Asymptotic Quantum Many-Body Localization from Thermal Disorder

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Abstract

We consider a quantum lattice system with infinite-dimensional on-site Hilbert space, very similar to the Bose-Hubbard model. We investigate many-body localization in this model, induced by thermal fluctuations rather than disorder in the Hamiltonian. We provide evidence that the Green-Kubo conductivity $\kappa(\beta)$, defined as the time-integrated current autocorrelation function, decays faster than any polynomial in the inverse temperature β as $\beta \to 0$. More precisely, we define approximations $\kappa_\tau(\beta)$ to $\kappa(\beta)$ by integrating the current-current autocorrelation function up to a large but finite time τ and we rigorously show that $\beta^{-n} \kappa_{\beta^{-m}}(\beta)$ vanishes as $\beta \to 0$, for any $n, m \in \mathbb{N}$ such that $m - n$ is sufficiently large.

1 Introduction

1.1 Localization and its characterization

The phenomenon of localization was introduced in the context of non-interacting electrons in random lattices in [1]. It is now widely accepted that in such systems, a delocalization-localization (or metal-insulator) transition occurs as the disorder strength is increased. This transition is often discussed by referring to the nature of the one-particle wavefunctions that are exponentially localized in space in the insulator, but delocalized in the metallic regime. The localized phase has been studied with mathematical rigor starting with [2], whereas for the delocalized regime, this has not been successful up to now.

The natural question how interactions modify this transition has received renewed attention lately. Both theoretical [3, 4] and numerical [5, 6] work suggests that the localization-delocalization transition persists, at least for short range interactions. When talking about models with interaction, most authors choose a model where the localization is manifest in the absence of interaction (whereas, in the original model of [1], it was a highly nontrivial result). For example, a simple model from [5] is the random-field Ising chain

$$H = \sum_{i=1}^{L} h_i S_i^z + J \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} \tag{1.1}$$

where $\boldsymbol{S}_i = (S_i^x, S_i^y, S_i^z)$ are the Pauli-matrices at site i and h_i are i.i.d. random variables with $\mathbb{E}(h_i) = 0$. We think of many-body localization as the property that a local in space excess of energy does not spread into the rest of the system. However, before formalizing this intuition, we give another possbile definition of many-body localization, used e.g. by [5, 7], in the model defined by (1.1). Let Ψ label eigenfunctions of H, then 'many-body localization' at infinite temperarure $\beta = 0$ (β is the inverse temperature) could be defined as the occurrence of the inequality

$$\lim_{L \to \infty} \frac{1}{2^L} \sum_{\Psi} \mathbb{E}_h(|\langle \Psi, S_{L/2}^z \Psi \rangle|^2) \neq \mathbb{E}_h(|\langle S_{L/2}^z \rangle_{\beta=0}|^2) = 0. \tag{1.2}$$

where $\langle \cdot \rangle_\beta$ on the right hand side refers to the thermal average and $\mathbb{E}_h(\cdot)$ refers to disorder average. Of course, one can also ask whether this inequality holds at $\beta > 0$, in which case the average over eigenfunctions $\frac{1}{2^L} \sum_\Psi$ on the left-hand side should be restricted to those eigenfunctions with an energy density corresponding to the inverse temperature β, and the right hand side does not automatically vanish. Depending on the disorder strength, the validity of (1.2) can then depend on the temperature as well. The appeal of the inequality (1.2) is that it violates the so-called Eigenstate Thermalization Hypothesis (ETH) which states that most eigenvectors of the Hamiltonian define an ensemble that is equivalent to the standard (micro)-canonical ensemble; i.e. with the notation as in (1.2), it states that, for for any $\delta > 0$, the bound

$$\left| \langle \Psi, S_{L/2}^z \Psi \rangle - \langle S_{L/2}^z \rangle_{\beta=0} \right| \leq \delta \tag{1.3}$$

is satisfied for fraction of eigenfunctions Ψ that approaches 1 as $L \to \infty$. Even though the ETH has not been proven for any interesting non-integrable system (the difficulty of doing so is related to the difficulty of proving delocalization), it has nevertheless been accepted by the theoretical physics community, starting with the works [8, 9]. It is however important to point out that the ETH also fails for ballistic systems like the ideal crystal for which there is surely no localization in the sense of non-spreading of energy excess.

There is at present no mathematical proof of many-body localization. Some progress was made for the (one-particle) Anderson model on a Cayley tree in [10], which is often quoted as a toy model for many-body localization and, recently, a proof via iterative perturbation theory for the model (1.1) was announced by [7].

As already indicated, we prefer a characterization that stresses the dynamics of energy fluctuations, and therefore we consider the Green-Kubo formula for the heat conductivity

$$\kappa(\beta) = \beta^2 \int_{-\infty}^{\infty} dt \lim_{L \to \infty} \sum_i \langle j_{L/2}(t) j_i(0) \rangle_\beta \tag{1.4}$$

where $j_i(t)$ are local energy currents at site i. Many-body localization is then understood as the vanishing of $\kappa(\beta)$. The picture underlying such a definition is that $\kappa(\beta) = 0$ means that energy excitations do not spread diffusively (or faster than diffusively) through the system. Let us bypass the question of the relation between these two characterizations of many-body localization; in the few cases where there exists up to date a convincing

argument for many-body localization, those arguments would imply $\kappa(\beta) = 0$, as well. In any case, it seems to us that the characterization via the conductivity is clearly physically relevant.

In classical mechanics, one can consider models of the same flavour: One-particle localization occurs in a chain of harmonic oscillators with random masses. Adding anharmoncity to this setup yields a model that is a candidate for many-body localization, but the expectation seems to be that these models do not exhibit strict many-body localization. However, the phenomenology can still manifest itself through the dependence of $\kappa(\beta, g)$ on the anharmonicity g. Form the works [11, 12, 13, 14], one conjectures that

$$\lim_{g \to 0} g^{-n} \kappa(g, \beta) \to 0, \qquad \text{for any } n > 0 \text{ and } \beta \sim 1. \tag{1.5}$$

In other words, the conductivity has a non-perturbative origin for small g. Below, we refer to this scenario as 'asymptotic localization'.

1.2   Thermal disorder instead of static disorder

Whereas all the models hinted at above have disorder in the Hamiltonian, this paper is concerned with the question whether one can in principle replace the disorder by thermal fluctuations, i.e. disorder due to the thermal Gibbs state. As far as we see, this question does not have any one-particle analogue but it is natural in many-body systems. Indeed, whereas disorder can model defects, it is also sometimes used as a model for slow degrees of freedom that are, in principle, influenced by the rest of the system.

The fact that randomness in the strict sense of the word is not necessary for localization had up to now been investigated by replacing the random field in the Hamiltonian by a quasi-random field, which is quite different from what we do. In the one-particle setup, this led to the study of models like the Aubry-André model [15], and recently it was argued [16] that also in the many-body setting, quasi-randomness suffices for many-body localization. To explain our setup and question, we now introduce our model. We consider a variant of the Bose-Hubbard model:

$$H = \sum_{i=1}^{L} N_i^q + g(a_i^* a_{i+1} + a_i a_{i+1}^*), \qquad q > 2 \tag{1.6}$$

where a_i, a_i^* are annihilation/creation operators of a boson at site i and $N_i = a_i^* a_i$. For $q = 2$, this model is exactly the Bose-Hubbard model. In fact, the model we study is slightly more general than (1.6) to avoid conceptual complications related to conserved quantities and nonequivalence of ensembles, see Section 2.2, however this is not relevant for the discussion here. W.r.t. the thermal state at $g = 0$, the occupations N_i behave as i.i.d. random variables whose distribution is given by

$$\text{Prob}(N_i = \eta_i) = \frac{1}{Z_0(\beta)} e^{-\beta \eta_i^q}, \qquad \text{with } Z_0(\beta) \text{ a normalizing constant} \tag{1.7}$$

We split our Hamiltonian as

$$H = H^{(0)} + H^{(1)}, \qquad \text{with} \quad H^{(0)} = \sum_x N_x^q \tag{1.8}$$

and we treat $H^{(1)}$ as a perturbation of $H^{(0)}$. Intuitively, a perturbative analysis is possible, if for a pair of eigenstates η, η' of $H^{(0)}$, we have the non-resonance condition

$$|\langle \eta, H^{(1)} \eta' \rangle| \ll |\langle \eta, H^{(0)} \eta \rangle - \langle \eta', H^{(0)} \eta' \rangle| \tag{1.9}$$

Since the distance between consecutive eigenvalues (level spacing) of the operator N_x^q grows roughly as N_x^{q-1} and the matrix elements of $H^{(1)}$, locally at site x, grow as N_x (since they are quadratic in the field operators), the condition (1.9) seems satisfied for most pairs η, η' if $q > 2$, that is, with high probability w.r.t. the probability measure (1.7) when β is sufficiently small. This is the basic intuition why this model should exhibit some localization effect at high temperature[1]. However, because of the many-body setup, it is not straightforward that the above claims make sense. In particular, it is certainly *false* that one could apply perturbation theory directly to the eigenstates η of $H^{(0)}$. Indeed, since the number of eigenstates should be thought of as C^L and the range of energies has width CL, the level spacing (difference between nearest levels) vanishes fast as $L \to \infty$.

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[1]One should not confuse this with the situation at $\beta = \infty$, where one expects a quantum phase transition between a conducting superfluid phase and an insulating Mott phase. This has nothing to do with our results.

Therefore, the locality of the operators is a crucial issue that should be used in making the above heuristics precise. In practice, we do this by restricting the condition (1.9) to a small region: Instead of having resonant and non-resonant configurations η, we will now assign to any η 'resonance spots', i.e. regions where the condition (1.9) fails for at least one of the nearby configurations η'. What exactly is a 'nearby' configuration, is determined by the order of perturbation theory that we consider.

Up to now, the heuristic reasoning is in fact no different from the one that would develop for the disordered Ising chain, except that we replaced the 'disorder distribution' by 'distribution in the uncoupled Gibbs state'. The difference kicks in when one realizes that the non-resonance condition is not static but it can change as the dynamics changes the occupations η. Therefore, it is not sufficient to argue that resonant spots are sparse, but one should investigate the dynamics of these resonance spots and exclude that this dynamics induces a current. The most intuitive part of this issue takes the form of a question in graph theory: The vertices of the graph are the configurations η and the edges are pairs of configurations that satisfy some resonance condition. If the connected components of this graph are small, i.e. they typically consist of a few configurations, then this hints at localization. The main problem to be overcome in the present article is to show that, indeed, typical graphs decompose into many small disconnected components. Our analysis is however only valid in the limit $\beta \to 0$, and for this reason, we do not know yet, even at an heuristic level, whether our model exhibits many-body localization in the strict sense, that is, whether the conductivity $\kappa(\beta) = 0$ for $\beta < \beta_c$ with $\beta_c > 0$, or whether the localization is only asymptotic as in (1.5), i.e.

$$\lim_{\beta \to 0} \beta^{-n} \kappa(\beta) = 0, \qquad \text{for any } n > 0 \tag{1.10}$$

In this paper, we give a strong indication why at least (1.10) should hold, even in higher dimensions $d > 1$, see Theorem 2.1. This is done by approximating the current-current correlation function by truncation at times that grow like an arbitrary polynomial in β^{-1} and proving (1.10) for these approximations.

Such a reasoning was developed earlier in [14] for disordered classical systems, and in [17], for classical systems where the setup is analogous to the present paper, i.e. disorder is replaced by thermal fluctuations.

1.3 Outline of the paper

In Section 2, we introduce the model in precise terms and we state our results. Section 4 deals with the iterative diagonalization of our Hamiltonian, excluding the resonant configurations (see explanation above). The sum of all terms that were not treated by iterative diagonalization is called 'the resonant Hamiltonian', indicated by the symbol Z. Sections 5 and 6 contain the analysis of the resonant Hamiltoninian Z. As such, they are fully independent of Section 4 and they form the main part of our work. In Section 7, we finally combine the results of Section 4 with the analysis of Sections 5 and 6 to prove our results. In the appendix, we establish exponential decay of correlations at small β for our model.

2 Model and result

2.1 Preliminaries

Let $\Lambda \subset \mathbb{Z}^d$ be a finite set. We define the Hilbert space

$$\mathcal{H} := \otimes_{x \in \Lambda} \ell^2(\mathbb{N}) \sim \ell^2(\mathbb{N}^\Lambda), \tag{2.1}$$

i.e. at each site there is an infinite-dimensional 'spin'-space. For an operator O acting on $\mathcal{H}_\Lambda$ we denote by $s(O)$ ('support' of O) the minimal set A such that $O = O_A \otimes \mathbb{1}_{\Lambda \setminus A}$ for some O_A acting on $\mathcal{H}_A$, and $\mathbb{1}_{A'}$ the identity on $\mathcal{H}_{A'}$ for any $A' \subset \Lambda$. We do not distinguish between O_A and O, and we will denote them by the same symbol.

Let a, a^* be the bosonic annihilation/creation operators on $\ell^2(\mathbb{N})$:

$$(af)(n) = \sqrt{n+1} f(n+1), \qquad (a^* f)(n+1) = \sqrt{n+1} f(n), \qquad \text{for } n \in \mathbb{N} \tag{2.2}$$

We write a_x, a_x^* for the annihilation/creation operators acting on site x, and, as announced above, we do not distinguish between a_x and $a_x \otimes \mathbb{1}_{\Lambda \setminus \{x\}}$. We also define the number operators

$$N_x := a_x^* a_x \tag{2.3}$$

The vectors diagonalizing the operators N_x play a distinguished role in our analysis. For a finite set A, we define the *phase space* $\Omega_A := \mathbb{N}^A$ with elements

$$\eta = (\eta(x))_{x \in A}, \qquad \eta(x) \in \mathbb{N} \tag{2.4}$$

such that $\mathcal{H}_A \sim \ell^2(\Omega_A)$ and we often use η as a label for the function δ_η i.e. $\delta_\eta(\eta') = \delta_{\eta,\eta'}$ for $\eta', \eta \in \Omega_A$.

