HAL open science

The area of a spectrally positive stable process stopped at zero

Julien Letemplier, Thomas Simon

▶ To cite this version:

Julien Letemplier, Thomas Simon. The area of a spectrally positive stable process stopped at zero. 2014. hal-01070220

HAL Id: hal-01070220
https://hal.science/hal-01070220v1

Preprint submitted on 30 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE AREA OF A SPECTRALLY POSITIVE STABLE PROCESS STOPPED AT ZERO

JULIEN LETEMPLIER AND THOMAS SIMON

Abstract. An identity in law for the area of a spectrally positive Lévy $\alpha-$stable process stopped at zero is established. Extending that of Lefebvre [14] for Brownian motion, it involves an inverse Beta random variable and the square of a positive stable random variable. This identity entails that the stopped area is distributed as the perpetuity of a spectrally negative Lévy process, and is hence self-decomposable. We also derive a convergent series representation for the density, whose behaviour at zero is shown to be Fréchet-like.

1. Introduction and statement of the results

Let $\{B_t,\, t \geq 0\}$ be a standard linear Brownian motion, starting from one, and let $T = \inf\{t > 0,\, B_t = 0\}$ be its first hitting time of zero. The random variable

$$\mathcal{A} \;=\; \int_0^T B_s\, ds$$

has been investigated by Lefebvre, who obtained in Theorem 2 of [14] the simple identity in law

$$\mathcal{A} \;\stackrel{d}{=}\; \frac{2}{9\mathbf{\Gamma}_{1/3}} \tag{1.1}$$

where, here and throughout, $\mathbf{\Gamma}_a$ stands for the Gamma random variable with density

$$\frac{x^{a-1}}{\Gamma(a)}\, e^{-x}\, \mathbf{1}_{\{x>0\}}.$$

The identity (1.1) is obtained as a consequence of the closed expression for the Laplace transform of the bivariate random variable $(T, \mathcal{A})$ in terms of the Airy function - see Theorem 1 in [14]. As observed in [13] p. 402, this latter expression can be easily derived thanks to the Feynman-Kac formula. Notice that Airy functions also appear in the expression of the Laplace transform of many other Brownian areas - see [11], whose laws are more complicated than (1.1).

In this paper, our concern is to generalize (1.1) to the random variables

$$\mathcal{A}_\alpha \;=\; \int_0^T L_s^{(\alpha)}\, ds$$

2010 *Mathematics Subject Classification.* 60E07, 60G51, 60G52.

Key words and phrases. Exponential functional - Hitting time - Integrated process - Moments of Gamma type - Self-decomposability - Series representation - Stable Lévy process.

where $\{L_t^{(\alpha)},\, t \geq 0\}$ is a strictly $\alpha-$stable Lévy process without negative jumps, starting from one, and $T = \inf\{t > 0,\, L_t^{(\alpha)} = 0\}$ is its first hitting time of zero. Without loss of generality we choose the normalization

$$\mathbb{E}[e^{-tL_1^{(\alpha)}}] = e^{t^\alpha}, \qquad t \geq 0,$$

where $\alpha \in [1,2]$ is the self-similarity parameter. We refer e.g. to Chapter 3 in [17] for more information on stable Lévy processes and the above normalization. The boundary cases $\alpha = 1, 2$ correspond to the unit drift resp. the Brownian motion with variance $\sqrt{2}$, so that we have

$$\mathcal{A}_1 \;=\; \frac{1}{2} \qquad \text{and} \qquad \mathcal{A}_2 \;\stackrel{d}{=}\; \frac{1}{9\mathbf{\Gamma}_{1/3}}\cdot \tag{1.2}$$

The above second identity, which is actually the precise statement of Theorem 2 in [14], follows from (1.1) and the self-similarity of Brownian motion. In the case $\alpha \in (1,2)$ the generator of the process $L^{(\alpha)}$ is non-local, and it seems unappropriate to appeal to Feynman-Kac formulæ in order to obtain a tractable expression for the Laplace transform for $\mathcal{A}_\alpha$. Moreover, the absence of transition densities written in closed form prevents from using explicit computations as in the Gaussian case - see [14] and the references therein, to handle the random variable $\mathcal{A}_\alpha$. Instead, we will compute the fractional moments of $\mathcal{A}_\alpha$ and exhibit a multiplicative identity in law. Introduce the Beta random variable $\mathbf{B}_{a,b}$ with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}\, \mathbf{1}_{(0,1)}(x),$$

and the positive $a-$stable random variable $\mathbf{Z}_a$ with Laplace transform

$$\mathbb{E}[e^{-\lambda \mathbf{Z}_a}] \;=\; e^{-\lambda^a}, \qquad \lambda \geq 0.$$

Our main observation is the following.

Theorem. *With the above notation, one has the independent factorization*

$$\mathcal{A}_\alpha \;\stackrel{d}{=}\; \left(\frac{\alpha+1}{4}\right) \times \mathbf{Z}^2_{\frac{2}{\alpha+1}} \;\times\; \mathbf{B}^{-1}_{\frac{1}{2},\frac{\alpha-1}{2(\alpha+1)}} \tag{1.3}$$

for every $\alpha \in (1,2)$.

Observe that (1.3) is in accordance with the two boundary cases: when $\alpha = 1$ the two random variables on the right-hand side boil down to one, whereas when $\alpha = 2$ the following identity obtained in Theorem 1 of [19]:

$$\mathbf{Z}^2_{\frac{2}{3}} \;\stackrel{d}{=}\; \frac{4}{27}\, \mathbf{\Gamma}^{-1}_{\frac{2}{3}} \;\times\; \mathbf{B}^{-1}_{\frac{1}{3},\frac{1}{6}},$$

combined with the elementary factorization $\mathbf{\Gamma}_a \stackrel{d}{=} \mathbf{\Gamma}_{a+b} \times \mathbf{B}_{a,b}$, allows to recover the second identity in (1.2). The proof of (1.3) relies on an identification of the fractional moments of $\mathcal{A}_\alpha$. The explicit computation of the latter in terms of Gamma functions - see (2.1) below - is made possible by the strong Markov property and some exact results on the stable Kolmogorov process recently obtained in [16].

