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► **To cite this version:**

Hubert Rubenthaler. Graded Lie algebras associated to a representation of a quadratic algebra. 2014.
hal-01070209v1

HAL Id: hal-01070209

<https://hal.science/hal-01070209v1>

Preprint submitted on 30 Sep 2014 (v1), last revised 18 Jan 2017 (v4)

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GRADED LIE ALGEBRAS ASSOCIATED TO A REPRESENTATION OF A QUADRATIC ALGEBRA

HUBERT RUBENTHALER

Preliminary version

ABSTRACT. Let (\mathfrak{g}_0, B_0) be a quadratic Lie algebra (i.e. a Lie algebra \mathfrak{g}_0 with a non degenerate symmetric invariant bilinear form B_0) and let $(\mathfrak{g}_0, \rho, V)$ be a finite dimensional representation of \mathfrak{g}_0 . We define on $\Gamma(\mathfrak{g}_0, B_0, V) = V^* \oplus \mathfrak{g}_0 \oplus V$ a structure of local Lie algebra in the sense of Kac ([2]). This implies the existence of two \mathbb{Z} -graded Lie algebras $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V))$ and $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ whose local part is $\Gamma(\mathfrak{g}_0, B_0, V)$. We investigate these graded Lie algebras, more specifically in the case where \mathfrak{g}_0 is reductive.

AMS classification: 17B70(17B20)

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1. INTRODUCTION

In this paper all gradings are \mathbb{Z} -gradings. If $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$ is a grading of a complex semi-simple Lie algebra, then it is well known that if B denotes the Killing form of \mathfrak{g} , then $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i + j \neq 0$. This allows to identify \mathfrak{g}_{-1} with the dual \mathfrak{g}_1^* . Moreover as B is invariant the bracket representation $(\mathfrak{g}_0, \mathfrak{g}_{-1})$ can be identified with the dual representation $(\mathfrak{g}_0, \mathfrak{g}_1^*)$. It is then a natural question to ask if any finite dimensional representation (\mathfrak{g}_0, V) can be embedded in a graded Lie algebra $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$ such that $(\mathfrak{g}_0, \mathfrak{g}_1) \simeq (\mathfrak{g}_0, V)$ and $(\mathfrak{g}_0, \mathfrak{g}_{-1}) \simeq (\mathfrak{g}_0, V^*)$. The first result of this paper is to prove that the answer is "yes" for any representation of a quadratic Lie algebra.

A quadratic Lie algebra is a pair (\mathfrak{g}_0, B_0) where \mathfrak{g}_0 is a Lie algebra and B_0 a non-degenerate invariant symmetric bilinear form on \mathfrak{g}_0 . We use a result of V. Kac ([2]) which asserts that in order to construct a graded Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ it suffices to construct the local part $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, which has to be endowed with a partial Lie bracket (see section 2 for details). Therefore once we have build the partial bracket on the local part $\Gamma(\mathfrak{g}_0, B_0, V) = V^* \oplus \mathfrak{g}_0 \oplus V$ (see Theorem 3.1.1) the existence of the "global" Lie algebra is just an application of the result of Kac (see Theorem 3.2.1). In fact Kac theory provides us with two such graded Lie algebras: a maximal one (denoted here $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V))$) and a minimal one (denoted $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$). Any graded Lie algebra with a given local

part is a quotient of the maximal algebra, and has a quotient isomorphic to the minimal one. Of course, in general, these algebras are infinite dimensional.

Lets us now give a more precise description of the paper.

In section 2 we give a brief account of the results of Kac that we will use.

The main result of section 3.1 is Theorem 3.1.1 where we construct the local Lie algebra structure on $\Gamma(\mathfrak{g}_0, B_0, V) = V^* \oplus \mathfrak{g}_0 \oplus V$. In this section we define the equivalence of the fundamental triplets (\mathfrak{g}_0, B_0, V) and we also investigate the dependance on B_0 and on the representation (\mathfrak{g}_0, V) of the local Lie algebra structure of $\Gamma(\mathfrak{g}_0, B_0, V) = V^* \oplus \mathfrak{g}_0 \oplus V$. This dependance may be strong.

In section 3.2 we apply Kac Theorem to obtain the minimal and maximal Lie algebras associated to the local data $\Gamma(\mathfrak{g}_0, B_0, V) = V^* \oplus \mathfrak{g}_0 \oplus V$. We also prove, that under some conditions, the reductive graded Lie algebras are always minimal graded Lie algebras (Proposition 3.2.3). We also explore in some detail the case of the usual "central" grading of $\mathfrak{sl}_{2n}(\mathbb{C})$, viewed as a local Lie algebra, when the form B_0 varies.

Section 3.3 deals with another important notion for graded Lie algebras due to Kac, namely the transitivity (see Definition 3.3.1). We give a necessary and sufficient condition for the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ (or the minimal Lie algebra $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$) to be transitive (Proposition 3.3.2 and Remark 3.3.3). We also prove that if \mathfrak{g}_0 is reductive, then under some conditions including the transitivity of $\Gamma(\mathfrak{g}_0, B_0, V)$, the fact that $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is finite dimensional implies that $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is semi-simple (see Proposition 3.3.5).

In section 3.4 we show that the form B_0 extends uniquely to an invariant symmetric bilinear form B on $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$. Moreover if the local part is transitive then the form B is nondegenerate (Proposition 3.4.2). This allows us to show that there exists a bijection between some equivalence classes of fundamental triplets and the equivalence classes of transitive graded Lie algebras endowed with a non-degenerate symmetric bilinear form B such that $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i \neq -j$ (Theorem 3.4.3).

In section 4 we prove a strong relationship between relative invariants and the existence of certain \mathfrak{sl}_2 -triples. A rational function R on V is said to be a relative invariant if there exists a connected complex Lie group G_0 with Lie algebra \mathfrak{g}_0 such that the representation (\mathfrak{g}_0, V) lifts to a representation of G_0 on V , and such that $R(g.x) = \chi(g)R(x)$, where χ is a character of G_0 . We prove, under some assumptions, that if V is irreducible, then the existence of

a non-trivial relative invariant implies that there exists in $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ an \mathfrak{sl}_2 -triple (Y, H_0, X) , where $y \in \mathfrak{g}_{-1}, X \in \mathfrak{g}_1$ and where H_0 is the grading element (i.e. $\mathfrak{g}_i = \{x \in \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) \mid H_0, X] = ix\}$).

Remark: This paper is a preliminary version. Forthcoming versions will include more examples and applications.

2. GRADED LIE ALGEBRAS AND LOCAL LIE ALGEBRAS

In this paper all the algebras are defined over the field \mathbb{C} of complex numbers. Let us first recall various definitions and results from [2].

Definition 2.1.

A Lie algebra \mathfrak{g} is said to be graded if:

- 1) \mathfrak{g} is a direct sum of subspaces: $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, such that $\dim \mathfrak{g}_i < +\infty$ and such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for all $i, j \in \mathbb{Z}$.
- 2) \mathfrak{g} is generated by $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

If \mathfrak{g} is a graded Lie algebra, the subspace $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called the *local part* of \mathfrak{g} .

Definition 2.2.

- 1) A local Lie algebra is a direct sum $\Gamma = \Gamma_{-1} \oplus \Gamma_0 \oplus \Gamma_1$ of finite dimensional subspaces such that if $|i + j| \leq 1$ there exists a bilinear anticommutative operation $\Gamma_i \times \Gamma_j \rightarrow \Gamma_{i+j}((x, y) \rightarrow [x, y])$ such that the Jacobi identity $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ holds each time the three terms of the identity are defined.
- 2) A symmetric bilinear form B_Γ on a local Lie algebra Γ is said to be invariant if the identity

$$B_\Gamma([x, y], z) = B_\Gamma(x, [y, z])$$

holds for $x, y, z \in \Gamma$ each time that the brackets are defined.

Of course the local part $\hat{\mathfrak{g}}$ of a graded Lie algebra \mathfrak{g} , endowed with the bracket of \mathfrak{g} is a local Lie algebra. A natural question is to know if, for a given local Lie algebra Γ , there exists a graded Lie algebra whose local part is Γ . The answer is "yes". More precisely we have:

Theorem 2.3. (Kac, [2], Proposition 4)

Let $\Gamma = \Gamma_{-1} \oplus \Gamma_0 \oplus \Gamma_1$ be a local Lie algebra.

- 1) There exists a unique graded Lie algebra $\mathfrak{g}_{max}(\Gamma)$ whose local part is Γ and which satisfies the following universal property.

Any morphism of local Lie algebras $\Gamma \rightarrow \hat{\mathfrak{g}}$ from Γ into the local part $\hat{\mathfrak{g}}$ of a graded Lie algebra \mathfrak{g} extends uniquely to a morphism of graded Lie algebras $\mathfrak{g}_{\max}(\Gamma) \rightarrow \mathfrak{g}$. (And hence any graded Lie algebra whose local part is isomorphic to Γ , is a quotient of $\mathfrak{g}_{\max}(\Gamma)$). Moreover we have

$$\mathfrak{g}_{\max}(\Gamma) = F(\mathfrak{g}_{-1}) \oplus \mathfrak{g}_0 \oplus F(\mathfrak{g}_1),$$

where $F(\mathfrak{g}_{-1})$ (resp. $F(\mathfrak{g}_1)$) is the free Lie algebra generated by \mathfrak{g}_{-1} (resp. \mathfrak{g}_1).

2) There exists a unique graded Lie algebra $\mathfrak{g}_{\min}(\Gamma)$ whose local part is Γ and which satisfies the following universal property.