2.2 Hamiltonian

We introduce the Hamiltonian of our model in finite volume $\Lambda \subset \mathbb{Z}^d$ and with free boundary conditions;

$$H_\Lambda = \sum_{x \in \Lambda} N_x^q + g_1(a_x + a_x^*)^2 + \sum_{x, x' \in \Lambda, x \sim x'} g_2(a_x^* a_{x'} + a_x a_{x'}^*) \tag{2.5}$$

where $N_x = a_x^* a_x$, $x \sim x'$ means that x, x' are nearest neighbours, and the exponent $q > 2$. The term $g_1(a_x + a_x^*)^2$ destroys the conservation of the total occupation number $\sum_{x \in \Lambda} N_x$. In the sequel, we will assume that $g_1, g_2 \sim 1$, so that total energy is the only conserved quantity. Nonetheless, all our results remain valid when g_1 or g_2 vanish. The reason why we find it important to destroy the second conserved quantity is that similar models with two conserved quantities typically exhibit non-equivalence of ensembles. As explained in [18], one expects in a microcanonical ensemble equilibrium states where a macroscopic part of the particles (the total number of particles would correspond to $\sum_{x \in \Lambda} N_x$ in our model) is concentrated on a single lattice site. We want to stress that this type of 'statistical localization' has nothing in common with the localisation mechanism in the present paper.

To avoid constants later on, we demand that $|g_1|, |g_2| \leq 1$. By standard methods (e.g. Kato-Rellich), one checks that H_Λ generates a unitary group e^{-itH_Λ} and we write

$$O(t) = e^{itH_\Lambda} O e^{-itH_\Lambda}, \qquad O \in \mathcal{B}(\mathcal{H}_\Lambda) \tag{2.6}$$

In the formula above, we assumed the observables O to be bounded operators, but this will not always be the case throughout the paper. Let $\mathcal{D}_\Lambda := \mathrm{Dom}(\sum_{x \in \Lambda} N_x^q)$, i.e. the domain of the unperturbed Hamiltonian. Then, the Kato-Rellich theorem implies that $\mathcal{D}_\Lambda = \mathrm{Dom}(H_\Lambda)$ and in particular $e^{-itH_\Lambda}\mathcal{D}_\Lambda = \mathcal{D}_\Lambda$. Let now O be an unbounded operator on $\mathcal{H}_\Lambda$ for which $\mathcal{D}_\Lambda$ is a core. For example, this could be any O that is a polynomial in creation/annihilation operators with degree not bigger than $2q$. Then $O(t)$ as defined by (2.6) is also closable on $\mathcal{D}_\Lambda$. This observation suffices to give a meaning to all time-evolved operators $O(t)$ in the article, and to their sums and integrals.

2.3 States

The thermal equilibrium state $\omega_{\beta,\Lambda}$ of the system at inverse temperature β and in finite volume Λ is defined as

$$\omega_{\beta,\Lambda}(O) = \frac{\mathrm{Tr}\, O e^{-\beta H_\Lambda}}{\mathrm{Tr}\, e^{-\beta H_\Lambda}}, \qquad O \in \mathcal{B}(\mathcal{H}_\Lambda) \tag{2.7}$$

We are interested in the high-temperature regime, where the finite-volume states $\omega_{\beta,\Lambda}$ have a unique infinite-volume limit (for, say, $\Lambda \nearrow \mathbb{Z}^d$ in the sense of Van Hove), independent of boundary conditions. Morally speaking, this results belongs to standard knowledge, but, literally, it does not, because of the infinite one-site Hilbert space. In principle, we deal with this issue in the appendix, but, since we in fact only need exponential decay of correlations, uniformly in Λ, we will not explicitly address the construction of the infinite volume state. We drop the volume Λ and inverse temperature β from the notation for the time being, writing simply $\omega(\cdot)$. It is understood that sums over x, x' are always restricted to the volume Λ.

2.4 Currents

We fix once and for all the vector $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$ and we study the current in this direction. First, we decompose the Hamiltonian as

$$H = \sum_x H_x \tag{2.8}$$

where

$$H_x = N_x^q + g_1(a_x + a_x^*)^2 + \tfrac{1}{2} \sum_{x' : x' \sim x} g_2(a_x^* a_{x'} + a_x a_{x'}^*) \tag{2.9}$$

We define local current operators J_x by

$$J_x = \mathrm{i} \sum_{x' : x_1' > x_1} [H_{x'}, H_x] \tag{2.10}$$

Since the operators H_x act on at most $2d+1$ sites, all x' that contribute a nonzero term to the sum in (2.10) are nearest neighbours of x. One way to convince oneself that this is a meaningful definition is to consider first the total current through the (restriction of a) hyperplane $\mathbb{H}_a = \{x \in \Lambda : x_1 = a\}$ as the time-derivative of the total energy to the left of this hyperplane, i.e.

$$J_{\mathbb{H}_a} := \mathrm{i}[H, H^{(\mathrm{L})}] = \frac{\mathrm{d}}{\mathrm{d}t} H^{(\mathrm{L})}(t)\big|_{t=0}, \qquad \text{with } H^{(\mathrm{L})} = \sum_{x : x_1 \le a} H_x \tag{2.11}$$

Then it follows that

$$J_{\mathbb{H}_a} = \sum_{x : x_1 = a} J_x \tag{2.12}$$

Note that, by the time-invariance of the equilibrium state, $\omega(O(t)) = \omega(O)$, we have

$$\omega(J_{\mathbb{H}_a}) = 0 \tag{2.13}$$

2.5 Green-Kubo formula

To study the Green-Kubo formula, we introduce an empiric average of the local current over space and time:

$$\mathcal{J}_\tau = \frac{1}{\sqrt{\tau |\Lambda|}} \int_0^\tau \mathrm{d}t \sum_x J_x(t) \tag{2.14}$$

where the scaling anticipates a central limit theorem, relying on the fact that the equilibrium expectation of $\mathcal{J}_\tau$ vanishes:

$$\omega(\mathcal{J}_\tau) = 0. \tag{2.15}$$

This follows directly from (2.13) by using the decomposition $\sum_x = \sum_a \sum_{x : x_1 = a}$. We introduce the finite-time conductivity

$$\kappa_\tau(\beta) = \beta^2 \lim_{\Lambda \nearrow \mathbb{Z}^d} \omega(\mathcal{J}_\tau^* \mathcal{J}_\tau) \tag{2.16}$$

A basic intuition in transport theory states that in systems with normal (diffusive) transport, the current-current correlations decay in an integrable way, resulting in the convergence of the finite-time conductivity to the conductivity $\kappa := \lim_{\tau \to \infty} \kappa_\tau$ with $0 < \kappa < \infty$. At present, this has however not been proven in any interacting Hamiltonian system. Instead, we study the behaviour of the approximants κ_τ for arbitrarily large τ (polynomial in β^{-1}) and we show that at all such times, the conductivity vanishes:

Theorem 2.1 (Conductivity in small β limit). *There is a $C > 0$ such that for any $0 < n < m - C$,*

$$\lim_{\beta \to 0} \beta^{-n} \kappa_{\beta^{-m}}(\beta) = 0 \tag{2.17}$$

As already explained in the introduction, we take this result as a strong indication that also

$$\lim_{\beta \to 0} \beta^{-n} \kappa(\beta) = 0, \qquad \text{for any } n > 0 \tag{2.18}$$

To make this precise, we should understand what type of processes dominate the dynamics after very long times, i.e. superpolynomial in β^{-1}. In [17], we argued for models of classical mechanics that in the case that the dynamics becomes chaotic at such large times, the conjecture (2.18) is definitely true. This was done by introducing an energy-conserving stochastic term in the dynamics of arbitrarily small strength and proving that the conductivity (which in that case can be shown to be finite) has the same order of magnitude as the stochastic term. This is not attempted in the present paper. On the other hand, without such a stochastic term, it remains an enormous task to prove that the conductivity is even finite and nonzero, see for example [19] for an exposition of this problem.

An alternative way to view our results, is to compare them to Nekhoroshev estimates in classical systems. Such estimates typically establish results very reminiscent of ours, but they are restricted to a finite number of degrees of freedom. We refer to [17] for a more thorough discussion of this point and for relevant references.

2.6 Splitting of the current

From a technical point of view, the key result in this paper is a splitting of the current $J_{\mathbb{H}_a}$ into an oscillatory part and a small part. To describe it, let us introduce a multi-dimensional strip (whose width is called $2r^2$) containing the hyperplane $\mathbb{H}_a$;

$$\mathbb{S}_{a,r^2} = \{x \in \Lambda : |x_1 - a| < r^2\} \tag{2.19}$$

and we often drop the parameters by simply writing $\mathbb{S} = \mathbb{S}_{a,r^2}$.

Theorem 2.2 (Splitting of current). *For any $r > 0$, and sufficiently small β, depending on r, the following holds uniformly in the volume Λ and the choice of a: There are collections of operators $(O_A)_{A \subset \mathbb{S}_{a,r^2}}, (I_A)_{A \subset \mathbb{S}_{a,r^2+2}}$, such that*

$$J_{\mathbb{H}_a} = \sum_{A \subset \mathbb{S}_{a,r^2}} \mathrm{i}[H, O_A] + \sum_{A \subset \mathbb{S}_{a,r^2+2}} I_A \tag{2.20}$$

and

1. *The operators O_A and I_A are supported in A, i.e. $s(I_A), s(O_A) \subset A$, and $O_A = I_A = 0$ whenever A is not connected.*

2. *O_A and I_A have zero average: $\omega(O_A) = \omega(I_A) = 0$*

3. *They are bounded as*

$$\omega(O_A^* O_A) \le C(r)\beta^{-C+c(r)|A|}, \qquad \omega(I_A^* I_A) \le C(r)\beta^{-C+cr+c(r)|A|} \tag{2.21}$$

Here, C, c denote constants with $C < \infty, c > 0$ that depend only on the dimension d, and the exponent q. The parameters $C(r), c(r)$ can additionally depend on r.

The relevance of this theorem in establishing asymptotic energy localization is explained in more details below.

3 Overview of the method

Before embarking into the proof of our results, let us informally describe the main steps leading to them. Let us first observe that Theorem 2.1 is readily deduced from Theorem 2.2, as detailed in Section 7.5. Indeed, to start with, the first sum in the right hand side of (2.20) just represents local energy oscillations; the contribution of such an oscillation to the current (2.14) is given by

$$\frac{1}{\sqrt{\tau}} \int_0^\tau i[H, O](t)\mathrm{d}t = \frac{O(\tau) - O(0)}{\sqrt{\tau}} \quad \to \quad 0 \qquad \text{as} \qquad \tau \to \infty.$$

Next, the terms in the second in sum in the right hand side of (2.20) possibly contribute to the conductivity, but are very small in the Hilbert-Schmidt norm $\|I_A\|_\omega := \omega(I_A^* I_A)^{1/2}$ based on the thermal state. In fact, they are seen to decay as an arbitrary large power in β, if r is taken large enough, thanks to the presence of the term 'cr' in the exponent of the bound in (2.21). Fianally, the terms $c(r)|A|$ in the exponents in (2.21) ensure that we can perform sums over the connected sets A.

We can thus now focus on the derivation of Theorem 2.2. Let us start by explaining the origin of the oscillatory term in (2.20). For the sake of the argument, let us consider a strongly localized solid. So we imagine that the unitary change of basis U that diagonalizes H is written as $U = \mathrm{e}^{-K}$, where the anti-hermitian matrix K is a sum of almost local terms (see Section 4.2 for a precise definition of what almost local means). The diagonalized Hamiltonian Δ then takes the form

$$\Delta = U^\dagger H U = \sum_x \Delta_x = \sum_x \left\{ f_1(N_x) + f_2(N_x, N_{x+1}) + f_3(N_{x-1}, N_x, N_{x+1}) + \ldots \right\} \tag{3.1}$$

where the terms f_k quickly decay to 0 as $k \to \infty$ (we took $d = 1$ in this formula to simplify writings). We now could say that $H^{(L)}$ defined in (2.11) was the naive left part of the total energy. We define

$$\widetilde{H}^{(L)} = U\Delta^{(L)}U^\dagger \qquad \text{with} \qquad \Delta^{(L)} = \sum_{x:x_1 \le a} \Delta_x.$$

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But then, from (2.11), we find

$$J_{\mathbb{H}_a} \; = \; i[H, H^{(L)}] \; = \; i[H, H^{(L)} - \widetilde{H}^{(L)}] + i[H, \widetilde{H}^{(L)}]. \tag{3.2}$$

On the one hand, the locality properties of U allow to conclude that $H^{(L)} - \widetilde{H}^{(L)}$ is localized near the hyperplane $\mathbb{H}_a$, so that the first term in this last equation may be identified with the first sum in the right hand side of (2.20). On the other hand $[H, \widetilde{H}^{(L)}]$ would here vanish. In reality, we will however not be able to fully diagonalize H, so that a "rest term" $\sum_A I_A$ appears in (2.20). Technically, this step consisting in deriving Theorem 2.2 once the change of basis U and the opertaor Δ are known, is performed in Sections 7.2-7.4. This leads to heavy computations, as the operator Δ that we manage to obtain is far less simple than (3.1); this issues from both conceptual (resonances) and technical questions (high energies).