The inverse Gamma random variable involved in Lefebvre's identity shares a number of distributional properties related to infinite divisibility. Recall that a non-negative random variable X is said to be self-decomposable if the following identities hold

$$X \stackrel{d}{=} cX + X_c$$

for every $c \in (0,1)$ with X_c independent of X, or equivalently if its log-Laplace transform is of the type

$$-\log \mathbb{E}[e^{-\lambda X}] = a\lambda + \int_0^\infty (1 - e^{-\lambda x}) \frac{k(x)}{x} dx, \qquad \lambda \geq 0$$

for some $a \geq 0$ and a non-negative, non-increasing function k integrating $1 \wedge x^{-1}$. See again Chapter 3 in [17] for details. The fact that all inverse Gamma random variables are self-decomposable can be observed either by a direct and non-trivial computation on the Laplace transform, or by Dufresne's celebrated identity for the perpetuity of a Brownian motion with drift - see Section 3.2 in [4] for details and references. In this paper we will use the second approach and show the same property for $\mathcal{A}_\alpha$.

Corollary 1. *For every $\alpha \in (1,2)$, the random variable $\mathcal{A}_\alpha$ is self-decomposable.*

More precisely, it will be proved that $\mathcal{A}_\alpha$ is distributed as the perpetuity of a spectrally negative Lévy process which drifts towards $+\infty$. The latter background integrated Lévy process turns out to be tightly connected with the dual process $-L^{(\alpha)}$ conditioned to stay positive, and also to the Fréchet distribution which is hidden in the factorization (1.3) and can be viewed as another perpetuity - see Remark 3 below.

It is known that inverse Gamma distributions also satisfy a property which is more stringent than self-decomposability. The law of a non-negative random variable X is called a generalized Gamma convolution ($X \in \mathcal{G}$ for short) if it is the weak limit of an independent sum of Gamma random variables, or equivalently if its log-Laplace transform reads

$$-\log \mathbb{E}[e^{-\lambda X}] = a\lambda + \int_0^\infty \log\left(\frac{x}{x+\lambda}\right) U(dx), \qquad \lambda \geq 0,$$

for some $a \geq 0$ and some non-negative measure U satisfying certain integrability conditions, which is called the Thorin measure of X. We refer to Chapter 3 in [2] for more details on this notion refining that of self-decomposability. The fact that inverse Gamma random variables are all in $\mathcal{G}$, with explicit Thorin measure, is a consequence of a computation on Bessel functions - see again Section 3.2 in [4] for details and references. Although we cannot write the Laplace transform of $\mathcal{A}_\alpha$ in any tractable way, we can prove the following result.

Corollary 2. *For every $\alpha \in (1,2)$, the law of $\sqrt{\mathcal{A}_\alpha}$ is a generalized Gamma convolution, with infinite Thorin measure.*

A conjecture by Bondesson - see [2] p. 97 and also Section 7 in [3] - states that the $\mathcal{G}-$property is stable by power transformation of order greater than one. If this conjecture is true, then Corollary 2 entails that $\mathcal{A}_\alpha \in \mathcal{G}$ as well, a reinforcement of Corollary 1. It will be shown in Section 3.1. below that $\mathcal{A}_{5/3} \in \mathcal{G}$, but we do not know as yet how to handle the other values of α.

The closed expression for the fractional moments of $\mathcal{A}_\alpha$ can be inverted in order to give a convergent series representation for its density $f_{\mathcal{A}_\alpha}$. Throughout this paper, we will set f_X for the density of an absolutely continuous random variable X.

Corollary 3. *The density of $\mathcal{A}_\alpha$ has a convergent series representation:*

$$f_{\mathcal{A}_\alpha}(x) \;=\; \Gamma\left(\frac{\alpha}{\alpha+1}\right) \times \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha+1)^{\frac{n+1}{\alpha+1}-1}x^{-\frac{n+1}{\alpha+1}-1}}{n!\,\Gamma(1-\frac{n+1}{\alpha+1})\Gamma(1-\frac{n+2}{\alpha+1})}, \qquad x>0.$$

Observe that when $\alpha=2$ the above summation is made over $n=3p$ only, and that further simplifications lead to

$$f_{\mathcal{A}_2}(x) \;=\; \frac{x^{-4/3}}{3^{2/3}\Gamma(1/3)} \sum_{p=0}^{\infty} \frac{(-1)^p(9x)^{-p}}{p!} \;=\; \frac{\Gamma(2/3)\,x^{-4/3}\,e^{-\frac{1}{9x}}}{2\pi\,3^{1/6}}, \tag{1.4}$$

which is the expression to be found in Theorem 2 of [14]. The above corollary also entails the first order asymptotics

$$f_{\mathcal{A}_\alpha}(x) \;\sim\; \frac{(\alpha+1)^{\frac{1}{\alpha+1}-1}x^{-\frac{1}{\alpha+1}-1}}{\Gamma(\frac{\alpha-1}{\alpha+1})} \qquad \text{as } x\to+\infty,$$

which has, up to the multiplicative constant, the same speed as that of the density of the factor $\mathbf{Z}^2_{2/(\alpha+1)}$ at infinity - see Formula (14.31) in [17]. On the other hand, it does not seem possible to deduce from the above series representation the exact behaviour of $f_{\mathcal{A}_\alpha}$ at zero. Nevertheless, using the identity (1.3) we can show the following estimate.