Any surjective morphism of local Lie algebras $\hat{\mathfrak{g}} \rightarrow \Gamma$ from the local part of a graded Lie algebra \mathfrak{g} into Γ extends uniquely to a (surjective) morphism of graded Lie algebras $\mathfrak{g} \rightarrow \mathfrak{g}_{\min}(\Gamma)$. (And hence $\mathfrak{g}_{\min}(\Gamma)$ is a quotient of any graded Lie algebra whose local part is isomorphic to Γ).

3. LOCAL AND GRADED LIE ALGEBRAS ASSOCIATED TO (\mathfrak{g}_0, B_0, V)

3.1. The local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$.

Remind that a *quadratic Lie algebra* is a finite dimensional Lie algebra endowed with an invariant nondegenerate symmetric bilinear form. The most obvious examples of such algebras are the semi-simple algebras (endowed with the Killing form), the commutative algebras (endowed with any symmetric nondegenerate bilinear form) or more generally the reductive Lie algebras. But there exist more sophisticated examples. General quadratic Lie algebras are obtained by a finite number of so-called *double extensions* of either a simple Lie algebra or a commutative Lie algebra. See [3].

Let \mathfrak{g}_0 be a quadratic Lie algebra. Let $(\mathfrak{g}_0, \rho, V)$ be a finite dimensional representation of \mathfrak{g}_0 . Let $(\mathfrak{g}_0, \rho^*, V^*)$ be the contragredient representation. We will often just denote these modules by (\mathfrak{g}_0, V) and (\mathfrak{g}_0, V^*) . Similarly, for $U \in \mathfrak{g}_0, X \in V, Y \in V^*$ we will often write $U.X$ and $U.Y$ instead of $\rho(U)X$ and $\rho^*(U)Y$. Put $\mathfrak{g}_{-1} = V^*$ and $\mathfrak{g}_1 = V$. Define also

$$\Gamma(\mathfrak{g}_0, B_0, V) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = V^* \oplus \mathfrak{g}_0 \oplus V.$$

Our aim is now to define a structure of local Lie algebra on $\Gamma(\mathfrak{g}_0, B_0, V)$, such that for $U \in \mathfrak{g}_0, X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}$, we have $[U, X] = U.X$ and $[U, Y] = U.Y$.

Theorem 3.1.1.

Let (\mathfrak{g}_0, V) be a finite dimensional representation of a quadratic Lie algebra \mathfrak{g}_0 . Let us denote by B_0 a symmetric invariant nondegenerate bilinear form which

is given on \mathfrak{g}_0 . As before we set:

$$\Gamma(\mathfrak{g}_0, B_0, V) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = V^* \oplus \mathfrak{g}_0 \oplus V.$$

For $U \in \mathfrak{g}_0, X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}$ define an anticommutative bracket by

$$a) [U, X] = U.X, [U, Y] = U.Y$$

b) The element $[X, Y]$ is the unique element of \mathfrak{g}_0 such that for all $U \in \mathfrak{g}_0$ the following identity holds:

$$B_0([X, Y], U) = Y(U.X) = -(U.Y)(X).$$

(The last equality is just the definition of the contragredient representation).

The preceding bracket defines a structure of a local Lie algebra on $\Gamma(\mathfrak{g}_0, B_0, V)$.

Proof. We must prove that the Jacobi identity is verified each times the brackets make sense. This means that we have to prove the following identities.

$$\alpha) \forall U_1, U_2 \in \mathfrak{g}_0, \forall X \in \mathfrak{g}_1, [X, [U_1, U_2]] = [[X, U_1], U_2] + [U_1, [X, U_2]].$$

$$\beta) \forall U_1, U_2 \in \mathfrak{g}_0, \forall Y \in \mathfrak{g}_{-1}, [Y, [U_1, U_2]] = [[Y, U_1], U_2] + [U_1, [Y, U_2]].$$

$$\gamma) \forall X \in \mathfrak{g}_1, \forall Y \in \mathfrak{g}_{-1}, \forall Z \in \mathfrak{g}_0, [Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]].$$

We have:

$$[X, [U_1, U_2]] = -[[U_1, U_2], X] = -[U_1, U_2].X = -U_1.(U_2.X) + U_2.(U_1.X).$$

On the other hand we have:

$$[[X, U_1], U_2] = [U_2, [U_1, X]] = U_2.(U_1.X) \text{ and } [U_1, [X, U_2]] = -[U_1, [U_2, X]] = -U_1.(U_2.X). \text{ This proves } \alpha). \text{ The proof of the identity } \beta) \text{ is similar.}$$

Let us now consider the identity $\gamma)$. We set $L = [Z, [X, Y]]$, $R_1 = [[Z, X], Y]$, $R_2 = [X, [Z, Y]]$, and $R = R_1 + R_2$. As $L, R_1, R_2, R \in \mathfrak{g}_0$, in order to prove $\gamma)$ it will be enough to show that for all $U \in \mathfrak{g}_0$, we have $B_0(L, U) = B_0(R, U)$.

Using the invariance of B_0 and definition b) we get:

$$\begin{aligned} B_0(L, U) &= B_0([Z, [X, Y]], U) = -B_0([[X, Y], Z], U) = -B_0([X, Y], [Z, U]) \\ &= -Y([Z, U].X) \\ &= -Y(Z.(U.X) - U.(Z.X)). \end{aligned}$$

On the other hand, using again definition b), we have also:

$$B_0(R_1, U) = B_0([[Z, X], Y], U) = Y(U.[Z, X]) = Y(U.(Z.X))$$

$$\text{and } B_0(R_2, U) = B_0([X, [Z, Y]], U) = [Z, Y](U.X) = Z.Y(U.X) = -Y(Z.(U.X)).$$

$$\text{Hence } B_0(R, U) = Y(U.(Z.X) - Z.(U.X)) = B_0(L, U).$$

□

Notation 3.1.2. It must be noted that the local Lie algebra described in Theorem 3.1.1 depends on B_0 and on ρ (see Proposition 3.1.7 and Proposition 3.1.8 below). In order to avoid confusion we will sometimes denote by $\Gamma(\mathfrak{g}_0, B_0, \rho)$

this local Lie algebra. Also, if necessary, we will denote by $[\cdot, \cdot]_{B_0}$ or $[\cdot, \cdot]_{B_0, \rho}$ the bracket in $\Gamma(\mathfrak{g}_0, B_0, \rho)$.

Definition 3.1.3. We call the triplet (\mathfrak{g}_0, B_0, V) a fundamental triplet. It consists of the following ingredients:

- a) a quadratic Lie algebra \mathfrak{g}_0
- b) a non degenerate symmetric invariant bilinear form B_0 on \mathfrak{g}_0
- c) a finite dimensional representation (ρ, V) of \mathfrak{g}_0 on the space V .

Definition 3.1.4. Let $(\mathfrak{g}_0^1, B_0^1, V_1)$ and $(\mathfrak{g}_0^2, B_0^2, V_2)$ be two fundamental triplets. Let $A \in \text{Hom}(\mathfrak{g}_0^1, \mathfrak{g}_0^2)$ is a isomorphism of Lie algebras and let $\gamma \in \text{Hom}(V_1, V_2)$ be an isomorphism of vector spaces. We say that the pair (A, γ) is an isomorphism of fundamental triplets if

- a) We have

$$\forall U, \forall V \in \mathfrak{g}_0^1, \quad B_0^2(A(U), A(V)) = B_0^1(U, V) \quad (3-1-1)$$

- b) The map γ intertwines the representations (ρ_1, V_1) and $(\rho_2 \circ A, V_1)$ of \mathfrak{g}_0 , in other words

$$\forall U \in \mathfrak{g}_0^1, \quad \rho_2(A(U)) \circ \gamma = \gamma \circ \rho_1(U) \quad (3-1-2)$$

Remark 3.1.5. In the definition above, condition a) coincides with the notion of isometric isomorphism, or i -isomorphism of quadratic Lie algebras introduced in [1]. This notion of i -isomorphism was already implicit in [3].

Theorem 3.1.6.

Any isomorphism of fundamental triplets

$$(A, \gamma) : (\mathfrak{g}_0^1, B_0^1, V_1) \longrightarrow (\mathfrak{g}_0^2, B_0^2, V_2)$$

extends uniquely to an isomorphism of local Lie algebras

$$\Psi_{(A, \gamma)} : \Gamma(\mathfrak{g}_0^1, B_0^1, V_1) \longrightarrow \Gamma(\mathfrak{g}_0^2, B_0^2, V_2).$$

Proof. Define $\gamma^* : V_1^* \longrightarrow V_2^*$ by $\gamma^* = {}^t\gamma^{-1}$ where t stands for the transposed map. Then it is easy to check that

$$\forall U \in \mathfrak{g}_0^1, \quad \rho_2^*(A(U)) \circ \gamma^* = \gamma^* \circ \rho_1^*(U) \quad (3-1-3)$$

Define now $\Psi_{(A, \gamma)}$ by setting:

$$\begin{aligned} \forall U \in \mathfrak{g}_0^1 : \quad \Psi_{(A, \gamma)}(U) &= A(U) \\ \forall X \in V_1 : \quad \Psi_{(A, \gamma)}(X) &= \gamma(X) \\ \forall Y \in V_1^* : \quad \Psi_{(A, \gamma)}(Y) &= \gamma^*(Y) \end{aligned}$$

It remains to prove that $\Psi_{(A, \gamma)}$ is an homomorphism of local Lie algebras.