We now need to find a change of variable U that will remove as much oscillations as possible, and then analyze the Hamiltonian in the new basis. The construction of the change of basis is performed in Section 4 (the notation U does not appear yet in Proposition 4.3; it only shows up in Section 4.5 when we restrict our attention to finite volumes). As already stressed in the introduction, the interaction between particles can be treated as a perturbation at high temperature, thanks to the choice $q > 2$ in the Hamiltonian (1.6): resonances are only met in some exceptional places in the solid (see figure 1). To make this a bit more transparent at this level of the discussion, we can rewrite H given by (1.6) as

$$H \; = \; E_0 + \tilde{g}V \qquad \text{with} \qquad E_0 \; = \; \sum_x N_x^q, \qquad \tilde{g} = \beta^{1-2/q}g, \qquad V \; = \; \beta^{-1+2/q}\sum_x (a_x^* a_{x+1} + a_x a_{x+1}^*).$$

With these notations, both typical self-energy differences and terms in V are of order $\beta^{-(1-1/q)}$, so that g' is indeed a perturbative parameter. We will however not explicitly make use of these notations in the proofs.
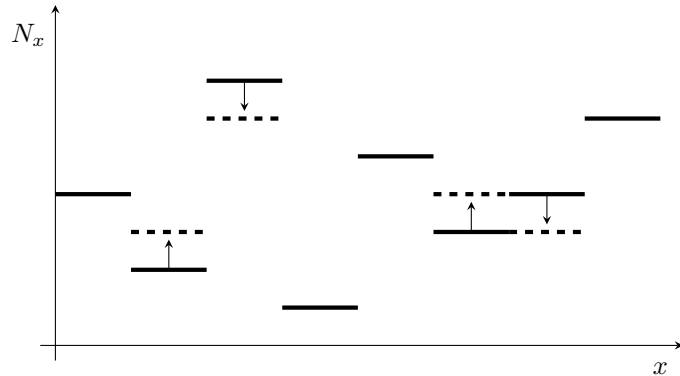

Figure 1: Resonances in the first order in perturbation. For simplicity we assume $d = 1$. The interaction on the left is typical at high temperature, and non-resonant, as the self-energy difference is much larger than the interaction energy: $\left(N_x^q + N_{x+1}^q\right) - \left((N_x + 1)^q + (N_{x+1} - 1)^q\right) \; >> \; g\sqrt{N_x(N_{x+1} - 1)}$. The interaction on the right is rare and resonant: the self-energy difference even vanishes in this case.

We construct U via an iterative KAM-like scheme, recently developed by Imbrie and Spencer [7] in the contex of quenched disordered systems. Naively, the scheme works as follows. In a first step, we determine U so that $H' := U^\dagger H U$ takes the form $H' = E_0' + \tilde{g}^2 V'$, for some new self energy $E_0' = E_0 + \mathcal{O}(\tilde{g}^2)$ and some new perturbation V'. For this, we write $U = \mathrm{e}^{-K}$ and, assuming that K is a sum of local terms of order $\tilde{g}$, we expand $U^\dagger H U$ in powers of $\tilde{g}$:

$$U^\dagger H U \; = \; \mathrm{e}^K(E_0 + \tilde{g}V)e^{-K} \; = \; E_0 + \left(\tilde{g}V + [K, E_0]\right) + \mathcal{O}(\tilde{g}^2). \tag{3.3}$$

Writing $V = \sum_x V_x$ and $K = \sum_x K_x$, we get rid of the first order in $\tilde{g}$ by setting

$$\langle\eta|K_x|\eta'\rangle \; = \; \tilde{g}\,\frac{\langle\eta|V_x|\eta'\rangle}{E_0(\eta) - E_0(\eta')} \qquad \Rightarrow \qquad \tilde{g}V + [K, E_0] \; = \; 0 \tag{3.4}$$

($\langle\eta|K_x|\eta'\rangle = 0$ if $\langle\eta|V_x|\eta'\rangle = 0$ by definition). The fact that resonances are rare precisely means that for most of the pairs of states η and η', the matrix element $\langle\eta|K_x|\eta'\rangle$ is well defined and of order $\tilde{g}$. Let us ignore resonances for the moment. We would then conclude that the expansion (3.3) was justified. So, we would also have determined a renormalized Hamiltonian H' with a perturbation of order $\tilde{g}^2$. This strategy could then be iterated, constructing a sequence of change of variables $e^{K^{(j)}}$, where $K^{(j)}$ is a sum of local terms of order $\tilde{g}^{2^{j-1}}$. Doing so, we would readily conclude that H is strongly localized in the sense of (3.1).

As an aside, let us observe that it is only possible to find K so that (3.4) holds if the pertubation V has no diagonal element. This explains that in principle we need to renormalize the self-energy ($E_0' = E_0 + \mathcal{O}(\tilde{g}^2)$ as mentionned), in order to absorbe all the diagonal part of the new Hamiltonian. It is however not what we do in practice, as we simply treat these extra diagonal elements as resonances. However, in order to investigate possible true localization in translation-invariant models, there would be a deeper reason to take account of the self-energy renormalization. Indeed, if this phenomenon is ignored, it is readily seen that, for fixed β, resonances would eventually become typical as one goes on in perturbation. Since, at higher orders, atoms may exchange more than one quantum of energy, the interaction could now just swap the levels of any two near atoms (whereas at the first order this was only possible if the energies were nearly the same as depicted on the right interaction of figure 1). So all particles would be in resonance with their neighboors, allowing energy to travel into the solid. On the other hand, such a drastic conclusion could not be reached if the renormalization of the self-energy was taken into account. It has in fact been suggested in [?] that this effect could guarantee that resonances rarefy as one moves to higher orders. To support this view, we indeed observe that the perturbative splitting of the levels could and should be exploited to show localization in the one-body Anderson model when the disorder only takes a finite numner of values, a model for which localization is clearly expected to hold.

Let us come back to the description of the scheme initiated in (3.3). It is clear that resonances, even if very rare, cannot just be ignored as we pretended up to now. We just do as much as we can: the perturbation V is splitted into a resonant and non-resonant term (see (4.30)), and (3.4) is only solved with V replaced by the non-resonant part of V. While the change of variables $e^{K^{(j)}}$ are now well defined and enjoy good decay properties, this replacement comes with a price. A first, technical, consequence is that the speed of the iteration procedure is much slowed down. Indeed, in this version of the scheme, we just let the resonant part as it is, so that at each step, resonant terms of order $\tilde{g}$ are present in the perturbation. Though they do not create any trouble as such, it is seen that, itarating the scheme once more, non-resonant terms are generated that would be too large for a superexponential bound like $\tilde{g}^{2^{j-1}}$ to survive. Instead, we can only obtain that $K^{(j)}$ is a sum of terms of order $\tilde{g}^j$ (so we do not progress faster than in usual perturbation theory).

The true problem is however that, after a large but finite number of iterations, we are left with a Hamiltonian containing still a perturbation of order $\tilde{g}$ (see the term $\mathscr{G}^{(r)}$ in (4.31)), and, later on, the resonant Hamiltonian Z defined in (5.1)). The resonant Hamiltonian is well sparse, but not as much as needed to get our results: a look at figure 1 shows indeed that the probability of two atoms to be resonant is at best bounded by $\beta^{1/q}$. Before indicating how we will get off the hook, let us stress here that the analysis of resonances reveals a fundamental difference between quenched and thermal disorder.

To see this, let us for example consider the first order resonances in a quenched disorder spin chain, as studied by [5] [7]. In this model, it is possible determine bonds on the lattice such that resonances can only occur on these bonds. Moreover, if the disorder is strong enough, these potentially resonant bonds form small isolated islands. In this case, it is then in fact possible to completely get rid of the resonant Hamiltonian at each step of the procedure. Indeed, one can diagonalize the Hamiltonian "on the resonant islands", meaning that we conjugate it with a change of basis that affects only the terms in H that act inside the islands. This rotation is non-perturbative, but does not entail any delocalization, as the resonant spots do not percolate. At the opposite, in the translation invariant set-up, it is no longer possible to visualize resonances on the physical lattice. Instead, we directly need to analyze a percolation problem in the full set of states (it should however be noticed that the eigenstates of the resonant Hamiltonian could still be localized even in the presence of a giant percolation cluster, but we are not aware of any convincing argument supporting this view). This is a rather delicate problem, illustrated on figure 2.

We will not attend to diagonalize the resonant Hamiltonian. Instead, the total energy will be separated into a left and right part, in a state dependent way, by a surface close to $\mathbb{H}_a$ that "slaloms" between the resonances (see Section 6). We so will come into a situation described again by (3.2): the second term in the right hand side of this equation will now be sparse enough current, while the first term still is just an oscillation.

To see how to define this surface, we need to analyze the motion of resonances (see Section 5). Let us first restrict the Hamiltonian to a large but fixed volume V around a point on $\mathbb{H}_a$ (a volume that will not be sent to
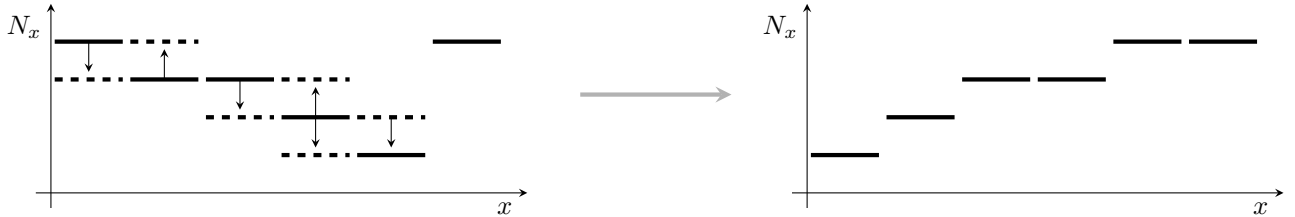
Figure 2: In translation invariant chains, resonances do travel into the system. If second neighboor hopping is allowed (what anyway occurs in second order in perturbation), the time evolution of the state on the left under the dynamics generated by the resonant Hamiltonian can have an overlap with the state on the right. We see that the most right atom can enter in resonance with the other ones, though it was not initially so.

infinity). We show the following. Let us pick up a state η in V, and let us collect all the other states in V that could have an overlap with the time evolution of η under the dynamics generated by the resonant Hamiltonian. We show that for an overwhelming majority of states η, there exists small isolated islands in V such that any of the state that we have collected, only differ from η on these islands. The set of states for which this does not hold is small enough to be neglected. On the one hand, we can convince ourselves of the validity of this statement by looking at figure 1. To simplify, let us assume that resonances are first order, and only occur when two levels are swapped as it is the case for the interaction on the right. Then on that example, it is seen that the only resonant island is located on the sites 5,6,7, assuming that particles have been labeled from 1 to 8. On the other hand, a look at figure 2 hints that this statement could be violated if V was sent to infinity for fixed β. Indeed, as the volume gets larger and larger, configurations that are rare locally, eventually occur. It is thereofre concivable that a big resonant spot starts invading the full space, connecting configurations that would have remained separted if the perturbation was confined to the volume V.

So we have found a way to construct the surface close to $\mathbb{H}_a$ in the volume V, but this is not completely satisfactory as we take the thermodynamic limit $\Lambda \to \infty$ before sending $\beta \to 0$. Two issues are raised. First, if the dimension is larger than one, we may take a volume V around each point in $\mathbb{H}_a$ and construct a piece of surface in each of these volumes, but we then have to glue them together. Second, even in one dimension, where the surface just reduces to a single point, we must analyze what extra-current is produced if the Hamiltonian is now defined on the full space. Let us bypass here the first question, that leads to intricate constructions (see Section 6), as the second one appears to us as more fundamental. We actually observe that the set of states for which an extra current is produced when reintroducing the interaction at the border is extremely small. Indeed, a non zero current could only be created if a small energy change at the border, induced by the perturbation, could completely modify the island picture up to the center of V. However, in most cases, the configuration of the islands is far less fragile: a very atypical configuration would be required for a single change at the border to propagate in the bulk of V (too few atoms appear on figure 1 to see this neatly, but one can be readily become convinced by adding a few sites). We thus see that the current is indeed very sparse.

This summarizes most of the conceptual points addressed in this article.

4 Perturbative diagonalization of H

In this section, we introduce the formalism of interaction potentials and we implement an iterative diagonalization scheme, acting on interaction potentials.

4.1 Energy cutoff

In our analysis, we find it convenient to introduce a high-energy cutoff, even though, in principle, the main reasoning of the paper is the more applicable, the higher the energy. Given a number $M > 0$ and an operator O with finite range $s(O)$, we set

$$\mathcal{P}_{\leq M}(O) := \left(\underset{x \in s(O)}{\otimes} \chi(N_x \leq M) \right) O \left(\underset{x \in s(O)}{\otimes} \chi(N_x \leq M) \right) \tag{4.1}$$

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and, analogously, we define $\mathcal{P}_{>M}(O)$ by replacing $N_x \leq M$ by $N_x > M$. Note that in general, $O \neq \mathcal{P}_{>M}(O) + \mathcal{P}_{\leq M}(O)$. The cutoff will be chosen, at the end of the analysis, to be $M = \beta^{-(1+\gamma_c)/q}$, for some small $\gamma_c > 0$

4.2 Interaction potentials

The Hamiltonian H is strictly local, i.e. it is a sum of terms that act on at most two lattice sites. When performing an iterative diagonalization, this will no longer be true and hence we first introduce a weaker notion of locality by introducing *interaction potentials*.

Definition 4.1. *An interaction potential $\mathscr{A}$ is a map from finite, connected sets $A \subset \mathbb{Z}^d$ to bounded operators $\mathscr{A}(A)$ on $\mathcal{H}_A$. A Hamiltonian in finite volume V associated to a potential $\mathscr{A}$ is defined by*

$$X_V(\mathscr{A}) = \sum_{A \text{ connected}: A \subset V} \mathscr{A}(A) \tag{4.2}$$

For simplicity, we henceforth assume that, for any interaction potential $\mathscr{A}$, $\mathscr{A}(A) = 0$ if A is not connected and we omit the restriction to connected A from sums like (4.2).

