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Hubert Rubenthaler

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Hubert Rubenthaler. Graded Lie algebras associated to a representation of a quadratic algebra. 2014.  
hal-01070209v3

**HAL Id: hal-01070209**

**<https://hal.science/hal-01070209v3>**

Preprint submitted on 17 Sep 2015 (v3), last revised 18 Jan 2017 (v4)

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# GRADED LIE ALGEBRAS ASSOCIATED TO A REPRESENTATION OF A QUADRATIC ALGEBRA

HUBERT RUBENTHALER

Preliminary version

ABSTRACT. Let  $(\mathfrak{g}_0, B_0)$  be a quadratic Lie algebra (i.e. a Lie algebra  $\mathfrak{g}_0$  with a non degenerate symmetric invariant bilinear form  $B_0$ ) and let  $(\rho, V)$  be a finite dimensional representation of  $\mathfrak{g}_0$ . We define on  $\Gamma(\mathfrak{g}_0, B_0, V) = V^* \oplus \mathfrak{g}_0 \oplus V$  a structure of local Lie algebra in the sense of Kac ([4]), where the bracket between  $\mathfrak{g}_0$  and  $V$  (resp.  $V^*$ ) is given by the representation  $\rho$  (resp.  $\rho^*$ ), and where the bracket between  $V$  and  $V^*$  depends on  $B_0$  and  $\rho$ . This implies the existence of two  $\mathbb{Z}$ -graded Lie algebras  $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V))$  and  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$  whose local part is  $\Gamma(\mathfrak{g}_0, B_0, V)$ . We investigate these graded Lie algebras, more specifically in the case where  $\mathfrak{g}_0$  is reductive. Roughly speaking, the map  $(\mathfrak{g}_0, B_0, V) \mapsto \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$  a bijection between triplets and a class of graded Lie algebras. We show that the existence of "associated  $\mathfrak{sl}_2$ -triples" is equivalent to the existence of non trivial relative invariants on some orbit, and we define the "graded Lie algebras of symplectic type" which give rise to some dual pairs.

AMS classification: 17B70(17B20)

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## 1. INTRODUCTION

In this paper all gradings are  $\mathbb{Z}$ -gradings. If  $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$  is a grading of a (finite dimensional) complex semi-simple Lie algebra, it is well known that if  $B$  denotes the Killing form of  $\mathfrak{g}$ , then  $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$  if  $i + j \neq 0$ . This allows us to identify  $\mathfrak{g}_{-1}$  with the dual  $\mathfrak{g}_1^*$ . Moreover as  $B$  is invariant the bracket representation  $(\mathfrak{g}_0, \mathfrak{g}_{-1})$  can be identified with the dual representation  $(\mathfrak{g}_0, \mathfrak{g}_1^*)$ . It is then a natural question to ask if any finite dimensional representation  $(\mathfrak{g}_0, \rho, V)$  of a finite dimensional Lie algebra  $\mathfrak{g}_0$  can be embedded in a graded Lie algebra  $\mathfrak{g} = \bigoplus_{i=-\infty}^{+\infty} \mathfrak{g}_i$  such that  $(\mathfrak{g}_0, \mathfrak{g}_1) \simeq (\mathfrak{g}_0, \rho, V)$  and  $(\mathfrak{g}_0, \mathfrak{g}_{-1}) \simeq (\mathfrak{g}_0, \rho^*, V^*)$ , and such that the bracket between  $V$  and  $V^*$  is non trivial.

The first result of this paper is to give a positive answer to this question for any representation of a quadratic Lie algebra. A quadratic Lie algebra is a pair  $(\mathfrak{g}_0, B_0)$  where  $\mathfrak{g}_0$  is a Lie algebra and  $B_0$  a non-degenerate invariant symmetric bilinear form on  $\mathfrak{g}_0$ . Of course the definition of the bracket between  $V$  and  $V^*$  will depend on  $B_0$  and  $\rho$ .

We will use a result of V. Kac ([4]) which asserts that in order to construct a graded Lie algebra  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$  it suffices to construct the local part  $\Gamma = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ , which has to be endowed with a partial Lie bracket (see section 2 for details). Therefore once we have build the partial bracket on the local part  $\Gamma(\mathfrak{g}_0, B_0, \rho) = V^* \oplus \mathfrak{g}_0 \oplus V$ , the existence of the "global" Lie algebra is just an application of a result of Kac. In fact Kac theory provides us with two such graded Lie algebras: a maximal one (denoted here  $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ ) and a minimal one (denoted  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ ). Any graded Lie algebra with a given local part is a quotient of the maximal algebra, and has a quotient isomorphic to the minimal one. Of course, in general, these algebras are infinite dimensional.

Lets us now give a more precise description of the paper.

In section 2.1 we give a brief account of the results of Kac that we will use.

In section 2.2 we prove a general result concerning this construction. Let  $\Gamma = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be a local Lie algebra and let  $\mathfrak{g}_{min}(\Gamma) = \bigoplus_{-\infty}^{+\infty} \mathfrak{g}_i$  be the minimal graded Lie algebra whith local part  $\Gamma$ . Let  $|n| \geq 2$ . We show that there exists a universal polynomial  $\mathcal{P}_n$  defined on the local part  $\Gamma$  such that  $\mathfrak{g}_n = \{0\}$  if and only if the identity  $\mathcal{P}_n = 0$  is satisfied on the local Lie algebra  $\Gamma$ .

In section 3.1 we define the local Lie algebra structure on  $\Gamma(\mathfrak{g}_0, B_0, \rho) = V^* \oplus \mathfrak{g}_0 \oplus V$  (see Theorem 3.1.1) . In section 3.2 we give necessary and sufficient conditions for the algebras  $\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)$  and  $\Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)$  corresponding to two fundamental data to be isomorphic. We also investigate the dependence of the local structure on  $B_0$  and  $\rho$  and show that any isomorphism between the "fundamental triplets"  $(\mathfrak{g}_0^1, B_0^1, (\rho_1, V_1))$  and  $(\mathfrak{g}_0^2, B_0^2, (\rho_1, V_2))$  can be extended to an isomorphism between  $\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)$  and  $\Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)$ . In section 3.3 we apply Kac Theorem to obtain the minimal and maximal Lie algebras associated to the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho) = V^* \oplus \mathfrak{g}_0 \oplus V$ . We also prove, that under some conditions, the reductive graded Lie algebras are always minimal graded Lie algebras (Proposition 3.3.3). Section 3.4 deals with another important notion for graded Lie algebras due to Kac, namely the transitivity (see Definition 3.4.1). We give a necessary and sufficient condition for the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  (or the minimal Lie algebra  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ ) to be transitive (Proposition 3.4.3). We also prove that if  $\mathfrak{g}_0$  is reductive, then under some conditions including the transitivity of  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ , the fact that  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is finite dimensional implies that  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is semi-simple (see Proposition 3.4.6). In section 3.5 we show that the form  $B_0$  extends uniquely to an invariant symmetric bilinear form  $B$  on  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ . Moreover if the

local part is transitive then the form  $B$  is nondegenerate (Proposition 3.5.2). This allows us to show that there exists a bijection between some equivalence classes of fundamental triplets and the equivalence classes of transitive graded Lie algebras endowed with a non-degenerate symmetric bilinear form  $B$  such that  $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$  if  $i \neq -j$  (Theorem 3.5.5).

In section 4.1 we give a necessary and sufficient condition for the existence of an  $\mathfrak{sl}_2$ -triple  $(Y, H_0, X)$  where  $Y \in V^*$ ,  $X \in V$  and where  $H_0$  is the "grading element" of the center of  $\mathfrak{g}_0$  defined by the condition  $\rho(H_0)|_V = 2\text{Id}_V$  (see Theorem 4.1.2). In section 4.2 we assume that the representation  $\rho$  lifts to a representation of a connected algebraic group  $G_0$  with Lie algebra  $\mathfrak{g}_0$ . We prove then that the existence of such an  $\mathfrak{sl}_2$ -triple is also equivalent to the existence of a  $G_0$ -orbit in  $V$  supporting a non trivial rational relative invariant (see Theorem 4.2.3).

Section 5 is devoted to the so-called graded Lie algebras of symplectic type. These algebras are defined in section 5.1 to be the minimal Lie algebras associated to  $\mathfrak{g}_0 = \mathfrak{gl}(W)$  and to the "natural" representation of  $\mathfrak{sl}(W) = \mathfrak{g}'_0$  on the space  $V = \mathbb{C}^p[W]$  of homogeneous polynomials of degree  $p$  on  $W$  (see Definition 5.1.1). We classify the finite dimensional Lie algebras of symplectic type (Proposition 5.1.3) and show that these algebras are always associated to  $\mathfrak{sl}_2$ -triples in the sense of section 4 (Theorem 5.1.4). Finally, in section 5.2 we show, under some assumptions, that if  $(A, W)$  is an irregular reductive regular prehomogeneous vector space, then the semi-simple part  $\mathfrak{a}'$  of the Lie algebra of  $A$  is the member of a dual pair in the Lie algebras of symplectic type associated to  $\mathbb{C}^p[W]$ , where  $p$  is the degree of the fundamental relative invariant of  $(A, W)$  (Theorem 5.2.4).

**Acknowledgment:** I would like to thank Michel Brion who communicated me Lemma 4.2.2.

## 2. GRADED LIE ALGEBRAS AND LOCAL LIE ALGEBRAS

In this paper all the algebras are defined over the field  $\mathbb{C}$  of complex numbers.

### 2.1. Maximal and minimal algebras.

Let us first recall various definitions and results from [4].

**Definition 2.1.1.**

A Lie algebra  $\mathfrak{g}$  is said to be graded if:

- 1)  $\mathfrak{g}$  is a direct sum of subspaces:  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ , such that  $\dim \mathfrak{g}_i < +\infty$  and such that  $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ , for all  $i, j \in \mathbb{Z}$ .
- 2)  $\mathfrak{g}$  is generated by  $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ .

If  $\mathfrak{g}$  is a graded Lie algebra, the subspace  $\Gamma(\mathfrak{g}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  is called the *local part* of  $\mathfrak{g}$ .

**Definition 2.1.2.**

- 1) A local Lie algebra is a direct sum  $\Gamma = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  of finite dimensional subspaces such that if  $|i + j| \leq 1$  there exists a bilinear anticommutative operation  $\mathfrak{g}_i \times \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j}((x, y) \rightarrow [x, y])$  such that the Jacobi identity  $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$  holds each time the three terms of the identity are defined.
- 2) A symmetric bilinear form  $B_\Gamma$  on a local Lie algebra  $\Gamma$  is said to be invariant if the identity

$$B_\Gamma([x, y], z) = B_\Gamma(x, [y, z])$$

holds for  $x, y, z \in \Gamma$  each time that the brackets are defined.

**Definition 2.1.3.** Let  $\mathfrak{g}^1 = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n^1$  and  $\mathfrak{g}^2 = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n^2$  be two graded Lie algebras. A homomorphism of graded Lie algebras from  $\mathfrak{g}^1$  to  $\mathfrak{g}^2$  is a map  $\Psi : \mathfrak{g}^1 \rightarrow \mathfrak{g}^2$  which is a homomorphism of Lie algebras such that  $\forall n \in \mathbb{N}, \Psi(\mathfrak{g}_n^1) \subset \mathfrak{g}_n^2$ . Homomorphisms of local Lie algebras are defined similarly.

Of course the local part  $\Gamma(\mathfrak{g})$  of a graded Lie algebra  $\mathfrak{g}$ , endowed with the bracket of  $\mathfrak{g}$  is a local Lie algebra. A natural question is to know if, for a given local Lie algebra  $\Gamma$ , there exists a graded Lie algebra whose local part is  $\Gamma$ . The answer is "yes". More precisely we have:

**Theorem 2.1.4.** (Kac, [4], Proposition 4)

Let  $\Gamma = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be a local Lie algebra.

- 1) There exists a unique graded Lie algebra  $\mathfrak{g}_{max}(\Gamma)$  whose local part is  $\Gamma$  and which satisfies the following universal property.

Any morphism of local Lie algebras  $\Gamma \rightarrow \Gamma(\mathfrak{g})$  from  $\Gamma$  into the local part  $\Gamma(\mathfrak{g})$  of a graded Lie algebra  $\mathfrak{g}$  extends uniquely to a morphism of graded Lie algebras  $\mathfrak{g}_{max}(\Gamma) \rightarrow \mathfrak{g}$ . (And hence any graded Lie algebra whose local part is isomorphic to  $\Gamma$ , is a quotient of  $\mathfrak{g}_{max}(\Gamma)$ ). Moreover we have

$$\mathfrak{g}_{max}(\Gamma) = F(\mathfrak{g}_{-1}) \oplus \mathfrak{g}_0 \oplus F(\mathfrak{g}_1),$$

where  $F(\mathfrak{g}_{-1})$  (resp.  $F(\mathfrak{g}_1)$ ) is the free Lie algebra generated by  $\mathfrak{g}_{-1}$  (resp.  $\mathfrak{g}_1$ ).

2) *There exists a unique graded Lie algebra  $\mathfrak{g}_{\min}(\Gamma)$  whose local part is  $\Gamma$  and which satisfies the following universal property.*

*Any surjective morphism of local Lie algebras  $\Gamma(\mathfrak{g}) \rightarrow \Gamma$  from the local part of a graded Lie algebra  $\mathfrak{g}$  into  $\Gamma$  extends uniquely to a (surjective) morphism of graded Lie algebras  $\mathfrak{g} \rightarrow \mathfrak{g}_{\min}(\Gamma)$ . (And hence  $\mathfrak{g}_{\min}(\Gamma)$  is a quotient of any graded Lie algebra whose local part is isomorphic to  $\Gamma$ ).*

*In fact  $\mathfrak{g}_{\max}(\Gamma)$  has a unique maximal graded ideal  $J_{\max}$  such that  $J_{\max} \cap \Gamma = \{0\}$ , and  $\mathfrak{g}_{\max}(\Gamma)/J_{\max} = \mathfrak{g}_{\min}(\Gamma)$ .*

## 2.2. When is $\dim(\mathfrak{g}_{\min}(\Gamma)) < +\infty$ ?

Let  $\Gamma$  be a local Lie algebra. Consider the minimal graded Lie algebra  $\mathfrak{g}_{\min}(\Gamma)$  associated to  $\Gamma$ . A natural question is to ask how one can see, only from the knowledge of the local Lie algebra  $\Gamma$ , whether or not  $\mathfrak{g}_{\min}(\Gamma)$  is finite dimensional. For example suppose that  $\Gamma$  is already a (3-graded) Lie algebra, but we do not know anything about the brackets between two elements of  $\mathfrak{g}_1$  or of  $\mathfrak{g}_{-1}$ . Then we have surely

$$\mathcal{P}_2(Y, X_1, X_2) = [[Y, X_1], X_2] + [X_1, [Y, X_2]] = 0 \quad (2 - 2 - 1)$$

for all  $Y \in \mathfrak{g}_{-1}$  and all  $X_1, X_2 \in \mathfrak{g}_{-1}$ . This is because this element is equal to

$$[Y, [X_1, X_2]] = 0 \quad (2 - 2 - 2)$$

in  $\mathfrak{g}_{\min}(\Gamma)$ . The first equation makes sense in  $\Gamma$ , but not the second. Conversely, if the relation (2 - 2 - 1) holds in the local Lie algebra  $\Gamma$ , then  $\Gamma = \mathfrak{g}_{\min}(\Gamma)$  is in fact a 3-graded Lie algebra.

We will show that there exists a "universal" polynomial identity  $\mathcal{P}_n = 0$  in any local Lie algebra  $\Gamma$  which is a necessary and sufficient condition for having  $\mathfrak{g}_n = \{0\}$  (see Theorem 2.2.4 below).

Let us denote by  $\mathcal{V}_n = \{Y_1, \dots, Y_{n-1}, X_1, \dots, X_n\}$  a set of  $2n - 1$  variables. Let  $F(\mathcal{V}_n)$  be the free Lie algebra on  $\mathcal{V}_n$ .

A *Lie monomial of degree one* in the variables  $\mathcal{V}_n$  is an element of  $\mathcal{V}_n$ . By induction a *Lie monomial of degree  $k$*  in the variables  $\mathcal{V}_n$  is an element of  $F(\mathcal{V}_n)$  of the form  $[u, v]$  where  $u$  is a Lie monomial of degree  $i$  ( $1 \leq i < k$ ) and  $v$  is a Lie monomial of degree  $k - i$ . A *Lie polynomial*  $\mathcal{P}$  in the variables  $\mathcal{V}_n$  is a linear combination of monomials (in other words it is just an element of  $F(\mathcal{V}_n)$ ).

### Proposition 2.2.1.

Let  $n \geq 2$ . The Lie monomial  $[Y_1, [Y_2, [\dots, [Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n] \dots]]]]$  is equal (as elements of the free algebra  $F(\mathcal{V}_n)$ ) to a Lie polynomial of the form

$$\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = \sum_{\alpha} U_{\alpha}(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n)$$

where each monomial  $U_{\alpha}$  makes sense in the local Lie algebra  $\Gamma$  when  $Y_i \in \mathfrak{g}_{-1}, i = 1, \dots, n-1$  and when  $X_j \in \mathfrak{g}_1, j = 1, \dots, n$ .

*Proof.*

As we have already noticed we have

$$[Y_1, [X_1, X_2]] = \mathcal{P}_2(Y_1, X_1, X_2) = [[Y_1, X_1], X_2] + [X_1, [Y_1, X_2]]$$

and the right hand side is defined in the local Lie algebra if  $Y_1 \in \mathfrak{g}_{-1}$ , and  $X_1, X_2 \in \mathfrak{g}_1$ .

Suppose now that the result is true for  $n-1$ . We start with the monomial

$$[Y_1, [Y_2, [\dots, [Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n] \dots]]]]] \quad (2-2-3)$$

Consider also the sub-monomial

$$[Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n] \dots]]$$

Using the Jacobi identity we get

$$\begin{aligned} & [Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n] \dots]] \\ &= \sum_{i=1}^n [X_1, [\dots, [[Y_{n-1}, X_i], [X_{i+1}, [\dots, [X_{n-1}, X_n] \dots]]]] \end{aligned}$$

Consider again the sub-monomial  $[[Y_{n-1}, X_i], [X_{i+1}, [\dots, [X_{n-1}, X_n] \dots]]$ , and set  $U_i = [Y_{n-1}, X_i]$  (notice that if  $Y_{n-1} \in \mathfrak{g}_{-1}$ , and  $X_i \in \mathfrak{g}_1$ , then  $U_i \in \mathfrak{g}_0$ ).