Let $X \in V_1$ and $Y \in V_1^*$. Let us denote by $[,]_1$ (resp. $[,]_2$) the bracket in $\Gamma(\mathfrak{g}_0^1, B_0^1, V_1)$ (resp. $\Gamma(\mathfrak{g}_0^2, B_0^2, V_2)$). From the definition, we see that the identity

$$\forall X \in V_1, Y \in V_2, \Psi_{(A,\gamma)}([X, Y]_1) = [\Psi_{(A,\gamma)}(X), \Psi_{(A,\gamma)}(Y)]_2 \quad (3-1-4)$$

is the same as

$$\forall X \in V_1, Y \in V_2, A([X, Y]_1) = [\gamma(X), \gamma^*(Y)]_2 \quad (3-1-5)$$

For all $U \in \mathfrak{g}_0^2$ we have:

$$\begin{aligned} B_0^2(A([X, Y]_1), U) &= B_0^2(A([X, Y]_1), AA^{-1}(U)) \\ &= B_0^1([X, Y]_1, A^{-1}(U)) \quad (\text{using } (3-1-1)) \\ &= Y(\rho_1(A^{-1}(U)(X)) \quad (\text{from Theorem 3.1.1}) \\ &= Y(\gamma^{-1}\rho_2(U)\gamma(X)) \quad (\text{using } (3-1-2)) \\ &= \gamma^*(Y)(\rho_2(U)\gamma(X)) \\ &= B_0^2([\gamma(X), \gamma^*(Y)]_2, U) \quad (\text{from Theorem 3.1.1}) \end{aligned}$$

This proves (3-1-5). We must also prove:

- a) $\forall U \in \mathfrak{g}_0^1, \forall X \in V_1 \quad \Psi_{(A,\gamma)}([U, X]_1) = [\Psi_{(A,\gamma)}(U), \Psi_{(A,\gamma)}(X)]_2$ and
- b) $\forall U \in \mathfrak{g}_0^1, \forall Y \in V_1^* \quad \Psi_{(A,\gamma)}([U, Y]_1) = [\Psi_{(A,\gamma)}(U), \Psi_{(A,\gamma)}(Y)]_2$

From the definition of $\Psi_{(A,\gamma)}$, a) is equivalent to $\gamma(\rho_1(U)X) = \rho_2(A(U))\gamma(X)$ and this is nothing else but (3-1-2). Similarly b) is just (3-1-3).

It remains to prove the uniqueness of the extension. But if

$$\Psi : \Gamma(\mathfrak{g}_0^1, B_0^1, V_1) \longrightarrow \Gamma(\mathfrak{g}_0^2, B_0^2, V_2)$$

is an isomorphism of local Lie algebras such that $\Psi|_{\mathfrak{g}_0} = A$, $\Psi|_V = \gamma$ and $\Psi|_{V^*} = \tilde{\gamma} : V^* \longrightarrow V^*$, then for $X \in V, Y \in V^*$, $A([X, Y]_1) = [\gamma(X), \tilde{\gamma}(Y)]_2$. Therefore $B_0^2(A([X, Y]_1), U) = B_0^2([\gamma(X), \tilde{\gamma}(Y)]_2, U)$, for all $U \in \mathfrak{g}_0$. And then the computation above proves that $\tilde{\gamma} = \gamma^*$.

□

We will now investigate the dependance of the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, \rho)$, under a change of the invariant form B_0 on \mathfrak{g}_0 . We make the following assumptions:

- $\mathfrak{g}_0 = L_1 \oplus L_2 \oplus \cdots \oplus L_k$, where the L_i 's are quadratic ideals.
- we denote by $B_{0,i}$ a nondegenerate invariant bilinear form on L_i ($i = 1, \dots, k$).
- the invariant bilinear form B_0 on \mathfrak{g}_0 is given by $B = B_{0,1} \oplus B_{0,2} \oplus \cdots \oplus B_{0,k}$.
- for $\lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{C}^*)^k$ we set

$$\lambda.B_0 = \lambda_1 B_{0,1} \oplus \cdots \oplus \lambda_k B_{0,k}$$

and for $u = u_1 + \cdots + u_k \in \mathfrak{g}_0$ ($u_i \in L_i$) we set

$$\lambda.u = \lambda_1 u_1 + \cdots + \lambda_k u_k.$$

Proposition 3.1.7.

1) Using the preceding notations, we have for $X \in V = \mathfrak{g}_1$ and for $Y \in V^* = \mathfrak{g}_{-1}$:

$$\lambda.[X, Y]_{\lambda.B_0} = [X, Y]_{B_0}$$

2) For $\mu \in \mathbb{C}^*$, the local Lie algebras $\Gamma(\mathfrak{g}_0, B_0, \rho)$ and $\Gamma(\mathfrak{g}_0, \mu B_0, \rho)$ attached respectively to B_0 and μB_0 are isomorphic (here μB_0 stands for the ordinary scalar multiplication).

Proof.

1) From the definition $[X, Y]_{\lambda.B_0}$ is the unique element in \mathfrak{g}_0 such that $\lambda.B_0([X, Y]_{\lambda.B_0}, U) = Y(U.X)$ for any $U \in \mathfrak{g}_0$. It is easy to see that from the definitions we have $\lambda.B_0([X, Y]_{\lambda.B_0}, U) = B_0(\lambda.[X, Y]_{\lambda.B_0}, U)$. Hence $\lambda.[X, Y]_{\lambda.B_0} = [X, Y]_{B_0}$.

2) Choose a square root $\sqrt{\mu}$ of μ . Define $\varphi_\mu : \Gamma(\mathfrak{g}_0, B_0, \rho) \longrightarrow \Gamma(\mathfrak{g}_0, \mu B_0, \rho)$ by

$$\forall X \in V, \varphi_\mu(X) = \sqrt{\mu}X, \quad \forall Y \in V^*, \varphi_\mu(Y) = \sqrt{\mu}Y, \quad \forall U \in \mathfrak{g}_0, \varphi_\mu(U) = U.$$

Then $[\varphi_\mu(X), \varphi_\mu(Y)]_{\mu B_0} = [\sqrt{\mu}X, \sqrt{\mu}Y]_{\mu B_0} = \mu[X, Y]_{\mu B_0} = [X, Y]_{B_0}$ (this is the particular case of 1) where $\mathfrak{g}_0 = L_1$). Hence $[\varphi_\mu(X), \varphi_\mu(Y)]_{\mu B_0} = \varphi_\mu([X, Y]_{B_0})$.

As $[U, X]_{\mu B_0} = [U, X]_{B_0} = U.X$ and $[U, Y]_{\mu B_0} = [U, Y]_{B_0} = U.Y$ for all $X \in V$, $Y \in V^*$ and $U \in \mathfrak{g}_0$, we have also $\varphi_\mu([U, X]_{B_0}) = [\varphi_\mu(U), \varphi_\mu(X)]_{\mu B_0}$ and $\varphi_\mu([U, Y]_{B_0}) = [\varphi_\mu(U), \varphi_\mu(Y)]_{\mu B_0}$. Therefore φ_μ is an isomorphism of local Lie algebras. □

We will also investigate the modification of the bracket in $\Gamma(\mathfrak{g}_0, B_0, \rho)$ under a slight change of ρ .

Suppose that $\mathfrak{g}_0 = Z \oplus L$ is a quadratic Lie algebra where Z is a central ideal and L is an ideal. For $\gamma \in \mathbb{C}^*$ we denote by $\gamma \square \rho$ the representation of \mathfrak{g}_0 on V given by $\gamma \square \rho(z + u) = \gamma \rho(z) + \rho(u)$, for $z \in Z$ and $u \in L$. If $U = z + u \in \mathfrak{g}_0$, and if we set $\gamma \square U = \gamma z + u$, we have $\gamma \square \rho(U) = \rho(\gamma \square U)$. Of course in the notations of Proposition 3.1.7 $\gamma \square B_0 = (\gamma, 0).B_0$ and $\gamma \square U = (\gamma, 0).U$

If $B_0 = B_{0,Z} + B_{0,L}$ where $B_{0,Z}$ and $B_{0,L}$ are forms on Z and L respectively, we define $\gamma \square B_0 = \gamma B_{0,Z} + B_{0,L}$.

The next proposition indicates the dependance of the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ if we change ρ into $\gamma \square \rho$.

Proposition 3.1.8.

1) Let us denote by $[\ , \]_{B_0, \rho}$ the bracket on $\Gamma(\mathfrak{g}_0, B_0, V)$ given by Theorem 3.1.1. Then, using the notations defined above, we have:

$$[X, Y]_{B_0, \gamma \square \rho} = \gamma \square [X, Y]_{B_0, \rho}$$

$$[U, X]_{B_0, \gamma \square \rho} = \gamma \square \rho(U)X, \quad [U, Y]_{B_0, \gamma \square \rho} = -\gamma \square \rho^*(U)Y$$

2) Suppose that $\lambda, \mu, \alpha, \beta \in \mathbb{C}^*$ verify the condition $\frac{\mu^2}{\lambda} = \frac{\beta^2}{\alpha}$. Then the local Lie algebras $\Gamma(\mathfrak{g}_0, \lambda \square B_0, \mu \square \rho)$ and $\Gamma(\mathfrak{g}_0, \alpha \square B_0, \beta \square \rho)$ are isomorphic. In particular $\Gamma(\mathfrak{g}_0, \lambda \square B_0, \rho)$ and $\Gamma(\mathfrak{g}_0, B_0, \frac{1}{\sqrt{\lambda}} \square \rho)$ are isomorphic.

Proof.

