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# Construction of the generalized Čech complex

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**Abstract**—In this paper, we introduce an algorithm which constructs the generalized Čech complex. The generalized Čech complex represents the topology of a wireless network whose cells are different in size. This complex is often used in many application to locate the boundary holes or to save energy consumption in wireless networks. The complexity of a construction of the Čech complex to analyze the coverage structure is found to be a polynomial time.

## I. INTRODUCTION

In recent years, homological topology is often used as an interesting tool in many algorithms to find the coverage holes and to save power in wireless networks. This tool was firstly introduced in [1] to analyze the topology of networks. After that, some algorithms which aim to locate coverage holes based on homological topology were developed in [2]–[4]. In [5], authors introduced a homological algorithm which can be applied to turn off redundant cells in wireless sensor networks. All these algorithms first require the computation of a simplicial complex that represents the geometric structure of network’s cells. There are two complexes that are often used: the Čech complex and the Vietoris-Rips complex. Let  $(X, d)$  be a metric space and  $\omega$  be a finite set of points in  $X$ . The Vietoris-Rips complex connects  $(k + 1)$  points in  $\omega$  with a  $k$ -simplex if the distance between any two of them is smaller than  $\epsilon_0$ . This complex doesn’t always describe exactly the coverage of networks. In this complex, some coverage holes may be considered as filled regions [6]. The Čech complex connect  $(k + 1)$  points in  $\omega$  with a  $k$ -simplex if there is a non-empty intersection of  $(k + 1)$  balls of radius  $\epsilon(\omega)$  centered at these points. If the  $\epsilon(\omega)$  is equal for every point in  $\omega$ , the Čech complex is standard. If  $\epsilon(\omega)$  is a sequence of real positive numbers, this complex is defined as the generalized Čech complex. By representing each cell of a network by a ball in the metric space, the Čech complex describes exactly the coverage of networks.

In [7], the authors introduced an algorithm to construct the Čech complex to use in computer and graphics science. This algorithm considers only the standard Čech complex whose parameter  $\epsilon$  is a constant for every ball. So, this algorithm is not appropriate for the wireless networks whose cells’ radii are different.

In this paper, we introduce an algorithm which constructs the generalized Čech complex whose parameter  $\epsilon(\omega)$  is different for each ball. This algorithm aims at representing the coverage structure of a network by a Čech complex. Then,

one can easily analyze the coverage structure through this Čech complex. Figure 1 gives an example of representing the coverage of network by the generalized Čech complex.

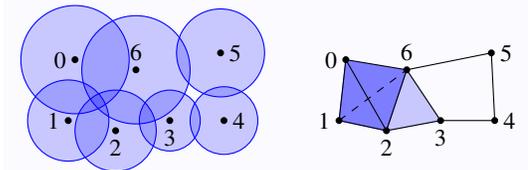


Fig. 1: Cells and their Čech representation.

The rest of this paper is organized as following. In section 2, we introduce the construction algorithm of the generalized Čech complex. In the next section, the complexity of this algorithm is computed. In the last section, some conclusions are drawn.

## II. CONSTRUCTION OF THE ČECH COMPLEX

We begin with the definition of the Čech complex:

*Definition 1 (Čech complex):* Given  $(X, d)$  a metric space,  $\omega$  a finite set of points in  $X$  and  $\epsilon(\omega)$  a sequence of real positive numbers, a Čech complex with parameter  $\epsilon(\omega)$  of  $\omega$ , denoted  $C_{\epsilon(\omega)}(\omega)$  is the abstract simplicial complex whose  $k$ -simplices correspond to non-empty intersection of  $(k + 1)$  balls of radius  $\epsilon(\omega)$  centered at the  $(k + 1)$  distinct vertices of  $\omega$ .

The construction of the Čech complex is a verification if a group of cells has a non-empty intersection. Obviously, each 0-simplex represents a cell, the set of 0-simplices, denoted  $S_0$ , is the list of cells. Each 1-simplex represents a pair of intersected cells. The construction of 1-simplices is the search for neighbors of each cell as in the Algorithm 1.