From the Jacobi identity we obtain

$$\begin{aligned} & [[Y_{n-1}, X_i], [X_{i+1}, [\dots, [X_{n-1}, X_n] \dots]] \\ &= [U_i, [X_{i+1}, [\dots, [X_{n-1}, X_n] \dots]] \\ &= \sum_{k=i+1}^n [X_{i+1}, [\dots, [[U_i, X_k], [\dots, [X_{n-1}, X_n] \dots]]] \\ &= \sum_{k=i+1}^n [X_{i+1}, [\dots, [\widetilde{X}_k^i, [\dots, [X_{n-1}, X_n] \dots]]] \end{aligned}$$

where  $\widetilde{X}_k^i = [U_i, X_k] = [[Y_{n-1}, X_i], X_k]$  is again a monomial which makes sense in  $\Gamma$ . Hence we have obtained

$$\begin{aligned} & [Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n] \dots]] \\ &= \sum_{i=1}^n \sum_{k=i+1}^n [X_1, [X_2, [\dots, [X_{i-1}, [X_{i+1}, [\dots, [\widetilde{X}_k^i, [\dots, [X_{n-1}, X_n] \dots]]]]]]] \end{aligned}$$

and finally if we report in (2-2-3) we get



$$\begin{aligned}
& [Y_1, [Y_2, [\dots, [Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n] \dots]]]] \\
&= \sum_{i=1}^n \sum_{k=i+1}^n [Y_1, [Y_2, \dots, [Y_{n-2}, [X_1, \dots, [X_{i-1}, [X_{i+1}, [\dots, [\widetilde{X}_k^i, [\dots, [X_{n-1}, X_n] \dots]]]]]]]]
\end{aligned}$$

But this last expression is a sum of monomials of type  $(2-2-3)$  with one "Y" and one "X" less and therefore

$$\begin{aligned}
& [Y_1, [Y_2, [\dots, [Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n] \dots]]]] \\
&= \sum_{i=1}^n \sum_{k=i+1}^n \mathcal{P}_{n-1}(Y_1, \dots, Y_{n-2}, X_1, \dots, X_{i-1}, X_{i+1}, \dots, \widetilde{X}_k^i, \dots, X_n).
\end{aligned}$$

As  $\widetilde{X}_k^i = [[Y_{n-1}, X_i], X_k]$  we obtain by induction that each  $\mathcal{P}_{n-1}(Y_1, \dots, Y_{n-2}, X_1, \dots, X_{i-1}, X_{i+1}, \dots, \widetilde{X}_k^i, \dots, X_n)$  is a sum of monomials which make sense in  $\Gamma$  if  $X_i \in \mathfrak{g}_1$  and  $Y_j \in \mathfrak{g}_{-1}$

□

Denote by  $\mathfrak{g}_{max}(\Gamma) = \bigoplus_{i \in \mathbb{Z}} (\mathfrak{g}_{max}(\Gamma))_i$  the grading in  $\mathfrak{g}_{max}(\Gamma)$ . Remember that  $(\mathfrak{g}_{max}(\Gamma))_i = \mathfrak{g}_i$  for  $i = -1, 0, 1$

**Lemma 2.2.2.** *Let  $n \geq 2$ . The polynomial identity  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = 0$  is satisfied in  $\Gamma$  for  $X_i \in \mathfrak{g}_1$  and  $Y_j \in \mathfrak{g}_{-1}$  if and only if the vector space*

$$J_n^+ = \sum_{k=0}^{n-2} (\text{ad } \mathfrak{g}_{-1})^k (\bigoplus_{i \geq n} (\mathfrak{g}_{max}(\Gamma))_i)$$

*is a graded ideal of  $\mathfrak{g}_{max}(\Gamma)$ , contained in  $\bigoplus_{i \geq 2} (\mathfrak{g}_{max}(\Gamma))_i$*

*Proof.*

Suppose that the identity  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = 0$  is satisfied in  $\Gamma$ . It is clear from the definition that  $J_n^+$  is stable under  $\text{ad}(\bigoplus_{i=0}^{\infty} (\mathfrak{g}_{max}(\Gamma))_i)$ . It remains to prove that  $\text{ad } Y(J_n^+) \subset J_n^+$  for  $Y \in \mathfrak{g}_{-1}$ . We have

$$[Y, (\text{ad } \mathfrak{g}_{-1})^k (\bigoplus_{i \geq n} (\mathfrak{g}_{max}(\Gamma))_i)] \subset (\text{ad } \mathfrak{g}_{-1})^{k+1} (\bigoplus_{i \geq n} (\mathfrak{g}_{max}(\Gamma))_i).$$

Therefore it is enough to show that

$$(\text{ad } \mathfrak{g}_{-1})^{n-1} (\mathfrak{g}_{max}(\Gamma))_n = \{0\} \quad (*)$$

From the definition of a graded Lie algebra,  $\mathfrak{g}_{max}(\Gamma)$  is generated by its local part  $\Gamma$ , and this implies that  $(\mathfrak{g}_{max}(\Gamma))_n$  is the space of linear combinations of Lie monomials of the form  $[X_1, [X_2, \dots, [X_{n-1}, X_n] \dots]]$  with  $X_i \in \mathfrak{g}_1$ . As the identity  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = [Y_1, [Y_2, [\dots, [Y_{n-1}, [X_1, [X_2, [\dots, [X_{n-1}, X_n] \dots]]]]]]$  holds in  $\mathfrak{g}_{max}(\Gamma)$ , we obtain condition (\*).

Conversely suppose that  $J_n^+$  is a graded ideal of  $\mathfrak{g}_{max}(\Gamma)$ . Then the identity (\*) above holds. This implies that the identity  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = 0$  is true.

□

**Remark 2.2.3.**

Of course, symmetrically, if the identity  $\mathcal{P}_n(X_1, \dots, X_{n-1}, Y_1, \dots, Y_n) = 0$  is verified for any elements  $X_i \in \mathfrak{g}_1$  and  $Y_j \in \mathfrak{g}_{-1}$ , then

$$J_n^- = \sum_{k=0}^{n-2} (\text{ad } \mathfrak{g}_1)^k (\oplus_{i \leq -n} (\mathfrak{g}_{max}(\Gamma))_i)$$

is a graded ideal of  $\mathfrak{g}_{max}(\Gamma)$ , contained in  $\oplus_{i \leq -2} (\mathfrak{g}_{max}(\Gamma))_i$

**Theorem 2.2.4.**

Let  $\Gamma$  be a local Lie algebra. Denote by  $\mathfrak{g}_{min}(\Gamma) = \oplus_{i \in \mathbb{Z}} (\mathfrak{g}_{min}(\Gamma))_i$  the grading in  $\mathfrak{g}_{min}(\Gamma)$ .

1) The following conditions are equivalent:

- a)  $(\mathfrak{g}_{min}(\Gamma))_i = \{0\}$  for  $i \geq n$
- b) The identity  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = 0$  holds in  $\Gamma$ , for all  $X_i \in \Gamma_1$  and all  $Y_j \in \Gamma_{-1}$ .

2) The following conditions are equivalent:

- a)  $(\mathfrak{g}_{min}(\Gamma))_i = \{0\}$  for  $i \leq -n$
- b) The identity  $\mathcal{P}_n(X_1, \dots, X_{n-1}, Y_1, \dots, Y_n) = 0$  holds in  $\Gamma$ , for all  $X_i \in \Gamma_1$  and all  $Y_j \in \Gamma_{-1}$ .

3)  $\dim \mathfrak{g}_{min}(\Gamma) < +\infty$  if and only there exist  $m, n \in \mathbb{Z}$  such that the identities  $\mathcal{P}_m(Y_1, \dots, Y_{m-1}, X_1, \dots, X_m) = 0$  and  $\mathcal{P}_n(X_1, \dots, X_{n-1}, Y_1, \dots, Y_n) = 0$  hold in  $\Gamma$ , for all  $X_i \in \Gamma_1$  and all  $Y_j \in \Gamma_{-1}$ .

*Proof.*

Lets us prove 1).

a)  $\implies$  b): if  $(\mathfrak{g}_{min}(\Gamma))_n = \{0\}$ , then  $[X_1, [\dots, [X_{n-1}, X_n]] = 0$ . And hence  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = [Y_1, [Y_2, [\dots, [Y_{n-1}, [X_1, [\dots, [X_{n-1}, X_n]]]]]] = 0$ .

b)  $\implies$  a): suppose that the identity  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = 0$  holds in  $\Gamma$ . Let  $J_n^+ = \sum_{k=0}^{n-2} (\text{ad } \mathfrak{g}_{-1})^k (\oplus_{i \geq n} \mathfrak{g}_{max}(\Gamma))_i$  the ideal that we considered in Lemma 2.2.2. Obviously  $J_n^+ \cap \Gamma = \{0\}$  and  $\oplus_{i \geq n} (\mathfrak{g}_{max}(\Gamma))_i \subset J_n^+$

Let  $J_{max}$  be the maximal graded ideal of  $\mathfrak{g}_{max}(\Gamma)$  which intersects  $\Gamma$  trivially (see Theorem 2.1.4). As  $\oplus_{i \geq n} (\mathfrak{g}_{max}(\Gamma))_i \subset J_n^+ \subset J_{max}$ , and as  $\mathfrak{g}_{min}(\Gamma) = \mathfrak{g}_{max}(\Gamma)/J_{max}$ , we obtain that  $(\mathfrak{g}_{min}(\Gamma))_i = \{0\}$  for  $i \geq n$ .

The proof of 2) is similar, using the ideal  $J_n^-$  introduced in Remark 2.2.3.

Assertion 3) is an immediate consequence of 1) and 2). □

**Example 2.2.5.** A direct calculus shows that

$$\begin{aligned} \mathcal{P}_3(Y_1, Y_2, X_1, X_2, X_3) &= [[Y_1, [[Y_2, X_1], X_2]], X_3] + [[[Y_2, X_1], X_2], [Y_1, X_3]] \\ &+ [[Y_1, X_2], [[Y_2, X_1], X_3]] + [X_2, [Y_1, [[Y_2, X_2], X_3]]] \\ &+ [[Y_1, X_1], [[Y_2, X_2], X_3]] + [X_1, [Y_1, [[Y_2, X_2], X_3]]] \\ &+ [[Y_1, X_1], [X_2, [Y_2, X_3]]] + [X_1, [Y_1, [X_2, [Y_2, X_3]]]]. \end{aligned}$$

### 3. LOCAL AND GRADED LIE ALGEBRAS ASSOCIATED TO $(\mathfrak{g}_0, B_0, \rho)$

#### 3.1. The local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .

Remind that a *quadratic Lie algebra* is a pair  $(\mathfrak{g}_0, B_0)$  where  $\mathfrak{g}_0$  is finite dimensional Lie algebra and where  $B_0$  is an invariant nondegenerate symmetric bilinear form on  $\mathfrak{g}_0$ . The most obvious examples of such algebras are the semi-simple algebras (endowed with the Killing form), the commutative algebras (endowed with any symmetric nondegenerate bilinear form) or more generally the reductive Lie algebras. But there exist more sophisticated examples. In general a quadratic Lie algebra, which is not a direct sum of two orthogonal ideals, is shown to be obtained by a finite number of so-called *double extensions* by either a simple Lie algebra or a one dimensional Lie algebra. For details see [7].

Let  $(\mathfrak{g}_0, B_0)$  be a quadratic Lie algebra. Let  $(\rho, V)$  be a finite dimensional representation of  $\mathfrak{g}_0$ . Let  $(\rho^*, V^*)$  be the contragredient representation. We will often just denote these modules by  $(\mathfrak{g}_0, V)$  and  $(\mathfrak{g}_0, V^*)$ . Similarly, for  $U \in \mathfrak{g}_0, X \in V, Y \in V^*$  we will often write  $U.X$  and  $U.Y$  instead of  $\rho(U)X$  and  $\rho^*(U)Y$ . Put  $\mathfrak{g}_{-1} = V^*$  and  $\mathfrak{g}_1 = V$ . Define also

$$\Gamma(\mathfrak{g}_0, B_0, \rho) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = V^* \oplus \mathfrak{g}_0 \oplus V.$$

Our aim is now to define a structure of local Lie algebra on  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ , such that for  $U \in \mathfrak{g}_0, X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}$ , we have  $[U, X] = U.X$  and  $[U, Y] = U.Y$ .

#### **Theorem 3.1.1.**

*Let  $(\mathfrak{g}_0, B_0)$  be a quadratic Lie algebra  $\mathfrak{g}_0$  and let  $(\rho, V)$  be a finite dimensional representation of  $\mathfrak{g}_0$ . As before we set:*

$$\Gamma(\mathfrak{g}_0, B_0, \rho) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = V^* \oplus \mathfrak{g}_0 \oplus V.$$

*For  $U \in \mathfrak{g}_0, X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}$  define an anticommutative bracket by*

a)  $[U, X] = U.X$ ,  $[U, Y] = U.Y$

b) The element  $[X, Y]$  is the unique element of  $\mathfrak{g}_0$  such that for all  $U \in \mathfrak{g}_0$  the following identity holds:

$$B_0([X, Y], U) = Y(U.X) = -(U.Y)(X).$$

(The last equality is just the definition of the contragredient representation).

The preceding bracket defines a structure of a local Lie algebra on  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .

*Proof.* We must prove that the Jacobi identity is verified each times the brackets make sense. This means that we have to prove the following identities.

$\alpha$ )  $\forall U_1, U_2 \in \mathfrak{g}_0, \forall X \in \mathfrak{g}_1, [X, [U_1, U_2]] = [[X, U_1], U_2] + [U_1, [X, U_2]]$ .

$\beta$ )  $\forall U_1, U_2 \in \mathfrak{g}_0, \forall Y \in \mathfrak{g}_{-1}, [Y, [U_1, U_2]] = [[Y, U_1], U_2] + [U_1, [Y, U_2]]$ .

$\gamma$ )  $\forall X \in \mathfrak{g}_1, \forall Y \in \mathfrak{g}_{-1}, \forall Z \in \mathfrak{g}_0, [Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]]$ .

We have:

$$[X, [U_1, U_2]] = -[[U_1, U_2], X] = -[U_1, U_2].X = -U_1.(U_2.X) + U_2.(U_1.X).$$

On the other hand we have:

$$[[X, U_1], U_2] = [U_2, [U_1, X]] = U_2.(U_1.X) \text{ and } [U_1, [X, U_2]] = -[U_1, [U_2, X]] = -U_1.(U_2.X). \text{ This proves } \alpha). \text{ The proof of the identity } \beta) \text{ is similar.}$$

Let us now consider the identity  $\gamma$ ). We set  $L = [Z, [X, Y]]$ ,  $R_1 = [[Z, X], Y]$ ,  $R_2 = [X, [Z, Y]]$ , and  $R = R_1 + R_2$ . As  $L, R_1, R_2, R \in \mathfrak{g}_0$ , in order to prove  $\gamma$ ) it will be enough to show that for all  $U \in \mathfrak{g}_0$ , we have  $B_0(L, U) = B_0(R, U)$ .

Using the invariance of  $B_0$  and definition b) we get:

$$\begin{aligned} B_0(L, U) &= B_0([Z, [X, Y]], U) = -B_0([[X, Y], Z], U) = -B_0([X, Y], [Z, U]) \\ &= -Y([Z, U].X) \\ &= -Y(Z.(U.X) - U.(Z.X)). \end{aligned}$$

On the other hand, using again definition b), we have also:

$$B_0(R_1, U) = B_0([[Z, X], Y], U) = Y(U.[Z, X]) = Y(U.(Z.X))$$

and  $B_0(R_2, U) = B_0([X, [Z, Y]], U) = [Z, Y](U.X) = Z.Y(U.X) = -Y(Z.(U.X))$ .

Hence  $B_0(R, U) = Y(U.(Z.X) - Z.(U.X)) = B_0(L, U)$ .

□

Hence we have associated a local Lie algebra to the data  $\mathfrak{g}_0, B_0, (\rho, V)$ .

**Notation 3.1.2.** For convenience we will sometimes denote this local algebra by  $\Gamma(\mathfrak{g}_0, B_0, V)$  instead of  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .

### 3.2. Isomorphisms of the local parts and dependence on $(B_0, \rho)$ .

We will first determine necessary and sufficient conditions on the data  $(\mathfrak{g}_0^i, B_0^i, \rho_i)$  ( $i = 1, 2$ ) for the corresponding local Lie algebras to be isomorphic.

Let  $\Psi : \Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1) \longrightarrow \Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)$  be an graded isomorphism between the two underlying vector spaces. Set  $A = \Psi|_{\mathfrak{g}_0^1} : \mathfrak{g}_0^1 \longrightarrow \mathfrak{g}_0^2$ ,  $\gamma = \Psi|_{V_1} : V_1 \longrightarrow V_2$  and  $\tilde{\gamma} = \Psi|_{V_1^*} : V_1^* \longrightarrow V_2^*$ . With these notations we will set  $\Psi = (\tilde{\gamma}, A, \gamma)$ . Our aim is to determine under which conditions on  $\tilde{\gamma}, A, \gamma$ , the map  $\Psi$  is an isomorphism of local Lie algebras. An obvious condition is of course that  $A$  is an isomorphism of Lie algebras.

**Proposition 3.2.1.**

Let  $\tilde{\gamma} : V_1^* \longrightarrow V_2^*$  and  $\gamma : V_1 \longrightarrow V_2$  be two isomorphisms of vector spaces and let  $A : \mathfrak{g}_0^1 \longrightarrow \mathfrak{g}_0^2$  be an isomorphism of Lie algebras. Then

$$\Psi = (\tilde{\gamma}, A, \gamma) : \Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1) \longrightarrow \Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)$$

is an isomorphism of local Lie algebras if and only the following three conditions hold (for all  $U_1 \in \mathfrak{g}_0^1, X_1 \in V_1, Y_1 \in V_1^*$ ):

- 1)  $\rho_2(A(U_1)) \circ \gamma = \gamma \circ \rho_1(U_1)$ ,
- 2)  $\rho_2^*(A(U_1)) \circ \tilde{\gamma} = \tilde{\gamma} \circ \rho_1^*(U_1)$ ,
- 3)  $B_0^2(A([X_1, Y_1]), A(U_1)) = B_0^1([\tilde{\gamma} \circ \gamma(X_1), Y_1], U_1)$ .

*Proof.*

$\Psi = (\tilde{\gamma}, A, \gamma)$  is an isomorphism if and only if the following three conditions hold for all  $U_1 \in \mathfrak{g}_0^1, X_1 \in V_1, Y_1 \in V_1^*$ :

- 1')  $\gamma([U_1, X_1]) = [A(U_1), \gamma(X_1)]$ ,
- 2')  $\tilde{\gamma}([U_1, Y_1]) = [A(U_1), \tilde{\gamma}(Y_1)]$ ,
- 3')  $A([X_1, Y_1]) = [\gamma(X_1), \tilde{\gamma}(Y_1)]$ .

But from the definition of the brackets in  $\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)$  and  $\Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)$ , conditions 1') and 2') are equivalent to conditions 1) and 2) respectively.

Consider now condition 3'). Again from the definition of the brackets it is equivalent to

$$B_0^2(A(X_1, Y_1), U_2) = B_0^2([\gamma(X_1), \tilde{\gamma}(Y_1)], U_2)$$

for all  $X_1 \in V_1, Y_1 \in V_1^*, U_2 \in \mathfrak{g}_0^2$ . Set  $U_2 = A(U_1)$ . Then

$$\begin{aligned} B_0^2(A(X_1, Y_1), A(U_1)) &= B_0^2([\gamma(X_1), \tilde{\gamma}(Y_1)], A(U_1)) \\ &= \tilde{\gamma}(Y_1)(\rho_2(A(U_1))(\gamma(X_1))) = Y_1({}^t\tilde{\gamma} \circ \rho_2(A(U_1)) \circ \gamma(X_1)) \end{aligned}$$

But from condition 2) it is easily seen that

$${}^t\tilde{\gamma} \circ \rho_2(A(U_1)) = \rho_1(U_1) \circ {}^t\tilde{\gamma} \tag{*}$$

Therefore

$$B_0^2(A(X_1, Y_1), A(U_1)) = Y_1(\rho_1(U_1) \circ {}^t\tilde{\gamma} \circ \gamma(X_1)) = B_0^1([\tilde{\gamma} \circ \gamma(X_1), Y_1], U_1),$$

which is condition 3).

Conversely suppose that  $\Psi = (\tilde{\gamma}, A, \gamma)$  satisfies conditions 1), 2) and 3). Obviously 1)  $\Leftrightarrow$  1') and 2)  $\Leftrightarrow$  2'). It remains to prove that 3') holds.

We have then:

$$\begin{aligned} B_0^1([\tilde{\gamma} \circ \gamma(X_1), Y_1], U_1) &= Y_1(\rho_1(U_1)({}^t\tilde{\gamma} \circ \gamma(X_1))) \\ &= Y_1({}^t\tilde{\gamma}\rho_2(A(U_1)\gamma(X_1))) \text{ (from (*))} \\ &= \tilde{\gamma}(Y_1)(\rho_2(A(U_1)\gamma(X_1))) \\ &= B_0^2([\gamma(X_1), \tilde{\gamma}(Y_1)], A(U_1)). \end{aligned}$$

And as 3) holds, we obtain that, for all  $X_1 \in V_1, Y_1 \in V_1^*, U_1 \in \mathfrak{g}_0^1$ , we have:

$$B_0^2(A([X_1, Y_1]), A(U_1)) = B_0^2([\gamma(X_1), \tilde{\gamma}(Y_1)], A(U_1)).$$

Therefore we obtain condition 3'):  $A([X_1, Y_1] = [\gamma(X_1), \tilde{\gamma}(Y_1)])$ . □

Now we give a specific criterion in the case where the representation  $\rho_1$  is irreducible.