1) The element $[X, Y]_{B_0, \gamma \square \rho}$ is by definition the unique element of \mathfrak{g}_0 such that, for all $U \in \mathfrak{g}_0$, $B_0([X, Y]_{B_0, \gamma \square \rho}, U) = Y(\gamma \square \rho(U)X) = Y(\rho(\gamma \square U)X) = B_0([X, Y]_{B_0, \rho}, \gamma \square U) = B_0(\gamma \square [X, Y]_{B_0, \rho}, U)$. Hence $[X, Y]_{B_0, \gamma \square \rho} = \gamma \square [X, Y]_{B_0, \rho}$. The other two identities are just the definitions of $[U, X]_{B_0, \gamma \square \rho}$ and $[U, Y]_{B_0, \gamma \square \rho}$.

2) Define $\varphi : \Gamma(\mathfrak{g}_0, \lambda \square B_0, \mu \square \rho) \longrightarrow \Gamma(\mathfrak{g}_0, \alpha \square B_0, \beta \square \rho)$ by

$$\begin{aligned} \varphi(U) &= \frac{\mu}{\beta} \square U, \text{ for all } U \in \mathfrak{g}_0 \\ \varphi(X) &= X \text{ for all } X \in V \\ \varphi(Y) &= Y \text{ for all } Y \in V^* \end{aligned}$$

We have $\varphi([U, X]_{\lambda \square B_0, \mu \square \rho}) = \mu \square \rho(U)X = \rho(\mu \square U)X$ and $[\varphi(U), \varphi(X)]_{\alpha \square B_0, \beta \square \rho} = [\frac{\mu}{\beta} \square U, X]_{\alpha \square B_0, \beta \square \rho} = \beta \square \rho(\frac{\mu}{\beta} \square U)X = \rho(\mu \square U)X$.

Hence $\varphi([U, X]_{\lambda \square B_0, \mu \square \rho}) = [\varphi(U), \varphi(X)]_{\alpha \square B_0, \beta \square \rho}$. Similarly one can prove that $\varphi([U, Y]_{\lambda \square B_0, \mu \square \rho}) = [\varphi(U), \varphi(Y)]_{\alpha \square B_0, \beta \square \rho}$.

We have also $\varphi([X, Y]_{\lambda \square B_0, \mu \square \rho}) = \frac{\mu}{\beta} \square [X, Y]_{\lambda \square B_0, \mu \square \rho} = \frac{\mu^2}{\beta} \square [X, Y]_{\lambda \square B_0, \rho}$ (use 1)) $= \frac{\mu^2}{\lambda \beta} \square [X, Y]_{B_0, \rho}$ (use Proposition 3.1.7).

On the other hand we have $[\varphi(X), \varphi(Y)]_{\alpha \square B_0, \beta \square \rho} = [X, Y]_{\alpha \square B_0, \beta \square \rho} = \frac{1}{\alpha} \square [X, Y]_{B_0, \beta \square \rho}$ (use Proposition 3.1.7) $= \frac{\beta}{\alpha} \square [X, Y]_{B_0, \rho}$ (use 1)). As $\frac{\mu^2}{\lambda} = \frac{\beta^2}{\alpha}$ we obtain that $\varphi([X, Y]_{\lambda \square B_0, \mu \square \rho}) = [\varphi(X), \varphi(Y)]_{\alpha \square B_0, \beta \square \rho}$. Finally we have proved that φ is an isomorphism of local Lie algebras. □

3.2. Graded Lie algebras with local part $\Gamma(\mathfrak{g}_0, B_0, V)$.

Let us now translate the result of Kac (Theorem 2.3) in the context of Theorem 3.1.1:

Theorem 3.2.1. *Let $(\mathfrak{g}_0, \rho, V)$ be a finite dimensional representation of a quadratic Lie algebra \mathfrak{g}_0 . Let $\Gamma(\mathfrak{g}_0, \rho, V)$ be the local Lie algebra constructed in Theorem 3.1.1.*

1) *There exists a unique graded Lie algebra $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V))$ whose local part is $\Gamma(\mathfrak{g}_0, B_0, V)$ and which satisfies the following universal property.*

Any morphism of local Lie algebras $\Gamma(\mathfrak{g}_0, B_0, V) \rightarrow \hat{\mathfrak{g}}$ from Γ into the local part $\hat{\mathfrak{g}}$ of a graded Lie algebra \mathfrak{g} extends uniquely to a morphism of graded Lie algebras $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V)) \rightarrow \mathfrak{g}$. (And hence any graded Lie algebra whose local part is isomorphic to $\Gamma(\mathfrak{g}_0, B_0, V)$, is a quotient of $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V))$). Moreover we have

$$\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V)) = F(V^*) \oplus \mathfrak{g}_0 \oplus F(V),$$

where $F(V^)$ (resp. $F(V)$) is the free Lie algebra generated by V^* (resp. V).*

2) *There exists a unique graded Lie algebra $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ whose local part is $\Gamma(\mathfrak{g}_0, B_0, V)$ and which satisfies the following universal property.*

Any surjective morphism of local Lie algebras $\hat{\mathfrak{g}} \rightarrow \Gamma(\mathfrak{g}_0, B_0, V)$ from the local part of a graded Lie algebra \mathfrak{g} into $\Gamma(\mathfrak{g}_0, B_0, V)$ extends uniquely to a (surjective) morphism of graded Lie algebras $\mathfrak{g} \rightarrow \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$. (And hence $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is a quotient of any graded Lie algebra whose local part is isomorphic to $\Gamma(\mathfrak{g}_0, B_0, V)$).

Remark 3.2.2. Let (\mathfrak{g}_0, B_0) be a quadratic Lie algebra. Suppose that $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$ is an orthogonal decomposition into ideals. Define $B_0^1 = B_0|_{\mathfrak{g}_0^1 \times \mathfrak{g}_0^1}$ and $B_0^2 = B_0|_{\mathfrak{g}_0^2 \times \mathfrak{g}_0^2}$. Suppose also that the representation (\mathfrak{g}, ρ, V) is a direct sum $(\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$. Then from the definitions we obtain that $\Gamma(\mathfrak{g}_0, B_0, V) = \Gamma(\mathfrak{g}_0^1, B_0^1, V_1) \oplus \Gamma(\mathfrak{g}_0^2, B_0^2, V_2)$, and therefore

$$\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0^1, B_0^1, V_1)) \oplus \mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0^2, B_0^2, V_2))$$

and

$$\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0^1, V_1)) \oplus \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^2, B_0^2, V_2)).$$

As an example let us show that graded reductive (finite dimensional) Lie algebras are always minimal graded Lie algebras.

Proposition 3.2.3. *Let \mathfrak{g} be a reductive (finite dimensional) Lie algebra. Suppose that we are given a \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$ such that \mathfrak{g} is generated by its local part $\Gamma(\mathfrak{g}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (in other words \mathfrak{g} is a graded Lie algebra in the sense of Definition 2.1). Let $B_{\Gamma(\mathfrak{g})}$ be a nondegenerate invariant symmetric bilinear form on $\Gamma(\mathfrak{g})$ such that $B_{\Gamma(\mathfrak{g})}(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ when $|i + j| \neq 0$. Then using*

$B_{\Gamma(\mathfrak{g})}$, the contragredient representation $(\mathfrak{g}_0, \mathfrak{g}_1^*)$ can be identified with $(\mathfrak{g}_0, \mathfrak{g}_{-1})$. If B_0 denotes the restriction of $B_{\Gamma(\mathfrak{g})}$ to \mathfrak{g}_0 , then we have:

$$\mathfrak{g} \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g})) \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \mathfrak{g}_1))$$

Proof.

From the definition we can identify \mathfrak{g}_1^* with \mathfrak{g}_{-1} by using $B_{\Gamma(\mathfrak{g})}$. Then for $U \in \mathfrak{g}_0$, $Y \in \mathfrak{g}_{-1}$ and $X \in \mathfrak{g}_1$ we have, from the invariance of $B_{\Gamma(\mathfrak{g})}$:

$$B_{\Gamma(\mathfrak{g})}(U.Y, X) = -B_{\Gamma(\mathfrak{g})}(Y, [U, X]) = B_{\Gamma(\mathfrak{g})}([U, Y], X)$$

and hence $[U, Y] = U.Y$ (here $U.Y$ stands for the contragredient action of $U \in \mathfrak{g}_0$ on $\mathfrak{g}_{-1} \simeq \mathfrak{g}_1^*$).

Similarly, we have also:

$$B_0([X, Y], U) = B_{\Gamma(\mathfrak{g})}([X, Y], U) = B_{\Gamma(\mathfrak{g})}(Y, [U, X]) = Y(U.X).$$

Hence the original bracket in $\Gamma(\mathfrak{g})$ is the bracket constructed in Theorem 3.1.1. Therefore $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g})) \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \mathfrak{g}_1))$.

From the universal property of $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}))$ there exists a graded ideal $I \subset \mathfrak{g}$ such that $\mathfrak{g}/I \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}))$. As \mathfrak{g} is reductive there exists an ideal $U \subset \mathfrak{g}$ such that $\mathfrak{g} = U \oplus I$. Hence $U \simeq \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}))$. But then the local part of U is $\Gamma(\mathfrak{g})$, and this contradicts the fact that \mathfrak{g} is generated by $\Gamma(\mathfrak{g})$ unless $I = \{0\}$. \square

Remark 3.2.4.