Corollary 4. *The asymptotic behaviour of the density of $\mathcal{A}_\alpha$ when $x\to 0+$ is*

$$f_{\mathcal{A}_\alpha}(x) \;\sim\; \kappa_\alpha\, x^{\frac{\alpha^2}{1-\alpha^2}}\, e^{-c_\alpha\, x^{\frac{1}{1-\alpha}}}, \tag{1.5}$$

with

$$\kappa_\alpha \;=\; \frac{\Gamma(\frac{\alpha}{\alpha+1})\sqrt{\frac{\alpha+1}{\alpha-1}}}{2\pi\,(\alpha+1)^{\frac{\alpha}{\alpha^2-1}}} \qquad \textit{and} \qquad c_\alpha \;=\; (\alpha-1)(\alpha+1)^{\frac{\alpha}{1-\alpha}}.$$

This shows that the behaviour of $f_{\mathcal{A}_\alpha}$ at zero is that of the generalized Fréchet density

$$f_{c_\alpha^{\alpha-1}\mathbf{\Gamma}^{1-\alpha}_{1/(\alpha+1)}}(x) \;=\; \tilde{\kappa}_\alpha\, x^{\frac{\alpha^2}{1-\alpha^2}}\, e^{-c_\alpha\, x^{\frac{1}{1-\alpha}}},$$

up to the normalizing constant

$$\tilde{\kappa}_\alpha \;=\; \frac{(\alpha-1)^{\frac{-\alpha}{\alpha+1}}}{\Gamma(\frac{1}{\alpha+1})\,(\alpha+1)^{\frac{\alpha}{\alpha^2-1}}}$$

which does not coincide with κ_α except for $\alpha=2$. Observe also that making $\alpha=2$ on the right-hand side of (1.5) yields the density in (1.4). It should be possible to obtain a full asymptotic expansion of $f_{\mathcal{A}_\alpha}$ at zero with our method - see Remark 5 below. But we have not adressed this issue, which is believed to be awfully technical, in the present paper.

2. Proofs

2.1. **Proof of the Theorem.** To simplify the notation we will set $L = L^{(\alpha)}$. Introducing the area process

$$A_t \;=\; \int_0^t L_s\, ds, \qquad t \geq 0,$$

recall that the bivariate process $X = \{(A_t, L_t),\, t \geq 0\}$ is strongly Markovian and denote by $\mathbb{P}_{(x,y)}$ its law starting from (x, y). Consider the stopping time

$$S \;=\; \inf\{t > 0,\; A_t = 0\}$$

and observe that under $\mathbb{P}_{(0,1)}$ one has a.s. $S > T, A_T > 0$, and $L_S < 0$. Setting $\{\mathcal{F}_t,\, t \geq 0\}$ for the natural completed filtration of X and applying the strong Markov property at T entails that for every $s \in \mathbb{R}$, one has

$$\begin{aligned}\mathbb{E}_{(0,1)}[|L_S|^{s-1}] \;=\; \mathbb{E}_{(0,1)}[\,\mathbb{E}[|L_S|^{s-1}|\mathcal{F}_T]] \;&=\; \mathbb{E}_{(0,1)}[\,\mathbb{E}_{(A_T,0)}[|L_S|^{s-1}]]\\ &=\; \mathbb{E}\,[\mathcal{A}_\alpha^{\frac{s-1}{\alpha+1}}] \,\times\, \mathbb{E}_{(1,0)}[|L_S|^{s-1}],\end{aligned}$$

possibly with infinite values on both sides, where the second equality follows from the absence of negative jumps for L, and the third equality from the self-similarity of L and A with respective indices $1/\alpha$ and $1 + 1/\alpha$. Applying Theorem B in [16] in the particular case $\rho = 1/\alpha$ (beware that we consider here the dual process, with no positive jumps), we get

$$\mathbb{E}_{(0,1)}[|L_S|^{s-1}] \;=\; \frac{\sin(\frac{\pi s}{\alpha+1})}{\sin(\frac{\pi\alpha s}{\alpha+1})}$$

and

$$\mathbb{E}_{(1,0)}[|L_S|^{s-1}] \;=\; \frac{(\alpha+1)^{\frac{1-s}{\alpha+1}}\Gamma(\frac{\alpha+2}{\alpha+1})\Gamma(\frac{1-s}{\alpha+1})\sin(\frac{\pi}{\alpha+1})}{\Gamma(\frac{s}{\alpha+1})\Gamma(1-s)\sin(\frac{\pi\alpha s}{\alpha+1})}$$

for all $|s| < 1 + 1/\alpha$. Dividing and simplifying with the help of the complement formula for the Gamma function, we deduce

$$\mathbb{E}\,[\mathcal{A}_\alpha^s] \;=\; (\alpha+1)^s \,\times\, \frac{\Gamma(\frac{\alpha}{\alpha+1})\Gamma(1-(\alpha+1)s)}{\Gamma(\frac{\alpha}{\alpha+1}-s)\Gamma(1-s)} \tag{2.1}$$

for all $s < 1/(\alpha + 1)$. Applying the Legendre-Gauss multiplication formula for the Gamma function entails

$$\mathbb{E}\,[\mathcal{A}_\alpha^s] \;=\; \left(\frac{\alpha+1}{4}\right)^s \times \frac{\Gamma(1-(\alpha+1)s)}{\Gamma(1-2s)} \,\times\, \frac{\Gamma(\frac{\alpha}{\alpha+1})\Gamma(\frac{1}{2}-s)}{\Gamma(\frac{1}{2})\Gamma(\frac{\alpha}{\alpha+1}-s)},$$

and we can conclude by a fractional moment identification, recalling (see e.g. Formula (25.5) in [17] for the second expression) that

$$\mathbb{E}\,[\mathbf{B}_{a,b}^s] \;=\; \frac{\Gamma(a+s)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+s)} \qquad \text{and} \qquad \mathbb{E}\,[\mathbf{Z}_a^s] \;=\; \frac{\Gamma(1-\frac{s}{a})}{\Gamma(1-s)}.$$

□

Remark 1. (a) It is well-known and easy to see - see e.g. Theorem 46.3 in [17] - that under $\mathbb{P}_{(0,1)}$, the random variable T is distributed as $\mathbf{Z}_{1/\alpha}$. The above theorem provides hence a connection between $\mathcal{A}_\alpha = A_T$ and the random variable $\mathbf{Z}_{2/(\alpha+1)}$. Notice that one can also derive from (2.1) another factorization:

$$\mathbf{Z}_{\frac{1}{\alpha+1}} \stackrel{d}{=} (\alpha+1)^{-1} \times \mathbf{\Gamma}^{-1}_{\frac{\alpha}{\alpha+1}} \times \mathcal{A}_\alpha.$$

However, it seems difficult with our method to obtain some valuable information on the Mellin transform of the bivariate random variable (T, A_T).