**Proposition 3.2.2.**

Suppose that the representation  $(\mathfrak{g}_0^1, \rho_1, V_1)$  is irreducible.

Let  $\tilde{\gamma} : V_1^* \rightarrow V_2^*$  and  $\gamma : V_1 \rightarrow V_2$  be two isomorphisms of vector spaces and let  $A : \mathfrak{g}_0^1 \rightarrow \mathfrak{g}_0^2$  be an isomorphism of Lie algebras. Then

$$\Psi = (\tilde{\gamma}, A, \gamma) : \Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1) \rightarrow \Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)$$

is an isomorphism of local Lie algebras if and only the following two conditions hold (for all  $U_1 \in \mathfrak{g}_0^1, X_1 \in V_1, Y_1 \in V_1^*$ ):

- 1)  $\rho_2(A(U_1)) \circ \gamma = \gamma \circ \rho_1(U_1)$ ,
- 2) There exists  $c \in \mathbb{C}^*$  such that:
  - a)  $\tilde{\gamma} = c({}^t\gamma^{-1})$
  - b)  $B_0^2(A([X_1, Y_1]), A(U_1)) = cB_0^1([X_1, Y_1], U_1)$ .

*Proof.* We have already remarked that if  $\Psi = (\tilde{\gamma}, A, \gamma)$  is an isomorphism of local Lie algebras, then condition 1) and the identity

$$\rho_2^*(A(U_1)) \circ \tilde{\gamma} = \tilde{\gamma} \circ \rho_1^*(U_1)$$

hold (this is just condition 2) in Proposition 3.2.1), and this again implies that

$${}^t\tilde{\gamma} \circ \rho_2(A(U_1)) = \rho_1(U_1) \circ {}^t\tilde{\gamma}.$$

Composing on the right by  $\gamma$  one gets:

$${}^t\tilde{\gamma} \circ \rho_2(A(U_1)) \circ \gamma = \rho_1(U_1) \circ {}^t\tilde{\gamma} \circ \gamma.$$

Now, using condition 1) we obtain

$$t\tilde{\gamma} \circ \gamma \circ \rho_1(U_1) = \rho_1(U_1) \circ {}^t\tilde{\gamma} \circ \gamma,$$

and hence  $t\tilde{\gamma} \circ \gamma$  is an intertwining operator for the irreducible representation  $\rho_1$ . By Schur's lemma we obtain now  $t\tilde{\gamma} \circ \gamma = c\text{Id}_{V_1}$ , with  $c \in \mathbb{C}^*$ . As  $t\tilde{\gamma}$  and  $\gamma$  are invertible, this is condition 2)a). From Proposition 3.2.1, we have also  $B_0^2(A([X_1, Y_1]), A(U_1)) = B_0^1({}^t\tilde{\gamma} \circ \gamma(X_1), Y_1, U_1)$ , and hence we obtain condition 2)b), namely

$$B_0^2(A([X_1, Y_1]), A(U_1)) = cB_0^1([X_1, Y_1], U_1).$$

Conversely suppose that conditions 1) and 2) hold. Let us start with condition 1) multiplied by the constant  $-c$ :

$$-c\rho_2(A(U_1)) \circ \gamma = -c\gamma \circ \rho_1(U_1).$$

Multiplying left and right by  $\gamma^{-1}$  we get

$$-c\gamma^{-1} \circ \rho_2(A(U_1)) = -\rho_1(U_1) \circ c\gamma^{-1}.$$

And hence

$$-{}^t\rho_2(A(U_1)) \circ c{}^t(\gamma^{-1}) = c{}^t(\gamma^{-1}) \circ -{}^t\rho_1(U_1),$$

which gives, using condition 2)a)

$$\rho_2^*(A(U_1)) \circ \tilde{\gamma} = \tilde{\gamma} \circ \rho_1^*(U_1).$$

This is condition 2) in Proposition 3.2.1.

Condition 2)b), namely  $B_0^2(A([X_1, Y_1]), A(U_1)) = cB_0^1([X_1, Y_1], U_1)$ , can be written by using condition 2)a) as

$$B_0^2(A([X_1, Y_1]), A(U_1)) = B_0^1({}^t\tilde{\gamma} \circ \gamma(X_1), Y_1, U_1),$$

which is condition 3) in Proposition 3.2.1. Therefore  $\Psi$  is an isomorphism of the corresponding local Lie algebras. □

### Definition 3.2.3.

We call the triplet  $(\mathfrak{g}_0, B_0, (\rho, V))$  a fundamental triplet. It consists of the following ingredients:

a) a quadratic Lie algebra  $(\mathfrak{g}_0, B_0)$

b) a finite dimensional representation  $(\rho, V)$  of  $\mathfrak{g}_0$  on the space  $V$ .

In order to simplify the notation, we will sometimes denote the triplet by  $(\mathfrak{g}_0, B_0, \rho)$  or  $(\mathfrak{g}_0, B_0, V)$ .

**Definition 3.2.4.**

Let  $(\mathfrak{g}_0^1, B_0^1, \rho_1)$  and  $(\mathfrak{g}_0^2, B_0^2, \rho_2)$  be two fundamental triplets. Let  $A \in \text{Hom}(\mathfrak{g}_0^1, \mathfrak{g}_0^2)$  is a isomorphism of Lie algebras and let  $\gamma \in \text{Hom}(V_1, V_2)$  be an isomorphism of vector spaces. We say that the pair  $(A, \gamma)$  is an isomorphism of fundamental triplets if

a)

$$\forall U, \forall V \in \mathfrak{g}_0^1, B_0^2(A(U), A(V)) = B_0^1(U, V)$$

b)

$$\forall U \in \mathfrak{g}_0^1, \rho_2(A(U)) \circ \gamma = \gamma \circ \rho_1(U)$$

**Remark 3.2.5.** In the definition above, condition a) coincides with the notion of isometric isomorphism, or  $i$ -isomorphism of quadratic Lie algebras introduced in [1]. This notion of  $i$ -isomorphism was already implicit in [7].

**Theorem 3.2.6.**

Any isomorphism of fundamental triplets

$$(A, \gamma) : (\mathfrak{g}_0^1, B_0^1, \rho_1) \longrightarrow (\mathfrak{g}_0^2, B_0^2, \rho_2)$$

extends to an isomorphism of local Lie algebras

$$\Psi_{(A, \gamma)} : \Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1) \longrightarrow \Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2).$$

Moreover, if the space  $\langle \rho_1(\mathfrak{g}_0^1).V_1 \rangle$  generated by the vectors  $\rho_1(U)X$  ( $X \in V_1, U \in \mathfrak{g}_0^1$ ) is equal to  $V_1$ , then the preceding extension is unique.

*Proof.*

Note first that condition a) in Definition 3.2.4 is exactly condition 1) in Proposition 3.2.1. Define  $\tilde{\gamma} : V_1^* \longrightarrow V_2^*$  by  $\tilde{\gamma} = {}^t\gamma^{-1}$  where  ${}^t$  stands for the transposed map. Then it is easy to check that

$$\forall U \in \mathfrak{g}_0^1, \rho_2^*(A(U)) \circ \tilde{\gamma} = \tilde{\gamma} \circ \rho_1^*(U)$$

and this is exactly condition 2) in Proposition 3.2.1.

As  ${}^t\tilde{\gamma} \circ \gamma = \text{Id}_{|V_1}$ , condition b) in Definition 3.2.4 is exactly condition 3) in Proposition 3.2.1. This Proposition implies then that  $\Psi_{(A, \gamma)} = (\tilde{\gamma}, A, \gamma)$  is an isomorphism of the corresponding local Lie algebras.

It remains to prove the uniqueness of the extension under the condition  $\langle \rho_1(\mathfrak{g}_0^1).V_1 \rangle = V_1$ . But if

$$\Psi : \Gamma(\mathfrak{g}_0^1, B_0^1, V_1) \longrightarrow \Gamma(\mathfrak{g}_0^2, B_0^2, V_2)$$



is an isomorphism of local Lie algebras such that  $\Psi|_{\mathfrak{g}_0} = A$ ,  $\Psi|_{V_1} = \gamma$  and  $\Psi|_{V_1^*} = \tilde{\gamma} : V_1^* \rightarrow V_2^*$ , then for  $X_1 \in V_1, Y_1 \in V_1^*$ ,  $A([X_1, Y_1]) = [\gamma(X_1), \tilde{\gamma}(Y_1)]$ . Therefore

$$B_0^2(A([X_1, Y_1]), U_2) = B_0^2([\gamma(X_1), \tilde{\gamma}(Y_1)], U_2),$$

for all  $U_2 \in \mathfrak{g}_0^2, X_1 \in V_1, Y_1 \in V_1^*$ . Then for  $U_2 = A(U_1), U_1 \in V_1$ ,

$$\begin{aligned} B_0^2(A([X_1, Y_1]), A(U_1)) &= B_0^1([X_1, Y_1], U_1) = Y_1(\rho_1(U_1)X_1) \\ &= B_0^2([\gamma(X_1), \tilde{\gamma}(Y_1)], A(U_1)) = \tilde{\gamma}(Y_1)(\rho_2(A(U_1)) \circ \gamma(X_1)) = Y(t\tilde{\gamma} \circ \rho_2(A(U_1)) \circ \gamma(X_1)) \\ &= Y_1(t\tilde{\gamma} \circ \gamma \circ \rho_1(U_1)X_1) \end{aligned}$$

Then from  $Y_1(\rho_1(U_1)X_1) = Y_1(t\tilde{\gamma} \circ \gamma \circ \rho_1(U_1)X_1)$ , we obtain that  $t\tilde{\gamma} \circ \gamma$  is the identity on the space  $\langle \rho_1(\mathfrak{g}_0^1).V_1 \rangle = V_1$ . Therefore  $\tilde{\gamma} = {}^t\gamma^{-1}$ . □

**Notation 3.2.7.** *The local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  described in Theorem 3.1.1 depends on  $B_0$  and on  $\rho$  (see Proposition 3.2.8, Proposition 3.2.9 and Example 3.3.7 below). If necessary, we will denote by  $[ , ]_{B_0}$  or  $[ , ]_{B_0, \rho}$  the bracket in  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .*

We will now investigate the dependance of the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ , under a change of the invariant form  $B_0$  on  $\mathfrak{g}_0$ . We make the following assumptions:

There exist quadratic Lie algebras  $(L_i, B_{0,i})$   $i = 1, \dots, k$  such that

–  $\mathfrak{g}_0 = L_1 \oplus L_2 \oplus \dots \oplus L_k$  (direct sum of ideals)

– the invariant bilinear form  $B_0$  on  $\mathfrak{g}_0$  is given by  $B_0 = B_{0,1} \oplus B_{0,2} \oplus \dots \oplus B_{0,k}$ .

Then for  $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{C}^*)^k$  we set

$$\lambda.B_0 = \lambda_1 B_{0,1} \oplus \dots \oplus \lambda_k B_{0,k}$$

and for  $u = u_1 + \dots + u_k \in \mathfrak{g}_0$  ( $u_i \in L_i$ ) we set

$$\lambda.u = \lambda_1 u_1 + \dots + \lambda_k u_k.$$

**Proposition 3.2.8.**

1) *Using the preceding notations, we have for  $X \in V = \mathfrak{g}_1$  and for  $Y \in V^* = \mathfrak{g}_{-1}$ :*

$$\lambda.[X, Y]_{\lambda.B_0} = [X, Y]_{B_0}$$

2) *For  $\mu \in \mathbb{C}^*$ , the local Lie algebras  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  and  $\Gamma(\mathfrak{g}_0, \mu B_0, \rho)$  attached respectively to  $B_0$  and  $\mu B_0$  are isomorphic (here  $\mu B_0$  stands for the ordinary scalar multiplication).*

*Proof.*

1) From the definition  $[X, Y]_{\lambda.B_0}$  is the unique element in  $\mathfrak{g}_0$  such that  $\lambda.B_0([X, Y]_{\lambda.B_0}, U) = Y(U.X)$  for any  $U \in \mathfrak{g}_0$ . It is easy to see that from the definitions we have  $\lambda.B_0([X, Y]_{\lambda.B_0}, U) = B_0(\lambda.[X, Y]_{\lambda.B_0}, U)$ . Hence  $\lambda.[X, Y]_{\lambda.B_0} = [X, Y]_{B_0}$ .

2) Choose a square root  $\sqrt{\mu}$  of  $\mu$ . Define  $\varphi_\mu : \Gamma(\mathfrak{g}_0, B_0, \rho) \longrightarrow \Gamma(\mathfrak{g}_0, \mu B_0, \rho)$  by

$$\forall X \in V, \varphi_\mu(X) = \sqrt{\mu}X, \quad \forall Y \in V^*, \varphi_\mu(Y) = \sqrt{\mu}Y, \quad \forall U \in \mathfrak{g}_0, \varphi_\mu(U) = U.$$

A direct computation or the use of the criterion of Proposition 3.2.1 shows easily that  $\varphi_\mu$  is an isomorphism of local Lie algebras. □

We will also investigate the modification of the bracket in  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  under a slight change of  $\rho$ .

Suppose that  $\mathfrak{g}_0 = Z \oplus L$  is a quadratic Lie algebra where  $Z$  is a central ideal and  $L$  is an ideal. For  $\gamma \in \mathbb{C}^*$  we denote by  $\gamma \square \rho$  the representation of  $\mathfrak{g}_0$  on  $V$  given by  $\gamma \square \rho(z + u) = \gamma \rho(z) + \rho(u)$ , for  $z \in Z$  and  $u \in L$ . If  $U = z + u \in \mathfrak{g}_0$ , and if we set  $\gamma \square U = \gamma z + u$ , we have  $\gamma \square \rho(U) = \rho(\gamma \square U)$ . If  $B_0 = B_{0,Z} + B_{0,L}$  where  $B_{0,Z}$  and  $B_{0,L}$  are forms on  $Z$  and  $L$  respectively, we define  $\gamma \square B_0 = \gamma B_{0,Z} + B_{0,L}$ .

Of course in the notations of Proposition 3.2.8  $\gamma \square B_0 = (\gamma, 1).B_0$  and  $\gamma \square U = (\gamma, 1).U$

The next proposition indicates the dependance of the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  if we change  $\rho$  into  $\gamma \square \rho$ .

**Proposition 3.2.9.**

1) Let us denote by  $[ , ]_{B_0, \rho}$  the bracket on  $\Gamma(\mathfrak{g}_0, B_0, V)$  given by Theorem 3.1.1. Then, using the notations defined above, we have:

$$[X, Y]_{B_0, \gamma \square \rho} = \gamma \square [X, Y]_{B_0, \rho}$$

$$[U, X]_{B_0, \gamma \square \rho} = \gamma \square \rho(U)X, \quad [U, Y]_{B_0, \gamma \square \rho} = -\gamma \square \rho^*(U)Y$$

2) Suppose that  $\lambda, \mu, \alpha, \beta \in \mathbb{C}^*$  verify the condition  $\frac{\mu^2}{\lambda} = \frac{\beta^2}{\alpha}$ . Then the local Lie algebras  $\Gamma(\mathfrak{g}_0, \lambda \square B_0, \mu \square \rho)$  and  $\Gamma(\mathfrak{g}_0, \alpha \square B_0, \beta \square \rho)$  are isomorphic. In particular  $\Gamma(\mathfrak{g}_0, \lambda \square B_0, \rho)$  and  $\Gamma(\mathfrak{g}_0, B_0, \frac{1}{\sqrt{\lambda}} \square \rho)$  are isomorphic.

*Proof.*

1) The element  $[X, Y]_{B_0, \gamma \square \rho}$  is by definition the unique element of  $\mathfrak{g}_0$  such that, for all  $U \in \mathfrak{g}_0$ ,  $B_0([X, Y]_{B_0, \gamma \square \rho}, U) = Y(\gamma \square \rho(U)X) = Y(\rho(\gamma \square U)X) = B_0([X, Y]_{B_0, \rho}, \gamma \square U) = B_0(\gamma \square [X, Y]_{B_0, \rho}, U)$ . Hence  $[X, Y]_{B_0, \gamma \square \rho} = \gamma \square [X, Y]_{B_0, \rho}$ . The other two identities are just the definitions of  $[U, X]_{B_0, \gamma \square \rho}$  and  $[U, Y]_{B_0, \gamma \square \rho}$ .

- 2) Define  $\varphi : \Gamma(\mathfrak{g}_0, \lambda \square B_0, \mu \square \rho) \longrightarrow \Gamma(\mathfrak{g}_0, \alpha \square B_0, \beta \square \rho)$  by
- $$\begin{aligned}\varphi(U) &= \frac{\mu}{\beta} \square U, \text{ for all } U \in \mathfrak{g}_0 \\ \varphi(X) &= X \text{ for all } X \in V \\ \varphi(Y) &= Y \text{ for all } Y \in V^*\end{aligned}$$

Again a direct computation or the criterion of Proposition 3.2.1 shows that  $\varphi$  is an isomorphism of local Lie algebras. □

**Remark 3.2.10. (inverse isomorphism)**

Let  $(\mathfrak{g}_0, B_0)$  be a quadratic Lie algebra. Consider a finite dimensional representation  $(\rho, V)$  of  $\mathfrak{g}_0$ . Set  $\mathfrak{g}_0^1 = \mathfrak{g}_0^2 = \mathfrak{g}_0$ ,  $\mathfrak{g}_1^1 = V$ ,  $\mathfrak{g}_{-1}^1 = V^*$ ,  $\mathfrak{g}_1^2 = V^*$ ,  $\mathfrak{g}_{-1}^2 = V$ . Hence  $\Gamma(\mathfrak{g}_0^1, B_0, \rho) = V^* \oplus \mathfrak{g}_0^1 \oplus V$  and  $\Gamma(\mathfrak{g}_0^2, B_0, \rho^*) = V \oplus \mathfrak{g}_0^2 \oplus V^*$ . The map

$$\theta : \Gamma(\mathfrak{g}_0, B_0, \rho) \longrightarrow \Gamma(\mathfrak{g}_0, B_0, \rho^*)$$

defined by  $\theta(U) = U$  for all  $U \in \mathfrak{g}_0^1 = \mathfrak{g}_0^2 = \mathfrak{g}_0$ ,  $\theta(X) = X$  for all  $X \in \mathfrak{g}_1^1 = V$  and  $\theta(Y) = Y$  for all  $Y \in \mathfrak{g}_0^2 = V^*$ , is a *non graded* isomorphism of local Lie algebras (it sends  $\mathfrak{g}_1^1$  onto  $\mathfrak{g}_{-1}^2$  and  $\mathfrak{g}_{-1}^1$  onto  $\mathfrak{g}_1^2$ ). Of course the isomorphism  $\theta$  extends uniquely to an non graded isomorphism, still denoted  $\theta$ , between  $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  and  $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho^*))$ .

**3.3. Graded Lie algebras with local part  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .**

Let us now translate the result of Kac (Theorem 2.1.4) in the context of Theorem 3.1.1:

**Theorem 3.3.1.**

*Let  $(\mathfrak{g}_0, B_0, \rho)$  be a fundamental triplet. Let  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  be the local Lie algebra constructed in Theorem 3.1.1.*

1) *There exists a unique graded Lie algebra  $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  whose local part is  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  and which satisfies the following universal property.*

*Any morphism of local Lie algebras  $\Gamma(\mathfrak{g}_0, B_0, \rho) \rightarrow \Gamma(\mathfrak{g})$  from  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  into the local part  $\Gamma(\mathfrak{g})$  of a graded Lie algebra  $\mathfrak{g}$  extends uniquely to a morphism of graded Lie algebras  $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho)) \rightarrow \mathfrak{g}$ . (And hence any graded Lie algebra whose local part is isomorphic to  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ , is a quotient of  $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ ).*

*Moreover we have*

$$\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = F(V^*) \oplus \mathfrak{g}_0 \oplus F(V),$$

*where  $F(V^*)$  (resp.  $F(V)$ ) is the free Lie algebra generated by  $V^*$  (resp.  $V$ ).*

2) *There exists a unique graded Lie algebra  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  whose local part is  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  and which satisfies the following universal property.*

Any surjective morphism of local Lie algebras  $\Gamma(\mathfrak{g}) \rightarrow \Gamma(\mathfrak{g}_0, B_0, \rho)$  from the local part of a graded Lie algebra  $\mathfrak{g}$  into  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  extends uniquely to a (surjective) morphism of graded Lie algebras  $\mathfrak{g} \rightarrow \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ . (And hence  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is a quotient of any graded Lie algebra whose local part is isomorphic to  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ ).