It must be noticed that if $\mathfrak{g} = \bigoplus_{i=-n}^n \mathfrak{g}_i$ is an arbitrary \mathbb{Z} -grading of a semi-simple Lie algebra \mathfrak{g} then \mathfrak{g} is in general not generated by its local part, and is therefore not a graded Lie algebra in the sense of Definition 2.1. Let us explain this briefly. It is well known that there exists always a grading element, that is an element $H \in \mathfrak{g}$ such that $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [H, X] = iX\}$. Let \mathfrak{h} be Cartan subalgebra of \mathfrak{g}_0 (which is also a Cartan subalgebra of \mathfrak{g}), containing H . Let Ψ be a set of simple roots of the root system $\Sigma(\mathfrak{g}, \mathfrak{h})$ such that $\alpha(H) \in \mathbb{N}$ (such a set of simple roots always exists). Hence we have associated a "weighted Dynkin diagram" to a grading. The subdiagram of roots of weight 0 corresponds to the semi-simple part of the Levi subalgebra \mathfrak{g}_0 . But if the weighted Dynkin diagram has weights equal to 1 and to $n > 1$, then it is easy to see that \mathfrak{g} cannot be generated by the local part $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Example 3.2.5. (Prehomogeneous spaces of parabolic type)

Let \mathfrak{g} be a simple complex Lie algebra. Let \mathfrak{h} be Cartan subalgebra. Denote as before by $\Sigma(\mathfrak{g}, \mathfrak{h})$ the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$. Let Ψ be a set of simple roots

for $\Sigma(\mathfrak{g}, \mathfrak{h})$. Let $\theta \subset \Psi$ be a subset and let $\langle \theta \rangle$ denote the subset of roots which are linear combinations of elements of θ . Let $\mathfrak{g}_0 = \mathfrak{l}_\theta$ be the Levi subalgebra corresponding to θ . That is $\mathfrak{g}_0 = \mathfrak{h} \oplus (\oplus_{\alpha \in \langle \theta \rangle} \mathfrak{g}^\alpha)$ where \mathfrak{g}^α is the root space corresponding to α . Let H_0 be the unique element in \mathfrak{h} such that $\alpha(H_0) = 0$ if $\alpha \in \theta$ and $\alpha(H_0) = 1$ if $\alpha \in \Psi \setminus \theta$. Define then $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [H_0, X] = iX\}$ (this definition of \mathfrak{g}_0 is coherent with the preceding one). One obtains this way a grading $\mathfrak{g} \oplus_{i=-n}^n \mathfrak{g}_i$. The representations $(\mathfrak{g}_0, \mathfrak{g}_1)$ are prehomogeneous vector spaces called prehomogeneous spaces of parabolic type. It is easy to see that they correspond to gradings whose weights in the sense of the the preceding remark are only 0 and 1. From Proposition 3.2.3 we obtain that in this case $\mathfrak{g} = \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \mathfrak{g}_1))$ where B_0 is the restriction of the Killing form of \mathfrak{g} to \mathfrak{g}_0 .

Example 3.2.6.

We will now examine the case of $\mathfrak{sl}_{2n}(\mathbb{C})$ which will be considered both as a graded Lie algebra and a local Lie algebra. This will show that the local Lie algebras $\Gamma(\mathfrak{g}_0, B_0, V)$ and the corresponding minimal Lie algebra $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ depend strongly on the choice of B_0 .

Consider first the classical 3-grading of $\mathfrak{sl}_{2n}(\mathbb{C})$ defined by:

$$\begin{aligned} \mathfrak{g}_{-1} = V^* &= \left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}, Y \in M_n(\mathbb{C}) \right\} \\ \mathfrak{g}_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A, B \in M_n(\mathbb{C}), Tr(A+B) = 0 \right\} \\ \mathfrak{g}_1 = V &= \left\{ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, X \in M_n(\mathbb{C}) \right\} \end{aligned}$$

We will use the letter U for elements in \mathfrak{g}_0 and the letters X, Y for elements in \mathfrak{g}_1 and \mathfrak{g}_{-1} respectively. And in order to simplify notations we will set $X = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$.

As invariant form on $\mathfrak{sl}_{2n}(\mathbb{C})$ we will take $B(\alpha, \beta) = Tr(\alpha\beta)$, $(\alpha, \beta \in \mathfrak{sl}_{2n}(\mathbb{C}))$. This is just a multiple of the Killing form. Let us call B_0 the restriction of B to \mathfrak{g}_0 . The form B_0 is of course nondegenerate. The representation $(\mathfrak{g}_0, \rho, V)$ is defined by the bracket. Therefore if $U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, we have $\rho(U)X = [U, X] =$

$$\begin{pmatrix} 0 & AX - XB \\ 0 & 0 \end{pmatrix} = AX - XB.$$

If we consider $\mathfrak{sl}_{2n}(\mathbb{C})$ as the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, \rho)$ we now from Proposition 3.2.3 that $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) = \mathfrak{sl}_{2n}(\mathbb{C})$.

Next we will modify the form B_0 in the following manner. Let $\lambda = (\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$ and set $B_0^\lambda \left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \right) = \lambda_1 \text{Tr}(AA') + \lambda_2 \text{Tr}(BB')$ (remember that $B_0 \left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \right) = \text{Tr}(AA') + \text{Tr}(BB')$).

We will decompose the form B_0^λ according to the decomposition of \mathfrak{g}_0 into ideals. Obviously

$$\mathfrak{g}_0 = \mathbb{C}H_0 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_2 \quad (*)$$

where $H_0 = \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix}$ and where $\mathfrak{a}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, A \in \mathfrak{sl}_n(\mathbb{C}) \right\}$ and $\mathfrak{a}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, B \in \mathfrak{sl}_n(\mathbb{C}) \right\}$.

The decomposition of $[Y, X]_{B_0}$ according to (*) is as follows:

$$\begin{aligned} [Y, X]_{B_0} &= -\frac{1}{n} \text{Tr}(YX) \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix} \\ &\quad + \begin{pmatrix} -XY + \frac{1}{n} \text{Tr}(YX) \text{Id}_n & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & YX - \frac{1}{n} \text{Tr}(YX) \text{Id}_n \end{pmatrix} \end{aligned}$$

An easy computation shows that $B_0^\lambda = \frac{\lambda_1 + \lambda_2}{2} B_{0|_{\mathbb{C}H_0}} + \lambda_1 B_{0|_{\mathfrak{a}_1}} + \lambda_2 B_{0|_{\mathfrak{a}_2}}$ (hence the form B_0^λ is nondegenerate if and only if $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_1 + \lambda_2 \neq 0$). Therefore $B_0^\lambda = \mu \cdot B_0$, where $\mu = \left(\frac{\lambda_1 + \lambda_2}{2}, \lambda_1, \lambda_2 \right)$. From Proposition 3.1.7 we obtain that

$$\begin{aligned} [Y, X]_{B_0^\lambda} &= -\frac{2}{n(\lambda_1 + \lambda_2)} \text{Tr}(YX) \begin{pmatrix} \text{Id}_n & 0 \\ 0 & -\text{Id}_n \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{\lambda_1} (-XY + \frac{1}{n} \text{Tr}(YX) \text{Id}_n) & 0 \\ 0 & \frac{1}{\lambda_2} (YX - \frac{1}{n} \text{Tr}(YX) \text{Id}_n) \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1}{\lambda_1} XY + \frac{\lambda_2 - \lambda_1}{n(\lambda_1 + \lambda_2)\lambda_1} \text{Tr}(YX) \text{Id}_n & 0 \\ 0 & \frac{1}{\lambda_2} YX + \frac{\lambda_2 - \lambda_1}{n(\lambda_1 + \lambda_2)\lambda_2} \text{Tr}(YX) \text{Id}_n \end{pmatrix} \quad (**) \end{aligned}$$

Suppose now that the local Lie algebras $\Gamma(\mathfrak{g}_0, B_0, \rho)$ and $\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho)$ are isomorphic. Then as $\Gamma(\mathfrak{g}_0, B_0, \rho) = \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) \simeq \mathfrak{sl}_{2n}(\mathbb{C})$, we should have that $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho)) \simeq \mathfrak{sl}_{2n}(\mathbb{C})$, and hence for $Y \in V^*$ and $X, X' \in V$ we have $[Y, [X, X']_{B_0^\lambda}]_{B_0^\lambda} = 0$. And then from the Jacobi identity the following identity should hold in $\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho)$:

$$[[Y, X]_{B_0^\lambda}, X']_{B_0^\lambda} + [X, [Y, X']_{B_0^\lambda}]_{B_0^\lambda} = 0 \quad (***)$$

A calculation, using (**) shows that the left member of (***) is equal to

$$\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) \left((XYX' - X'YX) + \frac{\lambda_2 - \lambda_1}{n(\lambda_1 + \lambda_2)} (Tr(YX')X - Tr(YX)X') \right)$$

(in the simplified notation explained at the beginning of the example).

It is easy to see that if $\lambda_1 \neq \lambda_2$ this element is not equal to zero for general $Y, X, X' \in M_n(\mathbb{C})$. Therefore the local Lie algebra $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0^\lambda, \rho))$ cannot be isomorphic to $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, \rho)) \simeq \mathfrak{sl}_{2n}(\mathbb{C})$.

3.3. Transitivity.

Let us also recall the notion of transitivity introduced by V. Kac.

Definition 3.3.1. (Kac [2], Definition 2)

Let \mathfrak{g} (resp. $\hat{\mathfrak{g}}$) be a graded Lie algebra (resp. a local Lie algebra). Then \mathfrak{g} (resp. $\hat{\mathfrak{g}}$) is said to be transitive if

- for $x \in \mathfrak{g}_i, i \geq 0, [x, \mathfrak{g}_{-1}] = \{0\} \Rightarrow x = 0$
- for $x \in \mathfrak{g}_i, i \leq 0, [x, \mathfrak{g}_1] = \{0\} \Rightarrow x = 0$.