(b) With the notation of our above proof, it is possible to derive the law of A_T under $\mathbb{P}_{(x,y)}$ for any $x \in \mathbb{R}$ and $y > 0$, by the self-similarity of $L^{(\alpha)}$. One finds

$$\mathcal{A}_\alpha \stackrel{d}{=} x + \left(\frac{(\alpha+1)\, y^{\alpha+1}}{4}\right) \times \mathbf{Z}^2_{\frac{2}{\alpha+1}} \times \mathbf{B}^{-1}_{\frac{1}{2},\frac{\alpha-1}{2(\alpha+1)}}.$$

2.2. **Proof of Corollary 1.** Let us first observe that $\mathcal{A}_\alpha$ is infinitely divisible, by a simple pathwise argument not relying on (1.3). Setting $T_x = \inf\{t > 0,\, L_t = x\}$ for all $x > 0$ and using the fact that L has no negative jumps, it is easy to see from the Markov property that under $\mathbb{P}_{(0,1)}$, for every $n \geq 2$, there is an independent decomposition

$$\mathcal{A}_\alpha = X_1^{(n)} + \cdots + X_n^{(n)}$$

where

$$X_i^{(n)} \stackrel{d}{=} A_{T_{\frac{n-i}{n}}} \text{ under } \mathbb{P}_{(0,\frac{n+1-i}{n})}$$

for every $i = 1, \ldots, n$. Moreover, one has $T_{1-1/n} \to 0$ a.s. under $\mathbb{P}_{(0,1)}$ as $n \to +\infty$ (by the well-known fact - see e.g. Theorem 47.1 in [17] - that L visits immediately the negative half-line when starting from 0), so that $X_1^{(n)} \to 0$ a.s. under $\mathbb{P}_{(0,1)}$ when $n \to +\infty$ as well. Last, it is straightforward that

$$\mathbb{P}[|X_i^{(n)}| > \varepsilon] \leq \mathbb{P}[|X_1^{(n)}| > \varepsilon]$$

for every $\varepsilon > 0$ and $i = 1, \ldots, n$. Putting everything together and applying Khintchine's theorem on triangular arrays - see e.g. Theorem 9.3 in [17], entails that $\mathcal{A}_\alpha$ is infinitely divisible.

□

Remark 2. The above argument does not make use of the self-similarity of L, and hence applies to any spectrally positive Lévy process which is not a subordinator.

We now proceed to the proof of the self-decomposability of $\mathcal{A}_\alpha$. We will use the same argument as in [4, 5], expressing $\mathcal{A}_\alpha$ as the perpetuity of a certain spectrally negative Lévy process. Setting

$$\Psi_\alpha(u) = \frac{u\, \mathbb{E}[\mathcal{A}_\alpha^{-(u+1)}]}{\mathbb{E}[\mathcal{A}_\alpha^{-u}]}$$

for every $u > 0$, we first deduce from (1.3) the formula $\Psi_\alpha(u) = \Phi_\alpha(s)$, with the notation $s = (\alpha+1)u$ and

$$\Phi_\alpha(s) \;=\; \frac{\Gamma(\alpha+s)}{\Gamma(s)}\cdot$$

Applying the Lemma in [4] with $t = 1$ shows, after some simplifications, that

$$\Phi_\alpha(s) \;=\; \Gamma(\alpha)\, s \;+\; \int_{-\infty}^{0} (e^{sx} - 1 - sx) f_\alpha(x)\, dx$$

with

$$f_\alpha(x) \;=\; \frac{e^{\alpha x}}{\Gamma(-\alpha)(1-e^x)^{\alpha+1}}\cdot$$

Since f_α integrates $x^2 \wedge 1$ on $(-\infty, 0)$, this entails that Ψ_α is the Laplace exponent of a spectrally negative Lévy process. Applying Bertoin-Yor's criterion for perpetuities, as stated in [5] pp. 8-9, shows that

$$\mathcal{A}_\alpha \;\stackrel{d}{=}\; \int_0^\infty e^{-Z_t^{(\alpha)}}\, dt \tag{2.2}$$

where $\{Z_t^{(\alpha)},\, t \geq 0\}$ is the spectrally negative Lévy process with Laplace exponent

$$\mathbb{E}[e^{\lambda Z_1^{(\alpha)}}] \;=\; e^{\Psi_\alpha(\lambda)} \;=\; e^{\Phi_\alpha((\alpha+1)\lambda)}, \qquad \lambda \geq 0.$$

It is then easy to see from the representation (2.2) and the spectral negativity of $Z^{(\alpha)}$ that $\mathcal{A}_\alpha$ is self-decomposable - see the end of the proof of the Theorem in [4].

□

Remark 3. (a) The above expression (2.2) extends to the boundary cases $\alpha = 1, 2$. When $\alpha = 1$, the Lévy process $Z^{(1)}$ has Laplace exponent 2λ, so that (2.2) boils down to

$$\mathcal{A}_1 \;\stackrel{d}{=}\; \int_0^\infty e^{-2t}\, dt \;=\; \frac{1}{2}\cdot$$

When $\alpha = 2$, the Lévy process $Z^{(2)}$ has Laplace exponent $3\lambda + 9\lambda^2$, and (2.2) reads

$$\mathcal{A}_2 \;\stackrel{d}{=}\; \int_0^\infty e^{3\sqrt{2}B_t - 3t}\, dt \;\stackrel{d}{=}\; \frac{1}{18}\int_0^\infty e^{B_t - t/6}\, dt \;\stackrel{d}{=}\; \frac{1}{9\mathbf{\Gamma}_{1/3}},$$

the third identity in law being a particular case of Dufresne's identity.

