the following process. Using $h = h_n$, where h_n is defined in (4.17), and $\varphi = T_k(u_\varepsilon)$ in the renormalized formulation (2.12) we have, for any $n > 0$ and any $k > 0$

$$(5.10) \quad \int_{\Omega} h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) dx \\ + \int_{\Omega} h'_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon T_k(u_\varepsilon) dx + \int_{\Omega} h_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) \nabla T_k(u_\varepsilon) dx \\ + \int_{\Omega} h'_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon T_k(u_\varepsilon) dx = \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) h_n(u_\varepsilon) dx.$$

We now pass to the limit as n goes to infinity. In view of the definition of h_n for any $n > k$ we have

$$\int_{\Omega} h_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) dx = \int_{\Omega} \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) dx$$

and

$$\int_{\Omega} h_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) \nabla T_k(u_\varepsilon) dx = \int_{\Omega} \Phi(x, u_\varepsilon) \nabla T_k(u_\varepsilon) dx.$$

Since u_ε is finite almost everywhere in Ω , the function $h_n(u_\varepsilon)$ converges to 1 in $L^\infty(\Omega)$ weak*, so that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) h_n(u_\varepsilon) dx = \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) dx.$$

Due to (2.11), we get

$$\lim_{n \rightarrow +\infty} \int_{\Omega} h'_n(u_\varepsilon) \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon T_k(u_\varepsilon) dx = 0.$$

It remains to control the behavior of the fourth term to the right hand side of (5.10). Since we have

$$\left| \int_{\Omega} h'_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon T_k(u_\varepsilon) dx \right| \leq \frac{k}{n} \int_{\Omega} |\Phi_\varepsilon(x, u_\varepsilon)| \times |\nabla T_k(u_\varepsilon)| dx$$

recalling (2.13) we obtain that

$$\lim_{n \rightarrow +\infty} \left| \int_{\Omega} h'_n(u_\varepsilon) \Phi_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon T_k(u_\varepsilon) dx \right| = 0.$$

It follows that passing to the limit as n goes to infinity in (5.10) leads to

$$(5.11) \quad \int_{\Omega} \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u_\varepsilon) dx + \int_{\Omega} \Phi(x, u_\varepsilon) \nabla T_k(u_\varepsilon) dx \\ = \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) dx$$

and then assumptions on \mathbf{a} , Φ_ε and f_ε give (5.7) and (5.8).

For the same reasons following Step 2 in the proof of Theorem 4.1, there exists a function u such that, up to a subsequence still indexed by ε ,

$$\begin{aligned} u_\varepsilon &\rightarrow u \text{ a.e. in } \Omega, \\ T_k(u_\varepsilon) &\rightharpoonup T_k(u) \text{ weakly in } W^{1,p}(\Omega), \\ \mathbf{a}(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) &\rightharpoonup \sigma_k \text{ weakly in } (L^{p'}(\Omega))^N \quad \forall k > 0, \end{aligned}$$

where σ_k belongs to $L^{p'}(\Omega)$ for any $k > 0$.

Using a similar process to one used to obtain (5.11) we get

$$\begin{aligned} (5.12) \quad \int_{\Omega} \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \frac{\nabla u_\varepsilon}{(1 + |u_\varepsilon|)^p} dx + \int_{\Omega} \Phi_\varepsilon(x, u_\varepsilon) \frac{\nabla u_\varepsilon}{(1 + |u_\varepsilon|)^p} dx \\ = \int_{\Omega} f_\varepsilon \Psi_p(u_\varepsilon) dx, \end{aligned}$$

where $\Psi_p(r) = \int_0^r \frac{1}{(1+|s|)^p} ds$. Therefore the arguments developed in Steps 3 and 4 imply that u is finite almost everywhere in Ω and lead to (5.9). Because the sequel of the proof uses mainly admissible test function in the renormalized formulation and the monotone character of the operator we can repeat the same arguments to show that u is a renormalized solution to (2.1) with null median. In particular following Steps 5 and 6 (see (4.24) in the proof of Theorem 4.1) allow to obtain that (5.6) hold. \square

Now we prove that if $\mathbf{a}(x, r, \xi)$ is a classical Leray-Lions operator verifying (3.4) and if $f \in L^q$ with $q \leq (p^*)'$ then any renormalized solution to (2.1) is also a weak solution to (2.1) belonging to $W^{1,p}(\Omega)$.