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### Algorithm 1 Construction of 1-simplices

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**Require:**  $S_0$  {collection of cells};  
1:  $S_1 = \emptyset$ ; {the set of 1-simplices}  
2:  $N \leftarrow |S_0|$ ;  
3: **for**  $i = 1 \rightarrow N - 1$  **do**  
4:     **for**  $j = i + 1 \rightarrow N$  **do**  
5:         **if** cell(i) intersected cell(j) **then**  
6:             add  $s = (i, j)$  to  $S_1$ ;  
7:         **end if**  
8:     **end for**  
9: **end for**  
10: **return**  $S_1$

---

The construction of  $k$ -simplices where  $k \geq 2$  is more complex. The rest of this section is devoted to the details of the construction of  $k$ -simplices where  $k \geq 2$ . From the definition of the Čech complex, each  $k$ -simplex represents a group of  $(k+1)$  cells which has a non-empty intersection. The number of combinations of  $(k+1)$  cells in all  $N$  cells is huge. We should first find out a candidate group that has an opportunity to be a  $k$ -simplex. Let us assume that  $u = \{v_0, v_1, \dots, v_k\}$  is a  $k$ -simplex. Then we can deduce that each pair  $(v_i, v_j)$ , where  $0 \leq i \neq j \leq k$ , are neighbors. This suggests that  $\hat{u} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$ , where  $\hat{v}_1, \dots, \hat{v}_k$  are neighbors of  $\hat{v}_0$ , is a candidate of the cell  $\hat{v}_0$  to be a  $k$ -simplex. If one of  $\hat{v}_1, \dots, \hat{v}_k$  is not neighbor of  $\hat{v}_0$ , then  $\{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$  can't be a  $k$ -simplex. We now need to verify if each candidate is a  $k$ -simplex. Let us denote  $x_{ij}$  an  $x_{ji}$  the two intersection points of cell  $v_i$  with cell  $v_j$ . Let  $\mathbb{X} = \{x_{ij}, x_{ji} | 0 \leq i < j \leq k\}$  be the set of intersection points for the candidate  $\hat{u}$ . Denote  $v_*$  is the smallest cell in candidate cells  $\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k$ . There are only three cases possible. The first case is that the smallest cell  $v_*$  is inside the others. We then conclude that  $\hat{u} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$  is a  $k$ -simplex.

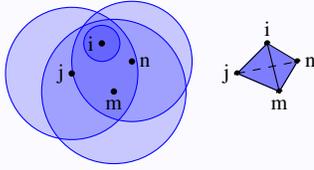


Fig. 2: Smallest cell is inside the other cells.

Look at the example from Figure 2, the cell  $i$  is the smallest cell and it is inside the others. So the four cells  $i, j, m,$  and  $n$  compose a 3-simplex  $(i, j, m, n)$ . If the first case is not satisfied, we consider the second case that the smallest cell  $v_*$  is not inside others and there exists a intersection point  $x_{ij} \in \mathbb{X}$  that is inside cell  $v_t$  for all  $0 \leq t \leq k$  and  $t \neq i, j$ . We then conclude that  $\hat{u} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$  is a  $k$ -simplex.

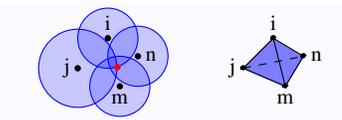


Fig. 3: One of the intersection points of cell  $i$  and cell  $j$  is inside the other cells.

In Figure 3, an intersection point of cell  $i$  and cell  $j$  is inside other cells  $m$  and  $n$ . This point is marked as a red point in Figure 3. So, we conclude that  $(i, j, m, n)$  is a 3-simplex. If both first case and second case are not satisfied, we conclude that  $\hat{u} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$  is not a  $k$ -simplex. The smallest cell  $v_*$  is not inside others and no intersection point  $x_{ij} \in \mathbb{X}$  is inside cell  $v_t$  for any  $0 \leq t \leq k$  and  $t \neq i, j$ . So there must exist  $i^*, j^*$  and  $m^*$  such that  $x_{i^*}$  and  $x_{j^*}$  are not inside the cell  $m^*$ . So  $(i^*, j^*, m^*)$  isn't a 2-simplex then it can not be part of a  $k$ -simplex with  $k \geq 2$ .

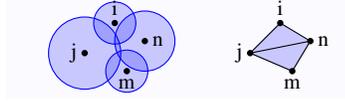


Fig. 4: No intersection point of cell  $i$  and cell  $j$  is inside the other cells.

In Figure 4, we can not find a pair of cells that one of their intersection points is inside all the other cells. In addition, both intersection points of cell  $i$  and cell  $j$  are not inside cell  $m$ . So the three cells  $i, j$  and  $m$  do not compose a 2-simplex. Then, four cells  $i, j, m$  and  $n$  do not compose a 3-simplex. The algorithm to verify a candidate if it's a  $k$ -simplex, where  $k \geq 2$ , is given here:

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**Algorithm 2** Verification a candidate of  $k$ -simplex