(b) It follows from Corollary 2 in [6] that the spectrally negative Lévy process $\{\xi_t^{\uparrow,n},\, t \geq 0\}$ appearing in the Lamperti transform of the dual process $-L^{(\alpha)}$ conditioned to stay positive, has log-Laplace exponent

$$-\log \mathbb{E}[e^{\lambda \xi_1^{\uparrow,n}}] \;=\; \Phi_\alpha(\lambda), \qquad \lambda \geq 0,$$

with the above notation for Φ_α. This can be shown from the second formula in Corollary 2 of [6] written in an appropriate way, bewaring the unusual notation (7) therein for the

negativity parameter and correcting a misprint (the + before c_- should be a $-$) in Formula (17) therein. This entails the curious identity in law

$$A_T \stackrel{d}{=} \int_0^\infty e^{-(\alpha+1)\xi_t^{\uparrow,n}}\, dt. \tag{2.3}$$

Recall that Brownian motion conditioned to stay positive is distributed as a three-dimensional Bessel process, whose Lamperti process $\xi^{\uparrow,n}$ is the drifted Brownian motion $\{B_t + t/2,\, t \geq 0\}$ - see [6] p. 969 and the references therein, so that by Dufresne's identity we obtain

$$\int_0^\infty e^{-3\xi_t^{\uparrow,n}}\, dt \stackrel{d}{=} \frac{2}{9\mathbf{\Gamma}_{\frac{1}{3}}} \stackrel{d}{=} \mathcal{A}.$$

This can be viewed as a particular case of (2.3), with the proper normalization. It is quite interesting to compare (2.3) with the identity

$$T \stackrel{d}{=} \int_0^\infty e^{-\alpha\xi_t^{\uparrow,n}}\, dt, \tag{2.4}$$

which follows from the above Remark 1 and Theorem 7 of [7]. We do not have any sensible interpretation for the structural relationship between (2.3) and (2.4). Notice last that the perpetuities of certain Lévy processes with positive mean and jumping density

$$\frac{K\, e^{bx}}{(1-e^x)^{\alpha+1}}\, \mathbf{1}_{\{x<0\}}$$

for some $K, b > 0$ and $\alpha \in (1,2)$ have been studied in [15]. Observe in particular that the factorization obtained in Theorem 4.6 (2) therein shares some similarities with (1.3).

(c) Combining the Kanter factorization - see Corollary 4.1 in [12] - and (1.3) shows the identity

$$\mathcal{A}_\alpha \stackrel{d}{=} \left(\frac{\alpha+1}{4}\right) \times \mathbf{\Gamma}_1^{1-\alpha} \times \left(\mathbf{B}_{\frac{1}{2},\frac{\alpha-1}{2(\alpha+1)}} \times \mathbf{K}^{\alpha+1}_{\frac{2}{\alpha+1}}\right)^{-1},$$

where $\mathbf{K}_a$ is the so-called Kanter random variable of index $a \in (0,1)$ - see Section 3 in [19] for more details about this random variable. Consider now the Fréchet random variable $\mathbf{\Gamma}_1^{1-\alpha}$ appearing in the above factorization of $\mathcal{A}_\alpha$. In [4], it was shown that $\mathbf{\Gamma}_1^{1-\alpha}$ is also distributed as the perpetuity of a spectrally negative Lévy process, and the latter turns out to be quite close to the above process $Z^{(\alpha)}$. More precisely, it follows from the Lemma in [4] with $t = 1$, the proof of the Theorem in [4] and a change of variable that

$$\mathbf{\Gamma}_1^{1-\alpha} \stackrel{d}{=} \int_0^\infty e^{-\tilde{Z}_t^{(\alpha)}}\, dt \tag{2.5}$$

where $\{\tilde{Z}_t^{(\alpha)},\, t \geq 0\}$ is the spectrally negative Lévy process having log-Laplace exponent

$$-\log \mathbb{E}[e^{\lambda \tilde{Z}_1^{(\alpha)}}] \;=\; (\alpha-1)^{-1}\Phi_\alpha((\alpha-1)\lambda)), \qquad \lambda \geq 0.$$

We do not have any explanation for this proximity between the two Laplace exponents of $Z^{(\alpha)}$ and $\tilde{Z}^{(\alpha)}$. Notice that the identity (2.5) can also been derived, in a complicated way, from Theorem 4.1 in [15].

2.3. **Proof of Corollary 2.** From (1.3) and a general result by Bondesson - see Theorem 1 in [3], in order to show $\sqrt{\mathcal{A}_\alpha} \in \mathcal{G}$ it is enough to prove that

$$\mathbf{B}^{-1/2}_{\frac{1}{2},\frac{\alpha-1}{2(\alpha+1)}} \in \mathcal{G} \qquad \text{and} \qquad \mathbf{Z}_{\frac{2}{\alpha+1}} \in \mathcal{G}.$$

The second fact is well-known - see Example 3.2.1 in [2], whereas the first one follows at once from Theorem 2 in [5]. Finally, the fact that $\sqrt{\mathcal{A}_\alpha}$ has infinite Thorin measure follows from Theorem 4.1.4 in [2] and the above formula (2.1), which entails that $\sqrt{\mathcal{A}_\alpha}$ has negative moments of all orders.

□

Remark 4. A combination of Corollaries 1 and 2 shows the following identities in law between three random integrals:

$$\int_0^\infty e^{-Z_t^{(\alpha)}} dt \stackrel{d}{=} \int_0^T L_t^{(\alpha)} dt \stackrel{d}{=} \left(\int_0^\infty a_t^{(\alpha)} d\Gamma_t \right)^2,$$

where $\{\Gamma_t,\, t \geq 0\}$ is the Gamma subordinator and $\{a_t^{(\alpha)},\, t \geq 0\}$ some deterministic function which is related to the Thorin measure of $\sqrt{\mathcal{A}_\alpha}$ - see Proposition 1.1 in [10].