Proposition 5.2. *Assume that (2.2), (2.3), (2.5), (2.6) and (3.4) hold. Let u be a renormalized solution to (2.1) with $\text{med}(u) = 0$. If $f \in L^q(\Omega)$ with $q \leq (p^*)'$ if $N > p$ and $q < +\infty$ if $N = p$ then u belongs to $W^{1,p}(\Omega)$ and*

$$\int_{\Omega} \mathbf{a}(x, u, \nabla u) \nabla v dx + \int_{\Omega} \Phi(x, u) \nabla v dx = \int_{\Omega} f v dx$$

for any $v \in W^{1,p}(\Omega)$.

Proof. Let u be a renormalized solution to (2.1). We can proceed as in the proof of Theorem 5.1 and we obtain (5.11). Then we have

$$(5.13) \quad \int_{\Omega} \mathbf{a}(x, u, \nabla u) \nabla T_k(u) dx + \int_{\Omega} \Phi(x, u) \nabla T_k(u) dx = \int_{\Omega} f T_k(u) dx.$$

Using (2.2), (2.5) and the regularity of f we obtain

$$\alpha \int_{\Omega} |\nabla T_k(u)|^p dx \leq \int_{\Omega} c(x) (1 + |u|^{p-1}) |\nabla T_k(u)| dx + \|f\|_{L^q(\Omega)} \|T_k(u)\|_{L^{q'}(\Omega)}.$$

Let $R > 0$ be a real number which will be chosen later and denote

$$E_R = \{x \in \Omega; |u(x)| > R\}.$$

Using again $\text{med}(T_k(u)) = 0$, Poincaré-Wirtinger inequality and Sobolev embedding Theorem we have

$$\begin{aligned} \alpha \int_{\Omega} |\nabla T_k(u)|^p dx &\leq \int_{\Omega} c(x) |\nabla T_k(u)| dx + \int_{E_R} c(x) |u|^{p-1} |\nabla T_k(u)| dx \\ &\quad + \int_{\Omega \setminus E_R} c(x) |u|^{p-1} |\nabla T_k(u)| dx + \|f\|_{L^q(\Omega)} \|T_k(u)\|_{L^{q'}(\Omega)}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\Omega} |\nabla T_k(u)|^p dx &\leq C \left(\|\nabla T_k(u)\|_{(L^p(\Omega))^N} + \|c\|_{L^q(E_R)} \|\nabla T_k(u)\|_{(L^p(\Omega))^N}^p \right. \\ &\quad \left. + R^{p-1} \|\nabla T_R(u)\|_{(L^p(\Omega))^N} \right) \end{aligned}$$

where $C > 0$ depends on $\alpha, f, N, p, \text{meas}(\Omega), c$ but is independent of k . Since u is finite a.e. in Ω , $\lim_{R \rightarrow +\infty} \text{meas}(E_R) = 0$. By the equi-integrability of c in $L^q(\Omega)$ we can choose $R > 0$ such that $C\|c\|_{L^q(E_R)}$ is sufficiently small enough so that

$$\int_{\Omega} |\nabla T_k(u)|^p dx \leq C \left(\|\nabla T_k(u)\|_{(L^p(\Omega))^N} + R^{p-1} \|\nabla T_R(u)\|_{(L^p(\Omega))^N} \right)$$

where $C > 0$ does not depend on k . It follows that

$$\int_{\Omega} |\nabla T_k(u)|^p dx \leq C$$

where $C > 0$ depends on $\alpha, f, N, p, \Omega, c, R$ but is independent of k . Since $\text{med}(T_k(u)) = 0$ Poincaré-Wirtinger inequality implies that $T_k(u)$ is bounded in $W^{1,p}(\Omega)$ uniformly with respect to k . Therefore we conclude that u belongs to $W^{1,p}(\Omega)$.

Using the renormalized formulation (2.12) with $h = h_n$ and passing to the limit as n goes to infinity leads to

$$(5.14) \quad \int_{\Omega} \mathbf{a}(x, u, \nabla u) \nabla v dx + \int_{\Omega} \Phi(x, u) \nabla v dx = \int_{\Omega} f v dx$$

for any $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. Due to growth assumptions (3.4) on \mathbf{a} and (2.5) on Φ we deduce that $\mathbf{a}(x, u, \nabla u)$ and $\Phi(x, u)$ belong to $(L^{p'}(\Omega))^N$. It follows that (5.14) holds for any $v \in W^{1,p}(\Omega)$. \square