**Remark 3.3.2.** Let  $(\mathfrak{g}_0, B_0)$  be a quadratic Lie algebra. Suppose that  $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$  is an orthogonal decomposition into ideals. Define  $B_0^1 = B_0|_{\mathfrak{g}_0^1 \times \mathfrak{g}_0^1}$  and  $B_0^2 = B_0|_{\mathfrak{g}_0^2 \times \mathfrak{g}_0^2}$ . Suppose also that the representation  $(\mathfrak{g}, \rho, V)$  is a direct sum  $(\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$ . Then from the definitions we obtain that  $\Gamma(\mathfrak{g}_0, B_0, V) = \Gamma(\mathfrak{g}_0^1, B_0^1, V_1) \oplus \Gamma(\mathfrak{g}_0^2, B_0^2, V_2)$ , and therefore

$$\mathfrak{g}_{\max}(\Gamma(\mathfrak{g}_0, B_0, \rho)) \simeq \mathfrak{g}_{\max}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)) \oplus \mathfrak{g}_{\max}(\Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2))$$

and

$$\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)) \oplus \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)).$$

As an example let us show that graded reductive (finite dimensional) Lie algebras are always minimal graded Lie algebras.

**Proposition 3.3.3.**

Let  $\mathfrak{g}$  be a reductive (finite dimensional) Lie algebra. Suppose that we are given a  $\mathbb{Z}$ -grading  $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$  such that  $\mathfrak{g}$  is generated by its local part  $\Gamma(\mathfrak{g}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$  (in other words  $\mathfrak{g}$  is a graded Lie algebra in the sense of Definition 2.1.1). Let  $B_{\Gamma(\mathfrak{g})}$  be a nondegenerate invariant symmetric bilinear form on  $\Gamma(\mathfrak{g})$  such that  $B_{\Gamma(\mathfrak{g})}(\mathfrak{g}_i, \mathfrak{g}_j) = 0$  when  $|i + j| \neq 0$ . Then, using  $B_{\Gamma(\mathfrak{g})}$ , the contragredient representation  $(\mathfrak{g}_0, \mathfrak{g}_1^*)$  can be identified with  $(\mathfrak{g}_0, \mathfrak{g}_{-1})$ . If  $B_0$  denotes the restriction of  $B_{\Gamma(\mathfrak{g})}$  to  $\mathfrak{g}_0$ , then we have:

$$\mathfrak{g} \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g})) \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \mathfrak{g}_1))$$

*Proof.*

From the definition we can identify  $\mathfrak{g}_1^*$  with  $\mathfrak{g}_{-1}$  by using  $B_{\Gamma(\mathfrak{g})}$ . Then for  $U \in \mathfrak{g}_0$ ,  $Y \in \mathfrak{g}_{-1}$  and  $X \in \mathfrak{g}_1$  we have, from the invariance of  $B_{\Gamma(\mathfrak{g})}$ :

$$B_{\Gamma(\mathfrak{g})}(U.Y, X) = -B_{\Gamma(\mathfrak{g})}(Y, [U, X]) = B_{\Gamma(\mathfrak{g})}([U, Y], X)$$

and hence  $[U, Y] = U.Y$  (here  $U.Y$  stands for the contragredient action of  $U \in \mathfrak{g}_0$  on  $\mathfrak{g}_{-1} \simeq \mathfrak{g}_1^*$ ).

Similarly, we have also:

$$B_{\Gamma(\mathfrak{g})}([X, Y], U) = B_{\Gamma(\mathfrak{g})}([X, Y], U) = B_{\Gamma(\mathfrak{g})}(Y, [U, X]) = Y(U.X).$$

Hence the original bracket in  $\Gamma(\mathfrak{g})$  is the bracket constructed in Theorem 3.1.1.

Therefore  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g})) \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \mathfrak{g}_1))$ .

From the universal property of  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}))$  there exists a graded ideal  $I \subset \mathfrak{g}$  such that  $\mathfrak{g}/I \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}))$ . As  $\mathfrak{g}$  is reductive there exists an ideal  $U \subset \mathfrak{g}$  such that  $\mathfrak{g} = U \oplus I$ . Hence  $U \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}))$ . But then the local part of  $U$  is  $\Gamma(\mathfrak{g})$ , and this contradicts the fact that  $\mathfrak{g}$  is generated by  $\Gamma(\mathfrak{g})$  unless  $I = \{0\}$ .  $\square$

**Remark 3.3.4.**

It must be noticed that if  $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$  is an arbitrary  $\mathbb{Z}$ -grading of a semi-simple Lie algebra  $\mathfrak{g}$  then  $\mathfrak{g}$  is in general not generated by its local part, and is therefore not a graded Lie algebra in the sense of Definition 2.1.1. Let us explain this briefly. It is well known that there exists always a grading element, that is an element  $H \in \mathfrak{g}$  such that  $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [H, X] = iX\}$ . Let  $\mathfrak{h}$  be Cartan subalgebra of  $\mathfrak{g}_0$  (which is also a Cartan subalgebra of  $\mathfrak{g}$ ), containing  $H$ . Let  $\Psi$  be a set of simple roots of the root system  $\Sigma(\mathfrak{g}, \mathfrak{h})$  such that  $\alpha(H) \in \mathbb{N}$  (such a set of simple roots always exists). Hence we have associated a "weighted Dynkin diagram" to a grading  $\cdot$ . The subdiagram of roots of weight 0 corresponds to the semi-simple part of the Levi subalgebra  $\mathfrak{g}_0$ . But if the weighted Dynkin diagram has weights equal to 1 and to  $n > 1$ , then it is easy to see that  $\mathfrak{g}$  cannot be generated by the local part  $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ .

**Example 3.3.5. (Prehomogeneous vector spaces of parabolic type)**

Let  $\mathfrak{g}$  be a semi-simple complex Lie algebra. Let  $\mathfrak{h}$  be Cartan subalgebra. Denote as before by  $\Sigma(\mathfrak{g}, \mathfrak{h})$  the set of roots of the pair  $(\mathfrak{g}, \mathfrak{h})$ . Let  $\Psi$  be a set of simple roots for  $\Sigma(\mathfrak{g}, \mathfrak{h})$ . Let  $\theta \subset \Psi$  be a subset and let  $\langle \theta \rangle$  denote the subset of roots which are linear combinations of elements of  $\theta$ . Let  $\mathfrak{g}_\theta = \mathfrak{l}_\theta$  be the Levi subalgebra corresponding to  $\theta$ . That is  $\mathfrak{g}_\theta = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \langle \theta \rangle} \mathfrak{g}^\alpha)$  where  $\mathfrak{g}^\alpha$  is the root space corresponding to  $\alpha$ . Let  $H_\theta$  be the unique element in  $\mathfrak{h}$  such that  $\alpha(H_\theta) = 0$  if  $\alpha \in \theta$  and  $\alpha(H_\theta) = 1$  if  $\alpha \in \Psi \setminus \theta$ . Define then  $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [H_\theta, X] = iX\}$  (this definition of  $\mathfrak{g}_0$  is coherent with the preceding one). One obtains this way a grading  $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$ . The representations  $(\mathfrak{g}_0, \mathfrak{g}_1)$  are prehomogeneous vector spaces called prehomogeneous spaces of parabolic type. It is easy to see that they correspond to gradings whose weights in the sense of the preceding Remark are only 0 and 1. From Proposition 3.3.3 we obtain that in this case  $\mathfrak{g} = \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \mathfrak{g}_1))$  where  $B_0$  is the restriction of the Killing form of  $\mathfrak{g}$  to  $\mathfrak{g}_0$ .

**Example 3.3.6. (Principal gradings of symmetrizable Lie algebras)**

We adopt here the same definitions and notations as in [6]. Let  $A = (a_{i,j})$  be a  $n \times n$  matrix with complex coefficients. Let  $(\mathfrak{h}, \Pi, \check{\Pi})$  be a realization of  $A$ . The Lie algebra  $\tilde{\mathfrak{g}}(A)$  is the Lie algebra with generators  $\mathfrak{h}$ ,  $e_i$ ,  $f_i$  ( $i = 1, \dots, n$ ) and the relations:

$$[e_i, f_j] = \delta_{i,j} \check{\alpha}_i, [h, e_i] = \alpha_i(h)e_i, [h, f_i] = -\alpha_i(h)f_i, [h, h'] = 0, (h, h' \in \mathfrak{h}).$$

Then the Lie algebra  $\mathfrak{g}(A)$  is defined by  $\tilde{\mathfrak{g}}(A)/\mathfrak{t}$  where  $\mathfrak{t}$  is the maximal ideal intersecting  $\mathfrak{h}$  trivially. Then  $\mathfrak{g}(A) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$  where  $\mathfrak{g}_i = \bigoplus_{ht(\alpha)=i} \mathfrak{g}_\alpha$  (if  $\alpha = \sum_{i=1}^n m_i \alpha_i$ ,  $ht(\alpha) = \sum_i m_i$ ). This is the principal grading of  $\mathfrak{g}(A)$ . We have  $\mathfrak{g}_{-1} = \bigoplus_{i=1}^n \mathbb{C}f_i$ ,  $\mathfrak{g}_0 = \mathfrak{h}$ ,  $\mathfrak{g}_1 = \bigoplus_{i=1}^n \mathbb{C}e_i$ , and it is easy to see that  $\mathfrak{g}(A) = \mathfrak{g}_{min}(\Gamma(A))$  where  $\Gamma(A) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ .

We suppose further that the matrix  $A$  is symmetrizable. This means that there exist a symmetric matrix  $B = (b_{i,j})$  and an invertible diagonal matrix  $D = \text{diag}(\epsilon_1, \dots, \epsilon_n)$  such that  $A = DB$ . The Lie algebra is then called a *symmetrizable Lie Algebra*. Then, according to Theorem 2.2. of [6], there exists a non-degenerate invariant symmetric bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{g}(A)$  such that the restriction of this form to  $\mathfrak{h}$  is non-degenerate and  $(\mathfrak{g}_i, \mathfrak{g}_j) = 0$  if  $i + j \neq 0$ . Therefore  $\mathfrak{g}_{-i}$  can be identified to  $\mathfrak{g}_i^*$  and the representation  $(\mathfrak{g}_0, \mathfrak{g}_{-1})$  is the contragredient representation of  $(\mathfrak{g}_0, \mathfrak{g}_1)$ . Let  $\rho$  denote the representation  $(\mathfrak{g}_0, \mathfrak{g}_1)$  and let  $B_0$  be the restriction of the form  $(\cdot, \cdot)$  to  $\mathfrak{g}_0 = \mathfrak{h}$ . Then, in the notations of section 3.3 we have

$$\mathfrak{g}(A) = \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho)).$$

**Example 3.3.7.** We will now examine the case of  $\mathfrak{sl}_{2n}(\mathbb{C})$  which will be considered both as a graded Lie algebra and a local Lie algebra. This will show that the local Lie algebras  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  and the corresponding minimal Lie algebra  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  depend strongly on the choice of  $B_0$ .

Consider first the classical 3-grading of  $\mathfrak{sl}_{2n}(\mathbb{C})$  defined by:

$$\begin{aligned} \mathfrak{g}_{-1} = V^* &= \left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}, Y \in M_n(\mathbb{C}) \right\} \\ \mathfrak{g}_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A, B \in M_n(\mathbb{C}), \text{Tr}(A+B) = 0 \right\} \\ \mathfrak{g}_1 = V &= \left\{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, X \in M_n(\mathbb{C}) \right\} \end{aligned}$$

We will use the letter  $U$  for elements in  $\mathfrak{g}_0$  and the letters  $X, Y$  for elements in  $\mathfrak{g}_1$  and  $\mathfrak{g}_{-1}$  respectively. And in order to simplify notations we will set  $X = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$  and  $Y = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$ .

As an invariant form on  $\mathfrak{sl}_{2n}(\mathbb{C})$  we will take  $B(\alpha, \beta) = \text{Tr}(\alpha\beta)$ ,  $(\alpha, \beta \in \mathfrak{sl}_{2n}(\mathbb{C}))$ . This is just a multiple of the Killing form. Let us call  $B_0$  the restriction of  $B$  to  $\mathfrak{g}_0$ . The form  $B_0$  is of course nondegenerate. The representation  $(\mathfrak{g}_0, \rho, V)$  is defined by the bracket. Therefore if  $U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ , we have  $\rho(U)X = [U, X] =$

$$\begin{pmatrix} 0 & AX - XB \\ 0 & 0 \end{pmatrix} = AX - XB.$$

If we consider  $\mathfrak{sl}_{2n}(\mathbb{C})$  as the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  we now from Proposition 3.3.3 that  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = \mathfrak{sl}_{2n}(\mathbb{C})$ .

Next we will modify the form  $B_0$  in the following manner. Let  $\lambda = (\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$  and set  $B_0^\lambda\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}\right) = \lambda_1 \text{Tr}(AA') + \lambda_2 \text{Tr}(BB')$  (remember that  $B_0\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}\right) = \text{Tr}(AA') + \text{Tr}(BB')$ ).

We will decompose the form  $B_0^\lambda$  according to the decomposition of  $\mathfrak{g}_0$  into ideals. Obviously

$$\mathfrak{g}_0 = \mathbb{C}H_0 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \quad (*)$$

where  $H_0 = \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix}$  and where  $\mathfrak{a}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, A \in \mathfrak{sl}_n(\mathbb{C}) \right\}$  and  $\mathfrak{a}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, B \in \mathfrak{sl}_n(\mathbb{C}) \right\}$ .

The decomposition of  $[Y, X]_{B_0}$  according to (\*) is as follows:

$$\begin{aligned} [Y, X]_{B_0} &= -\frac{1}{n} \text{Tr}(YX) \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix} \\ &\quad + \begin{pmatrix} -XY + \frac{1}{n} \text{Tr}(YX) \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & YX - \frac{1}{n} \text{Tr}(YX) \text{Id}_n \end{pmatrix} \end{aligned}$$

An easy computation shows that  $B_0^\lambda = \frac{\lambda_1 + \lambda_2}{2} B_{0|_{\mathbb{C}H_0}} + \lambda_1 B_{0|_{\mathfrak{a}_1}} + \lambda_2 B_{0|_{\mathfrak{a}_2}}$  (hence the form  $B_0^\lambda$  is nondegenerate if and only if  $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_1 + \lambda_2 \neq 0$ ). Therefore  $B_0^\lambda = \mu \cdot B_0$ , where  $\mu = \left(\frac{\lambda_1 + \lambda_2}{2}, \lambda_1, \lambda_2\right)$  (and with the definition of  $\mu \cdot B_0$  given just before Proposition 3.2.8). From Proposition 3.2.8 we obtain that

$$\begin{aligned}
[Y, X]_{B_0^\lambda} &= -\frac{2}{n(\lambda_1 + \lambda_2)} \text{Tr}(YX) \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix} \\
&\quad + \begin{pmatrix} \frac{1}{\lambda_1}(-XY + \frac{1}{n}\text{Tr}(YX)\text{Id}_n) & 0 \\ 0 & \frac{1}{\lambda_2}(YX - \frac{1}{n}\text{Tr}(YX)\text{Id}_n) \end{pmatrix} \\
&= \begin{pmatrix} \frac{-1}{\lambda_1}XY + \frac{\lambda_2 - \lambda_1}{n(\lambda_1 + \lambda_2)\lambda_1} \text{Tr}(YX)\text{Id}_n & 0 \\ 0 & \frac{1}{\lambda_2}YX + \frac{\lambda_2 - \lambda_1}{n(\lambda_1 + \lambda_2)\lambda_2} \text{Tr}(YX)\text{Id}_n \end{pmatrix} \quad (**).
\end{aligned}$$

Suppose now that the local Lie algebras  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  and  $\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho)$  are isomorphic. Then as  $\Gamma(\mathfrak{g}_0, B_0, \rho) = \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) \simeq \mathfrak{sl}_{2n}(\mathbb{C})$ , we should have that  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho)) \simeq \mathfrak{sl}_{2n}(\mathbb{C})$ , and hence for  $Y \in V^*$  and  $X, X' \in V$  we have  $[Y, [X, X']_{B_0^\lambda}]_{B_0^\lambda} = 0$ . And then from the Jacobi identity the following identity should hold in  $\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho)$ :

$$[[Y, X]_{B_0^\lambda}, X']_{B_0^\lambda} + [X, [Y, X']_{B_0^\lambda}]_{B_0^\lambda} = 0 \quad (***)$$

(This is just the identity  $\mathcal{P}_2 = 0$  seen in section 2.2).

A calculation, using (\*\*) shows that the left member of (\*\*\*) is equal to

$$\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) \left( (XYX' - X'YX) + \frac{\lambda_2 - \lambda_1}{n(\lambda_1 + \lambda_2)} (\text{Tr}(YX')X - \text{Tr}(YX)X') \right)$$

(in the simplified notation explained at the beginning of the example).

It is easy to see that if  $\lambda_1 \neq \lambda_2$  this element is not equal to zero for general  $Y, X, X' \in M_n(\mathbb{C})$ . Therefore the local Lie algebra  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho))$  cannot be isomorphic to  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) \simeq \mathfrak{sl}_{2n}(\mathbb{C})$ . In fact, using Proposition 3.4.6 below, one can prove that actually  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho))$  is infinite dimensional if  $\lambda_1 \neq \lambda_2$ .

### 3.4. Transitivity.

Let us also recall the notion of transitivity introduced by V. Kac.

**Definition 3.4.1.** (Kac [4], Definition 2)

Let  $\mathfrak{g}$  (resp.  $\Gamma$ ) be a graded Lie algebra (resp. a local Lie algebra). Then  $\mathfrak{g}$  (resp.  $\Gamma$ ) is said to be transitive if

- for  $x \in \mathfrak{g}_i$ ,  $i \geq 0$ ,  $[x, \mathfrak{g}_{-1}] = \{0\} \Rightarrow x = 0$
- for  $x \in \mathfrak{g}_i$ ,  $i \leq 0$ ,  $[x, \mathfrak{g}_1] = \{0\} \Rightarrow x = 0$ .

In particular if  $\mathfrak{g}$  (or  $\Gamma$ ) is transitive, then the modules  $(\mathfrak{g}_0, \mathfrak{g}_{-1})$  and  $(\mathfrak{g}_0, \mathfrak{g}_1)$  are faithful.



**Remark 3.4.2.**

It is easy to see that if a graded Lie algebra  $\mathfrak{g}$  is transitive then its center  $Z(\mathfrak{g})$  is trivial.

If  $A$  is a subset of a vector space  $V$ , we denote by  $\langle A \rangle$  the subspace of  $V$  generated by  $A$ .

**Proposition 3.4.3.**

Let  $(\mathfrak{g}_0, B_0, (\rho, V))$  be a fundamental triplet.

- 1) The local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  (or the minimal algebra  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ ) is transitive if and only if  $(\rho, V)$  is faithful and  $\langle \mathfrak{g}_0.V \rangle = V$  and  $\langle \mathfrak{g}_0.V^* \rangle = V^*$ .
- 2) If the representation  $(\rho, V)$  is completely reducible, then the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  (or the minimal algebra  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ ) is transitive if and only if  $(\rho, V)$  is faithful and  $V$  does not contain the trivial module.