In the literature, one almost always uses the notation $H_V(\mathscr{A})$ but we have chosen $X_V(\mathscr{A})$ to avoid confusion with the Hamiltonian H_Λ defined in (2.5). Obviously, the denomination 'Hamiltonian' is a misnomer in case the operators $\mathscr{A}(A)$ are not Hermitian. For a potential $\mathscr{A}$, we define the cutoff potential

$$(\mathcal{P}_{\leq M}(\mathscr{A}))(A) := \mathcal{P}_{\leq M}(\mathscr{A}(A)) \tag{4.3}$$

and analogously for $\mathcal{P}_{>M}(\mathscr{A})$. An important example of a potential is the potential $\mathscr{E}$ specifying our model Hamiltonian itself, with an energy cutoff. It is defined by

$$\mathscr{E}(A) := \begin{cases} \mathcal{P}_{\leq M}(H_x) & \text{if } A = \{x' \in \Lambda : |x' - x| \leq 1\} \text{ for some } x \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

We also define the potential of the free Hamiltonian

$$\mathscr{E}_0(\{x\}) = \mathcal{P}_{\leq M}(N_x^q), \qquad \text{and} \qquad \mathscr{E}_0(A) = 0, \quad \text{whenever} \quad |A| > 1. \tag{4.5}$$

so that indeed

$$X_\Lambda(\mathscr{E}) = \sum_{x \in \Lambda} \mathcal{P}_{\leq M}(H_x), \qquad X_\Lambda(\mathscr{E}_0) = \sum_{x \in \Lambda} \mathcal{P}_{\leq M}(N_x^q). \tag{4.6}$$

Note however that other choices are possible for $\mathscr{E}$; different potentials can define the same Hamiltonian.

4.2.1 Norms

Note that interaction potentials form a linear space under the addition $(\mathscr{A} + \mathscr{A}')(A) := \mathscr{A}(A) + \mathscr{A}'(A)$. We introduce a family of suitable norms on interaction potentials, based on the following weighted operator norms: For an operator O on $\mathcal{H}_A$, we define an associated operator $\breve{O}$ on $\mathcal{H}_A$ by

$$\langle \eta, \breve{O}\eta' \rangle := |\langle \eta, O\eta' \rangle|, \qquad \eta, \eta' \in \Omega_A \tag{4.7}$$

such that, in particular, $\|O\| \leq \|\breve{O}\|$ where $\|\cdot\|$ is the standard operator norm. Further, for $\kappa > 1$, we set

$$\|O\|_\kappa := \sup_{\substack{w \in \mathbb{R}_+^A \\ \kappa^{-1} \leq w(x) \leq \kappa}} \|w^N \breve{O} w^{-N}\|, \qquad \text{with } w^N = \prod_{x \in A} w(x)^{N_x} \tag{4.8}$$

For $\kappa = 1$, we define simply $\|O\|_1 := \|O\|$ and we note that

$$\|O\|_{\kappa'} \leq \|O\|_\kappa, \qquad \text{for } 1 \leq \kappa' \leq \kappa \tag{4.9}$$

Note that these definitions are independent of A provided $s(O) \subset A$. For $\kappa > 1$, the $\|\cdot\|_\kappa$-norm penalizes off-diagonal elements in the number basis. The corresponding class of norms on interaction potentials is

$$\|\mathscr{A}\|_{\kappa_1, \kappa_2} := \sup_{x \in \mathbb{Z}^d} \sum_{A: A \ni x} \kappa_1^{|A|} \|\mathscr{A}(A)\|_{\kappa_2}, \qquad \|\mathscr{A}\|_\kappa := \|\mathscr{A}\|_{\kappa, \kappa} \tag{4.10}$$

There is no compelling reason to consider $\kappa_1 = \kappa_2$, but we often do so for reasons of simplicity.

4.3 Operations on interaction potentials

Given two interaction potentials $\mathscr{A}, \mathscr{B}$ we define a new potential

$$[\mathscr{A}, \mathscr{B}](A) := \sum_{A_1, A_2 : A_1 \cup A_2 = A} [\mathscr{A}(A_1), \mathscr{B}(A_2)] \tag{4.11}$$

and we note that every term in the sum on the right hand side vanishes unless $A_1 \cap A_2 \neq \emptyset$. In particular, if $\mathscr{A}, \mathscr{B}$ assign zero to every non-connected set A, then so does $[\mathscr{A}, \mathscr{B}]$. The motivation for this definition is of course that, for any volume V

$$X_V([\mathscr{A}, \mathscr{B}]) = [X_V(\mathscr{A}), X_V(\mathscr{B})] \tag{4.12}$$

Often, we prefer to use the notation

$$\mathrm{ad}_{\mathscr{A}}(\mathscr{B}) = -\mathrm{ad}_{\mathscr{B}}(\mathscr{A}) = [\mathscr{A}, \mathscr{B}] \tag{4.13}$$

If one imagines that $\mathrm{i} X_V(\mathscr{A})$ is an anti-Hermitian operator and hence that it generates a time evolution, then one might ask how this time-evolution affects a potential $\mathscr{B}$. To address such questions, we define (for the moment as a formal series)

$$\mathrm{e}^{\mathrm{ad}_{\mathscr{A}}}(\mathscr{B}) := \sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}_{\mathscr{A}}^n(\mathscr{B}) \tag{4.14}$$

Provided this series converges (in one of the norms $\|| \cdot \||_{\kappa}$), we can conclude that

$$X_V(\mathrm{e}^{\mathrm{i} \mathrm{ad}_{\mathscr{A}}}(\mathscr{B})) = \mathrm{e}^{\mathrm{i} X_V(\mathscr{A})} X_V(\mathscr{B}) \mathrm{e}^{-\mathrm{i} X_V(\mathscr{A})} \tag{4.15}$$

In particular, for any time t, we can consider the time-evolution

$$\mathscr{B}_t := \mathrm{e}^{\mathrm{i} t \mathrm{ad}_{\mathscr{A}}}(\mathscr{B}) \tag{4.16}$$

The intuition that $\mathscr{B}_t$ is still a bonafide interaction potential, though with range growing with t, is captured by the so-called Lieb-Robinson bounds that have received a lot of attention lately [20]. In some sense, we rederive such bounds in the following lemma (in particular 3)), which helps us to handle multiple commutators of potentials. We do not require Hermiticity, but we are restricted to small potentials, corresponding to small time t in the setup above.

Lemma 4.1. *Let $\kappa_1 > \kappa_1' \geq 1$ and $\kappa, \kappa_2 \geq 1$, let $\mathscr{A}, \mathscr{B}$ be interaction potentials and let O_1, O_2 be bounded operators. In all inequalities below, both sides can be infinite.*

1.

$$\|O_1 O_2\|_{\kappa} \leq \|O_1\|_{\kappa} \|O_2\|_{\kappa} \tag{4.17}$$

2.

$$\||\mathrm{ad}_{\mathscr{A}}(\mathscr{B})\||_{\kappa_1', \kappa_2} \leq 4 (\log(\kappa_1/\kappa_1'))^{-1} \|| \mathscr{A} \||_{\kappa_1, \kappa_2} \||\mathscr{B}\||_{\kappa_1, \kappa_2} \tag{4.18}$$

3. If $4(\log(\kappa_1/\kappa_1'))^{-1} \|| \mathscr{A} \||_{\kappa_1, \kappa_2} < 1$, then, for any bounded sequence $|g(k)| \leq 1, k \in \mathbb{N}$

$$\|| \sum_{k \geq 0} \frac{g(k)}{k!} \mathrm{ad}_{\mathscr{A}}^k(\mathscr{B}) \||_{\kappa_1', \kappa_2} \leq \frac{1}{1 - 4(\log(\kappa_1/\kappa_1'))^{-1} \|| \mathscr{A} \||_{\kappa_1, \kappa_2}} \|| \mathscr{B} \||_{\kappa_1, \kappa_2} \tag{4.19}$$

In particular, by choosing $g(k) = 1$, the potential on the left hand side equals $\mathrm{e}^{\mathrm{ad}_{\mathscr{A}}}(\mathscr{B})$.

Proof. Point 1) is trivial. To address points 2), 3), we introduce some more structure. Let us first define, for a function $F \geq 0$ on finite subsets of $\mathbb{Z}^d$, the norm on potentials

$$\||\mathscr{A}\||_F := \sup_x \sum_{A : A \ni x} F(A) \|\mathscr{A}(A)\| \tag{4.20}$$

The following class of functions F will be of relevance:

$$F_{m, \kappa}(A) := |A|^{-m} \kappa^{|A|}, \qquad m \geq 0. \tag{4.21}$$

We establish

Lemma 4.2. *For any $m \geq 0$,*

$$\||\mathrm{ad}_{\mathscr{A}}(\mathscr{B})\||_{F_{m+1,\kappa}} \leq 4 \, \||\, \mathscr{A} \, \||_{F_{0,\kappa}} \, \||\mathscr{B}\||_{F_{m,\kappa}} \tag{4.22}$$

Proof.

$$\||\mathrm{ad}_{\mathscr{A}}(\mathscr{B})\||_{F_{m+1,\kappa}} \leq \sup_x \sum_{A_1:A_1 \ni x} \sum_{x' \in A_1} \sum_{A_2:A_2 \ni x'} F_{m+1,\kappa}(A_1 \cup A_2) \left(\||[\mathscr{A}(A_1), \mathscr{B}(A_2)]\|| + \||[\mathscr{A}(A_2), \mathscr{B}(A_1)]\|| \right) \tag{4.23}$$

To deal with the first term and second term, we dominate, respectively,

$$F_{m+1,\kappa}(A_1 \cup A_2) \leq F_{0,\kappa}(A_1) F_{m,\kappa}(A_2) |A_1|^{-1} \tag{4.24}$$

$$F_{m+1,\kappa}(A_1 \cup A_2) \leq F_{m,\kappa}(A_1) F_{0,\kappa}(A_2) |A_1|^{-1} \tag{4.25}$$

and $\||[\mathscr{A}(A), \mathscr{B}(A')]\|| \leq 2\|\mathscr{A}(A)\|\|\mathscr{B}(A')\|$. The claim follows. $\qquad\square$

In the same spirit, we now estimate, for $1 \leq \kappa' < \kappa$,

$$
\begin{aligned}
\||\sum_{k \geq 0} \frac{g(k)}{k!} \mathrm{ad}_{\mathscr{A}}^k(\mathscr{B})\||_{F_{0,\kappa'}} &\leq \sum_{k \geq 0} \sup_x \sum_{A \ni x} \frac{1}{k!} \left(\frac{\kappa'}{\kappa}\right)^{|A|} F_{0,\kappa}(A) \|(\mathrm{ad}_{\mathscr{A}}^k(\mathscr{B}))(A)\| \\
&\leq \sum_{k \geq 0} \sup_x \sum_{A \ni x} (\log(\kappa/\kappa')|A|)^{-k} F_{0,\kappa}(A) \|(\mathrm{ad}_{\mathscr{A}}^k(\mathscr{B}))(A)\| \\
&\leq \sum_{k \geq 0} (\log(\kappa/\kappa'))^{-k} \, \||\, \mathrm{ad}_{\mathscr{A}}^k(\mathscr{B})\||_{F_{k,\kappa}} \\
&\leq \sum_{k \geq 0} (\log(\kappa/\kappa'))^{-k} 4^k \, \||\, \mathscr{A} \, \||_{F_{0,\kappa}}^k \, \||\mathscr{B}\||_{F_{0,\kappa}} \\
&\leq (1 - 4(\log(\kappa/\kappa'))^{-1} \, \||\, \mathscr{A} \, \||_{F_{0,\kappa}})^{-1} \, \||\, \mathscr{B}\||_{F_{0,\kappa}} \tag{4.26}
\end{aligned}
$$

where the second inequality follows from

$$\sup_{a > 0} a^k \mathrm{e}^{-a} \leq k!, \qquad k \in \mathbb{N} \tag{4.27}$$

and the fourth inequality follows by k applications of Lemma 4.2.

This means that we have obtained items 2), 3) for $\kappa_2 = 1$ because $\|\cdot\|_{F_{0,\kappa}} = \|\cdot\|_{\kappa,1}$. More precisely, for 2), take $m = 0$ in (4.22) and use that

$$\||\mathrm{ad}_{\mathscr{A}}(\mathscr{B})\||_{\kappa',1} \leq (\log(\kappa/\kappa'))^{-1} \, \||\, \mathrm{ad}_{\mathscr{A}}(\mathscr{B})\||_{F_{1,\kappa}}, \qquad \text{for } 1 \leq \kappa' < \kappa. \tag{4.28}$$

By inspection of the above estimates we see that the reasoning applies just as well with $\kappa_2 > 1$, so that 2), 3) are proven.