---

**Require:**  $\hat{u} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$  a candidate;

- 1:  $v^* \leftarrow$  the smallest cell of  $\{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$ ;
- 2: **if** cell  $v^*$  is inside cell  $\hat{v}_i$  for all  $\hat{v}_i \neq v^*$  **then**
- 3:     **return true**;
- 4: **end if**
- 5:  $\mathbb{X} = \emptyset$ ;
- 6: **for**  $i = 0 \rightarrow k$  **do**
- 7:     **for**  $j = i + 1 \rightarrow k + 1$  **do**
- 8:          $\{x_{ij}, x_{ji}\} =$  intersection points of cells  $i, j$ ;
- 9:         add  $\{x_{ij}, x_{ji}\}$  into  $\mathbb{X}$ ;
- 10:     **end for**
- 11: **end for**
- 12: **if** there exists  $x_{ij}$  inside cell  $\hat{v}_t$  for all  $t \neq j, j$  **then**
- 13:     **return true**;
- 14: **else**
- 15:     **return false**;
- 16: **end if**

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The construction of the Čech complex can be summarized as in the Algorithm 3:

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**Algorithm 3** Construction of the Čech complex

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**Require:**  $S_0$  and  $S_1$ ;

- 1:  $k = 2$ ;
- 2: **while** (1) **do**
- 3:      $S_k = \emptyset$ ; {collection of  $k$ -simplices}
- 4:     **for each**  $\hat{v}_0 \in S_0$  **do**
- 5:          $S^* \leftarrow$  a set of candidate  $\{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\}$  of  $\hat{v}_0$ ;
- 6:         **for each**  $\hat{u} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_k\} \in S^*$  **do**
- 7:             **if** verification( $\hat{u}$ ) = **true** **then**
- 8:                 add  $\hat{u}$  into  $S_k$ ;
- 9:             **end if**
- 10:         **end for**
- 11:     **end for**
- 12:     **if**  $S_k \neq \emptyset$  **then**
- 13:          $k = k + 1$ ;
- 14:     **else**
- 15:         **break**;
- 16:     **end if**
- 17: **end while**
- 18: **return** sequence of  $S_k$  for all  $k \geq 2$ ;

---

### III. COMPLEXITY

At the beginning of these algorithms, the initial Čech complex is constructed. The construction of Čech complex is a verification if a group of cells has a non-empty intersection. The 0-simplices are obviously a collection of vertices. Computing 1-simplices is the search for neighbors of each cell. Its complexity is  $C_N^2$ , where  $N$  is the number of cells. To compute 2-simplices, for each cell we take two of its neighbors and verify if this cell and the two neighbors have a non-empty intersection. Let  $n$  be the average number of neighbors of each cell, the complexity to compute 2-simplices for each cell is  $C_n^2$  on average. The complexity of the 2-simplices computation for all cells is  $NC_n^2$ . Similarly, to compute the  $k$ -simplices for each cell, we take  $k$  of its neighbors and verify if this cell and the neighbors have a non-empty intersection. The complexity of  $k$ -simplices computation of one cell is  $C_n^k$  and for all cells is  $NC_n^k$ . The complexity to construct the initial Čech complex is:  $CP(\text{initial Čech}) = C_N^2 + N \sum_{k=2}^{d_{\max}} C_n^k$ , where  $d_{\max}$  is the highest dimension of the Čech complex. The complexity to construct the Čech complex up to dimension 2 is  $O(N^2 + Nn^2)$ . The sum  $\sum_{k=2}^{d_{\max}} C_n^k$  can be approximated by  $2^n$  if  $d_{\max}$  is high enough. The complexity to construct the Čech complex up to highest dimension is  $O(N^2 + N2^n)$ .

### IV. EXPERIMENT

We have done a demonstration experiment. We deployed the cells following a Poisson point process with the density one on the square  $6 \times 6$ . The cells have radius which vary from 0.5 to 1. The Figure 5 shows the cells with the corresponding Čech complex.

In this figure, the darker color indicates the higher dimension simplex, the lighter color indicates the lower dimension simplex. In this figure, there is one coverage hole indicated by the white space surrounded by colored simplices.

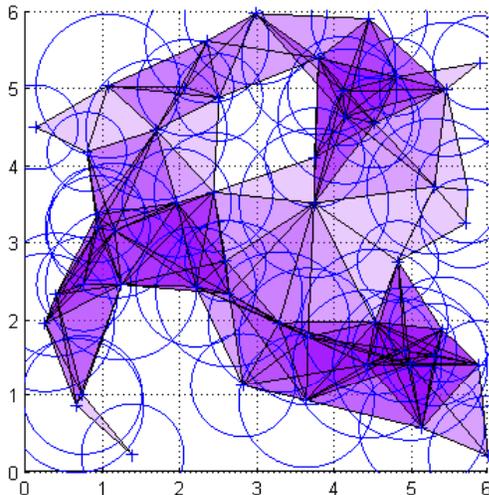


Fig. 5: Random cells and their Čech complex.

### V. CONCLUSION

In this paper, we propose an algorithm which constructs the Čech complex specified to analyze the coverage structure of wireless networks. The cells of this network can have different sizes. The complexity to build this Čech complex is polynomial.

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