*Proof.*

As a minimal graded Lie algebra with a transitive local part is transitive ([4], Prop. 5 page 1278), it is enough to prove the proposition for the local part.

1) Suppose that  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is transitive. We have already remarked that then the representation  $(\rho, V)$  is faithful (and hence  $(\rho^*, V^*)$  is faithful too). If  $\langle \mathfrak{g}_0.V \rangle \neq V$ , then there exists  $Y \in V^*, Y \neq 0$  such that  $Y(\mathfrak{g}_0.V) = 0$ . From the definition of the bracket we obtain that  $B_0([V, Y], \mathfrak{g}_0) = 0$ . Hence  $[Y, V] = \{0\}$ . This contradicts the transitivity. Similarly one proves that transitivity implies  $\langle \mathfrak{g}_0.V^* \rangle = V^*$ . Conversely suppose that  $(\rho, V)$  is faithful and  $\langle \mathfrak{g}_0.V \rangle = V$  and  $\langle \mathfrak{g}_0.V^* \rangle = V^*$ . The first of these assumptions is one of the conditions needed for the transitivity. Suppose also that  $[X, V^*] = \{0\}$  for an  $X \in V$ . Then  $B_0([X, Y], U) = 0$  for all  $U \in \mathfrak{g}_0$  and all  $Y \in V^*$ . Hence  $Y(U.X) = -U.Y(X) = 0$ . Therefore, as  $\langle \mathfrak{g}_0.V^* \rangle = V^*$ , we have  $V^*(X) = 0$  and hence  $X = 0$ . The same proof, using the identity  $\langle \mathfrak{g}_0.V \rangle = V$ , shows that  $[Y, V] = \{0\}$  implies  $Y = 0$ . Hence  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is transitive.

2) Let  $V = \bigoplus_{i=1}^k V_i$  be a decomposition of  $V$  into irreducibles. If  $V_i$  is not the trivial module we have of course  $\langle \mathfrak{g}_0.V_i \rangle = V_i$ . It is then easy to see that the preceding condition " $\langle \mathfrak{g}_0.V \rangle = V$  and  $\langle \mathfrak{g}_0.V^* \rangle = V^*$ " is equivalent to the condition " $V$  does not contain the trivial module".

□

**Remark 3.4.4.**

1) Suppose that there exists an element  $H \in \mathfrak{g}_0$  such that  $H.X = X$  for all  $X \in V$ . Such an element is called *grading element*. Then obviously the conditions

$\langle \mathfrak{g}_0.V \rangle = V$  and  $\langle \mathfrak{g}_0.V^* \rangle = V^*$  are satisfied. In this case the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, V)$  is transitive if and only if  $(\rho, V)$  is faithful.

2) Suppose that the representation  $(\rho, V)$  is faithful and completely reducible. Let  $V = \bigoplus_{i=1}^k V_i$  be a decomposition of  $V$  into irreducibles. Then by Schur's Lemma we obtain that  $\dim Z(\mathfrak{g}_0) \leq k$  ( $Z(\mathfrak{g}_0)$  denotes the center of  $\mathfrak{g}_0$ ). Hence if the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is transitive, then  $\dim Z(\mathfrak{g}_0) \leq k$ . In particular if  $V$  is irreducible, then  $\dim Z(\mathfrak{g}_0) \leq 1$ .

The next proposition describes the structure of the algebra  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  in the non transitive case, under some assumptions.

**Proposition 3.4.5.** (see Notation 3.1.2)

Let  $\mathfrak{g}_0$  be a reductive Lie algebra, and let  $B_0$  be a non degenerate invariant symmetric bilinear form on  $\mathfrak{g}_0$ . Let  $(\rho, V)$  be a finite dimensional completely reducible representation. Denote by  $\mathfrak{g}_0^k$  the kernel of the representation. We suppose also that the restriction of  $B_0$  to  $Z(\mathfrak{g}_0^k) = Z(\mathfrak{g}_0) \cap \mathfrak{g}_0^k$  is non-degenerate. Then if we denote by  $\mathfrak{g}_0^f$  the ideal of  $\mathfrak{g}_0$  orthogonal to  $\mathfrak{g}_0^k$  we have  $\mathfrak{g}_0 = \mathfrak{g}_0^k \oplus \mathfrak{g}_0^f$ . Denote also by  $V_0$  the isotypic component of the trivial module in  $V$  and by  $V_1$  the  $\mathfrak{g}_0$ -invariant supplementary subspace to  $V_0$  ( $V = V_1 \oplus V_0$ ).

Then:

1) a)

$$\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0^f, B_0|_{\mathfrak{g}_0^f} V)) \oplus \mathfrak{g}_0^k \quad \text{and}$$

b)

$$\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_0|_{\mathfrak{g}_0^f} V)) \oplus \mathfrak{g}_0^k.$$

2) Moreover

$$\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_0|_{\mathfrak{g}_0^f} V_1)) \oplus (V_0 \oplus V_0^*) \oplus \mathfrak{g}_0^k$$

where  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_0|_{\mathfrak{g}_0^f}, V_1))$  is transitive and where the non defined brackets in  $(\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_0|_{\mathfrak{g}_0^f}, V_1)) \oplus (V_0 \oplus V_0^*))$  are given by

$$[V_0, V_0^*] = [\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_0|_{\mathfrak{g}_0^f}, V_1)), V_0] = [\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_0|_{\mathfrak{g}_0^f}, V_1)), V_0^*] = \{0\} ,$$

$$\text{and } [V_0, V_0] = [V_0^*, V_0^*] = \{0\} .$$

3)

$$Z(\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))) = V_0 \oplus V_0^* \oplus Z(\mathfrak{g}_0^k)$$

*Proof.*

1) Define  $Z(\mathfrak{g}_0^f) = (Z(\mathfrak{g}_0^k))^\perp \cap Z(\mathfrak{g}_0)$ . From the assumptions we obtain that

$$Z(\mathfrak{g}_0) = Z(\mathfrak{g}_0^k) \oplus Z(\mathfrak{g}_0^f)$$

and

$$\mathfrak{g}_0^f = Z(\mathfrak{g}_0^f) \oplus (\mathfrak{g}'_0)^f$$

(where  $(\mathfrak{g}'_0)^f = \mathfrak{g}'_0 \cap \mathfrak{g}_0^f$  is a product of some of the simple ideals of  $\mathfrak{g}'_0$ ).

Let  $X \in V$ ,  $Y \in V^*$ ,  $U \in \mathfrak{g}_0^k$ . The bracket  $[X, Y]$  in  $\Gamma(\mathfrak{g}_0, B_0, V)$  verifies the identity  $B_0([X, Y], U) = Y(U.X) = 0$ . Hence  $[X, Y] \in \mathfrak{g}_0^f$ . This shows that the bracket of  $X$  and  $Y$  is the same in  $\Gamma(\mathfrak{g}_0, B_0, V)$  and in  $\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}}, V)$ . This proves that we have the following isomorphism of local Lie algebras:

$$\Gamma(\mathfrak{g}_0, B_0, V) \simeq \Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}}, V) \oplus \mathfrak{g}_0^k.$$

But then, from Theorem 2.1.4, we have

$$\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}}, V) \oplus \mathfrak{g}_0^k) = \mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}} V)) \oplus \mathfrak{g}_0^k \quad \text{and}$$

$$\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}}, V) \oplus \mathfrak{g}_0^k) = \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}} V)) \oplus \mathfrak{g}_0^k.$$

This proves a) and b).

2) As the representation  $(\mathfrak{g}_0^f, V_1)$  is faithful and as  $V_1$  does not contain the trivial module, Proposition 3.4.3 implies that  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}} V_1))$  is transitive.

Let  $X \in V_0$ ,  $Y \in V_0^*$ . Then the bracket of  $X$  and  $Y$  in  $\Gamma(\mathfrak{g}_0, B_0, V)$  is the unique element  $[X, Y] \in \mathfrak{g}_0$ , such that for any  $U \in \mathfrak{g}_0$ ,  $B_0([X, Y], U) = Y(U.X) = 0$ . Hence  $[X, Y] = 0$ .

This proves that, with the given brackets,  $(\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}} V_1)) \oplus (V_0 \oplus V_0^*) \oplus \mathfrak{g}_0^k)$  becomes a graded Lie algebra, whose local part is

$$\Gamma(\mathfrak{g}_0, B_0, V) = \Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}} V_1) \oplus (V_0 \oplus V_0^*) \oplus \mathfrak{g}_0^k.$$

This is a direct sum of local Lie algebras where  $V_0 \oplus V_0^*$  and  $\mathfrak{g}_0^k$  are already Lie algebras. This proves that

$$\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^f, B_{0|_{\mathfrak{g}_0^f}} V_1)) \oplus (V_0 \oplus V_0^*) \oplus \mathfrak{g}_0^k.$$

3) As the center of  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$  is trivial (Remark 3.4.2), the third assertion is now clear. □

### Proposition 3.4.6.

Let  $(\mathfrak{g}_0, B_0, \rho)$  be a fundamental triplet where  $\mathfrak{g}_0$  is reductive and where  $\rho$  is completely reducible. Suppose that the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is transitive (see Proposition 3.4.3 above). Then if  $\dim(\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho)))$  is finite, the Lie algebra  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is semi-simple.

*Proof.*

It suffices to prove that if  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  cannot be decomposed as in Remark 3.3.2, then  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is simple. We know from [4] (Prop. 5, p.1278) that  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is transitive. Denote by  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$  the grading. Let  $\mathfrak{a}$  be a non zero ideal of  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ . Let  $a \in \mathfrak{a}$ ,  $a \neq 0$ . Let  $a = a_{-i} + a_{-i+1} + \dots + a_j$  be the decomposition of  $a$  according to the grading of  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ , where either  $-i \leq 0$  and  $a_{-i} \neq 0$ , or  $j \geq 0$ , and  $a_j \neq 0$ . Suppose for example that  $-i \leq 0$  and  $a_{-i} \neq 0$ . From the transitivity of  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ , we know that there exists  $x_1^i \in \mathfrak{g}_1$  such that  $[a_{-i}, x_1^i] \neq 0$ . Then  $[a, x_1^i] = [a_{-i}, x_1^i] + \dots + [a_j, x_1^i] \in \mathfrak{a}$  therefore there exists an element  $a' \in \mathfrak{a}$ , such that  $a' = a'_{-i+1} + \dots + a'_{j+1}$  ( $a'_k \in \mathfrak{g}_k$ ) and  $a'_{-i+1} \neq 0$ . By induction we prove that there exists an element  $x = x_0 + x_1 + \dots \in \mathfrak{a}$  such that  $x_0 \neq 0$  and also an element  $y = y_1 + \dots \in \mathfrak{a}$  such that  $y_1 \neq 0$ . Let  $\mathcal{T}_k = \bigoplus_{n \geq k} \mathfrak{g}_n$  ( $k \geq 0$ ). Denote by  $\tilde{\mathfrak{a}}_0$  (resp.  $\tilde{\mathfrak{a}}_1$ ) the projection of  $\mathfrak{a} \cap \mathcal{T}_0$  on  $\mathfrak{g}_0$  (resp. the projection of  $\mathfrak{a} \cap \mathcal{T}_1$  on  $\mathfrak{g}_1$ ).

The preceding considerations show that  $\tilde{\mathfrak{a}}_0 \neq \{0\}$  and  $\tilde{\mathfrak{a}}_1 \neq \{0\}$ . As  $\mathfrak{a}$  is an ideal,  $\tilde{\mathfrak{a}}_0$  is an ideal of  $\mathfrak{g}_0$  and  $\tilde{\mathfrak{a}}_1$  is a sub- $\mathfrak{g}_0$ -module of  $\mathfrak{g}_1 = V$ . Let  $\tilde{\mathfrak{b}}_0$  be the orthogonal of  $\tilde{\mathfrak{a}}_0$  in  $\mathfrak{g}_0$  with respect to  $B_0$ , and let  $\tilde{\mathfrak{b}}_1$  be a  $\mathfrak{g}_0$ -invariant supplementary space to  $\tilde{\mathfrak{a}}_1$  in  $\mathfrak{g}_1$ . That is  $\mathfrak{g}_1 = \tilde{\mathfrak{a}}_1 \oplus \tilde{\mathfrak{b}}_1$ . As  $\mathfrak{a}$  is an ideal we obtain  $[\tilde{\mathfrak{a}}_0, \tilde{\mathfrak{b}}_1] = \{0\}$ . Let now  $B$  be the extended form as defined in Proposition 3.5.2 below. Then, as  $[\tilde{\mathfrak{a}}_1, \mathfrak{g}_{-1}] \subset \tilde{\mathfrak{a}}_0$ , we have for all  $Y \in \mathfrak{g}_{-1}$ ,  $B([\tilde{\mathfrak{b}}_0, \tilde{\mathfrak{a}}_1], Y) = B(\tilde{\mathfrak{b}}_0, [\tilde{\mathfrak{a}}_1, Y]) = \{0\}$ . This shows that  $[\tilde{\mathfrak{b}}_0, \tilde{\mathfrak{a}}_1]$  is orthogonal to  $\mathfrak{g}_{-1}$ . Therefore  $[\tilde{\mathfrak{b}}_0, \tilde{\mathfrak{a}}_1] = \{0\}$ . Let  $x \in \tilde{\mathfrak{a}}_0 \cap \tilde{\mathfrak{b}}_0$ . Then  $[x, \tilde{\mathfrak{a}}_1 + \tilde{\mathfrak{b}}_1] = [x, \mathfrak{g}_1] = \{0\}$ . As  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is transitive we obtain that  $x = 0$ . Hence  $\mathfrak{g}_0 = \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{b}}_0$ .

We have supposed that  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is not decomposable in the sense of Remark 3.3.2. Then  $\tilde{\mathfrak{b}}_0 = \{0\}$  and  $\tilde{\mathfrak{b}}_1 = \{0\}$ , and  $\mathfrak{g}_0 = \tilde{\mathfrak{a}}_0$  and  $\mathfrak{g}_1 = \tilde{\mathfrak{a}}_1$ .

As  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is finite dimensional we can write  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = \bigoplus_{i=-n}^n \mathfrak{g}_i$ . As  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is generated by its local part, we obtain that any  $X_n \in \mathfrak{g}_n$  is a linear combination of elements of the form  $[\dots [X_1^1, X_1^2] \dots] X_1^n$  where  $X_1^1, \dots, X_1^n \in \mathfrak{g}_1$ . But as  $\mathfrak{g}_1 = \tilde{\mathfrak{a}}_1$ , we obtain that  $X_n \in \mathfrak{g}_n \cap \mathfrak{a}$ , and hence  $\mathfrak{g}_n \cap \mathfrak{a} = \mathfrak{g}_n$

From the transitivity, we know that there exist  $Y_1^1, \dots, Y_1^n \in \mathfrak{g}_{-1}$  such that  $[Y_1^n, [Y_1^{n-1}, \dots [Y_1^1, X_n] \dots]] \neq 0$ . This proves that  $\mathfrak{a}_1 = \mathfrak{a} \cap \mathfrak{g}_1 \neq \{0\}$  and  $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0 \neq \{0\}$ . Then the same reasoning as above shows that  $\mathfrak{a}_0 = \mathfrak{g}_0$  and  $\mathfrak{a}_1 = \mathfrak{g}_1$ . As  $[\mathfrak{g}_0, \mathfrak{g}_{-1}] = \mathfrak{g}_1$  (this is again the transitivity condition), we have also that  $\mathfrak{a}_{-1} = \mathfrak{a} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$ .

Finally we have proved that  $\Gamma(\mathfrak{g}_0, B_0, \rho) \subset \mathfrak{a}$ . Hence  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = \mathfrak{a}$ .

□

**Corollary 3.4.7.**

Suppose that  $\mathfrak{g}_0$  is reductive and that  $(\rho, V)$  is a faithful completely reducible  $\mathfrak{g}_0$ -module which does not contain the trivial module.

Let  $k$  denote the number of irreducible components of  $V$ .

If  $\dim Z(\mathfrak{g}_0) < k$ , then  $\dim \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = +\infty$ .

*Proof.*

First remember from Remark 3.4.4 2) that, as the representation is faithful, we have  $\dim Z(\mathfrak{g}_0) \leq k$ .

From Proposition 3.4.6 we know that if  $\dim \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) < +\infty$  then the Lie algebra  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is semi-simple. Then, if  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = \bigoplus_{i=-n}^{i=n} \mathfrak{g}_i$  is a grading, the Lie algebra  $\mathfrak{g}_0$  is a Levi sub-algebra of  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  (see Remark 3.3.4 and Remark 3.3.5). But then  $\dim Z(\mathfrak{g}_0) = k$  (see for example Proposition 4.4.2 d) in [9]).

□

**3.5. Invariant bilinear forms.**

Consider again a general quadratic Lie algebra  $\mathfrak{g}_0$  with a non degenerate symmetric bilinear form  $B_0$ . We will first show that  $B_0$  extends to an invariant form on the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .

Define a symmetric bilinear form  $B$  on  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  by setting:

- $\forall u, v \in \mathfrak{g}_0, B(u, v) = B_0(u, v)$
- $\forall u \in \mathfrak{g}_0, \forall X \in \mathfrak{g}_1 = V, \forall Y \in \mathfrak{g}_{-1} = V^*$

$$B(u, X) = B(X, u) = B(u, Y) = B(Y, u) = 0 \quad (*)$$

- $\forall X \in \mathfrak{g}_1 = V, \forall Y \in \mathfrak{g}_{-1} = V^*, B(X, Y) = B(Y, X) = Y(X)$  (\*\*)

**Lemma 3.5.1.**

a) The form  $B$  is a non degenerate invariant form on  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  (the definition of an invariant form on a local Lie algebra is analogous to the Lie algebra case).

b) Suppose that there exists a grading element in  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ , that is an element  $H_0 \in \mathfrak{g}_0$  such that  $[H_0, x] = ix$  for  $x \in \mathfrak{g}_i, i = -1, 0, 1$ , then the preceding form  $B$  is the only invariant extension of  $B_0$  to  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .

*Proof.*

a) Of course as  $B|_{\mathfrak{g}_0 \times \mathfrak{g}_0} = B_0$ , the invariance is verified on  $\mathfrak{g}_0$ . Let  $X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}, u, v \in \mathfrak{g}_0$ . From the definition of  $[X, Y]$  (see Theorem 3.1.1), we have  $B([X, Y], u) = Y(u.X)$ . On the other hand we have  $B(X, [Y, u]) =$

$B(X, -u.Y) = -u.Y(X) = Y(u.X)$ . Hence  $B([X, Y], u) = B(X, [Y, u])$ . We have also  $B([u, v], X) = 0 = B(u, v.X) = B(u, [v, X])$ . Similarly  $B([u, v], Y) = B(u, [v, Y]) = 0$ .

b) Suppose that  $x \in \mathfrak{g}_i$  and  $y \in \mathfrak{g}_j$  with  $i + j \neq 0$ . Then  $B([H_0, x], y) = B(ix, y) = iB(x, y) = -B(x, [H_0, y]) = -jB(x, y)$ . Therefore  $(i+j)B(x, y) = 0$ , and hence  $B(x, y) = 0$ . We also have for  $X \in V = \mathfrak{g}_1$  and  $Y \in V^* = \mathfrak{g}_{-1}$ :  $B([X, Y], H_0) = Y(X) = B(X, [Y, H_0]) = B(X, Y)$ . Hence conditions (\*) and (\*\*) are satisfied. □

**Proposition 3.5.2.**

- 1) Let  $(\rho, V)$  be a representation of the quadratic Lie algebra  $\mathfrak{g}_0$ . The bilinear form  $B$  on  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  defined in Lemma 3.5.1 extends uniquely to a invariant symmetric bilinear form (still denoted  $B$ ) such that  $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$  if  $i \neq -j$  on any graded Lie algebra whose local part is  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .
- 2) Moreover if  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is transitive, then the extended form  $B$  on  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is non-degenerate.
- 3) The extended form  $B$  on  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  is also non-degenerate in the case where  $\mathfrak{g}_0$  is reductive, the representation  $V$  is completely reducible, and the restriction of  $B_0$  to  $Z(\mathfrak{g}_0) \cap \mathfrak{g}_0^k$  is non degenerate.