2.4. **Proof of Corollary 3.** We will reason along the same lines as in Proposition 2 in [18], and omit some details. Applying the Mellin inversion formula yields first

$$f_{\mathcal{A}_\alpha}(x) \;=\; \frac{1}{2\pi x} \int_{\mathbb{R}} \mathcal{M}_\alpha(s)\, x^{-\mathrm{i}s}\, ds,$$

with the notation

$$\mathcal{M}_\alpha(s) \;=\; \mathbb{E}\left[\mathcal{A}_\alpha^{\mathrm{i}s}\right] \;=\; (\alpha+1)^{\mathrm{i}s} \;\times\; \frac{\Gamma(\frac{\alpha}{\alpha+1})\Gamma(1-(\alpha+1)\mathrm{i}s)}{\Gamma(\frac{\alpha}{\alpha+1}-\mathrm{i}s)\Gamma(1-\mathrm{i}s)}$$

for every $s \in \mathbb{R}$. Suppose first $x > 1$. We evaluate the above integral with the help of the residue theorem applied to the contour joining $-R$ to R along the real axis, and R to $-R$ along the half-circle plotted in the lower half-plane. It is easy to see that the integral along this half-circle vanishes as $R \to +\infty$, so that it remains to consider the singularities inside the big contour, which are located at $t_n = -\mathrm{i}(n+1)/(\alpha+1), n \geq 0$. Computing

$$\mathrm{Res}_{t_n}(\mathcal{M}_\alpha(s)x^{-\mathrm{i}s}) \;=\; -\mathrm{i}\,\Gamma\left(\frac{\alpha}{\alpha+1}\right) \times \frac{(-1)^n(\alpha+1)^{\frac{n+1}{\alpha+1}-1}x^{-\frac{n+1}{\alpha+1}}}{n!\,\Gamma(1-\frac{n+1}{\alpha+1})\Gamma(1-\frac{n+2}{\alpha+1})}$$

we deduce

$$f_{\mathcal{A}_\alpha}(x) \;=\; \Gamma\left(\frac{\alpha}{\alpha+1}\right) \times \sum_{n=0}^\infty \frac{(-1)^n(\alpha+1)^{\frac{n+1}{\alpha+1}-1}x^{-\frac{n+1}{\alpha+1}-1}}{n!\,\Gamma(1-\frac{n+1}{\alpha+1})\Gamma(1-\frac{n+2}{\alpha+1})}$$

for every $x > 1$, and hence for every $x > 0$ by analytic continuation (Stirling's formula shows indeed that the series on the right-hand side converges absolutely for every $x > 0$).

□

2.5. **Proof of Corollary 4.** We will work on the random variable $\mathcal{X}_\alpha = \mathcal{A}_\alpha^{\frac{1}{1-\alpha}}$, in order to simplify the notation. Changing the variable, the required estimate is tantamount to

$$f_{\mathcal{X}_\alpha}(x) \sim \frac{\Gamma(\frac{\alpha}{\alpha+1})\sqrt{\alpha^2-1}}{2\pi\,(\alpha+1)^{\frac{\alpha}{\alpha^2-1}}}\, x^{-\frac{\alpha}{\alpha+1}}\, e^{-c_\alpha x}, \qquad x \to 0+. \tag{2.6}$$

Evaluating with (2.1) the positive entire moments

$$\mathbb{E}[\mathcal{X}_\alpha^n] \;=\; (\alpha+1)^{\frac{n}{1-\alpha}} \times \frac{\Gamma(\frac{\alpha}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1}+\frac{n}{\alpha-1})} \times \frac{\Gamma(1+\frac{(\alpha+1)n}{\alpha-1})}{\Gamma(1+\frac{n}{\alpha-1})}$$

for every $n \geq 0$, and applying Stirling's formula shows that

$$\frac{(\mathbb{E}[\mathcal{X}_\alpha^n])^{\frac{1}{n}}}{n} \;\to\; \frac{(\alpha+1)^{\frac{\alpha}{\alpha-1}}}{\mathrm{e}(\alpha-1)} \qquad \text{as } n \to \infty.$$

By a theorem of Davies-Kasahara (see Corollary 4.12.5 in [1], or Lemma 3.2 in [8] for a more appropriate formulation), we deduce that

$$x^{-1}\log \mathbb{P}[\mathcal{X}_\alpha > x] \;\to\; -c_\alpha \qquad \text{as } x \to +\infty.$$

This yields the required asymptotic behaviour, at the logarithmic scale, for the survival function of $\mathcal{X}_\alpha$. Moreover, writing down via Fubini's theorem the moment generating function

$$\mathbb{E}[e^{x\mathcal{X}_\alpha}] \;=\; \sum_{n=0}^{\infty} a_n\, x^n, \qquad x > 0,$$

with

$$\begin{aligned} a_n \;&=\; (\alpha+1)^{\frac{n}{1-\alpha}} \times \frac{\Gamma(\frac{\alpha}{\alpha+1})\Gamma(1+\frac{(\alpha+1)n}{\alpha-1})}{n!\,\Gamma(\frac{\alpha}{\alpha+1}+\frac{n}{\alpha-1})\Gamma(1+\frac{n}{\alpha-1})} \\ &\sim\; \frac{\Gamma(\frac{\alpha}{\alpha+1})\sqrt{\alpha^2-1}}{2\pi\,(\alpha-1)^{\frac{1}{\alpha+1}}}\, c_\alpha^{-n}\, n^{\frac{-\alpha}{\alpha+1}} \quad \text{as } n \to \infty, \end{aligned}$$

and applying Karamata's theorem for power series - see Corollary 1.7.3 in [1] - shows that

$$\mathbb{E}[e^{c_\alpha x\mathcal{X}_\alpha}] \sim \frac{\Gamma(\frac{1}{\alpha+1})\,\Gamma(\frac{\alpha}{\alpha+1})\sqrt{\alpha^2-1}}{2\pi\,(\alpha-1)^{\frac{1}{\alpha+1}}(1-x)^{\frac{1}{\alpha+1}}} \qquad \text{as } x \to 1-. \tag{2.7}$$

At this stage, it is worth mentioning that (2.7) can be obtained from (2.6) by integration. However, it does not seem that we can infer the reverse inclusion without any further assumption, such as the existence of a meromorphic extension in the neighbourhood of c_α for the moment generating function - see Theorem 4 in [9], or a monotonicity condition on $f_{\mathcal{X}_\alpha}$ at infinity - see Theorem 4.12.11 in [1], which we both could not prove *a priori*.