$\qquad\square$

4.4 Perturbative diagonalization

Let us define the cut-off *phase-space*, for finite $A \subset \mathbb{Z}^d$

$$\Omega_A^{(M)} = \{0, 1, 2, \ldots M\}^A, \qquad \text{with } M \text{ as in Section 4.1}$$

Slightly abusing notation, we denote its elements by η, η' and we recall that they index eigenvectors of the free Hamiltonian $\sum_{x \in A} N_x^q$, with eigenvalues

$$E_A(\eta) = \sum_{x \in A} \langle \eta, N_x^q \eta \rangle = \sum_{x \in A} \eta_x^q \tag{4.29}$$

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Moreover, we will decompose interaction potentials in *resonant* and *non-resonant* parts. For this purpose, we fix some small *resonance threshold* $0 < \delta < 1$ (that will be related to the cutoff M in Section 5) and we define

$$\mathrm{Res}_A := \big\{(\eta,\eta') \in \Omega_A^{(M)} \times \Omega_A^{(M)} : |E_A(\eta) - E_A(\eta')| \leq \delta^{-1}M\big\},$$

$$\mathrm{NRes}_A := \big\{(\eta,\eta') \in \Omega_A^{(M)} \times \Omega_A^{(M)} : |E_A(\eta) - E_A(\eta')| > \delta^{-1}M\big\}.$$

and the linear maps on interaction potentials

$$(\mathcal{P}_{\mathrm{Res}}(\mathscr{A}))(A) := \sum_{(\eta,\eta')\in\mathrm{Res}_A} P_\eta \mathscr{A}(A) P_{\eta'}, \qquad \mathcal{P}_{\mathrm{NRes}}(\mathscr{A})(A) := \sum_{(\eta,\eta')\in\mathrm{NRes}_A} P_\eta \mathscr{A}(A) P_{\eta'} \qquad (4.30)$$

where $P_\eta \in \mathcal{B}(\mathcal{H}_A)$ is the one-dimensional orthogonal projection on the space spanned by the vector η, i.e., by $\delta_\eta(\cdot)$, see Section 2.1. The following proposition is inspired by [7]:

Proposition 4.3 (Perturbative diagonalization)**.** *For any $r = 0, 1, 2, \ldots$ and sufficiently small $\delta > 0$, depending on r, we find interaction potentials $\mathscr{F}^{(r)}, \mathscr{G}^{(r)}, \mathscr{K}^{(r)}$ such that*

$$\mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(r)}}} \ldots \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(2)}}} \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(1)}}}(\mathscr{E}) = \mathscr{E}_0 + \mathscr{F}^{(r)} + \mathscr{G}^{(r)} \qquad (4.31)$$

(where the left hand side is understood to be $\mathscr{E}$ for $r = 0$), and the following properties hold with

$$\nu = \frac{1}{4(2d+3)}, \qquad e(r) = (2r-1)/3, \qquad (4.32)$$

1. *All potentials have the M-cutoff;*

$$\mathcal{P}_{\leq M}(\mathscr{F}^{(r)}) = \mathscr{F}^{(r)}, \qquad \mathcal{P}_{\leq M}(\mathscr{G}^{(r)}) = \mathscr{G}^{(r)}, \qquad \mathcal{P}_{\leq M}(\mathscr{K}^{(r)}) = \mathscr{K}^{(r)} \qquad (4.33)$$

2. *The $\mathscr{F}^{(r)}$-potential is small and nonresonant*

$$\||\mathscr{F}^{(r)}\||_{\delta^{-\nu}} \leq C(r)M\delta^{e(r)}, \qquad \mathcal{P}_{\mathrm{Nres}}(\mathscr{F}^{(r)}) = \mathscr{F}^{(r)}. \qquad (4.34)$$

3. *The $\mathscr{G}^{(r)}$-potential is 'not too big' and resonant*

$$\||\mathscr{G}^{(r)}\||_{\delta^{-\nu}} \leq C(r)\delta^{e(0)}M, \qquad \mathcal{P}_{\mathrm{Res}}(\mathscr{G}^{(r)}) = \mathscr{G}^{(r)} \qquad (4.35)$$

4. *The $\mathscr{K}^{(r)}$-potential is small;*

$$\||\mathscr{K}^{(r+1)}\||_{\delta^{-\nu}} \leq C(r)\delta^{e(r)+1} \qquad (4.36)$$

Before giving the proof, we slightly reformulate this theorem to put it in the form in which it will be used. To that order, let us define two additional operations on interaction potentials: First the operation $\mathscr{A} \mapsto \mathcal{D}(\mathscr{A})$ that selects only the diagonal terms

$$(\mathcal{D}(\mathscr{A}))(A) := \sum_{\eta\in\Omega_A} P_\eta \mathscr{A}(A) P_\eta \qquad (4.37)$$

and $\mathscr{A} \mapsto \mathcal{R}_n(\mathscr{A})$ for some $n > 0$, the restriction to terms of range not larger than n on the lattice and in the number-operator basis

$$\mathcal{R}_n(\mathscr{A})(A) := \chi(|A| \leq n) \sum_{\eta,\eta'\in\Omega_A} \chi(|\eta - \eta'| \leq n) P_\eta \mathscr{A}(A) P_{\eta'} \qquad (4.38)$$

where $|\eta|^2 := \sum_{x\in A} |\eta(x)|^2$. Now we define a new decomposition of potentials:

$$\mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(r)}}} \ldots \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(2)}}} \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(1)}}}(\mathscr{E}) = \Big(\mathcal{D}\mathcal{R}_r(\mathscr{E}_0 + \mathscr{G}^{(r)})\Big) + \Big((1-\mathcal{D})\mathcal{R}_r(\mathscr{G}^{(r)})\Big) + \Big(\mathscr{F}^{(r)} + (1-\mathcal{R}_r)(\mathscr{G}^{(r)})\Big)$$

$$=: \mathscr{D}^{(r)} + \hat{\mathscr{G}}^{(r)} + \hat{\mathscr{F}}^{(r)} \qquad (4.39)$$

This is indeed a decomposition since $\mathcal{R}_r(\mathscr{E}_0) = \mathscr{E}_0$ and $\mathcal{D}(\mathscr{E}_0) = \mathscr{E}_0$. Then

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Corollary 4.4.

$$\||\hat{\mathscr{F}}^{(r)}\||_{\delta^{-\nu/2}} \leq C(r)M(\delta^{e(0)+(\nu r)/2} + \delta^{e(r)}), \tag{4.40}$$

Proof. By Proposition 4.3 2), $\mathscr{F}^{(r)}$ satisfies this estimate and we only need to check $(1 - \mathcal{R}_r)(\mathscr{G}^{(r)})$. We note that, in general, for $n \geq 1$,

$$\|(1 - \mathcal{R}_n)(\mathscr{A})(A)\|_{\kappa'} \leq \chi(|A| > n)\,\|\mathscr{A}(A)\|_{\kappa'} + \chi(|A| \leq n)\|\sum_{\eta,\eta'}\chi(|\eta - \eta'| > n)P_\eta\mathscr{A}(A)P_{\eta'}\|_{\kappa'}, \tag{4.41}$$

To analyze the last term, we introduce, for $\sigma \in \{1, -1\}^A$,

$$O_\sigma := \sum_{\eta,\eta'}\chi(|\eta - \eta'| > n)\big(\prod_{x \in A}\chi(\mathrm{sgn}(\eta(x) - \eta'(x)) = \sigma(x))\big)P_\eta\mathscr{A}(A)P_{\eta'} \tag{4.42}$$

where we use the signum function $\mathrm{sgn}(a) := a/|a|$ for $a \in \mathbb{R}_0$ and $\mathrm{sgn}(0) = 1$. Note that

$$\sum_{\eta,\eta'}\chi(|\eta - \eta'| > n)P_\eta\mathscr{A}(A)P_{\eta'} = \sum_{\sigma \in \{1,-1\}^A}O_\sigma.$$

The advantage of the operators O_σ is that we can explicitly perform the supremum over $w \in \mathbb{R}_+^A$ in (4.8) to obtain, for $\kappa \geq \kappa' > 1$,

$$\|O_\sigma\|_{\kappa'} = \|\sum_{\eta,\eta'}(\kappa')^{|\eta-\eta'|_1}P_\eta\breve{O}_\sigma P_{\eta'}\| \tag{4.43}$$

$$\leq \big(\max_{f \in \mathbb{R}^A: |f| \geq n}(\kappa'/\kappa)^{|f|_1}\big)\|\sum_{\eta,\eta'}\kappa^{|\eta-\eta'|_1}P_\eta\breve{O}_\sigma P_{\eta'}\| = (\kappa'/\kappa)^n\|O_\sigma\|_\kappa \tag{4.44}$$

where we put $|g|_1 := \sum_x |g(x)|$ for functions $g \in \mathbb{R}^A$ and we recall the notation $|g|^2 = \sum_x |g(x)|^2$ so that $|g| \leq |g|_1$, which we used in the last equality for $g = f$. Hence the last term on the right hand side of (4.41) is bounded by

$$2^n(\kappa'/\kappa)^n\|\mathscr{A}(A)\|_\kappa, \qquad \text{for } \kappa > \kappa'. \tag{4.45}$$

because the number of σ's is no larger than 2^n. Therefore, (4.41) yields

$$\||(1 - \mathcal{R}_n)(\mathscr{A})\||_{\kappa'} \leq (1 + 2^n)(\kappa'/\kappa)^n\,\||\mathscr{A}\||_\kappa \tag{4.46}$$

We apply this with $\mathscr{A} = \mathscr{G}^{(r)}$, $n = r$, and $\kappa = (\kappa')^2 = \delta^{-\nu}$ and we use the bound of Proposition 4.3 3). $\qquad\square$

Proof of Proposition 4.3. Our proof is by induction, but of a slightly different statement than that given in the proposition; namely we replace the norm $\|\cdot\|_{\delta^{-\nu}}$ by $\|\cdot\|_{(1+2^{-r})\delta^{-\nu}}$ such that at each induction step, we can reduce the decay parameter in the norm. This is necessary in view point 3) of Lemma 4.1, i.e. the necessity of $\kappa - \kappa' > 0$. Throughout the proof, we denote the potential on the right hand side of (4.31) by $\mathscr{H}^{(r)}$.

To save some writing in the formulas, we abbreviate

$$\|\cdot\|_{m(r)} = \|\cdot\|_{(1+2^{-r})\delta^{-\nu}} \tag{4.47}$$

For $r = 0$, we set

$$\mathscr{G}^{(0)} := \mathcal{P}_{\mathrm{Res}}(\mathscr{E} - \mathscr{E}_0), \qquad \mathscr{F}^{(0)} := \mathcal{P}_{\mathrm{NRes}}(\mathscr{E} - \mathscr{E}_0), \qquad \mathscr{K}^{(0)} := 0 \tag{4.48}$$

We choose $e(0)$ and ν such that

$$\||(\mathscr{E} - \mathscr{E}_0)\||_{m(0)} \leq C\delta^{e(0)}M \tag{4.49}$$

To satisfy this, note that $\||(\mathscr{E} - \mathscr{E}_0)\||_\kappa \leq CM\kappa^{2d+6}$, hence we need the condition

$$\delta^{-\nu(2d+6)} \leq \delta^{e(0)} \quad \Rightarrow \quad \nu(2d+6) + e(0) < 0 \tag{4.50}$$

Then the bounds are satisfied because $\mathcal{P}_{\mathrm{NRes}}, \mathcal{P}_{\mathrm{Res}}$ are contractions. This establishes the induction hypothesis for $r = 0$.

We now assume that the result holds for a given $r \geq 0$ and we show it for $r + 1$. We consider a transformation

$$\mathscr{H}^{(r+1)} := \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(r+1)}}}\left(\mathscr{H}^{(r)}\right)$$

such that, to lowest order in $\mathscr{K}^{(r+1)}$, the nonresonant potential $\mathscr{F}^{(r)}$ is eliminated.

$$[\mathscr{K}^{(r+1)}, \mathscr{E}_0] = -\mathscr{F}^{(r)}. \tag{4.51}$$

A possible choice is

$$\langle \eta, \mathscr{K}^{(r+1)}(A)\eta' \rangle \; := \; \frac{\langle \eta, \mathscr{F}^{(r)}(A)\eta' \rangle}{E_A(\eta) - E_A(\eta')} \tag{4.52}$$

where the right hand side is defined to be 0 whenever $E_A(\eta) = E_A(\eta')$. It follows[2] that for any $\kappa > 0$

$$\|\mathscr{K}^{(r+1)}(A)\|_\kappa \leq \frac{\delta}{M}\|\mathscr{F}^{(r)}(A)\|_\kappa \tag{4.53}$$

hence in particular

$$\||\mathscr{K}^{(r+1)}\||_{m(r)} \leq \frac{\delta}{M}\;\||\,\mathscr{F}^{(r)}\,\||_{m(r)}\;. \tag{4.54}$$

Now we calculate

$$\mathscr{H}^{(r+1)} \;=\; \mathscr{E}_0 \;+\; \sum_{k \geq 1}\frac{1}{k!}\mathrm{ad}^k_{\mathscr{K}^{(r+1)}}(\mathscr{E}_0) + \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(r+1)}}}(\mathscr{G}^{(r)}) + \sum_{k \geq 0}\frac{1}{k!}\mathrm{ad}^k_{\mathscr{K}^{(r+1)}}(\mathscr{F}^{(r)})$$

$$=\; \mathscr{E}_0 \;+\; \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(r+1)}}}(\mathscr{G}^{(r)}) + \sum_{k \geq 0}\frac{(k+1)}{(k+2)!}\mathrm{ad}^{k+1}_{\mathscr{K}^{(r+1)}}(\mathscr{F}^{(r)})$$

where we used (4.51) to get the last line. We define

$$\mathscr{G}^{(r+1/2)} := \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(r+1)}}}(\mathscr{G}^{(r)}),$$

$$\mathscr{F}^{(r+1/2)} := \sum_{k \geq 0}\frac{(k+1)}{(k+2)!}\mathrm{ad}^{k+1}_{\mathscr{K}^{(r+1)}}(\mathscr{F}^{(r)})$$

$$\mathscr{G}^{(r+1)} \;:= \mathcal{P}_{\mathrm{Res}}\left(\mathscr{G}^{(r+1/2)} \;+\; \mathscr{F}^{(r+1/2)}\right)$$

$$\mathscr{F}^{(r+1)} \;:= \mathcal{P}_{\mathrm{NRes}}\left(\mathscr{G}^{(r+1/2)} \;+\; \mathscr{F}^{(r+1/2)}\right)$$

so that indeed $\mathscr{H}^{(r+1)} = \mathscr{E}_0 + \mathscr{F}^{(r+1)} + \mathscr{G}^{(r+1)}$. It remains to verify the bounds. Let us first consider $\mathscr{F}^{(r+1)}$:
Note that

$$\mathscr{F}^{(r+1/2)} = \sum_{k \geq 0}\frac{g(k)}{k!}\mathrm{ad}^k_{\mathscr{K}^{(r+1)}}(\mathrm{ad}_{\mathscr{K}^{(r+1)}}(\mathscr{F}^{(r)}))$$

for a bounded sequence $|g(k)| \leq 1$. Therefore, by Lemma 4.1 2), 3),

$$\||\mathcal{P}_{Nres}(\mathscr{F}^{(r+1/2)})\||_{m(r+1)} \leq \||\mathscr{F}^{(r+1/2)}\||_{m(r+1)}$$

$$\leq (1 - C(r)\, \||\,\mathscr{K}^{(r+1)}\||_{m(r+1/2)})^{-1}\,\||\,\mathrm{ad}_{\mathscr{K}^{(r+1)}}\mathscr{F}^{(r)}\||_{m(r+1/2)}$$

$$\leq \frac{C(r)}{1 - C(r)\,\||\,\mathscr{K}^{(r+1)}\||_{m(r)}}\,\||\,\mathscr{K}^{(r+1)}\,\||_{m(r)}\,\||\mathscr{F}^{(r)}\||_{m(r)} \tag{4.55}$$

where we also used $\|| \cdot \||_{\kappa'} \leq \|| \cdot \||_\kappa$ for $1 \leq \kappa' \leq \kappa$.