*Proof.*

- 1) is due to V. Kac: see Proposition 7 p. 1279 of [4].
- 2) Suppose now that  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is transitive. Let us denote the grading by  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ . We must prove that if  $X \in \mathfrak{g}_i$  is such that  $B(X, Y) = 0$  for all  $Y \in \mathfrak{g}_{-i}$ , then  $X = 0$ . We will first prove the result by induction for  $i \geq 0$ .  
From the definition of  $B$  on  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  we see that the result is true for  $i = 0$  and  $i = 1$ . Suppose now that the result is true for  $i < k$ . Let  $x_k \in \mathfrak{g}_k$  such that  $B(x_k, \mathfrak{g}_{-k}) = 0$ . Then for all  $x_{-1} \in \mathfrak{g}_{-1}$  and all  $x_{-k+1} \in \mathfrak{g}_{-k+1}$  we have  $B(x_k, [x_{-1}, x_{-k+1}]) = 0$ . And hence  $B([x_k, x_{-1}], x_{-k+1}) = 0$ . From the induction hypothesis we get  $[x_k, x_{-1}] = 0$  for all  $x_{-1} \in \mathfrak{g}_{-1}$ . But we know from [4] (Prop. 5, p. 1278) that a minimal graded Lie algebra with a transitive local part is transitive. This implies that  $x_k = 0$ . The same proof works for  $i \leq 0$ .
- 3) This is a consequence of 2) and the explicit form of  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  given in this case in Proposition 3.4.5 2). □

**Corollary 3.5.3.**

Let  $(\mathfrak{g}_0, B_0)$  be a quadratic algebra where  $\mathfrak{g}_0$  is reductive. Suppose also that the restriction of  $B_0$  to  $Z(\mathfrak{g}_0) \cap \mathfrak{g}_0^k$  is non degenerate. Let  $(\rho, V)$  be a completely reducible representation of  $\mathfrak{g}_0$ . Recall that we denote the grading in  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$  by  $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ . Then

- a)  $\dim \mathfrak{g}_i = \dim \mathfrak{g}_{-i}$  for all  $i \in \mathbb{Z}$ .
- b)  $\dim \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) < +\infty$  if and only if there exists an integer  $n \geq 2$  such that the identity  $\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n)$  holds in the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ , where  $\mathcal{P}_n$  is the Lie polynomial defined in Proposition 2.2.1. More precisely in that case  $\mathfrak{g}_i = \{0\}$  for  $|i| \geq n$ .

*Proof.*

- a) From Proposition 3.5.2 3) we know that the extended form  $B$  defines a non-degenerate duality between  $\mathfrak{g}_i$  and  $\mathfrak{g}_{-i}$ . Hence  $\dim \mathfrak{g}_i = \dim \mathfrak{g}_{-i}$ .
- b) We already know from Theorem 2.2.4 that the identity

$$\mathcal{P}_n(Y_1, \dots, Y_{n-1}, X_1, \dots, X_n) = 0$$

implies  $\mathfrak{g}_i = \{0\}$  for  $i \geq n$ . Then the result follows from a). □

#### Definition 3.5.4.

- 1) A quadratic graded Lie algebra is a pair  $(\mathfrak{g}, B)$  where  $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$  is a graded Lie algebra, and where  $B$  is a non-degenerate symmetric invariant bilinear form on  $\mathfrak{g}$  such that  $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$  if  $i \neq -j$ .
- 2) Let  $(\mathfrak{g}^1, B^1)$  and  $(\mathfrak{g}^2, B^2)$  be two quadratic graded Lie algebras. An isomorphism from  $(\mathfrak{g}^1, B^1)$  onto  $(\mathfrak{g}^2, B^2)$  is an isomorphism of graded Lie algebras  $\Psi : \mathfrak{g}^1 \rightarrow \mathfrak{g}^2$  such that:

$$\forall x, y \in \mathfrak{g}^1, B^2(\Psi(x), \Psi(y)) = B^1(x, y).$$

#### Theorem 3.5.5.

Let  $\mathcal{T}$  be the set of equivalence classes of fundamental triplets such that the representation  $(\rho, V)$  is faithful and such that  $\langle \mathfrak{g}_0.V \rangle = V$  and  $\langle \mathfrak{g}_0.V^* \rangle = V^*$ . Let  $\mathcal{G}$  be the set of isomorphism classes of transitive quadratic graded Lie algebras. The map

$$\tau : \mathcal{T} \rightarrow \mathcal{G}$$

defined by  $\tau(\overline{(\mathfrak{g}_0, B_0, \rho)}) = \overline{(\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)), B)}$  is a bijection (here  $B$  is the form defined in Proposition 3.5.2 and the "overline" denotes the equivalence class).

*Proof.*

If  $T$  is a fundamental triplet satisfying the given assumptions, then its local part  $\Gamma(T)$  is transitive from Proposition 3.4.3. And hence  $\mathfrak{g}_{min}(\Gamma(T))$  is transitive from [4], Prop. 5 b), p. 1278. Moreover, with the form  $B$  defined in Proposition 3.5.2, the Lie algebra  $\mathfrak{g}_{min}(\Gamma(T))$  becomes a quadratic graded Lie algebra.

Let  $T_1$  and  $T_2$  be two fundamental triplets and let  $\Gamma(T_1)$  and  $\Gamma(T_2)$  be the corresponding local Lie algebras. Suppose that  $T_1 \simeq T_2$ . Then from Theorem 3.2.6, we have  $\Gamma(T_1) \simeq \Gamma(T_2)$ , and hence  $\mathfrak{g}_{min}(T_1) \simeq \mathfrak{g}_{min}(T_2)$  (isomorphism of graded Lie algebras).

Let  $T_1 = \Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)$ , and  $T_2 = \Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2)$  the explicit triplets we consider. Let  $\Psi : \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)) \rightarrow \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2))$  be the preceding isomorphism. We will prove that  $\Psi$  is in fact an isomorphism of quadratic graded Lie algebras. Therefore we must prove that if  $x \in \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1))_i$  ( $i \geq 0$ ) and  $y \in \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1))_{-i}$ , then

$$B^2(\Psi(x), \Psi(y)) = B^1(x, y) \quad (*)$$

where  $B^1$  and  $B^2$  are the extended forms defined in Proposition 3.5.2. We will prove (\*) by induction on  $i$ . It is clear that (\*) is true for  $i = 0$  (see condition a) in definition 3.2.4). We need also to prove this for  $i = 1$ . Set  $\Psi|_{V_1} = \gamma$ . Then from the proof of Theorem 3.2.6 we have  $\Psi|_{V_{-1}} = {}^t\gamma^{-1}$ . Therefore for  $x \in V_1, y \in V_1^*$ , we have:

$$B^2(\Psi(x), \Psi(y)) = B^2(\gamma(x), {}^t\gamma^{-1}(y)) = {}^t\gamma^{-1}(y)(\gamma(x)) = y(x) = B^1(x, y)$$

This proves (\*) for  $i = 1$ .

Suppose now that (\*) is true for  $0 \leq i < k$ . In the rest of the proof the elements  $x_i$  or  $y_i$  belong always to  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1))_i$ . Any  $x \in \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1))_k$  is a linear combination of elements of the form  $[x_{k-1}, x_1]$  and any  $y \in \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1))_{-k}$  is a linear combination of elements of the form  $[y_{-k+1}, y_{-1}]$ . Then from the proof of Proposition 7 p. 1279 of [4] the extended form is defined inductively by

$$B^1([x_{k-1}, x_1], [y_{-k+1}, y_{-1}]) = B^1([x_{k-1}, x_1], [y_{-k+1}, y_{-1}])$$

and the same is true for  $B^2$ .

Therefore

$$\begin{aligned} B^2(\Psi([x_{k-1}, x_1]), \Psi([y_{-k+1}, y_{-1}])) &= B^2([\Psi(x_{k-1}), \Psi(x_1)], [\Psi(y_{-k+1}), \Psi(y_{-1})]) \\ &= B^2([\Psi(x_{k-1}), \Psi(x_1)], \Psi(y_{-k+1}), \Psi(y_{-1})) \\ &= B^2(\Psi([x_{k-1}, x_1], y_{-k+1}), \Psi(y_{-1})) \\ \text{(by induction:)} &= B^1([x_{k-1}, x_1], [y_{-k+1}, y_{-1}]) \\ &= B^1([x_{k-1}, x_1], [y_{-k}, y_{-1}]) \end{aligned}$$



Therefore (\*) is proved. Hence if  $T_1 \simeq T_2$ , then the algebras  $\mathfrak{g}_{min}(T_1)$  and  $\mathfrak{g}_{min}(T_2)$  are isomorphic as quadratic graded Lie algebras. In other words the map  $\tau$  is well defined.

Let  $(\mathfrak{g}, B)$  be a transitive quadratic graded Lie algebra, with local part  $\Gamma(\mathfrak{g}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ . As a transitive graded algebra is minimal ([4], Prop. 5 a), p. 1278) and has of course a transitive local part, we have  $\mathfrak{g} = \mathfrak{g}_{min}(\Gamma(\mathfrak{g}))$ . Moreover the properties of  $B$  imply immediately that if  $V = \mathfrak{g}_1$  denotes the corresponding  $\mathfrak{g}_0$ -module then  $\mathfrak{g}_{-1} = V^*$ , the dual  $\mathfrak{g}_0$ -module. And then, from the transitivity we obtain that the representation  $(\rho, V)$  is faithful and that  $\langle \mathfrak{g}_0.V \rangle = V$  and  $\langle \mathfrak{g}_0.V^* \rangle = V^*$  (Proposition 3.4.3). This proves that the map  $\tau$  is surjective.

Suppose now that  $\Psi : \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, \rho_1)) \rightarrow \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^2, B_0^2, \rho_2))$  is now an isomorphism of quadratic graded Lie algebra. Set  $\Psi|_{\mathfrak{g}_0^1} = A$  and  $\Psi|_{\mathfrak{g}_1^1} = \gamma$ . Then  $A$  is a Lie algebra isomorphism from  $\mathfrak{g}_0^1$  onto  $\mathfrak{g}_0^2$  and  $\gamma$  is an isomorphism from  $\mathfrak{g}_1^1$  onto  $\mathfrak{g}_1^2$  which satisfy:

- $\forall U, V \in \mathfrak{g}_0^1, B^2(A(U), A(V)) = B^1(U, V)$  and
- $\forall U \in \mathfrak{g}_0^1$  and  $X \in \mathfrak{g}_1^1, \Psi([U, X]) = [\Psi(U), \Psi(X)] = [A(U), \gamma(X)]$  and this means exactly, that in the usual notations,  $\gamma \circ \rho_1(U)X = \rho_2(A(U)) \circ \gamma(X)$ .

Hence  $(A, \gamma)$  is an isomorphism of fundamental triplets (see Definition 3.2.4).

This proves that the map  $\tau$  is injective. □

#### 4. $\mathfrak{sl}_2$ -TRIPLES

Let us consider the following assumption on the fundamental triplet  $(\mathfrak{g}_0, B_0, (\rho, V))$ .

**Assumption (H):**

- a) The Lie algebra  $\mathfrak{g}_0$  is reductive with a one dimensional center:  $\mathfrak{g}_0 = Z(\mathfrak{g}_0) \oplus \mathfrak{g}'_0$  where  $\mathfrak{g}'_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$  and  $\dim Z(\mathfrak{g}_0) = 1$ .
- b) We suppose also that  $Z(\mathfrak{g}_0)$  acts by a non trivial character (i.e.  $\rho(Z(\mathfrak{g}_0)) = \mathbb{C}Id_V$ ).

Then there exists  $H_0 \in Z(\mathfrak{g}_0)$  such that  $\rho(H_0) = 2Id_V$  (and  $\rho^*(H_0) = -2Id_{V^*}$ ). Recall also that in a Lie algebra, or in a local Lie algebra, a triple of elements  $(y, h, x)$  is called an  $\mathfrak{sl}_2$ -triple if  $[h, x] = 2x$ ,  $[h, y] = -2y$  and  $[y, x] = h$ .

##### 4.1. Associated $\mathfrak{sl}_2$ -triple.

**Definition 4.1.1.** Let  $(\mathfrak{g}_0, B_0, \rho)$  be a fundamental triplet satisfying assumption (H). We say that the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ , or the graded Lie algebra

$\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ , is associated to an  $\mathfrak{sl}_2$ -triple if there exists  $X \in V \setminus \{0\}$ ,  $Y \in V^* \setminus \{0\}$  such that  $(Y, H_0, X)$  is an  $\mathfrak{sl}_2$ -triple.

**Theorem 4.1.2.**

Let  $(\mathfrak{g}_0, B_0, \rho)$  be a fundamental triplet satisfying assumption **(H)**. Then  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  is associated to an  $\mathfrak{sl}_2$ -triple if and only if there exists  $X \in V$  such that  $X \notin \mathfrak{g}'_0 \cdot X$  where  $\mathfrak{g}'_0 \cdot X = \{U \cdot X, U \in \mathfrak{g}'_0\}$ . The set  $\{X \in V, X \notin \mathfrak{g}'_0 \cdot X\}$  is exactly the set of elements in  $V$  which belong to an associated  $\mathfrak{sl}_2$ -triple.

*Proof.*

Recall from Lemma 3.5.1 that the form  $B_0$  extends to an invariant form  $B$  on  $\Gamma(\mathfrak{g}_0, B_0, \rho)$ .

Suppose that  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  has an associated  $\mathfrak{sl}_2$ -triple  $(Y, H_0, X)$ . Then  $Y(\mathfrak{g}'_0 \cdot X) = B(Y, \mathfrak{g}'_0 \cdot X) = B(Y, [\mathfrak{g}'_0, X]) = B([Y, X], \mathfrak{g}'_0) = B(H_0, \mathfrak{g}'_0) = \{0\}$ . Hence the form  $Y$  is zero on  $\mathfrak{g}'_0 \cdot X$ . On the other hand  $B(Y, X) = Y(X) = \frac{1}{2}B(Y, [H_0, X]) = -\frac{1}{2}B([Y, X], H_0) = -\frac{1}{2}B(H_0, H_0) = -\frac{1}{2}B_0(H_0, H_0) \neq 0$ . Therefore  $X \notin \mathfrak{g}'_0 \cdot X$ . Conversely suppose that  $X \notin \mathfrak{g}'_0 \cdot X$ . We choose  $Y \in V^*$  such that  $Y(\mathfrak{g}'_0 \cdot X) = \{0\}$  and  $Y(X) \neq 0$ . Then  $B([Y, X], \mathfrak{g}'_0) = B(Y, \mathfrak{g}'_0 \cdot X) = Y(\mathfrak{g}'_0 \cdot X) = \{0\}$ . Therefore  $[Y, X] \in (\mathfrak{g}'_0)^\perp = \mathbb{C}H_0$ . Set  $[Y, X] = \lambda H_0$ . We have also  $Y(X) = B(Y, X) = \frac{1}{2}B(Y, [H_0, X]) = -\frac{1}{2}B([Y, X], H_0) = -\frac{1}{2}B(\lambda H_0, H_0) = -\frac{1}{2}\lambda B(H_0, H_0) \neq 0$ . Hence  $\lambda \neq 0$ . Define  $\tilde{Y} = \frac{1}{\lambda}Y$ . Then  $(\tilde{Y}, H_0, X)$  is an  $\mathfrak{sl}_2$ -triple. □

**Corollary 4.1.3.**

- a) Let  $(\mathfrak{g}_0, B_0, \rho)$  be a fundamental triplet satisfying assumption **(H)**. The existence of a  $\mathfrak{sl}_2$ -triple associated to the local Lie algebra  $\Gamma(\mathfrak{g}_0, B_0, \rho)$  does not depend on the invariant form  $B_0$  on  $\mathfrak{g}_0$ , but only on the representation  $(\mathfrak{g}'_0, \rho|_{\mathfrak{g}'_0}, V)$ .  
b) Suppose that the module  $(\rho, V)$  satisfies the following property **(P)**:

$$\textbf{(P)} \quad \text{There exists } X \in V \text{ such that } X \notin \mathfrak{g}'_0 \cdot X.$$

Then the dual module  $(\rho^*, V^*)$  satisfies the same property.

*Proof.*

Assertion a) is an immediate consequence of Theorem 4.1.2. For assertion b) it is enough to remark that for a given representation  $(\rho, V)$  of a semi-simple Lie algebra  $\mathfrak{g}'_0$ , one can add a one dimensional center  $\mathbb{C}H_0$ , with an obvious action on  $V$ , such that the fundamental triplet  $(\mathfrak{g}_0 = \mathbb{C}H_0 \oplus \mathfrak{g}'_0, \rho, V)$  satisfies assumption **(H)**. Hence there exists an associate  $\mathfrak{sl}_2$ -triple, and this is a symmetric condition. □

#### 4.2. Property **(P)**, relative invariants and $\mathfrak{sl}_2$ -triples.

In this section we will assume that the fundamental triplet  $(\mathfrak{g}_0, B_0, (\rho, V))$  satisfies assumption **(H)**. (But of course any finite dimensional representation  $(\rho, V)$  of  $\mathfrak{g}'_0$  can be extended to a fundamental triplet  $(\mathfrak{g}_0, B_0, (\rho, V))$  satisfying assumption **(H)**).

We will also suppose that the representation  $(\rho, V)$  of  $\mathfrak{g}_0$  lifts to a representation  $\pi$  of a connected algebraic group  $G_0$  whose Lie algebra is  $\mathfrak{g}_0$  (in other words  $\rho$  is the derived representation  $d\pi$ ).

**Definition 4.2.1.** *Let  $(G_0, \pi, V)$  be a finite dimensional representation of a connected complex algebraic group  $G_0$ . Let  $X \in V$  and consider its  $G_0$ -orbit  $\mathcal{O}_X = G_0 \cdot X$ . Then  $\mathcal{O}_X$  is open in its closure  $\overline{\mathcal{O}_X}$  which is an irreducible affine variety. Let  $R$  be an element of the field  $\mathbb{C}(\overline{\mathcal{O}_X})$  of rational functions on  $\mathcal{O}_X$ . The function  $R$  is called a relative invariant on  $\mathcal{O}_X$  if there exists a rational character  $\chi$  of  $G_0$  such that*

$$\forall x \in \mathcal{O}_X, \quad R(\pi(g)x) = \chi(g)R(x) \quad (4-2-1)$$

For convenience we will often write  $g.x$  ( $g \in G_0, x \in V$ ) instead of  $\pi(g)x$  and also  $A.x$  ( $A \in \mathfrak{g}_0, x \in V$ ) instead of  $\rho(A)x$ .

We will now prove that for a fundamental triplet satisfying assumption **(H)** the existence of an associated  $\mathfrak{sl}_2$ -triple  $(X, H_0, Y)$  is equivalent to the existence of a non-trivial relative invariant on the  $G_0$ -orbit of  $X$ .