6. OPERATOR WITH A ZERO ORDER TERM

In this section we consider Neumann problems which are similar to (1.1) with a zero order term. Precisely let us consider the following Neumann problem

$$(6.1) \quad \begin{cases} \lambda(x, u) - \text{div}(\mathbf{a}(x, u, \nabla u) + \Phi(x, u)) = f & \text{in } \Omega, \\ (\mathbf{a}(x, u, \nabla u) + \Phi(x, u)) \cdot \underline{n} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda : \Omega \times \mathbb{R}$ is a Carathéodory function verifying

$$(6.2) \quad \lambda(x, s)s \geq 0,$$

$$(6.3) \quad \forall k > 0, \exists c_k > 0 \text{ such that } |\lambda(x, s)| \leq c_k \quad \forall |s| \leq k, \text{ a.e. in } \Omega,$$

$$(6.4) \quad \forall s \in \mathbb{R} \quad |\lambda(x, s)| \geq g(s) \text{ a.e. in } \Omega$$

where g is function such that $\lim_{s \rightarrow \pm\infty} g(s) = +\infty$.

If f belongs to $L^1(\Omega)$ and without additional growth assumptions on g we cannot expect to have in general a solution (in whatever sense) lying in $L^1(\Omega)$ and then we have similar difficulties to deal with (6.1). In particular the presence of $\lambda(x, u)$ does not help to deal with the term $-\operatorname{div}(\Phi(x, u))$ and we cannot follow the approach of [3, 15, 16, 25] which use the mean value. However the “median” tool and some modifications of the proof of Theorem 4.1 allow to show that there exists at least a renormalized solution to (6.1):

Theorem 6.1. *Assume (2.2)–(2.6) and (6.2)–(6.4). If the datum f belongs to $L^1(\Omega)$ then there exists at least one renormalized solution u to problem (6.1).*

Sketch of proof. As in Theorem 3.1, a fixed point theorem and classical results of Leray-Lions give the existence of u_ε belonging to $W^{1,p}(\Omega)$ verifying

$$(6.5) \quad \varepsilon \int_{\Omega} |u_\varepsilon|^{p-2} u_\varepsilon v dx + \int_{\Omega} \lambda(x, T_{1/\varepsilon}(u_\varepsilon)) v dx \\ + \int_{\Omega} \mathbf{a}(x, T_{1/\varepsilon}(u_\varepsilon), \nabla u_\varepsilon) \nabla v dx \\ + \int_{\Omega} \Phi(x, T_{1/\varepsilon}(u_\varepsilon)) \nabla v dx = \int_{\Omega} T_{1/\varepsilon}(f) v dx$$

for any v lying in $W^{1,p}(\Omega)$. Due to the zero order term $\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon + \lambda(x, T_{1/\varepsilon}(u_\varepsilon))$ in the equation, we do not need any compatibility condition on f . The counter part is that we cannot expect to have (or to fix) $\operatorname{med}(u_\varepsilon) = 0$ and then it yields another difficulties. In particular Steps 3 and 4 (see the proof of Theorem 4.1) which use strongly the fact that the solution has a null median should be adapted in the case of the approximated problem (6.5). Step 2 is unchanged and we have the following and additional estimate

$$(6.6) \quad T_{1/\varepsilon}(g(u_\varepsilon)) \text{ bounded in } L^1(\Omega).$$

Due to the behavior at infinity of the function g we deduce that

$$(6.7) \quad \lim_{A \rightarrow +\infty} \sup_{\varepsilon > 0} \operatorname{meas}\{x \in \Omega; |u_\varepsilon(x)| > A\} = 0,$$

$$(6.8) \quad \forall \varepsilon > 0 \quad |\operatorname{med}(u_\varepsilon)| \leq M$$

where M is a positive real number independent of ε . It follows (after extracting appropriate subsequence, see Step 2) that there exists a measurable

function u which is finite almost everywhere in Ω such that

$$u_\varepsilon \rightarrow u \text{ a.e. in } \Omega,$$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } W^{1,p}(\Omega), \quad \forall k > 0.$$