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[2] Here (and only here) we exploit the fact that the weighted norm $\| \cdot \|_\kappa$ was defined in Section 4.2.1 by replacing an operator O by $\breve{O}$

Next, we estimate the contribution to $\mathscr{F}^{(r+1)}$ from $\mathcal{P}_{\mathrm{NRes}}\mathscr{G}^{(r+1/2)}$. Proceeding as above, we get, for some sequence $|g(k)| \leq 1$,

$$\|\mathcal{P}_{\mathrm{NRes}}\mathscr{G}^{(r+1/2)}\|_{m(r+1)} \leq \sum_{k\geq 1} \frac{1}{k!} \|\mathrm{ad}^k_{\mathscr{K}^{(r+1)}}(\mathscr{G}^{(r)})\|_{m(r+1)}$$

$$\leq \sum_{k\geq 0} \frac{g(k)}{k!} \|\mathrm{ad}^k_{\mathscr{K}^{(r+1)}}(\mathrm{ad}_{\mathscr{K}^{(r+1)}}(\mathscr{G}^{(r)}))\|_{m(r+1)}$$

$$\leq \frac{C(r)}{1 - C(r)\|\mathscr{K}^{(r+1)}\|_{m(r)}} \|\mathscr{K}^{(r+1)}\|_{m(r)} \|\mathscr{G}^{(r)}\|_{m(r)} \tag{4.56}$$

The first inequality follows because the induction hypothesis $\mathcal{P}_{\mathrm{NRes}}(\mathscr{G}^{(r)}) = 0$ allows to drop the $k = 0$ term. By the induction hypothesis and (4.54), we have $\|\mathscr{K}^{(r+1)}\|_{m(r)} \leq C(r)\delta^{e(r)+1}$ and therefore the denominators in the above formulae are of order 1 since

$$1 + e(r) > 0. \tag{4.57}$$

Adding the two contributions (4.55) and (4.56), we get

$$\|\mathscr{F}^{(r+1)}\|_{m(r+1)} \leq C(r)\left(\delta^{2e(r)+1} + \delta^{e(0)+e(r)+1}\right)$$

and hence the bound on $\mathscr{F}^{(r+1)}$ holds because

$$e(r+1) \leq \min(2e(r)+1, e(r)+e(0)+1) \tag{4.58}$$

The bound on the potential $\mathscr{G}^{(r+1)}$ is derived in by analogous (though simpler) reasoning. $\qquad\square$

4.5 Transformations and spatial truncations

Proposition 4.3 is set in the language of transformed potentials. We investigate the question how accurately such transformations can be restricted to small volumes.

First, if two potentials $\mathscr{A}, \mathscr{K}$ are finite in one of the $\|\cdot\|_{\kappa_1,\kappa_2}$-norms, then the equality

$$\mathrm{e}^{-\mathrm{ad}_{\mathscr{K}}}\mathrm{e}^{\mathrm{ad}_{\mathscr{K}}}(\mathscr{A}) = \mathscr{A} \tag{4.59}$$

holds (in a weaker norm). This can be checked explicitly by manipulating the defining series (4.14). Let us abbreviate

$$\mathcal{K}(\mathscr{A}) = \mathcal{K}^{(r)}(\mathscr{A}) := \mathrm{e}^{-\mathrm{ad}_{\mathscr{K}^{(1)}}}\mathrm{e}^{-\mathrm{ad}_{\mathscr{K}^{(2)}}}\ldots\mathrm{e}^{-\mathrm{ad}_{\mathscr{K}^{(r)}}}(\mathscr{A}). \tag{4.60}$$

with $\mathscr{K}^{(j)}$ as given in Proposition 4.3. Then, by (4.59), we can invert the operator $\mathcal{K}$:

$$\mathcal{K}^{-1}(\mathscr{A}) = (\mathcal{K}^{(r)})^{-1}(\mathscr{A}) = \mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(r)}}}\ldots\mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(2)}}}\mathrm{e}^{\mathrm{ad}_{\mathscr{K}^{(1)}}}(\mathscr{A}). \tag{4.61}$$

By repeated application of Lemma 4.1 3) and Proposition 4.3 4), one shows that

$$\|\mathcal{K}(\mathscr{A})\|_{\frac{\kappa_1}{2},\kappa_2} \leq C(r)\|\mathscr{A}\|_{\kappa_1,\kappa_2}, \qquad \|\mathcal{K}^{-1}(\mathscr{A})\|_{\frac{\kappa_1}{2},\kappa_2} \leq C(r)\|\mathscr{A}\|_{\kappa_1,\kappa_2}. \tag{4.62}$$

for $\kappa_1, \kappa_2 \leq \delta^{-\nu}$ and δ small enough, depending on r. For a finite set D, we define

$$U_D := \mathrm{e}^{X_D(\mathscr{K}^{(r)})}\ldots\mathrm{e}^{X_D(\mathscr{K}^{(2)})}\mathrm{e}^{X_D(\mathscr{K}^{(1)})} \tag{4.63}$$

Note that U_D is unitary since $X_D(\mathscr{K}^{(j)})$ are anti-Hermitian matrices, as one checks by inspecting the definitions of $\mathscr{K}^{(j)}$ and $\mathscr{F}^{(j)}$. By repeated application of (4.15) we derive

$$X_D(\mathcal{K}(\mathscr{A})) = U_D X_D(\mathscr{A}) U_D^* \tag{4.64}$$

In what follows, we will interpret an operator O as a potential $\mathscr{A}_O$ such that

$$\mathscr{A}_O(A) = \begin{cases} O & A = A_O \\ 0 & A \neq A_O \end{cases} \tag{4.65}$$

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for some connected set A_O such that $s(O) \subset A_O$. If $A_O \subset D$, the identity (4.64) reads

$$X_D(\mathcal{K}(O)) = U_D O U_D^* \tag{4.66}$$

where, as announced, $\mathcal{K}(O) = \mathcal{K}(\mathscr{A}_O)$ on the left hand side. From now on, we write O for $\mathscr{A}_O$ without further comment. To quantify the dependence on the set D in the above formula, it is helpful to define first the restriction of a potential to some volume: Let

$$\mathcal{I}_D(\mathscr{A})(A) := \chi(A \cap D \neq \emptyset)\mathscr{A}(A). \tag{4.67}$$

then we have, for $A_O \subset D \subset V$

$$X_V(\mathcal{K}(O)) = X_V(\mathcal{I}_{D^c}\mathcal{K}(O)) + U_D O U_D^* \tag{4.68}$$

The upcoming Lemma 4.5 provides some bounds. In what follows, we will stop keeping track of the precise value of exponents like ν. We will also set $\kappa_2 = 1$ in the norm $\|\cdot\|_{\kappa_1,\kappa_2}$ for simplicity, because, once $\hat{\mathscr{F}}, \hat{\mathscr{G}}$ have been defined, the parameter κ_2 plays no role anymore.

Lemma 4.5. Let $\operatorname{dist}(D^c, A_O) > c|A_O|$ for some $c > 0$, then

$$\||\mathcal{I}_{D^c}\mathcal{K}(O)\||_{\delta-c',1} \leq C(r)\delta^{c''\operatorname{dist}(D^c,A_O)}\|O\| \tag{4.69}$$

for some $c', c'' > 0$.

Proof. Trivially, for any $c_1 > 0$

$$\||O\||_{\delta-c_1,1} \leq \delta^{-c_1|A_O|}\|O\| \tag{4.70}$$

and hence, by (4.62), for $c_1 > 0$ small enough,

$$\||\mathcal{K}(O)\||_{\frac{\delta-c_1}{2},1} \leq C(r)\delta^{-c_1|A_O|}\|O\|. \tag{4.71}$$

Furthermore, if $(\mathcal{K}(O))(A) \neq 0$, then $A_O \subset A$ and hence, if $(\mathcal{I}_{D^c}\mathcal{K}(O))(A) \neq 0$, then $|A| \geq \operatorname{dist}(D^c, A_O) + |A_O|$. Therefore

$$\||\mathcal{I}_{D^c}\mathcal{K}(O)\||_{\delta-c',1} \leq \delta^{c_2(\operatorname{dist}(D^c,A_O)+|A_O|)}\sum_A \delta^{-(c'+c_2)|A|}\|(\mathcal{I}_{D^c}\mathcal{K}(O))(A)\| \tag{4.72}$$

$$\leq \delta^{c_2(\operatorname{dist}(D^c,A_O)+|A_O|)}\||\mathcal{K}(O)\||_{\delta-(c'+c_2),1} \tag{4.73}$$

$$\leq C(r)\delta^{c_2(\operatorname{dist}(D^c,A_O)+|A_O|)}\delta^{-(1+c_3)(c'+c_2)|A_O|}\|O\|. \tag{4.74}$$

$$\leq C(r)\delta^{c_2\operatorname{dist}(D^c,A_O)}\delta^{-((1+c_3)c'+c_3c_2)|A_O|}\|O\|. \tag{4.75}$$

The second inequality follows from $\||\mathcal{I}_{D^c}(\mathscr{A})\||_{\kappa_1,\kappa_2} \leq \||\mathscr{A}\||_{\kappa_1,\kappa_2}$ and the third inequality follows from (4.71), for δ small enough such that $\delta^{-(c'+c_2)} \leq (1/2)\delta^{-(1+c_3)(c'+c_2)}$. The claim now follows from (4.75) by using $\operatorname{dist}(D^c, A_O) > c|A_O|$ and choosing c', c_3 small enough such that

$$c'' := c_2 - (1/c)((1+c_3)c' + c_3c_2) > 0. \tag{4.76}$$

$\square$

Obviously, changing the volume D far away from A_O leads to small changes in $U_D O U_D^*$, as we show next. We denote the symmetric difference of sets by $D\Delta D' := (D \cup D') \setminus (D \cap D')$.

Lemma 4.6. If $\operatorname{dist}(D\Delta D', A_O) \geq c|A_O|$ for some $c > 0$, then

$$\|U_D O U_D^* - U_{D'} O U_{D'}^*\| \leq C(r)\delta^{c'\operatorname{dist}(D\Delta D',A_O)}\|O\|. \tag{4.77}$$

for some $c' > 0$.

Proof. Note that this lemma is not restricted to the case $A_O \subset (D \cap D')$. Let us however first treat this case. Then (4.68), applied to both D and D', yields for large enough V,

$$U_D O U_D^* - U_{D'} O U_{D'}^* = X_V(\mathcal{I}_{(D')^c}\mathcal{K}(O)) - X_V(\mathcal{I}_{D^c}\mathcal{K}(O)) = \sum_{A \subset V : A \cap (D\Delta D') \neq \emptyset} \varsigma(A) \times (\mathcal{K}(O))(A) \qquad (4.78)$$

where $\varsigma(A) = \pm 1$. The operator norm of the left-most expression is trivially bounded by

$$\|\|\mathcal{I}_{(D\Delta D')}\mathcal{K}(O)\|\|_{\kappa,1}, \qquad \text{for any } \kappa > 1, \qquad (4.79)$$

and hence the claim follows by Lemma 4.5 with $(D\Delta D')$ in the role of D^c.

Next, we consider the case where $G := A_O \setminus (D \cap D')$ is not empty. Note that $G \cap (D \cup D') = \emptyset$ since $(D\Delta D') \cap A_O = \emptyset$. Set

$$\tilde{D} := D \cup G, \qquad \tilde{D}' := D' \cup G.$$

and define modified potentials $\tilde{\mathcal{K}}^{(j)}$ by

$$\tilde{\mathcal{K}}^{(j)}(A) := \begin{cases} \mathcal{K}^{(j)}(A) & A \subset (D \cup D') \\ 0 & A \not\subset (D \cup D') \end{cases} \qquad (4.80)$$

and let $\tilde{U}_A$ (for a set A) by the modified version of U_A obtained by replacing $\mathcal{K}^{(j)}$ by $\tilde{\mathcal{K}}^{(j)}$. Then it is clear that

$$\tilde{U}_{\tilde{D}} = U_D, \qquad \tilde{U}_{\tilde{D}'} = U_{D'} \qquad (4.81)$$

such that in particular

$$U_D O U_D^* - U_{D'} O U_{D'}^* = \tilde{U}_{\tilde{D}} O \tilde{U}_{\tilde{D}}^* - \tilde{U}_{\tilde{D}'} O \tilde{U}_{\tilde{D}'}^*. \qquad (4.82)$$

For the second expression, the above proof still applies since $A_O \subset \tilde{D} \cap \tilde{D}'$ and hence we conclude that its norm is bounded by

$$C(r)\delta^{c' \operatorname{dist}(\tilde{D}\Delta\tilde{D}', A_O)}\|O\| \qquad (4.83)$$

Since however $\tilde{D}\Delta\tilde{D}' = D\Delta D'$, we have obtained the claim of the lemma. $\square$

5 Analysis of the resonant Hamiltonian: Invariant subspaces

We define the resonant Hamiltonian

$$Z = Z^{(r)} := X_{\mathbb{S}}(\mathscr{D}^{(r)}) + X_{\mathbb{S}}(\hat{\mathscr{G}}^{(r)}) \qquad (5.1)$$

where $\mathscr{D}^{(r)}, \hat{\mathscr{G}}^{(r)}$ were defined preceding Corollary 4.4. Note that the potential $\hat{\mathscr{G}}^{(r)}$ depends on the resonance threshold δ that we choose as

$$\delta = M^{-\gamma_1}, \qquad \text{for some } 0 < \gamma_1 < q - 2 \qquad (5.2)$$

The main point of the analysis below is to show that the non-diagonal terms in the Hamiltonian Z are *sparse*, and therefore, transport induced by this Hamiltonian is small. This goal will be achieved in Proposition 6.2 and one can consider the Sections 5 and 6 as the proof of this result.