It will be easier to specialize the property **(P)** at a point  $x$ . For  $x \in V$ , the property  $(\mathbf{P})_x$  is defined as follows:

$$(\mathbf{P})_x : x \notin \mathfrak{g}'_0 \cdot x$$

**Lemma 4.2.2.** (M. Brion)<sup>1</sup>

Let  $(\mathfrak{g}_0, B_0, (\rho, V))$  be a fundamental triplet satisfying assumption **(H)**, and let  $G_0$  be a connected reductive group whose Lie algebra is  $\mathfrak{g}_0$  on which the representation  $\rho$  lifts to  $\pi$ . Let  $G'_0 = [G_0, G_0]$  be the commutator subgroup of  $G_0$ . Then:

$$\text{non}(\mathbf{P})_x \iff \text{the } G'_0\text{-orbit } G'_0 \cdot x \text{ is stable under scalar multiplication.} \quad (*)$$

or equivalently:

$$\text{non}(\mathbf{P})_x \iff G'_0 \cdot x = G_0 \cdot x \quad (**)$$

---

<sup>1</sup>This lemma was communicated to me by Michel Brion

*Proof.*

Let us consider the property

$$\text{non}(\mathbf{P})_x : x \in \mathfrak{g}'_0 \cdot x.$$

Let  $N_x = \{g \in G'_0, \pi(g)(\mathbb{C}x) \subset \mathbb{C}x\}$  be the normalizer of the line  $\mathbb{C}x$  in  $G'_0$ . Condition  $\text{non}(\mathbf{P})_x$  means that there exists  $z$  in the Lie algebra  $\mathfrak{n}_x$  of  $N_x$  such that  $z \cdot x = x$ , and hence there exists  $z \in \mathfrak{n}_x$  such that  $z \cdot x = tx$ , for any  $t \in \mathbb{C}$  (just by replacing  $z$  by  $tz$ ). Therefore  $\text{non}(\mathbf{P})_x$  is equivalent to: for any  $t \in \mathbb{C}^*$ , there exists  $g \in G'_0$  such that  $g \cdot x = tx$ . So (\*) is proved. But as the one dimensional center of  $G_0$  acts by a surjective character (assumption **(H)**), the equivalence of (\*) and (\*\*) is clear.  $\square$

**Theorem 4.2.3.**

Let  $(\rho, V)$  be a finite dimensional representation of a semi-simple Lie algebra  $\mathfrak{g}'_0$ . Extend this representation to a fundamental triplet  $(\mathfrak{g}_0, B_0, \rho)$  satisfying assumption **(H)**, and let  $G_0$  be a connected reductive group whose Lie algebra is  $\mathfrak{g}_0$ , on which the representation  $\rho$  lifts.

Let  $X \in V$ . The following three conditions are equivalent:

- 1)  $(\mathbf{P})_X : X \notin \mathfrak{g}'_0 \cdot X$ .
- 2) There exists a non trivial relative invariant on the  $G_0$ -orbit  $\mathcal{O}_X = G_0 \cdot X$ .
- 3)  $X$  belongs to an associated  $\mathfrak{sl}_2$ -triple  $(X, H_0, Y) \in \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ .

*Proof.*

The equivalence of 1) and 3) has already been proved in Theorem 4.1.2.

Assume that condition 2) holds. Let  $R$  a non trivial relative invariant on  $\mathcal{O}_X$ . Let  $t_1 = R(X)$  and  $t_2$  be two distinct values taken by  $R$ . Define  $\mathcal{O}_i = \{x \in \mathcal{O}_X \mid R(x) = t_i\}$  for  $i = 1, 2$ . Then  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are two  $G'_0$ -stable subsets of  $\mathcal{O}_X$  such that  $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ . Hence  $G'_0 \cdot X \neq G_0 \cdot X$ . From Lemma 4.2.2 (\*\*) we obtain that 2)  $\implies$  1).

Conversely, let us assume that  $(\mathbf{P})_X$  holds. Then from Lemma 4.2.2 we know that the  $G_0$ -orbit  $\mathcal{O}_X$  splits into several  $G'_0$ -orbits.

Suppose that one of these, say  $G'_0 \cdot v$ , is open in  $\mathcal{O}_X$ . Denote by  $G_{0_v}$  be the stabilizer of  $v$  in  $G_0$  and by  $G'_{0_v} = G_{0_v} \cap G'_0$  the stabilizer of  $v$  in  $G'_0$ . Then the subgroup  $G'_0 \cdot G_{0_v}$  is open in  $G_0$  as the inverse image of  $G'_0 \cdot v$  under the orbital map  $g \mapsto g \cdot v$ . As  $G_0$  is connected we have  $G_0 = G'_0 \cdot G_{0_v}$ . Then  $\mathcal{O}_X = G_0 \cdot v = G'_0 \cdot G_{0_v} \cdot v = G'_0 \cdot v$  and this is not possible as we should have several  $G'_0$ -orbits in  $\mathcal{O}_X$ . Hence if we assume that  $(\mathbf{P})_X$  holds, there is no open

$G'_0$ -orbits in  $\mathcal{O}_X$ . In particular  $\dim \mathcal{O}_X - \dim G'_0.X > 0$ . Therefore

$$\begin{aligned} \dim \mathcal{O}_X - \dim G'_0.X &= \dim G_0 - \dim G_{0_X} - \dim G'_0.X \\ &= \dim G_0 - \dim G_{0_X} - \dim G'_0 + \dim G'_{0_X} \\ &= \dim(G_0/G'_0) - \dim(G_{0_X}/G'_{0_X}) \\ &= 1 - \dim(G_{0_X}/G'_{0_X}) > 0 \end{aligned}$$

This implies that  $\dim(G_{0_X}/G'_{0_X}) = 0$  (hence at the Lie algebra level we have  $\mathfrak{g}_{0_X} = \mathfrak{g}'_{0_X}$ ). Then  $\det(\pi(G_{0_X}))$  is a finite group. Therefore there exists  $p \in \mathbb{N}$  such that  $\det^p(\pi(G_{0_X})) = 1$ , but from assumption **(H)** we know that the character  $g \mapsto \det^p(\pi(g))$  is non trivial. Then the function  $R : \mathcal{O}_X \rightarrow \mathbb{C}$  defined by

$$\forall g \in G_0, R(\pi(g)X) = \det^p(\pi(g))$$

is a non-trivial relative invariant on  $\mathcal{O}_X$ , and this is condition 2). □

**Remark 4.2.4.** It is known that for irreducible prehomogeneous vector spaces of parabolic type the existence of a relative invariant on the open orbit is equivalent to the existence of an associated  $\mathfrak{sl}_2$ -triple (see [10] or [9]). But in the preceding Theorem the representation does not need to be irreducible and there is no assumption of prehomogeneity. It works for any representation of  $\mathfrak{g}'_0$ .

**Corollary 4.2.5.**

*Consider the particular case where  $\mathfrak{g} = \bigoplus_{i=-n}^{i=n} \mathfrak{g}_i$  is a grading of a semi-simple (finite dimensional) Lie algebra  $\mathfrak{g}$  (see example 3.3.5) such that the representation  $(\mathfrak{g}_0, \mathfrak{g}_1)$  is irreducible. Then the representation  $(G_0, \mathfrak{g}_1)$  is known to be prehomogeneous. Denote by  $\Omega$  its open orbit. Let  $x \in \mathfrak{g}_1$ .*

*If the prehomogeneous vector space is regular, then*

$$x \in \mathfrak{g}'_0.x = [\mathfrak{g}'_0, x] \iff x \notin \Omega \quad (*)$$

*If the prehomogeneous vector space is not regular, then for each  $x \in \mathfrak{g}_1$ , we have  $x \in [\mathfrak{g}'_0, x]$ . Moreover except for the open orbit in the regular case, all  $G_0$ -orbits are  $G'_0$ -orbits.*

*Proof.*

Suppose that  $x \in \mathfrak{g}'_0.x = [\mathfrak{g}'_0, x]$ . This means that property  $\text{non}(\mathbf{P})_x$  holds. According to Theorem 4.2.3, this is equivalent to the fact that  $x$  does not belong to an associated  $\mathfrak{sl}_2$ -triple. From [10] (Corollaire 4.3.3 p. 134), or from [9], the only elements which belong to an associated  $\mathfrak{sl}_2$ -triple are those in the open orbit in the regular case. This proves the two first assertions. If  $x \in \mathfrak{g}_1$  is not an element of the open orbit in the regular case, then  $x$  does not belong to an

associated  $\mathfrak{sl}_2$ -triple, and therefore  $x \in \mathfrak{g}'_0 \cdot x = [\mathfrak{g}'_0, x]$  (Theorem 4.2.3). Then, from Lemma 4.2.2, we obtain  $G'_0 \cdot x = G_0 \cdot x$ .  $\square$

**Remark 4.2.6.** The equivalence (\*) in Corollary 4.2.5 was first proved in the particular case of so-called "Heisenberg gradings" of a simple Lie-algebra (over any field) by M. Slupinski and R. Stanton (see [11], Lemma 3.3 p. 164). These gradings are special gradings of length 5. It is worth noticing that it is also possible to prove (\*) by combining results of V. Kac [5] (more specially Lemma 1.1 p.193) and of the author ([10] or [9]).

Let  $X \in \mathfrak{g}_1 = V$  satisfying condition  $(\mathbf{P})_X : X \notin \mathfrak{g}'_0 \cdot X$ . Then any element  $x$  belonging to the orbit  $\mathcal{O}_X = G_0 \cdot X$  satisfies condition  $(\mathbf{P})_x$  and therefore belongs to an associated  $\mathfrak{sl}_2$ -triple. Let  $T_x = \mathfrak{g}_0 \cdot x = [\mathfrak{g}_0, x]$  denote the tangent space at  $x$  to  $\mathcal{O}_X$ . Using the canonical isomorphism  $T_x^* \simeq V^*/(T_x)^\perp$ , any element of  $T_x^*$  can be considered as a class modulo  $(T_x)^\perp$  in  $V^* = \mathfrak{g}_{-1}$ .

**Proposition 4.2.7.** (notations as before).

Define, for  $x \in \mathcal{O}_X$ :

$$\varphi(x) = \{y \in \mathfrak{g}_{-1}, (y, H_0, x) \text{ is a } \mathfrak{sl}_2\text{-triple}\}.$$

Then

a)  $\varphi(x)$  is a class modulo  $(T_x)^\perp$  in  $V^* = \mathfrak{g}_{-1}$ . Hence the map

$$x \longmapsto \varphi(x) \in V^*/(T_x)^\perp \simeq T_x^*$$

is a section of the cotangent bundle  $T^*(\mathcal{O}_X)$ .

b) The preceding section is equivariant:

$$\forall g \in G_0, \varphi(\pi(g) \cdot x) = \pi^*(g)\varphi(x)$$

*Proof.*

a) Let  $y_0 \in \varphi(x)$  and let  $z \in (T_x)^\perp$ . Then for  $u \in \mathfrak{g}_0$ , we have  $B_0([z, x], u) = -z([u, x]) = 0$ , as  $[u, x] \in T_x$ . Hence  $y_0 + (T_x)^\perp \subset \varphi(x)$ .

Conversely let  $y \in \varphi(x)$ . Then  $[y - y_0, x] = 0$ , and therefore  $B_0([y - y_0, x], u) = -(y - y_0)([u, x]) = 0$  for all  $u \in \mathfrak{g}_0$ . Hence  $y \in y_0 + (T_x)^\perp$ .

b) Suppose that  $(y, H_0, x)$  is a  $\mathfrak{sl}_2$ -triple. Then for  $u \in \mathfrak{g}_0$ :

$$\begin{aligned} B_0([\pi^*(g)y, \pi(g)x], u) &= -\pi^*(g)y(u \cdot \pi(g)x) \\ &= -y(\pi(g^{-1})u \cdot \pi(g)x) = -y((\text{Ad } g^{-1}u) \cdot x) \\ &= B_0([y, x], \text{Ad } g^{-1}u) = B_0(H_0, u). \end{aligned}$$

Hence  $[\pi^*(g)y, \pi(g)x] = H_0$ . Therefore  $\pi^*(g)y \in \varphi(\pi(g)x)$  or equivalently  $\pi^*(g)\varphi(x) \subset \varphi(\pi(g)x)$ . As  $\pi^*(g)(T_x)^\perp = (T_{\pi(g)x})^\perp$  and  $\varphi(\pi(g)x)$  is a class modulo  $(T_x)^\perp$ , we obtain  $\pi^*(g)\varphi(x) = \varphi(\pi(g).x)$ . □

Let  $X \in V = \mathfrak{g}_1$  which belongs to an associated  $\mathfrak{sl}_2$ -triple. Then the orbit  $G_0.X = \mathcal{O}_X$  has a non trivial relative invariant  $R$  by Theorem 4.2.3. The next proposition shows how one can built an associated  $\mathfrak{sl}_2$ -triple containing  $x \in \mathcal{O}_X$  from the knowledge of  $R$ .

For this we will now consider the "logarithmic differential" (or "gradlog") of  $R$  given by  $\varphi_R(x) = \frac{dR(x)}{R(x)} \in T_x^*$  as a class in  $V^*/(T_x)^\perp$ . In particular  $\varphi_R(x) = \frac{dR(x)}{R(x)}$  is a subset of  $V^*$ .

**Proposition 4.2.8.**

*Let  $(\rho, V)$  be a finite dimensional representation of a semi-simple Lie algebra  $\mathfrak{g}'_0$ . Extend this representation to a fundamental triplet  $(\mathfrak{g}_0, B_0, \rho)$  satisfying assumption **(H)**, and let  $G_0$  be a connected reductive group whose Lie algebra is  $\mathfrak{g}_0$ , on which the representation  $\rho$  lifts.*

*Suppose that the orbit  $\mathcal{O}_X = G_0.X$  has a non trivial relative invariant  $R$ . Let  $x \in \mathcal{O}_X$ . Then, for any element  $y$  in the class  $\varphi_R(x)$ ,  $(x, H_0, -\frac{B(H_0, H_0)}{d\chi(H_0)}y)$  is an associated  $\mathfrak{sl}_2$ -triple. Therefore, in the notation of Proposition 4.2.3, we have  $\varphi(x) = -\frac{B(H_0, H_0)}{d\chi(H_0)}\varphi_R(x)$*

*Proof.*

For  $A \in \mathfrak{g}_0$  and  $x \in \mathcal{O}_X$ , let us derive the identity  $R(\pi(\exp tA)x) = \chi(\exp tA)R(x)$  with respect to  $t$ , at  $t = 0$ . We obtain

$$dR(x)d\pi(A)x = dR(x)\rho(A)x = d\chi(A)R(x).$$

Using the extended form  $B$  defined in Proposition 3.5.2, we observe that  $B(y, [A, x])$  does only depend on the class  $\varphi_R(x)$  of  $y$ . Therefore the preceding equation can be written:

$$B\left(\frac{dR(x)}{R(x)}, \rho(A)x\right) = B(\varphi_R(x), [A, x]) = d\chi(A).$$

And as  $B$  is invariant we obtain:

$$\forall A \in \mathfrak{g}_0, \quad -B([\varphi_R(x), x], A) = d\chi(A).$$

As  $B$  is non-degenerate and as  $d\chi(\mathfrak{g}'_0) = 0$ ,  $[\varphi_R(x), x]$  is a fixed vector (as  $x$  varies) orthogonal to  $\mathfrak{g}'_0$ . Hence  $[\varphi_R(x), x] = cH_0$  ( $c \in \mathbb{C}$ ). If  $A = H_0$ , one obtains  $-B([\varphi_R(x), x], H_0) = d\chi(H_0) \neq 0$  (because  $\chi$  is non trivial). Therefore  $-cB(H_0, H_0) = d\chi(H_0)$  and  $c = -\frac{d\chi(H_0)}{B(H_0, H_0)} \neq 0$ .

Then  $(-\frac{B(H_0, H_0)}{d\chi(H_0)}\varphi_R(x), H_0, x)$  is an  $\mathfrak{sl}_2$ -triple (this means that for  $y$  in the class of  $-\frac{B(H_0, H_0)}{d\chi(H_0)}\varphi_R(x)$  in  $V^*/(T_x)^\perp$ ,  $(y, H_0, x)$  is an associated  $\mathfrak{sl}_2$ -triple).  $\square$

**Remark 4.2.9.**

It is worth noticing that from the preceding result, the "gradlog" section  $x \mapsto \frac{dR(x)}{R(x)} = \varphi_R(x)$  satisfies the same equivariance property as  $\varphi$ :

$$\varphi_R(\pi(g)x) = \pi^*(g)(\varphi_R(x)).$$

## 5. LIE ALGEBRAS OF SYMPLECTIC TYPE AND DUAL PAIRS

### 5.1. Lie algebras of symplectic type.

In this section we will deal with a particular kind of minimal graded Lie algebras of the form  $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho))$ .

**Definition 5.1.1.**

Let  $W$  be a finite dimensional vector space over  $\mathbb{C}$ , and let  $\mathfrak{gl}(W)$  be the Lie algebra of endomorphisms of  $W$ . Let also  $\mathbb{C}^p[W]$  be the vector space of homogeneous polynomials of degree  $p$  on  $W$ . We set  $\mathfrak{g}_0 = \mathfrak{gl}(W)$  and  $V = \mathbb{C}^p[W]$ . For  $\lambda \in \mathbb{C}^*$  we define the representation  $\rho_\lambda$  of  $\mathfrak{gl}(W)$  on  $\mathbb{C}^p[W]$  by saying that  $\rho_\lambda|_{\mathfrak{sl}(W)}$  is the natural representation of  $\mathfrak{sl}(W)$  on  $\mathbb{C}^p[W]$  and  $\rho_\lambda(\text{Id}_{\mathfrak{gl}(W)}) = \lambda \text{Id}_{\mathbb{C}^p[W]}$ . Let  $B_0$  be a non degenerate bilinear symmetric form on  $\mathfrak{gl}(W)$ . With these notations, the Lie algebra of symplectic type  $\mathfrak{sp}^p(W, B_0)$  is defined by

$$\mathfrak{sp}^p(W, B_0, \lambda) = \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho_\lambda)).$$

**Remark 5.1.2.**

- 1) Note that from Proposition 3.2.9, we have  $\mathfrak{sp}^p(W, B_0, \lambda) \simeq \mathfrak{sp}^p(W, \mu \square B_0, \frac{\lambda}{\sqrt{\mu}})$ .
- 2) Note from Lemma 3.5.1 and Proposition 3.5.2 that the form  $B_0$  extends uniquely to a non-degenerate invariant form  $B$  on  $\mathfrak{sp}^p(W, B_0, \lambda)$  such that  $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$  if  $i + j \neq 0$ .



3) Remember also that  $\mathfrak{g}_{-1} = (\mathbb{C}^p[W])^* \simeq \mathbb{C}^p[W^*]$ . Let  $Q \in \mathbb{C}^p[W^*]$ . Define a differential operator  $Q(\partial)$  on  $W$  by setting:

$$Q(\partial)e^{\langle x,y \rangle} = Q(y)e^{\langle x,y \rangle} \quad \text{for all } x \in W \text{ and } y \in W^*.$$

Then the isomorphism between  $\mathbb{C}^p[W^*]$  and  $(\mathbb{C}^p[W])^*$  sends  $Q$  on the linear form  $P \mapsto Q(\partial)P(0) = Q(\partial)P$ .

**Proposition 5.1.3.**

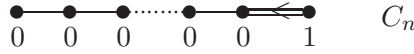
*Up to isomorphism the only finite dimensional Lie algebras of symplectic type are:*

1)  $\mathfrak{sl}(n+1, \mathbb{C}) \simeq \mathfrak{sp}^1(\mathbb{C}^n, B_0, n)$ , where, up to a multiplicative constant,  $B_0(U, V) = \text{tr}(U) \text{tr}(V) + \text{tr}(UV)$  ( $U, V \in \mathfrak{gl}(\mathbb{C}^n)$ ), and where the grading of  $\mathfrak{sl}(n+1, \mathbb{C})$  is defined by the diagram:



(see Remark 3.3.4 for the definition of the corresponding grading).

2)  $\mathfrak{sp}(n, \mathbb{C}) \simeq \mathfrak{sp}^2(\mathbb{C}^n, B_0, 2)$ , where, up to a multiplicative constant,  $B_0(U, V) = \text{tr}(UV)$ , ( $U, V \in \mathfrak{gl}(\mathbb{C}^n)$ ), and where the grading of  $\mathfrak{sp}(n, \mathbb{C})$  is defined by the diagram:



3)  $G_2 \simeq \mathfrak{sp}^3(\mathbb{C}^2, B_0, 1)$  where, up to a multiplicative constant,  $B_0(U, V) = 3 \text{tr}(UV) - \text{tr}(U) \text{tr}(V)$  ( $U, V \in \mathfrak{gl}(\mathbb{C}^2)$ ) and where the grading of  $G_2$  is defined by the diagram:



*Proof.*

We know from proposition 3.4.6 that if  $\dim(\mathfrak{sp}^p(W, B_0)) < +\infty$ , then  $\mathfrak{sp}^p(W, B_0)$  is semi-simple. Hence the grading of  $\dim(\mathfrak{sp}^p(W, B_0))$  is defined by a weighted Dynkin diagram with weights equal to 0 or 1, see Remark 3.3.4. As the representation is irreducible there must be only one 1 among the weights (see Example 3.3.5). Remember also from Example 3.3.5 that  $\mathfrak{g}'_0$  corresponds to the sub-diagram of roots of weight 0. As in our case  $\mathfrak{g}'_0 = \mathfrak{sl}(W)$ , we are looking here for a connected Dynkin diagram with  $\dim W$  vertices, and where the subdiagram of vertices of weight 0 is of type  $A_{(\dim W)-1}$ . Hence the weighted Dynkin diagram of  $\mathfrak{sp}^p(W, B_0, \lambda)$  must be of the following type:



where the boldface edge stands for one of the standard allowed edges (possibly with an arrow) in the Dynkin diagram.