In particular if \mathfrak{g} (or $\hat{\mathfrak{g}}$) is transitive, then the modules $(\mathfrak{g}_0, \mathfrak{g}_{-1})$ and $(\mathfrak{g}_0, \mathfrak{g}_1)$ are faithful.

If A is a subset of a vector space V , we denote by $\langle A \rangle$ the subspace of V generated by A .

Proposition 3.3.2.

1) Let (\mathfrak{g}_0, V) be a representation of the quadratic Lie algebra \mathfrak{g}_0 . The local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive if and only if (\mathfrak{g}_0, V) is faithful and $\langle \mathfrak{g}_0.V \rangle = V$ and $\langle \mathfrak{g}_0.V^* \rangle = V^*$.

2) If the representation (\mathfrak{g}_0, V) is completely reducible, then the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive if and only if (\mathfrak{g}_0, V) is faithful and V does not contain the trivial module.

Proof. 1) Suppose that $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive. We have already remarked that then the representation (\mathfrak{g}_0, V) is faithful (and hence (\mathfrak{g}_0, V^*) is faithful too). If $\langle \mathfrak{g}_0.V \rangle \neq V$, then there exists $Y \in V^*, Y \neq 0$ such that $Y(\mathfrak{g}_0.V) = 0$. From the definition of the bracket we obtain that $B_0([V, Y], \mathfrak{g}_0) = 0$. Hence $[Y, V] = \{0\}$. This contradicts the transitivity. Similarly one proves that transitivity implies $\langle \mathfrak{g}_0.V^* \rangle = V^*$. Conversely suppose that (\mathfrak{g}_0, V) is faithful and $\langle \mathfrak{g}_0.V \rangle = V$ and $\langle \mathfrak{g}_0.V^* \rangle = V^*$. The first of these assumptions is one of the conditions needed for the transitivity. Suppose also that $[X, V^*] = \{0\}$ for an $X \in V$. Then $B_0([X, Y], U) = 0$ for all $U \in \mathfrak{g}_0$ and all $Y \in V^*$. Hence

$Y(U.X) = -U.Y(X) = 0$. Therefore, as $\langle \mathfrak{g}_0.V^* \rangle = V^*$, we have $V^*(X) = 0$ and hence $X = 0$. The same proof, using the identity $\langle \mathfrak{g}_0.V \rangle = V$, shows that $[Y, V] = \{0\}$ implies $Y = 0$. Hence $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive.

2) Let $V = \bigoplus_{i=1}^k V_i$ be a decomposition of V into irreducibles. If V_i is not the trivial module we have of course $\langle \mathfrak{g}_0.V_i \rangle = V_i$. It is then easy to see that the preceding condition " $\langle \mathfrak{g}_0.V \rangle = V$ and $\langle \mathfrak{g}_0.V^* \rangle = V^*$ " is equivalent to the condition " V does not contain the trivial module". \square

Remark 3.3.3.

As a minimal graded Lie algebra with a transitive local part is transitive ([2], Prop. 5 page 1278), the Proposition above gives also necessary and sufficient conditions for the transitivity of $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$.

Remark 3.3.4.

1) Suppose that there exists an element $H \in \mathfrak{g}_0$ such that $H.X = X$ for all $X \in V$. Such an element is called *grading element*. Then obviously the conditions $\langle \mathfrak{g}_0.V \rangle = V$ and $\langle \mathfrak{g}_0.V^* \rangle = V^*$ are satisfied. In this case the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive if and only if (\mathfrak{g}_0, V) is faithful.

2) Suppose that \mathfrak{g}_0 is reductive and that the representation (\mathfrak{g}_0, V) is not faithful. Then $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$, where \mathfrak{g}_0^i are ideals and where \mathfrak{g}_0^2 is the kernel of the representation. Then from the universal property of $\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V))$ and $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ (see Th.2.3) it is easy to see that

$$\mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{max}(\Gamma(\mathfrak{g}_0^1, B_0|_{\mathfrak{g}_0^1} V)) \oplus \mathfrak{g}_0^2 \quad \text{and}$$

$$\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) \simeq \mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0^1, B_0|_{\mathfrak{g}_0^1} V)) \oplus \mathfrak{g}_0^2.$$

3) Suppose that (\mathfrak{g}_0, V) is completely reducible and that the representation (\mathfrak{g}_0, V) is faithful. Let $V = \bigoplus_{i=1}^k V_i$ be a decomposition of V into irreducibles. Then by Schur's Lemma we obtain that $\dim Z(\mathfrak{g}_0) \leq k$ ($Z(\mathfrak{g}_0)$ denotes the center of \mathfrak{g}_0). Hence if the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive, then $\dim Z(\mathfrak{g}_0) \leq k$. In particular if V is irreducible $\dim Z(\mathfrak{g}_0) \leq 1$.

Proposition 3.3.5.

Let \mathfrak{g}_0 be a reductive Lie algebra. Let B_0 be a non-degenerate invariant symmetric bilinear form on \mathfrak{g}_0 and let (ρ, V) be a finite dimensional completely reducible representation of \mathfrak{g}_0 . Suppose that the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive (see Proposition 3.3.2 above). Then if $\dim(\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)))$ is finite, the Lie algebra $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is semi-simple.

Proof.

It suffices to prove that if $\Gamma(\mathfrak{g}_0, B_0, V)$ cannot be decomposed as in Remark 3.2.2, then $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is simple. We know from [2] (Prop. 5, p.1278) that $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is transitive. Denote by $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ the grading. Let \mathfrak{a} be a non zero ideal of $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$. Let $a \in \mathfrak{a}$, $a \neq 0$. Let $a = a_{-i} + a_{-i+1} + \dots + a_j$ be the decomposition of a according to the grading of $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$, where either $-i \leq 0$ and $a_i \neq 0$, or $j \geq 0$, and $a_j \neq 0$. Suppose for example that $-i \leq 0$ and $a_i \neq 0$. From the transitivity of $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$, we know that there exists $x_1^i \in \mathfrak{g}_1$ such that $[a_{-i}, x_1^i] \neq 0$. Then $[a, x_1^i] = [a_i, x_1^i] + \dots + [a_j, x_1^i] \in \mathfrak{a}$ therefore there exists an element $a' \in \mathfrak{a}$, such that $a' = a'_{-i+1} + \dots + a'_{j+1}$ ($a'_k \in \mathfrak{g}_k$) and $a'_{-i+1} \neq 0$. By induction we prove that there exists an element $x = x_0 + x_1 + \dots \in \mathfrak{a}$ such that $x_0 \neq 0$ and also an element $y = y_1 + \dots \in \mathfrak{a}$ such that $y_1 \neq 0$. Let $\mathcal{T}_k = \bigoplus_{n \geq k} \mathfrak{g}_i$ ($k \geq 0$). Denote by $\tilde{\mathfrak{a}}_0$ (resp. $\tilde{\mathfrak{a}}_1$) the projection of $\mathfrak{a} \cap \mathcal{T}_0$ on \mathfrak{g}_0 (resp. the projection of $\mathfrak{a} \cap \mathcal{T}_1$ on \mathfrak{g}_1). The preceding considerations show that $\tilde{\mathfrak{a}}_0 \neq \{0\}$ and $\tilde{\mathfrak{a}}_1 \neq \{0\}$. As \mathfrak{a} is an ideal, $\tilde{\mathfrak{a}}_0$ is an ideal of \mathfrak{g}_0 and $\tilde{\mathfrak{a}}_1$ is a sub- \mathfrak{g}_0 -module of $\mathfrak{g}_1 = V$. Let $\tilde{\mathfrak{b}}_0$ be the orthogonal of $\tilde{\mathfrak{a}}_0$ in \mathfrak{g}_0 with respect to B_0 , and let $\tilde{\mathfrak{b}}_1$ be a \mathfrak{g}_0 -invariant supplementary space to $\tilde{\mathfrak{a}}_1$ in \mathfrak{g}_1 . That is $\mathfrak{g}_1 = \tilde{\mathfrak{a}}_1 \oplus \tilde{\mathfrak{b}}_1$. And, as \mathfrak{g}_0 is reductive, we have also $\mathfrak{g}_0 = \tilde{\mathfrak{a}}_0 \oplus \tilde{\mathfrak{b}}_0$ and $\tilde{\mathfrak{b}}_0$ is an ideal of \mathfrak{g}_0 . As \mathfrak{a} is an ideal we obtain $[\tilde{\mathfrak{a}}_0, \tilde{\mathfrak{b}}_1] = \{0\}$. Let now B the extended form as defined in Proposition 3.4.2 below. Then, as $[\tilde{\mathfrak{a}}_1, \mathfrak{g}_{-1}] \subset \tilde{\mathfrak{a}}_0$, we have for all $Y \in \mathfrak{g}_{-1}$, $B([\tilde{\mathfrak{b}}_0, \tilde{\mathfrak{a}}_1], Y) = B(\tilde{\mathfrak{b}}_0, [\tilde{\mathfrak{a}}_1, Y]) = \{0\}$. This shows that $[\tilde{\mathfrak{b}}_0, \tilde{\mathfrak{a}}_1]$ is orthogonal to \mathfrak{g}_{-1} . Therefore $[\tilde{\mathfrak{b}}_0, \tilde{\mathfrak{a}}_1] = \{0\}$. We have supposed that $\Gamma(\mathfrak{g}_0, B_0, V)$ is not decomposable in the sense of Remark 3.2.2. Then $\tilde{\mathfrak{b}}_0 = \{0\}$ and $\tilde{\mathfrak{b}}_1 = \{0\}$, and $\mathfrak{g}_0 = \tilde{\mathfrak{a}}_0$ and $\mathfrak{g}_1 = \tilde{\mathfrak{a}}_1$.