5.1 Setup and definition

In the present section 5, our analysis will depend on a volume $V \subset \Lambda$ that should be thought of as being much smaller than Λ. Even though this is not necessary for most of the statements below, we will always assume that $|V| \leq (2r)^{2d}$, as will anyhow be done in Section 6. We mostly drop the dependence on r, for example writing $Z = Z^{(r)}$, but we write $C(r), c(r)$ for constants $C(r) < \infty, c(r) > 0$ that can depend on r. Recall that $\Omega_V^{(M)}$ is the phase space in V with a cutoff at M. In what follows we often abbreviate $\Omega_V = \Omega_V^{(M)}$ because the high-energy cutoff is always in place.

To write the Hamiltonian Z in a more explicit way, we introduce

Definition 5.1 (Moves). *For a volume $V \subset \mathbb{S}$ and $r \in \mathbb{N}$, we set*

$$\mathcal{P}^{(V)} := \{\rho \in \mathbb{Z}^V : 1 \leq |\rho| \leq r, |s(\rho)| \leq r\}$$

where $s(\rho) = \{x : \rho(x) \neq 0\}$. We also define the 'dependence set' of a move

$$S(\rho) := \bigcup_{\substack{A \subset \mathbb{S} \\ |A| \leq r, s(\rho) \subset A}} A \tag{5.3}$$

such that, in particular, $\operatorname{diam}(S(\rho)) \leq 2r$.

To recast the Hamiltonian Z in terms of 'moves', we first introduce the 'move'-operators

$$W_\rho := \sum_{A \subset \mathbb{S}} \sum_{\eta \in \Omega_{\mathbb{S}}} P_\eta \hat{\mathscr{G}}^{(r)}(A) P_{\eta+\rho} \tag{5.4}$$

Remark by W: **I omitted the sentence here since it was more confusing than clarifying *EOR*** They satisfy

1. the high-energy cutoff $\mathcal{P}_{\leq M}(W_\rho) = W_\rho$

2. the locality property $s(W_\rho) \subset S(\rho)$ (In particular, the sum over A in (5.4) can be restricted to $S(\rho)$).

3. the bound $\|W_\rho\| \leq C(r) M^C$.

4. a resonance condition: $\langle \eta, W_\rho \eta' \rangle = 0$ unless $|E_{S(\rho)}(\eta) - E_{S(\rho)}(\eta')| \leq M^{1+\gamma_1}$.

This is easily checked relying on the locality and bounds on $\hat{\mathscr{G}}^{(r)}$, and (5.2). We can now recast the Hamiltonian Z as

$$Z = X_{\mathbb{S}}(\mathscr{D}) + X_{\mathbb{S}}(\hat{\mathscr{G}}) = X_{\mathbb{S}}(\mathscr{D}) + \sum_{\rho \in \mathcal{P}^{(\mathbb{S})}} W_\rho \tag{5.5}$$

Moreover, we recall that $\mathscr{D}(A) = 0$ unless $|A| \leq r$.

In the remaining part of this section, we will not need the strip $\mathbb{S}$, nor the Hamiltonian Z. Instead, we focus on the (joint) structure of the operators W_ρ with $\rho \in \mathcal{P}^{(V)}$. Whenever confusion is excluded, we drop V from our notation.

Definition 5.2 (Partition). *Let $\eta, \eta' \in \Omega_V$. Define*

$$\eta \underset{\rho}{\sim} \eta' \qquad \Leftrightarrow \qquad \left(\eta' - \eta \in \{-\rho, \rho\} \quad and \quad |E_V(\eta) - E_V(\eta')| \leq M^{1+\gamma_1}\right) \tag{5.6}$$

and

$$\eta \sim \eta' \qquad \Leftrightarrow \qquad \left(\eta \underset{\rho}{\sim} \eta' \quad for \ some \quad \rho \in \mathcal{P}\right) \tag{5.7}$$

Note that the relation $\sim$ is an adjacency relation, hence it induces a partition of Ω_V **Remark by W:** ***into connected components*** ***EOR*** *. We call this partition $\mathcal{F} = \mathcal{F}^{(V)}$ and its elements are denoted by $\mu, \mu', \ldots \in \mathcal{F}$.* We write

$$\mathcal{P}(\mu) = \mathcal{P}^{(V)}(\mu) := \{\rho \in \mathcal{P} \mid \exists \eta, \eta' \in \mu : \eta \underset{\rho}{\sim} \eta'\} \tag{5.8}$$

and, for $A \subset V$,

$$\mathcal{P}_A(\mu) = \mathcal{P}_A^{(V)}(\mu) := \{\rho \in \mathcal{P}(\mu) : s(\rho) \subset A\}. \tag{5.9}$$

and we also write $\mathcal{P}_A(\eta) = \mathcal{P}_A(\mu(\eta))$ where $\mu(\eta)$ is the unique $\mu \in \mathcal{F}$ such that $\eta \in \mu$.

From Definition 5.2, it is immediate that

$$[P_\mu, W_\rho] = 0, \qquad \text{with } P_\mu = \sum_{\eta \in \mu} P_\eta \quad \text{and } \rho \in \mathcal{P}^{(V)}, \mu \in \mathcal{F}^{(V)}$$

and, since $\mathscr{D}(A)$ is diagonal in the η-basis, also $[\mathscr{D}(A), P_\mu] = 0$. Note also that for $\rho \in \mathcal{P}^{(V)}$, ***Remark by W:*** **it is not guaranteed that $s(W_\rho) \subset V$ *EOR*** because $s(\rho) \subset V$ does not imply $S(\rho) \subset V$.

5.2 Structure of the partition $\mathcal{F}$

The main virtue of this construction is that the partition $\mathcal{F}$ is rather fine, as we state now

Proposition 5.1. *Assume that $0 < \gamma_1 < q - 2$ in the resonance condition (5.2). Then, for any $\gamma_2 > 0$, there is $C_0(r) < \infty$ such that, for any r,*

$$\max_{\mu \in \mathcal{F}} \max_{\eta, \eta' \in \mu} |\eta - \eta'| \leq C_0(r) M^{\gamma_2} \tag{5.10}$$

Remark by W: **Recall that we have assumed $|V| \leq (2r)^{2d}$. Therefore, there is no explicit dependence on V in the bound on the right hand side.** *EOR*

We define the scalar product

$$\langle \eta, \eta' \rangle = \sum_{x \in V} \eta(x) \eta'(x), \qquad \eta, \eta' \in \mathbb{N}^V \tag{5.11}$$

corresponding to the norm $|\eta|^2 = \sum_{x \in V} |\eta(x)|^2$. In what follows, x always ranges over V and we drop this from the notation. Let us consider a sequence $(\eta_n)_{n \geq 1} \subset \Omega_V$ such that, for every $n \geq 1$, we can write

$$\eta_{n+1} = \eta_n + \rho_n \qquad \text{with} \qquad \rho_n \in \mathcal{P} = \mathcal{P}^{(V)}, \tag{5.12}$$

and,

$$\left| E_V(\eta_{n+1}) - E_V(\eta_n) \right| \leq M^{1+\gamma_1}. \tag{5.13}$$

It is clear from the definition of the partition $\mathcal{F}$ that, if two configurations η, η' belong to the same partitioning set μ, then there must be a sequence as above, with $\eta_1 = \eta, \eta_l = \eta'$, for some $l \in \mathbb{N}$. We will now show in a series of lemma's that for any $\gamma_2 > 0$ there is a $C_0(r) < \infty$ such that, for any such sequence (in particular, for any l),

$$\left| \eta_l - \eta_1 \right| \leq C_0(r) M^{\gamma_2} \tag{5.14}$$

which implies the proposition.

To make the reasoning more transparent, we perform a change of variables; define $\theta_n \in (\mathbb{R}_+)^V$ by

$$\theta_n(x) = \left(\frac{\eta_n(x)}{M} \right)^{q-1} \quad \text{for} \quad x \in V, n \geq 1. \tag{5.15}$$

Lemma 5.2. *Let $0 < \gamma_3 < \min(1, (q-2) - \gamma_1)$, then, for M large enough,*

$$\left| \langle \theta_n, \rho_n \rangle \right| \leq M^{-\gamma_3}. \tag{5.16}$$

Proof. Let us write for simplicity $\eta = \eta_n$ and $\eta' = \eta_{n+1}$. First, by the fundamental theorem of calculus,

$$\begin{aligned}
q \left| \sum_x \rho(x) \eta(x)^{q-1} \right| &\leq \left| q \sum_x \rho(x) \eta(x)^{q-1} - (E_V(\eta') - E_V(\eta)) \right| + \left| E_V(\eta') - E_V(\eta) \right| \\
&\leq C(r) \left(1 + \sum_x \eta(x)^{q-2} \right) + \left| E_V(\eta') - E_V(\eta) \right| \\
&\leq C(r) M^{q-2} + M^{1+\gamma_1}. \tag{5.17}
\end{aligned}$$

Dividing by M^{q-1} yields the claim. $\square$

Henceforth, we assume that M is chosen large enough so that Lemma 5.2 holds.

We will now define regions $\mathcal{Z}(m)$ in $\mathbb{R}_+^{|V|}$ (think: the space containing θ_n) such that for all $h \in \mathcal{Z}(m)$, the condition $\left| \langle h, \rho \rangle \right| \leq M^{-\gamma_3}$ is 'nearly satisfied' for m linearly independent 'moves' $\rho^1, \ldots, \rho^m$, but far from satisfied for any move ρ that is not contained in $\mathrm{Span}\{\rho^1, \ldots, \rho^m\}$. The construction depends on a parameter $L > 2$ that will be chosen to be large enough later on. Note that in the case $q = 2$ the geometric picture

Remark by W: **I replace θ with h to avoid later on some very microscopic clash between θ as a general variable and $\theta = \theta_k$ as used in the lemmas. I have no strong opinion** *EOR* *Remark by W:* **I lost the remark about the case $q = 2$. As is stood, it did not really make sense (eg Proposition 5.1 is empty), but of course it might still be helpful.** *EOR*

Definition 5.3. *Let $\mathcal{Z}(m) \subset \mathbb{R}_+^{|V|}$ with $1 \leq m \leq |V|$ be the set consisting of those h for which there is a linearly independent collection $\{\rho^1, \ldots, \rho^m\} \subset \mathcal{P}$ such that*

1. $|\langle h, \rho^j \rangle| \leq L^{m-1} M^{-\gamma_3}$ for $j = 1, \ldots, m$.

2. $|\langle h, \rho \rangle| > L^m M^{-\gamma_3}$ for any $\rho \in \mathcal{P} \setminus \mathrm{span}\{\rho^1, \ldots, \rho^m\}$

For $m = 0$, we let $\mathcal{Z}(0) \subset \mathbb{R}_+^{|V|}$ be the set of h such that $|\langle h, \rho \rangle| > M^{-\gamma_3}$ for any $\rho \in \mathcal{P}$ (recall that $|\rho| \geq 1$).

Note that $(\mathbb{R}_+)^V = \cup_{j=0}^{|V|} \mathcal{Z}(j)$ but the regions $\mathcal{Z}(m)$ are in general not disjoint. In what follows we say that $h \in \mathcal{Z}(m)$ by virtue of $\rho^1, \ldots, \rho^m$ if $\{\rho^1, \ldots, \rho^m\} \subset \mathcal{P}$ is one of the linearly independent collections for which the above conditions 1 and 2 hold.