Let us denote by  $\omega$  the fundamental weight corresponding to the "last root on the right" in the sub-diagram of type  $A_{(\dim W)-1}$ , which is the last root on the right with weight 0. Then the representation  $(\mathfrak{g}'_0, V) = (\mathfrak{sl}(W), \mathbb{C}^p[W])$  has lowest weight  $-p\omega$ . This implies that the boldface edge in the diagram above consists of  $p$  lines. (for the general calculation of the lowest weight of  $(\mathfrak{g}'_0, V)$  from the Dynkin diagram in prehomogeneous vector spaces of parabolic type we refer to [10] p. 135). From the list of the Dynkin diagrams we obtain exactly the three cases in the Proposition.

We must now determine a possible representation  $\rho_\lambda$  and a possible form  $B_0$  which correspond through an isomorphism to the graded algebras  $\mathfrak{sl}(n+1, \mathbb{C})$ ,  $\mathfrak{sp}(n, \mathbb{C})$  and  $G_2$  (remember that we have always several choices due to the isomorphisms  $(\mathfrak{sp}^p(W, B_0, \lambda) \simeq \mathfrak{sp}^p(W, \mu \square B_0, \frac{\lambda}{\sqrt{\mu}}))$ ). For the two first cases the usual matrix realizations of  $\mathfrak{sl}(n+1, \mathbb{C})$  and  $\mathfrak{sp}(n, \mathbb{C})$  lead to the indicated  $B_0$  and  $\lambda$ .

For the  $G_2$  case one can proceed as follows. Let  $K_{G_2}$  (resp.  $K_{\mathfrak{g}'_0}$ ) be the Killing form of  $G_2$  (resp.  $\mathfrak{g}'_0$ ). We know from Proposition 3.3.3 that  $G_2 \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, K_{G_2}, \mathfrak{g}_1))$ . If  $\alpha$  is the simple root of weight zero in  $G_2$ , then one shows easily that  $K_{G_2}(H_\alpha, H_\alpha) = 48$  and  $K_{\mathfrak{g}'_0}(H_\alpha, H_\alpha) = 8$ . Hence  $K_{G_2}|_{\mathfrak{g}'_0 \times \mathfrak{g}'_0} = 6K_{\mathfrak{g}'_0}$ . Let  $H_0$  be the grading element in  $G_2$  for the grading corresponding to the given diagram (see Remark 3.3.4 and Example 3.3.5). As  $\dim \mathfrak{g}_1 = 4$  and  $\dim \mathfrak{g}_2 = 1$ , we have  $K_{G_2}(H_0, H_0) = 16$ . Due to the choice " $\lambda = 1$ " we have made in the proposition, the grading element in  $\mathfrak{sp}^3(\mathbb{C}^2, B_0, 1)$  is the identity matrix  $\text{Id}_2$ .

Now if  $\Psi : \mathfrak{sp}^3(\mathbb{C}^2, B_0, 1) \rightarrow G_2$  is a graded isomorphism, it is easy to see that  $\Psi(\text{Id}_2) = H_0$ . Moreover, as we can choose  $B_0$  up to a constant, and as in both algebras we have  $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$ , Proposition 3.2.2 implies that we can suppose  $K_{G_2}(\Psi(U), \Psi(V)) = B_0(U, V)$ , for all  $U, V \in \mathfrak{gl}(2)$ .

If  $U = U' + \lambda \text{Id}_2$  and  $V = V' + \mu \text{Id}_2$ , with  $U', V' \in \mathfrak{sl}(2)$ , and if  $K_{\mathfrak{sl}(2)}$  denotes the Killing form on  $\mathfrak{sl}(2)$ , we obtain

$$\begin{aligned} B_0(U, V) &= K_{G_2}(\Psi(U), \Psi(V)) \\ &= K_{G_2}(\Psi(U'), \Psi(V')) + \lambda\mu K_{G_2}(H_0, H_0) \\ &= 6K_{\mathfrak{g}'_0}(\Psi(U'), \Psi(V')) + 16\lambda\mu \\ &= 6K_{\mathfrak{sl}(2)}(U', V') + 16\lambda\mu \end{aligned}$$

(Because the pullback of the Killing form by an isomorphism is the Killing form). As  $U' = U - \frac{1}{2} \text{tr}(U) \text{Id}_2$ ,  $\lambda = \frac{1}{2} \text{tr}(U)$ ,  $V' = V - \frac{1}{2} \text{tr}(V) \text{Id}_2$  and  $\mu =$

$\frac{1}{2} \operatorname{tr}(V)$ , and as  $K_{\mathfrak{sl}(2)}(\cdot, \cdot) = 4 \operatorname{tr}(\cdot, \cdot)$  we obtain finally that

$$B_0 = 24 \operatorname{tr}(UV) - 8 \operatorname{tr}(U) \operatorname{tr}(V).$$

As the algebra  $\mathfrak{sp}^p(W, B_0, \lambda) \simeq \mathfrak{sp}^p(W, \epsilon B_0, \lambda)$  for  $\epsilon \neq 0$  (Proposition 3.2.8, 2)), we can take for  $B_0$  the form given in the Proposition.  $\square$

**Theorem 5.1.4.**

Let  $\dim W > 1$ . Then, except for the case  $p = 1$ , all graded Lie algebras of symplectic type  $\mathfrak{sp}^p(W, B_0, \lambda)$  are associated to an  $\mathfrak{sl}_2$ -triple.

*Proof.*

Remember from Theorem 4.1.2, that there exists an associated  $\mathfrak{sl}_2$ -triple if and only if there exists  $X \in V \setminus \{0\}$  such that  $X \notin \mathfrak{g}'_0.X$ . We identify  $W$  with  $\mathbb{C}^n$ , where  $n = \dim W$ . Define  $P \in \mathbb{C}^p[\mathbb{C}^n]$  by

$$P(x) = x_1^p + x_2^p + \cdots + x_n^p.$$

By the natural representation,  $\mathfrak{sl}(n)$  acts on  $\mathbb{C}^p[\mathbb{C}^n]$  by the classical vector fields:

$$U = (a_{i,j}) \mapsto \sum_{i,j} a_{i,j} x_i \frac{\partial}{\partial x_j}.$$

If  $U.P = P$ , for some  $U \in \mathfrak{sl}(n)$ :

$$\begin{aligned} \left( \sum_{i,j} a_{i,j} x_i \frac{\partial}{\partial x_j} \right) P(x) &= \left( \sum_{i,j} a_{i,j} x_i \frac{\partial}{\partial x_j} \right) \left( \sum_k x_k^p \right) \\ &= \sum_{i,j,k} a_{i,j} x_i \left( \frac{\partial}{\partial x_j} x_k^p \right) = \sum_{i,j,k} a_{i,j} x_i p x_k^{p-1} \delta_{j,k} \\ &= \sum_{i,j} a_{i,j} x_i p x_j^{p-1} = \sum_k x_k^p. \end{aligned}$$

For  $p > 1$ , this implies that  $a_{i,i} = \frac{1}{p}$  for  $i = 1, \dots, n$ , and then  $U$  cannot be in  $\mathfrak{sl}(n)$ . Therefore the Lie algebras of symplectic type  $\mathfrak{sp}^p(W, B_0, \lambda)$  are associated to an  $\mathfrak{sl}_2$ -triple for  $p > 1$ .

Suppose now  $p = 1$ . From Corollary 4.1.3 b) it is enough to prove that for any  $X \in \mathbb{C}^n \setminus \{0\}$ , there exists  $U \in \mathfrak{sl}(n)$  such that  $U.X = X$ . But as  $\mathbb{C}^n \setminus \{0\}$  is a single orbit under the group  $SL(n, \mathbb{C})$ , the map  $U \mapsto U.X$  from  $\mathfrak{sl}(n)$  to  $\mathbb{C}^n$  is surjective.  $\square$

**5.2. Prehomogeneous vector spaces and dual pairs.**

In Theorem 5.1.4 we have shown that if  $n = \dim W > 1$  and  $p > 1$ , then the polynomial  $X = x_1^p + \dots + x_n^p$  is the nil-positive element of an  $\mathfrak{sl}(2)$ -triple associated to the algebra  $\mathfrak{sp}^p(W, B_0, \lambda)$ . We will now show that, under some assumptions, if  $W$  is the space of an irreducible regular prehomogeneous vector space, and if  $P$  is the corresponding fundamental invariant, then if  $p = \partial^\circ(P)$ , the nil-positive element of an associated  $\mathfrak{sl}(2)$ -triple of  $\mathfrak{sp}^p(W, B_0, \lambda)$  can also be taken to be  $P$ .

**Proposition 5.2.1.**

*Let  $W$  be a finite dimensional vector space over  $\mathbb{C}$ . Let  $A \subset GL(W)$  be a connected reductive algebraic group with a one dimensional center. Denote by  $\mathfrak{a}$  its Lie algebra. Suppose that  $(A, W)$  is an irreducible regular prehomogeneous vector space. Let  $P$  be the corresponding fundamental invariant. Let  $p = \partial^\circ(P)$ . We also make the following assumption*

$$\begin{aligned} \mathfrak{a} &= \{U \in \mathfrak{gl}(W) \mid U.P = \mu(U)P, \mu \in \mathfrak{a}^*\} \\ \text{or equivalently} & & (5 - 2 - 1) \\ \mathfrak{a}' &= \{U \in \mathfrak{gl}(W) \mid U.P = 0\} \end{aligned}$$

*Then, for any  $B_0$ ,  $P$  is the nil-positive element of an associated  $\mathfrak{sl}(2)$ -triple in  $\mathfrak{sp}^p(W, B_0, \lambda)$ .*

*Proof.*

Suppose that there exists  $U \in \mathfrak{sl}(W)$ , such that  $U.P = P$ . Then, if  $H = \frac{1}{\lambda} \text{Id}_W$  is the grading element we would have  $(H - U).P = 0$ . Then from (5 - 2 - 1), we obtain that  $(H - U) \in \mathfrak{a}'$  and therefore  $H \in \mathfrak{a}' \subset \mathfrak{sl}(W)$ . This is not true. Then the proposition is a consequence of Theorem 4.1.2.

□

**Remark 5.2.2.**

Condition (5 - 2 - 1) is always satisfied if  $A$  is the structure group of a simple Jordan algebra  $W$  over  $\mathbb{C}$ . See [2], Chapter VIII, exercise 5 p. 160-161. This case corresponds to the so-called prehomogeneous vector spaces of commutative parabolic type (see [10], ch. 5 or [8]).

But it is also satisfied for many others prehomogeneous vector spaces.

**Definition 5.2.3.**

*Let  $\mathfrak{g}$  be Lie algebra. A pair  $(\mathfrak{g}_1, \mathfrak{g}_2)$  of Lie subalgebras of  $\mathfrak{g}$  is called a dual pair if  $\mathfrak{g}_1$  is the centralizer of  $\mathfrak{g}_2$  in  $\mathfrak{g}$  and vice versa.*

Let  $P^*$  be the fundamental invariant of the dual prehomogeneous vector space  $(A, W^*)$ . Of course  $P^*$  is only defined up to a multiplicative constant.

**Theorem 5.2.4.**

The notations are as in Proposition 5.2.1, and we suppose  $\lambda \neq 0$  and that condition  $(5 - 2 - 1)$  is satisfied.

Let  $\tilde{\mathfrak{b}} = Z_{\mathfrak{sp}^p(W, B_0, \lambda)}(\mathfrak{a}')$  be the centralizer of  $\mathfrak{a}'$  in  $\mathfrak{sp}^p(W, B_0, \lambda)$ .

1) One can choose  $P^*$  such that  $(P^*, H_0 = \frac{2}{\lambda} \text{Id}_W, P)$  is an  $\mathfrak{sl}(2)$ -triple (associated to the graded Lie algebra  $\mathfrak{sp}^p(W, B_0, \lambda)$ ). Then, if  $\mathfrak{b}$  is the Lie subalgebra of  $\mathfrak{sp}^p(W, B_0, \lambda)$  isomorphic to  $\mathfrak{sl}(2)$  generated by this triple, we have

$$\mathfrak{b} = \tilde{\mathfrak{b}} \cap \Gamma(\mathfrak{sp}^p(W, B_0, \lambda))$$

2) Moreover  $(\mathfrak{a}', \tilde{\mathfrak{b}})$  is a dual pair in  $\mathfrak{sp}^p(W, B_0, \lambda)$ .

*Proof.*

1) As  $\tilde{\mathfrak{b}} = Z_{\mathfrak{sp}^p(W, B_0, \lambda)}(\mathfrak{a}')$  and as  $\mathfrak{a}' \subset \mathfrak{gl}(W) = \mathfrak{g}_0$ , we see that  $\tilde{\mathfrak{b}}$  is a graded subalgebra of  $\mathfrak{sp}^p(W, B_0, \lambda)$ .

Hence

$$\tilde{\mathfrak{b}} \cap \Gamma(\mathfrak{sp}^p(W, B_0, \lambda)) = \tilde{\mathfrak{b}} \cap \mathfrak{g}_{-1} \oplus \tilde{\mathfrak{b}} \cap \mathfrak{g}_0 \oplus \tilde{\mathfrak{b}} \cap \mathfrak{g}_1.$$

But as  $W$  is irreducible under  $\mathfrak{a}$ , we obtain that

$$\tilde{\mathfrak{b}} \cap \mathfrak{g}_1 = \mathbb{C}.P \text{ and } \tilde{\mathfrak{b}} \cap \mathfrak{g}_{-1} = \mathbb{C}.P^*.$$

From the Schur Lemma we get also

$$\tilde{\mathfrak{b}} \cap \mathfrak{g}_0 = \{U \in \mathfrak{gl}(W), [U, \mathfrak{a}'] = 0\} = \mathbb{C}.\text{Id}_W.$$

As  $[P^*, P] \in \tilde{\mathfrak{b}} \cap \mathfrak{g}_0$ , we have  $[P^*, P] = \gamma \text{Id}_W$ ,  $\gamma \in \mathbb{C}$ .

Suppose that  $[P^*, P] = 0$ . Let  $B$  be the non-degenerate invariant bilinear form which extends  $B$  (see section 3.4). Then

$$0 = B([P^*, P], \text{Id}_W) = B(P^*, [P, \text{Id}_W]) = -\lambda B(P^*, P) = -\lambda P^*(\partial)P(0)$$

But it is well known from the theory of prehomogeneous space that  $P^*(\partial)P(0) \neq 0$  (see for example the computation on p. 19 in [10]). Hence  $(\frac{2}{\lambda\gamma}P^*, \frac{2}{\lambda}\text{Id}_W, P)$  is an  $\mathfrak{sl}(2)$ -triple and  $\mathfrak{b} = \tilde{\mathfrak{b}} \cap \Gamma(\mathfrak{sp}^p(W, B_0, \lambda))$ .

2) For the second assertion, we have just to prove that  $Z_{\mathfrak{sp}^p(W, B_0, \lambda)}(\tilde{\mathfrak{b}}) \subset \mathfrak{a}'$ . As  $\mathfrak{b} = \tilde{\mathfrak{b}} \cap \Gamma(\mathfrak{sp}^p(W, B_0, \lambda))$ , we have

$$Z_{\mathfrak{sp}^p(W, B_0, \lambda)}(\tilde{\mathfrak{b}}) \subset Z_{\mathfrak{sp}^p(W, B_0, \lambda)}(\mathfrak{b}) \subset Z_{\mathfrak{sp}^p(W, B_0, \lambda)}(\frac{2}{\lambda}\text{Id}_W) = \mathfrak{gl}(W).$$

Therefore  $Z_{\mathfrak{sp}^p(W, B_0, \lambda)}(\tilde{\mathfrak{b}}) \subset Z_{\mathfrak{gl}(W)}(P) = \mathfrak{a}'$  (condition  $(5 - 2 - 1)$ ).

□

**Example 5.2.5. The dual pair  $(\mathfrak{o}(n), \mathfrak{sl}(2))$** 

Let  $O(n)$  be the orthogonal group over  $\mathbb{C}$  of size  $n$  and let  $\mathfrak{o}(n)$  be its Lie algebra. Define  $A = \mathbb{C}^* \times O(n)$  and  $W = \mathbb{C}^n$ . The natural representation  $(A, W)$  is an irreducible regular prehomogeneous vector space whose fundamental relative invariant can be chosen to be the quadratic form  $P(x) = x_1^2 + \cdots + x_n^2$ . Condition  $(5 - 2 - 1)$  is satisfied as  $A$  is the structure group of the Jordan algebra  $\mathbb{C}^n$  (cf. Remark 5.2.2). Consider the trace form on  $\mathfrak{gl}(n)$  defined by  $B_0(U, V) = \text{tr}(UV)$ . Then, in the preceding notations, the algebra  $\mathfrak{sp}^2(W, \text{tr}, -2)$  is the ordinary symplectic algebra  $\mathfrak{sp}(n, \mathbb{C})$  with the grading defined as follows (here  $S_n(\mathbb{C})$  stands for the  $n \times n$  symmetric matrices):

$$\begin{aligned} V^* \simeq \mathbb{C}^2[(\mathbb{C}^n)^*] \simeq S_n(\mathbb{C}) \simeq \mathfrak{g}_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}, Y \in S_n(\mathbb{C}) \right\} \\ \mathfrak{gl}(n) \simeq \mathfrak{g}_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^t A \end{pmatrix}, A \in \mathfrak{gl}(n) \right\} \\ V \simeq \mathbb{C}^2[\mathbb{C}^n] \simeq S_n(\mathbb{C}) \simeq \mathfrak{g}_1 &= \left\{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, X \in S_n(\mathbb{C}) \right\} \end{aligned}$$

Here the quadratic form  $P$  can be identified with the matrix  $\begin{pmatrix} 0 & \text{Id}_n \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_1$ . In this case, keeping the preceding notations, we obtain the dual pair  $(\mathfrak{a}', \tilde{\mathfrak{b}})$  where  $\mathfrak{a}' = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A \in \mathfrak{o}(n) \right\} \simeq \mathfrak{o}(n)$ , and where  $\tilde{\mathfrak{b}} = \mathfrak{b} = \left\{ \begin{pmatrix} a\text{Id}_n & b\text{Id}_n \\ c\text{Id}_n & -a\text{Id}_n \end{pmatrix}, a, b, c \in \mathbb{C} \right\}$ . This is the archetype of a dual pair in  $\mathfrak{sp}(n, \mathbb{C})$  (see [3], p. 556).

Through our construction this pair appears to be associated to the prehomogeneous vector space  $(O(n) \times \mathbb{C}^*, \mathbb{C}^n)$ .

**Remark 5.2.6.** In the notations of the preceding example, we could now take  $(\mathfrak{g}_0, \mathfrak{g}_1)$  as the starting prehomogeneous space  $(\mathfrak{a}, W)$ , and do the same construction as in Theorem 5.2.4. But then, as the degree of the fundamental relative invariant (the determinant of the symmetric matrices) is of degree  $n$ , we are led to the algebras  $\mathfrak{sp}^n(\mathfrak{gl}(\frac{n(n+1)}{2}), B_0, \lambda)$ , which are of infinite dimension, according to Proposition 5.1.3.

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