In order to show (2.6) rigorously and finish the proof, we will use the following power transformation of (1.3):

$$\mathcal{X}_\alpha \stackrel{d}{=} \left(\frac{4}{\alpha+1}\right)^{\frac{1}{\alpha-1}} \times \mathbf{B}_{\frac{1}{2},\frac{\alpha-1}{2(\alpha+1)}}^{\frac{1}{\alpha-1}} \times \mathbf{Z}_{\frac{2}{\alpha+1}}^{\frac{2}{1-\alpha}}.$$

The multiplicative convolution formula and a change of variable entails

$$f_{\mathcal{X}_\alpha}(x) = \int_0^\infty \left(\frac{1}{1+y}\right) f_{\mathbf{U}_\alpha}\left(\frac{1}{1+y}\right) f_{\mathbf{V}_\alpha}(x+xy)\,dy, \tag{2.8}$$

where we have set

$$\mathbf{U}_\alpha = \mathbf{B}_{\frac{1}{2},\frac{\alpha-1}{2(\alpha+1)}}^{\frac{1}{\alpha-1}} \qquad \text{and} \qquad \mathbf{V}_\alpha = \left(\frac{4}{\alpha+1}\right)^{\frac{1}{\alpha-1}} \times \mathbf{Z}_{\frac{2}{\alpha+1}}^{\frac{2}{1-\alpha}}.$$

On the one hand, we have

$$\left(\frac{1}{1+y}\right) f_{\mathbf{U}_\alpha}\left(\frac{1}{1+y}\right) = \frac{(\alpha-1)^{\frac{\alpha-1}{2(\alpha+1)}}\,\Gamma(\frac{\alpha}{\alpha+1})}{\sqrt{\pi}\,\Gamma(\frac{\alpha-1}{2(\alpha+1)})}\, y^{\frac{\alpha-1}{2(\alpha+1)}-1}(1+O(y)) \qquad \text{as } y \to 0+.$$

On the other hand, Formula (14.35) in [17] entail after a change of variable and several simplifications

$$f_{\mathbf{V}_\alpha}(z) = \frac{\sqrt{\alpha-1}\,(\alpha+1)^{\frac{\alpha^2}{2(\alpha^2-1)}-\frac{1}{\alpha-1}}}{2\sqrt{\pi}}\, z^{-1/2}\, e^{-c_\alpha z}(1+O(z^{-1/2})) \qquad \text{as } z \to +\infty.$$

Plugging these two first order expansions in the integral (2.8), and making further simplifications, yields finally the required asymptotic behaviour (2.6).

□

Remark 5. The two asymptotic expansions for the above $f_{\mathbf{U}_\alpha}$ and $f_{\mathbf{V}_\alpha}$ can be continued at every order - see again Formula (14.35) in [17] for the second function. This could be used to obtain a refined expansion for $f_{\mathcal{X}_\alpha}$ at infinity, or equivalently for $f_{\mathcal{A}_\alpha}$ at zero. Displaying the full asymptotic expansion of $f_{\mathcal{A}_\alpha}$ at zero seems however to be a very painful task.

3. Final remarks

3.1. **On the $\mathcal{G}-$property.** As mentioned in the introduction, the fact that $\sqrt{\mathcal{A}_\alpha} \in \mathcal{G}$ gives some credit to the $\mathcal{G}-$property for $\mathcal{A}_\alpha$ itself. This refinement of Corollary 1 would also shed some analytic light on the self-decomposability of $\mathcal{A}_\alpha$. We can show this property for $\alpha = 5/3$, where a combination of (1.3) and Theorem 1 in [19] entails the identity in law

$$\mathcal{A}_{5/3} \stackrel{d}{=} \left(\frac{3}{2^{13/3}}\right) \times \mathbf{\Gamma}_{\frac{3}{4}}^{-\frac{2}{3}} \times \mathbf{B}_{\frac{1}{4},\frac{1}{12}}^{-\frac{2}{3}} \times \mathbf{B}_{\frac{1}{2},\frac{1}{6}}^{-\frac{2}{3}} \times \mathbf{B}_{\frac{1}{2},\frac{1}{8}}^{-1}.$$

Indeed, all random variable on the right-hand side are in $\mathcal{G}$ by Theorems 2 and 4 in [5], and we can conclude by Theorem 1 in [3]. Notice that the same kind of argument would show the property for all $\alpha \in (1,2)$ if we could prove that $\mathbf{Z}_\beta^2 \in \mathcal{G}$ for all $\beta \in (2/3,1)$ and that by the stability of the $\mathcal{G}-$property with respect to weak convergence - see Theorem 3.1.5 in [2], it is enough to consider β rational. Again, Theorem 1 in [19] shows a certain

factorisation of $\mathbf{Z}_\beta^2$ with β rational into the product of one Fréchet and several Beta random variables at a unique negative power, which unfortunately becomes too small when $\alpha \neq 5/3$, so that cannot use Theorem 5.2 in [5] as above. Observe also from Section 3.1 in [5] that small negative powers of Beta random variables may not be in $\mathcal{G}$. To show the plausible fact that $\mathbf{Z}_\beta^2 \in \mathcal{G}$ for all $\beta \in (0,1)$ rational, it could be useful to write down the density of the Beta-Gamma product given in Formula (2.3) of [19] as a Meijer G-function - see Theorem 9 in [21], although it does not seem easy at first sight to express the Laplace transform in a tractable way.