As $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is finite dimensional we can write $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) = \bigoplus_{i=-n}^n \mathfrak{g}_i$. As $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is generated by its local part, we obtain that any $X_n \in \mathfrak{g}_n$ is a linear combination of elements of the form $[\dots [X_1^1, X_1^2] \dots] X_1^n$ where $X_1^1, \dots, X_1^n \in \mathfrak{g}_1$. But as $\mathfrak{g}_1 = \tilde{\mathfrak{a}}_1$, we obtain that $X_n \in \mathfrak{g}_n \cap \mathfrak{a}$, and hence $\mathfrak{g}_n \cap \mathfrak{a} = \mathfrak{g}_n$.

From the transitivity, we know that there exist $Y_1^1, \dots, Y_1^n \in \mathfrak{g}_{-1}$ such that $[Y_1^n, [Y_1^{n-1}, \dots [Y_1^1, X_n] \dots]] \neq 0$. This proves that $\mathfrak{a}_1 = \mathfrak{a} \cap \mathfrak{g}_1 \neq \{0\}$ and $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0 \neq \{0\}$. Then the same reasoning as above shows that $\mathfrak{a}_0 = \mathfrak{g}_0$ and $\mathfrak{a}_1 = \mathfrak{g}_1$. As $[\mathfrak{g}_0, \mathfrak{g}_{-1}] = \mathfrak{g}_1$ (this is again the transitivity condition), we have also that $\mathfrak{a}_{-1} = \mathfrak{a} \cap \mathfrak{g}_{-1} = \mathfrak{g}_{-1}$.

Finally we have proved that $\Gamma(\mathfrak{g}_0, B_0, V) \subset \mathfrak{a}$. Hence $\mathfrak{g}_{min}(\Gamma(\mathfrak{g}_0, B_0, V)) = \mathfrak{a}$. \square

3.4. Invariant bilinear forms.

Consider again a general quadratic Lie algebra \mathfrak{g}_0 with a non degenerate symmetric bilinear form B_0 . We will first show that B_0 extends to an invariant form on the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$.

Define a symmetric bilinear form B on $\Gamma(\mathfrak{g}_0, B_0, V)$ by setting:

- $\forall u, v \in \mathfrak{g}_0, B(u, v) = B_0(u, v)$
- $\forall u \in \mathfrak{g}_0, \forall X \in \mathfrak{g}_1 = V, \forall Y \in \mathfrak{g}_{-1} = V^*$

$$B(u, X) = B(X, u) = B(u, Y) = B(Y, u) = 0 \quad (*)$$

$$- \forall X \in \mathfrak{g}_1 = V, \forall Y \in \mathfrak{g}_{-1} = V^*, B(X, Y) = B(Y, X) = Y(X) \quad (**)$$

Lemma 3.4.1.

a) *The form B is a non degenerate invariant form on $\Gamma(\mathfrak{g}_0, B_0, V)$ (the definition of an invariant form on a local Lie algebra is analogous to the Lie algebra case).*

b) *Suppose that there exists a grading element in $\Gamma(\mathfrak{g}_0, B_0, V)$, that is an element $H_0 \in \mathfrak{g}_0$ such that $[H_0, x] = ix$ for $x \in \mathfrak{g}_i, i = -1, 0, 1$, then the preceding form B is the only invariant extension of B_0 to $\Gamma(\mathfrak{g}_0, B_0, V)$.*

Proof.

a) Of course as $B|_{\mathfrak{g}_0 \times \mathfrak{g}_0} = B_0$, the invariance is verified on \mathfrak{g}_0 . Let $X \in \mathfrak{g}_1, Y \in \mathfrak{g}_{-1}, u, v \in \mathfrak{g}_0$. From the definition of $[X, Y]$ (see Theorem 3.1.1), we have $B([X, Y], u) = Y(u.X)$. On the other hand we have $B(X, [Y, u]) = B(X, -u.Y) = -u.Y(X) = Y(u.X)$. Hence $B([X, Y], u) = B(X, [Y, u])$. We have also $B([u, v], X) = 0 = B(u, v.X) = B(u, [v, X])$. Similarly $B([u, v], Y) = B(u, [v, Y]) = 0$.

b) Suppose that $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$ with $i + j \neq 0$. Then $B([H_0, x], y) = B(ix, y) = iB(x, y) = -B(x, [H_0, y]) = -jB(x, y)$. Therefore $(i+j)B(x, y) = 0$, and hence $B(x, y) = 0$. We also have for $X \in V = \mathfrak{g}_1$ and $Y \in V^* = \mathfrak{g}_{-1}$: $B([X, Y], H_0) = Y(X) = B(X, [Y, H_0]) = B(X, Y)$. Hence conditions (*) and (**) are satisfied. □

Proposition 3.4.2.

1) *Let (\mathfrak{g}_0, V) be a representation of the quadratic Lie algebra \mathfrak{g}_0 . The bilinear form B on $\Gamma(\mathfrak{g}_0, B_0, V)$ defined in Lemma 3.4.1 extends uniquely to a invariant symmetric bilinear form (still denoted B) such that $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i \neq -j$ on any graded Lie algebra whose local part is $\Gamma(\mathfrak{g}_0, B_0, V)$.*

2) *Moreover if $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive, then the extended form B on $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is non-degenerate.*

Proof. 1) is due to V. Kac: see Proposition 7 p. 1279 of [2].

2) Suppose now that $\Gamma(\mathfrak{g}_0, B_0, V)$ is transitive. Let us denote the grading by $\mathfrak{g}_{\min}(\mathfrak{g}_0, V) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. We must prove that if $X \in \mathfrak{g}_i$ is such that $B(X, Y) = 0$ for all $Y \in \mathfrak{g}_{-i}$, then $X = 0$. We will first prove the result by induction for $i \geq 0$.

From the definition of B on $\Gamma(\mathfrak{g}_0, B_0, V)$ we see that the result is true for $i = 0$ and $i = 1$. Suppose now that the result is true for $i < k$. Let $x_k \in \mathfrak{g}_k$ such that $B(x_k, \mathfrak{g}_{-k}) = 0$. Then for all $x_1 \in \mathfrak{g}_{-1}$ and all $x_{-k+1} \in \mathfrak{g}_{-k+1}$ we have $B(x_k, [x_{-1}, x_{-k+1}]) = 0$. And hence $B([x_k, x_{-1}], x_{-k+1}) = 0$. From the induction hypothesis we get $[x_k, x_{-1}] = 0$ for all $x_{-1} \in \mathfrak{g}_{-1}$. But we know from [2] (Prop. 5, p. 1278) that a minimal graded Lie algebra with a transitive local part is transitive. This implies that $x_k = 0$. The same proof works for $i \leq 0$. \square

Theorem 3.4.3.

Let \mathcal{T} be the set of equivalence classes of fundamental triplets such that the representation $(\mathfrak{g}_0, \rho, V)$ is faithful and such that $\langle \rho(\mathfrak{g}_0)V \rangle = V$ and $\langle \rho^*(\mathfrak{g}_0)V^* \rangle = V^*$. Let \mathcal{G} be the set of equivalence classes of transitive graded Lie algebras $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$ which are endowed with a non-degenerate symmetric invariant bilinear form B such that $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i \neq -j$. The map

$$\tau : \mathcal{T} \longrightarrow \mathcal{G}$$

defined by $\tau(\mathfrak{g}_0, B_0, V) = \mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, V))$ is a bijection.

Let \mathcal{T}_1 be the set of equivalence classes of fundamental triplets such that the representation $(\mathfrak{g}_0, \rho, V)$ is completely reducible and faithful and such that V does not contain the trivial module.

Let \mathcal{G}_1 be the subset of \mathcal{G} consisting of graded algebras where the representation $(\mathfrak{g}_0, \mathfrak{g}_1)$ is completely reducible.

By restriction the map τ defines also a bijection between \mathcal{T}_1 and \mathcal{G}_1 .

Proof. This is a consequence of Theorem 3.1.6, Proposition 3.3.2, Proposition 3.4.2 and of the fact that a transitive graded Lie algebra is minimal ([2], Prop. 5 a), p. 1278). \square

4. \mathfrak{sl}_2 -TRIPLES

In this section we will make the following assumptions. The Lie algebra \mathfrak{g}_0 is reductive with a one dimensional center: $\mathfrak{g}_0 = \mathfrak{z} \oplus \mathfrak{g}'_0$ where $\mathfrak{g}'_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ and

$\dim \mathfrak{z} = 1$. Moreover V is an irreducible \mathfrak{g}_0 -module on which \mathfrak{z} acts non trivially. Then there exists $H_0 \in \mathfrak{z}$ such that $\rho(H_0) = 2\text{Id}_{|_V}$ (and $\rho^*(H_0) = -2\text{Id}_{|_{V^*}}$). Recall also that in a Lie algebra, or in a local Lie algebra, a triple of elements (x, h, y) is called an \mathfrak{sl}_2 -triple if $[h, x] = 2x$, $[h, y] = -2y$ and $[y, x] = h$.