Step 4 which is crucial in dealing with renormalized solutions consists here in proving that

$$(6.9) \quad \lim_{n \rightarrow +\infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_n(u_\varepsilon) dx = 0$$

using the test function $T_n(u_\varepsilon)$ in (6.5). Due to the sign condition (6.2) the contribution of the zero order terms

$$\varepsilon \int_{\Omega} |u_\varepsilon|^{p-2} u_\varepsilon T_n(u_\varepsilon) dx + \int_{\Omega} \lambda(x, T_{1/\varepsilon}(u_\varepsilon)) T_n(u_\varepsilon) dx$$

is positive. It follows that the inequality (4.9) holds:

$$\begin{aligned} \frac{1}{n} \int_{\Omega} \mathbf{a}(x, u_\varepsilon, \nabla u_\varepsilon) \nabla T_n(u_\varepsilon) dx &\leq \frac{1}{n} \int_{\Omega} |T_{1/\varepsilon}(f)| \times |T_n(u_\varepsilon)| dx \\ &+ \frac{1}{n} \int_{\Omega} c(x) (1 + |T_n(u_\varepsilon)|^{p-1}) |\nabla T_n(u_\varepsilon)| dx. \end{aligned}$$

Because we do not have in the present case the property $\text{med}(u_\varepsilon) = \text{med}(T_n(u_\varepsilon)) = 0$ we have to modify the estimate of the term

$$(6.10) \quad \frac{1}{n} \int_{\Omega} c(x) (1 + |T_n(u_\varepsilon)|^{p-1}) |\nabla T_n(u_\varepsilon)| dx.$$

In view of (6.8) we have for any $n > 0$ and for any $\varepsilon > 0$ $|\text{med}(T_n(u_\varepsilon))| \leq M$. It follows that by writing $T_n(u_\varepsilon) = T_n(u_\varepsilon) - \text{med}(T_n(u_\varepsilon)) + \text{med}(T_n(u_\varepsilon))$ we obtain

$$(6.11) \quad \begin{aligned} \frac{1}{n} \int_{\Omega} c(x) (1 + |T_n(u_\varepsilon)|^{p-1}) |\nabla T_n(u_\varepsilon)| dx \\ \leq \frac{C}{n} \int_{\Omega} c(x) (1 + |T_n(u_\varepsilon) - \text{med}(T_n(u_\varepsilon))|^{p-1}) |\nabla T_n(u_\varepsilon)| dx \end{aligned}$$

where $C > 0$ is a constant independent of ε and n . Poincaré-Wirtinger inequality (2.8), similar arguments to the ones developed in Step 4 and (6.8) then allow conclude that (6.9) holds.

As far as Step 5 is concerned, it is sufficient to remark that the Lebesgue Theorem yields that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} h_n(u_\varepsilon) |u_\varepsilon|^{p-2} u_\varepsilon (T_k(u_\varepsilon) - T_k(u)) dx &= 0 \\ \lim_{n \rightarrow +\infty} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h_n(u_\varepsilon) \lambda(x, T_{1/\varepsilon}(u_\varepsilon)) (T_k(u_\varepsilon) - T_k(u)) dx &= 0. \end{aligned}$$

Since Step 6 remains unchanged, in Step 7 we pass to the limit as ε goes to zero in

$$\begin{aligned} & \varepsilon \int_{\Omega} h(u_{\varepsilon}) |u_{\varepsilon}|^{p-2} u_{\varepsilon} \varphi dx + \int_{\Omega} h(u_{\varepsilon}) \lambda(x, T_{1/\varepsilon}(u_{\varepsilon})) \varphi dx \\ & + \int_{\Omega} h(u_{\varepsilon}) \mathbf{a}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla \varphi dx + \int_{\Omega} h'(u_{\varepsilon}) \mathbf{a}(x, u_{\varepsilon}, \nabla u_{\varepsilon}) \nabla u_{\varepsilon} \varphi dx \\ & + \int_{\Omega} h(u_{\varepsilon}) \Phi(x, u_{\varepsilon}) \nabla \varphi dx + \int_{\Omega} h'(u_{\varepsilon}) \Phi(x, u_{\varepsilon}) \nabla u_{\varepsilon} \varphi dx \\ & = \int_{\Omega} T_{1/\varepsilon}(f) \varphi h(u_{\varepsilon}) dx \end{aligned}$$

where h is a Lipschitz continuous function with compact support and where φ lies in $W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$. Since the Lebesgue Theorem gives that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Omega} h(u_{\varepsilon}) |u_{\varepsilon}|^{p-2} u_{\varepsilon} \varphi dx &= 0 \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} h(u_{\varepsilon}) \lambda(x, T_{1/\varepsilon}(u_{\varepsilon})) \varphi dx &= \int_{\Omega} h(u) \lambda(x, u) \varphi dx \end{aligned}$$

the attentive reader may convince by himself that we obtain the existence of a renormalized solution to equation (6.1). \square

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