3.2. **On the bell-shape property.** It follows from Corollaries 3 and 4 that the density function of $\mathcal{A}_\alpha$ is real analytic on $(0,+\infty)$, and that all its derivatives vanish at zero and at infinity. Moreover, a consequence of Corollary 2, Wolfe-Yamazato's theorem - see e.g. Theorem 51.3 in [17], and the principle of isolated zeroes, is the strict unimodality of this density function, that is its first derivative vanishes only once on $(0,+\infty)$. By Rolle's theorem, we deduce that its $n-$th derivative vanishes at least n times on $(0,+\infty)$, and one can ask whether it vanishes exactly n times for all $n \geq 1$. Such a property, which is called the bell-shape in the literature, has been conjectured in [20] for all positive self-decomposable distributions having an infinite spectral function at zero - see Conjecture 1 therein. Observe that $\mathcal{A}_\alpha$ has such a self-decomposable law by Corollary 2, Corollary 4 and a theorem by Zolotarev - see Remark 28.6 in [17]. Observe also that drawing the density with the help of the series representation of Corollary 3 and some plotting software exhibits the visual bell-shape property for $f_{\mathcal{A}_\alpha}$, whose second derivative does seem to vanish only twice on $(0,+\infty)$.

3.3. **Stable processes with negative jumps.** The present paper studies the law of the random variable

$$\int_0^T L_s^{(\alpha)}\, ds$$

where $\{L_t^{(\alpha)},\, t \geq 0\}$ is a spectrally positive stable process. It is natural to ask if (1.3) could be extended in the presence of negative jumps, with T defined as the first passage time below zero. In this case, it is known that the stable process, when starting positive, crosses zero by a negative jump with an explicit expression for the law of the undershoot - see e.g. Remark 42.18 in [17] and the references therein. In order to apply our method, it would be hence necessary to find a closed expression of $\mathbb{E}_{(x,y)}[|L_S|^{s-1}]$ with $x > 0$ and $y < 0$, with the notation of our above proof. When $y = 0$, Theorem B in [16] provides a formula for these fractional moments in terms of the Gamma function, in full generality on the stable process. However, it seems hard to get a tractable formula when $y < 0$.

References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation.* Cambridge University Press, Cambridge, 1987.

[2] L. Bondesson. *Generalized Gamma convolutions and related classes of distributions and densities.* Lect. Notes Stat. **76**, Springer-Verlag, New York, 1992.

[3] L. Bondesson. A class of probability distributions that is closed with respect to addition as well as multiplication of independent random variables. To appear in *Journal of Theoretical Probability.*
[4] P. Bosch and T. Simon. On the self-decomposability of the Fréchet distribution. *Indagationes Math.* **24**, 626-636, 2013.
[5] P. Bosch and T. Simon. On the infinite divisibility of inverse Beta distributions. To appear in *Bernoulli.*
[6] M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.* **43** (4), 967-983, 2006.
[7] L. Chaumont, A. E. Kyprianou and J.-C. Pardo. Some explicit identities associated with positive self-similar Markov processes. *Stoch. Proc. Appl.* **119** (3), 980-1000, 2009.
[8] M. Csörgö, Z. Shi and M. Yor. Some asymptotic properties of the local time of the uniform empirical process. *Bernoulli* **5** (6), 1035-1058, 1999.
[9] P. Flajolet, X. Gourdon and P. Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoret. Comput. Sci.* **144**, 3-58, 1995.
[10] L. F. James, B. Roynette and M. Yor. Generalized Gamma Convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surveys* **5**, 346-415, 2008.
[11] S. Janson. Brownian excursion area, Wright's constants in graph enumeration, and other Brownian areas. *Probab. Surveys* **4**, 80-145, 2007.
[12] M. Kanter. Stable densities under change of scale and total variation inequalities. *Ann. Probab.* **3**, 697-707, 1975.
[13] A. Lachal. Sur le premier instant de passage de l'intégrale du mouvement brownien. *Ann. Inst. H. Poincaré Probab. Stat.* **27** (3), 385-405, 1991.
[14] M. Lefebvre. First-passage densities of a two-dimensional process. *SIAM J. Appl. Math.* **49** (5), 1514-1523, 1989.
[15] P. Patie. Exponential functional of a new family of Lévy processes and self-similar continuous state branching processes with immigration. *Bull. Sci. Math.* **133** (4), 355-382, 2009.
[16] C. Profeta and T. Simon. Persistence of integrated stable processes. To appear in *Probability Theory and Related Fields.*
[17] K. Sato. *Lévy processes and infinitely divisible distributions.* Cambridge University Press, Cambridge, 1999.
[18] T. Simon. Hitting densities for spectrally positive stable processes. *Stochastics* **83** (2), 203-214, 2011.
[19] T. Simon. Comparing Fréchet and positive stable laws. *Electron. J. Probab.* **19** (16), 1-25, 2014.
[20] T. Simon. Positive stable densities and the bell-shape. To appear in *Proceedings of the American Mathematical Society.*
[21] M. D. Springer and W. E. Thompson. The distribution of products of Beta, Gamma and Gaussian random variables. *SIAM J. Appl. Math.* **18** (4), 721-737, 1970.

Laboratoire Paul Painlevé, Université Lille 1, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex. *Email* : `ju.letemplier@gmail.com`

Laboratoire Paul Painlevé, Université Lille 1, Cité Scientifique, F-59655 Villeneuve d'Ascq Cedex. Laboratoire de physique théorique et modèles statistiques, Université Paris Sud, Bâtiment 100, F-91405 Orsay Cedex. *Email* : `simon@math.univ-lille1.fr`