4.1. Associated \mathfrak{sl}_2 -triple.

Definition 4.1.1. *We say that the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$, or the graded Lie algebra $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, V))$, is associated to an \mathfrak{sl}_2 -triple if there exists $X \in V \setminus \{0\}$, $Y \in V^* \setminus \{0\}$ and $H_0 \in Z$ such that (X, H_0, Y) is an \mathfrak{sl}_2 -triple.*

Theorem 4.1.2. *Let \mathfrak{g}_0 be a reductive Lie algebra with a one dimensional center, and let V be an irreducible \mathfrak{g}_0 -module. We suppose that the center does not act trivially on V . Then $\Gamma(\mathfrak{g}_0, B_0, V)$ is associated to an \mathfrak{sl}_2 -triple if and only if there exists $X \in V \setminus \{0\}$ such that $X \notin \mathfrak{g}'_0 \cdot X$ where $\mathfrak{g}'_0 \cdot X = \{U \cdot X, U \in \mathfrak{g}'_0\}$. The set $\{X \in V, X \notin \mathfrak{g}'_0 \cdot X\}$ is exactly the set of elements in V which belong to an associated \mathfrak{sl}_2 -triple.*

Proof.

Recall from Lemma 3.4.1 that the form B_0 extends to an invariant form B on $\Gamma(\mathfrak{g}_0, B_0, V)$.

Suppose that $\Gamma(\mathfrak{g}_0, B_0, V)$ has an associated \mathfrak{sl}_2 -triple (X, H_0, Y) . Then $Y(\mathfrak{g}'_0 \cdot X) = B(Y, \mathfrak{g}'_0 \cdot X) = B(Y, [\mathfrak{g}'_0, X]) = B([Y, X], \mathfrak{g}'_0) = B(H_0, \mathfrak{g}'_0) = \{0\}$. Hence the form Y is zero on $\mathfrak{g}'_0 \cdot X$. On the other hand $B(Y, X) = Y(X) = \frac{1}{2}B(Y, [H_0, X]) = -\frac{1}{2}B([Y, X], H_0) = -\frac{1}{2}B(H_0, H_0) = -\frac{1}{2}B_0(H_0, H_0) \neq 0$. Therefore $X \notin \mathfrak{g}'_0 \cdot X$. Conversely suppose that $X \notin \mathfrak{g}'_0 \cdot X$. We choose $Y \in V^*$ such that $Y(\mathfrak{g}'_0 \cdot X) = \{0\}$ and $Y(X) \neq 0$. Then $B([Y, X], \mathfrak{g}'_0) = B(Y, \mathfrak{g}'_0 \cdot X) = Y(\mathfrak{g}'_0 \cdot X) = \{0\}$. Therefore $[Y, X] \in (\mathfrak{g}'_0)^\perp = \mathbb{C}H_0$. Set $[Y, X] = \lambda H_0$. We have also $Y(X) = B(Y, X) = \frac{1}{2}B(Y, [H_0, X]) = -\frac{1}{2}B([Y, X], H_0) = -\frac{1}{2}B(\lambda H_0, H_0) = -\frac{1}{2}\lambda B(H_0, H_0) \neq 0$. Hence $\lambda \neq 0$. Define $\tilde{Y} = \frac{1}{\lambda}Y$. Then (\tilde{Y}, H_0, X) is an \mathfrak{sl}_2 -triple. \square

Corollary 4.1.3.

a) *The existence of a \mathfrak{sl}_2 -triple associated to the local Lie algebra $\Gamma(\mathfrak{g}_0, B_0, V)$ does not depend on the invariant form B_0 on \mathfrak{g}_0 , but only on the representation (\mathfrak{g}_0, V) .*

b) *If the irreducible module (\mathfrak{g}_0, V) has the property that there exists $X \in V \setminus \{0\}$ such that $X \notin \mathfrak{g}'_0 \cdot X$, then the dual module (\mathfrak{g}_0, V^*) has the same property.*

4.2. Relative invariants and \mathfrak{sl}_2 -triples.

Definition 4.2.1. Let (G_0, ρ, V) be a finite dimensional representation of a connected complex Lie group G_0 . We denote by $d\rho$ the corresponding derived representation of the Lie algebra \mathfrak{g}_0 of G_0 . Let R be a rational fraction on V . The function R is called a relative invariant of (G_0, ρ, V) if there exists a Zariski open subset Ω of V , which is G_0 -invariant, and a rational character χ of G_0 such that

$$\forall g \in G_0, \forall X \in \Omega, R(\rho(g)X) = \chi(g)R(X) \quad (4-2-1)$$

We will now prove that under some conditions, the existence of a relative invariant implies the existence of an associate \mathfrak{sl}_2 -triple.

Theorem 4.2.2.

Let $(\mathfrak{g}_0, d\rho, V)$ be a finite dimensional irreducible representation of a reductive Lie algebra \mathfrak{g}_0 . We suppose that \mathfrak{g}_0 has a one dimensional center \mathfrak{z} which acts non trivially on V . Let H_0 be the unique element of \mathfrak{z} such that $d\rho(H_0)|_{\mathfrak{g}_i} = 2i\text{Id}_{\mathfrak{g}_i}$ (where the \mathfrak{g}_i 's are the homogeneous components of $\mathfrak{g}_{\min}(\Gamma(\mathfrak{g}_0, B_0, V))$). Let G_0 be a connected complex Lie group such the representation $d\rho$ lifts to a representation ρ of G_0 .

Then, if (G_0, ρ, V) has a relative invariant R with a non trivial character, there exists an associated \mathfrak{sl}_2 -triple (Y, H_0, X) , ($Y \in \mathfrak{g}_{-1}, X \in \mathfrak{g}_1$).

Proof.

Let us first remark that as the center \mathfrak{z} acts by multiplication by a scalar, the function R is homogeneous of degree $k \in \mathbb{Z}^*$.

If we derive equation (4-2-1) we obtain

$$dR(\rho(g)x) \circ \rho(g) = \rho^*(g^{-1}) \circ dR(\rho(g)x) = \chi(g)dR(x).$$

(Here $dR : \Omega \rightarrow V^*$) Hence

$$\frac{dR(\rho(g)x)}{R(\rho(g)x)} = \rho^*(g) \frac{\chi(g)dR(x)}{R(\rho(g)x)} = \rho^*(g) \frac{dR(x)}{R(x)}.$$

Hence if we define $\varphi_R(x) = \frac{dR(x)}{R(x)}$ ($\varphi_R : \Omega \rightarrow V^*$), we have:

$$\varphi_R(\rho(g)x) = \rho^*(g)\varphi_R(x)$$

For $A \in \mathfrak{g}_0$ and $x \in \Omega$, we will now derive the identity $R(\rho(\exp tA)x) = \chi(\exp tA)R(x)$ with respect to t , at $t = 0$. We obtain

$$dR(x)d\rho(A)x = d\chi(A)R(x).$$

Using the extended form B defined in Proposition 3.4.2 this can be written

$$B(dR(x), d\rho(A)x) = d\chi(A)R(x)$$

Dividing by $R(x)$ gives

$$\forall A \in \mathfrak{g}_0, \quad B(\varphi_R(x), [A, x]) = d\chi(A).$$

And as B is invariant we obtain:

$$\forall A \in \mathfrak{g}_0, \quad -B([\varphi_R(x), x], A) = d\chi(A).$$

As B is non-degenerate and as $d\chi(\mathfrak{g}'_0) = 0$, $[\varphi_R(x), x]$ is a fixed vector orthogonal to \mathfrak{g}'_0 . Hence $[\varphi_R(x), x] = cH_0$ ($c \in \mathbb{C}$). If $A = H_0$, one obtains $-B([\varphi_R(x), x], H_0) = d\chi(H_0) \neq 0$ (because χ is non trivial). Therefore $-cB(H_0, H_0) = d\chi(H_0)$ and $c \neq 0$. But from the Euler equation we get:

$$-B([\varphi_R(x), x], H_0) = -cB(H_0, H_0) = B(\varphi_R(x), [H_0, x]) = 2B(\varphi_R(x), x) = 2k$$

(remember that k is the homogeneity degree of R).

But then $(-\frac{B(H_0, H_0)}{2k}\varphi_R(x), H_0, x)$ is an \mathfrak{sl}_2 -triple.

□

Remark 4.2.3. It is known that for irreducible prehomogeneous vector spaces of parabolic type the existence of a relative invariant implies the existence of an associated \mathfrak{sl}_2 -triple (see [5] or [4]). But in the preceding Theorem there is no assumption of prehomogeneity. It works for any irreducible representation.

REFERENCES

- [1] M. T. DUONG , G. PINCZON , R. USHIROBIRA, — *A New Invariant of Quadratic Lie Algebras*, *Algebr. Represent. Theory* 15 (2012), $n^\circ 6$, 1163-1203. – cited on p. 7
- [2] V. KAC — *Simple irreducible graded Lie algebras of finite growth*, *Math. USSR Izv.* 2 (1968), 1271-1311. – cited on p. 1, 2, 4, 15, 16, 17, 19
- [3] A. MEDINA- P. REVOY — *Algèbres de Lie et produit scalaire invariant*, [Lie algebras and invariant scalar products] *Ann. Sci. École Norm. Sup. (4)* 18 (1985), no. 3, 553-561 – cited on p. 5, 7
- [4] H. RUBENTHALER — *Espaces vectoriel préhomogènes, sous-groupes paraboliques et \mathfrak{sl}_2 – triplets*, *C.R. Acad. Sci. Paris Sér. A-B* 290 (1980), $n^\circ 3$, A127-A129. – cited on p. 22
- [5] H. RUBENTHALER — *Algèbres de Lie et espaces préhomogènes*, in: *Travaux en cours*, Hermann, Paris, 1992. – cited on p. 22

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