If $h \in \mathcal{Z}(m)$ by virtue of $\rho^1, \ldots, \rho^m$, then we easily deduce from condition 1) above, that

$$|P_{\mathrm{Span}\{\rho^1, \ldots, \rho^m\}} h| \leq C(\rho^1, \ldots, \rho^m) L^{m-1} M^{-\gamma_3}.$$

with $P_{\mathcal{V}}$ the orthogonal projection on the vector space $\mathcal{V}$. Since $|\mathcal{P}| \leq C(r)$, we can bound $C(\rho^1, \ldots, \rho^m) \leq C(r)$ on the right hand side. This observation will be used in the following form: There is a $C_1(r) < \infty$ such that for any $u \in \mathrm{span}\{\rho^1, \ldots, \rho^m\}$ with $|u| = 1$ and $h \in \mathcal{Z}(m)$ by virtue of $\rho^1, \ldots, \rho^m$,

$$|\langle h, u \rangle| \leq C_1(r) L^{m-1} M^{-\gamma_3}, \tag{5.18}$$

We also introduce the closed set $G_{\rho^1, \ldots, \rho^m} \subset \mathbb{R}^{|V'|}$ consisting of h such that, for all $x \in V$

$$h(x) = \nu(x) v(x), \qquad \text{with } \nu(x) \geq 0, v \in \mathrm{Span}\{\rho^1, \ldots, \rho^m\}. \tag{5.19}$$

Lemma 5.3. *Let us abbreviate (only in this lemma) $\mathcal{R} = \{\rho^1, \ldots, \rho^m\}$, then*

$$\inf_{h \in G_{\mathcal{R}}, |h| = 1} \sup_{u \in \mathrm{Span}\mathcal{R}, |u| = 1} |\langle h, u \rangle| \geq c(\mathcal{R}) \geq c(r) \tag{5.20}$$

Proof. Assume there is no such $c(\mathcal{R})$. Since the intersection of $G_{\mathcal{R}}$ with the unit sphere is compact, it follows that there is a $h \in G_{\mathcal{R}}, |h| = 1$ such that $h \perp \mathrm{Span}\mathcal{R}$. However, this is false because

$$\langle h, v \rangle = \sum_x \nu(x)(v(x))^2 > 0 \tag{5.21}$$

where ν, v are related to h as in the definition of $G_{\mathcal{R}}$, in particular $v \in \mathrm{Span}\mathcal{R}$. Finally, $c(\mathcal{R}) \geq c(r)$ follows because the number of possible collections $\mathcal{R}$ is $C(r)$. $\qquad \square$

Lemma 5.4. *Let $(\theta_n)_{n \geq 1}$ be the sequence defined in (5.15). We fix $k, k' \in \mathbb{N}$ with $k < k'$ and we abbreviate $\theta = \theta_k$ and $\theta' = \theta_{k'}$. Assume that $\theta \in \mathcal{Z}(m)$ for some $1 \leq m \leq |V|$ by virtue of $\rho^1, \ldots, \rho^m$. If*

$$|\theta_n - \theta| \leq \frac{L}{2r} L^{m-1} M^{-\gamma_3} \quad \text{for every } n \text{ with} \quad k \leq n \leq k', \tag{5.22}$$

then

$$\theta' - \theta \in G_{\rho^1, \ldots, \rho^m}$$

Proof. By (5.22), for every $\rho \in \mathcal{P} \setminus \mathrm{span}\{\rho^1, \ldots, \rho^m\}$,

$$|\langle \theta_n, \rho \rangle| \geq |\langle \theta, \rho \rangle| - |\langle \theta_n - \theta, \rho \rangle| > L^m M^{-\gamma_3} - \frac{|\rho|}{2r} L^m M^{-\gamma_3} \geq M^{-\gamma_3}, \tag{5.23}$$

because $|\rho| \leq r$ and $L \geq 2$. It follows therefore that condition (5.16) does not hold for ρ, hence

$$\eta_{n+1} = \eta_n + \rho_n \quad \text{with} \quad \rho_n \in \mathcal{P} \cap \mathrm{Span}\{\rho^1, \ldots, \rho^m\}.$$

and, since this holds for any n with $k \leq n \leq k'$, we have

$$\eta' = \eta + v \quad \text{for some} \quad v \in \mathrm{span}\{\rho^1, \ldots, \rho^m\}.$$

Since the function $t \mapsto t^{q-1}$ is non-decreasing for $t \geq 0$, this implies that for each x

$$(\eta'(x))^{q-1} = (\eta(x))^{q-1} + \nu(x) v(x) \qquad \text{for some } \nu(x) > 0$$

which proves the claim because $\theta(x) = (\eta(x)/M)^{q-1}, \theta'(x) = (\eta'(x)/M)^{q-1}$ $\qquad \square$

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Lemma 5.5. *Let θ, θ' be as in Lemma 5.4 and assume that all conditions of Lemma 5.4 hold. If additionally*

$$|\theta' - \theta| \geq \frac{L}{4r} L^{m-1} M^{-\gamma_3},$$

then $\theta' \in \mathcal{Z}(s)$ for some $s < m$.

Proof. The inequalities in (5.23) already show that, for L large enough,

$$|\langle \theta', \rho \rangle| > L^{m-1} M^{-\gamma_3} \quad \text{for any} \quad \rho \in \mathcal{P} \setminus \text{span}\{\rho^1, \ldots, \rho^m\}. \tag{5.24}$$

So to conclude, by (5.18), it is enough to find $u \in \text{span}\{\rho^1, \ldots, \rho^m\}, |u| = 1$ for which (5.18) is violated (with $h = \theta'$). Using (5.18) for θ, we have

$$|\langle \theta', u \rangle| \geq |\langle \theta' - \theta, u \rangle| - |\langle \theta, u \rangle| \geq |\langle \theta' - \theta, u \rangle| - C(r) L^{m-1} M^{-\gamma_3}$$

and hence, by choosing L large, it suffices to find u such that, for some $c(r) > 0$,

$$|\langle \theta' - \theta, u \rangle| \geq c(r) |\theta' - \theta|, \tag{5.25}$$

which follows from Lemma 5.3 because $\theta' - \theta \in G_{\rho^1, \ldots, \rho^m}$. by Lemma 5.4. $\qquad \square$

We are now ready to conclude the

5.2.1 Proof of Proposition 5.1

We show that, for L sufficiently large,

$$|\theta_l - \theta_1| \leq \frac{L^{|V|}}{2r} M^{-\gamma_3}, \quad \text{for any } l \in \mathbb{N} \tag{5.26}$$

which implies (5.14) if we take $\gamma_2 > (q - 1)\gamma_3$ (and hence it ends the proof). Indeed, since γ_3 can be taken arbitrarily small, so can γ_2.

Note first that

$$|\theta_{n+1} - \theta_n| \leq C(r). \tag{5.27}$$

If $\theta_1 \in \mathcal{Z}(0)$, then $\theta_j = \theta_1$ for every $j \geq 1$, and the claim follows. Next, assume that $\theta_1 \in \mathcal{Z}(m_1)$ for some $m_1 \in \{1, \ldots, |V|\}$. Set $n_1 := 1$. Because of (5.27), there is an n_2 such that the assumptions of Lemma 5.4 and Lemma 5.5 hold with $k = n_1, k' = n_2$. Then Lemma 5.5 implies that $\theta_{n_2} \in \mathcal{Z}(m_2)$ with $m_2 \leq m_1 - 1$ and

$$|\theta_{n_1} - \theta_{n_2}| \leq (1/2r) L^{m_1 - 1} M^{-\gamma_3}.$$

We repeat the reasoning starting now from θ_{n_2}, i.e. we find a $n_3 > n_2$ such that the assumptions of Lemma's 5.4 and 5.5 hold with $k = n_2, k' = n_3$. We repeat this procedure until we get $m_s = 0$, which occurs for $s \leq m_1 \leq |V|$ Colleting all the bounds we get

$$|\theta_{n_s} - \theta_1| \leq \sum_{j=1}^{s} |\theta_{n_{j+1}} - \theta_{n_j}| \leq \sum_{m=1}^{|V|} \frac{L^{m-1}}{2r} M^{-\gamma_3} \leq \frac{L^{|V|}}{2r} M^{-\gamma_3}. \tag{5.28}$$

and, since $\theta_n = \theta_{n_s}$ for any $n \geq n_s$, this proves (5.26). $\qquad \square$

5.2.2 Sparseness of resonances

Whereas in the previous section we dealt with the overall structure of the partitions $\mathcal{F}$ defined by the resonances, most configurations η have no resonance at all, i.e. $\mu(\eta) = \{\eta\}$ (as long as the volume V does not grow with M, of course). This captures the main intuition of our proof. It suffices to pick a single 'move' ρ since $|\mathcal{P}$, the total number of moves in V with $|V| = C(r)$, is $C(r)$ anyhow. ***Remark by W: added explanation EOR***

Lemma 5.6. *Fix $\rho \in \mathcal{P}^{(\mathbb{S})}$. For any γ_1 satisfying $0 < \gamma_1 < q - 2$, there is a $\gamma_4 \geq 0$ such that*

$$\frac{1}{M^{|s(\rho)|}} \sum_{\eta \in \Omega_{s(\rho)}^{(M)}} \chi(|E_{s(\rho)}(\eta + \rho) - E_{s(\rho)}(\eta)| \leq M^{1+\gamma_1}) \leq C(r) M^{-\gamma_4 |s(\rho)|} \tag{5.29}$$

Note that $M^{|s(\rho)|} = |\Omega^{(M)}_{s(\rho)}|$ so this estimate indeed means that the resonance condition is satisfied only for a small fraction of configurations. The proof of Lemma 5.6 is a straightforward calculation; by Lemma 5.2, it boils down to showing that, for any $\gamma_3 > 0$, there is a $\gamma_4 > 0$ such that

$$\int_{[0,1]^{s(\rho)}} \mathrm{d}\theta \, \big(\prod_{x \in s(\rho)} \theta(x) \big)^{-\frac{q-2}{q-1}} \chi(|\langle \rho, \theta \rangle| \leq M^{-\gamma_3}) \leq C(r) M^{-\gamma_4 |s(\rho)|}. \tag{5.30}$$

and (5.30) follows from a Hölder inequality. The Jacobian $(\cdot)^{-\frac{q-2}{q-1}}$ originates in the $\eta \to \theta$ change of variables.

5.3   Locality of the partition $\mathcal{F}$

The aim of this section is to control the set of moves $\mathcal{P}^{(V)}_A(\eta)$ *locally in* $A \subset V$, i.e. without knowing the configuration η outside of A. As such, this is impossible because $\mathcal{P}^{(V)}_A(\eta)$ is determined globally in V (**Remark by W: refer to introduction for illustration *EOR***). However, we can still achieve this control if we impose a condition on the boundary of the set A, roughly saying that no element of $\mathcal{P}^{(V)}(\eta)$ is supported there. This is the content of Lemma 5.7. Such locality statements become powerful when combined with an argument that tells us that it is easy to find regions A such that this condition on the boundary holds.

To state a convenient boundary condition, we introduce an set $\mathcal{P}'_A(\eta)$ that is bigger than $\mathcal{P}^{(V)}_A(\eta)$ but easier to control. Let

$$\mathcal{P}'_A(\eta) := \bigcup_{\eta' : |\eta - \eta'| \leq C_0(r) M^{\gamma_2}} \{\rho : s(\rho) \subset A \text{ and } \exists \eta'' : \eta' \underset{\rho}{\sim} \eta''\} \tag{5.31}$$

with $C_0(r)$ as in Proposition 5.1 and for some $\gamma_2 > 0$ that will later, in the proof of Lemma 5.8, be chosen small enough. Proposition 5.1 immediately yields

$$\mathcal{P}^{(V)}_A(\mu) \subset \mathcal{P}'_A(\eta), \qquad \text{for any } \eta \in \mu, \mu \in \mathcal{F}^{(V)} \text{ and any } V \supset A \text{ with } |V| \leq (2r)^{2d}. \tag{5.32}$$

Note that $\mathcal{P}'_A(\eta)$ is defined locally, which is the reason that it will be indeed easy to control.

We introduce the boundary set

$$\partial_k A := \{x \in A, \mathrm{dist}(x, A^c) \leq k\}. \tag{5.33}$$

and we consider two volumes V, V' with $|V|, |V'| \leq (2r)^{2d}$.

Lemma 5.7. *Assume* $A \subset V \cap V'$. *Let* $\eta \in \Omega_V, \eta' \in \Omega_{V'}$ *be such that* $\eta_A = \eta'_A$ *and*

$$\mathcal{P}'_{\partial_{2r} A}(\eta) = \emptyset \tag{5.34}$$

Then,

$$\mathcal{P}^{(V)}_A(\eta) = \mathcal{P}^{(V')}_A(\eta'). \tag{5.35}$$

Proof. From (5.34) and (5.32), we get

$$\mathcal{P}^{(V)}_{\partial_{2r} A}(\eta) = \emptyset, \qquad \mathcal{P}^{(V')}_{\partial_{2r} A}(\eta') = \emptyset. \tag{5.36}$$

Call $\tilde{A} := A \setminus \partial_r A$, then for any $\rho \in \mathcal{P}^{(V)}_A(\eta)$, we have either $s(\rho) \subset \tilde{A}$ or $s(\rho) \subset \tilde{A}^c$. This follows because $\mathcal{P}^{(V)}_{\partial_{2r} A}(\eta) = \emptyset$ and $|s(\rho)| \leq r$.

Let $\rho \in \mathcal{P}^{(V)}_A(\eta)$. As already used below Proposition 5.1, this is equivalent to the existence of a sequence $\rho_1, \ldots, \rho_n$ with $\rho_n = \rho$ such that

$$\eta_j \sim_{\rho_j} \eta_{j+1}, \qquad \text{for } j = 1, \ldots, n \text{ and } \eta_{j+1} = \eta_j + \rho_j, \eta_0 = \eta \tag{5.37}$$

where moreover it holds (without further assumptions) that $\rho_j \in \mathcal{P}^{(V)}_A(\eta)$.

We observe that we can in fact always find a sequence as above such that $s(\rho_j) \subset \tilde{A}$. Indeed, the validity of the relation

$$\eta_j \sim_{\rho_j} \eta_{j+1} \tag{5.38}$$

depends on the values of η_j in the region $s(\rho_j)$ only, therefore the presence of a $\rho_{j'}, j' < j$ in the sequence with $s(\rho_{j'}) \subset \tilde{A}^c$ (which influences the configurations η_j in $\tilde{A}^c$ only) does not influence the validity of (5.38). Hence, one can omit all ρ_j with $s(\rho_j) \subset \tilde{A}^c$ and obtain a (shorter) sequence that still satisfies (5.37). We assume hence that the sequence $\rho_1, \dots, \rho_n$ was chosen such that $s(\rho_j) \subset \tilde{A}$. For such a sequence we check that

$$\eta_j' \sim_{\rho_j} \eta_{j+1}', \qquad \text{for } j = 1, \dots, n \text{ and } \eta_{j+1}' = \eta_j' + \rho_j, \eta_0 = \eta' \tag{5.39}$$

Indeed, since the validity of $\eta_0' \sim_{\rho_1} \eta_1'$ depends on the configurations η_0' in $s(\rho_1) \subset \tilde{A}$ only, and since $\eta_0|_{\tilde{A}} = \eta_0'|_{\tilde{A}}$ (because $\tilde{A} \subset A$) and $\eta_0 \sim_{\rho_1} \eta_1$, we see that $\eta_0' \sim_{\rho_1} \eta_1'$ holds and moreover $\eta_1|_{\tilde{A}} = \eta_1'|_{\tilde{A}}$. We can iterate this argument to obtain (5.39) together with $\eta_j|_{\tilde{A}} = \eta_j'|_{\tilde{A}}$ for $j = 1, \dots, n$. Hence we have proven in particular $\rho \in \mathcal{P}_A^{(V')}(\eta')$, hence $\mathcal{P}_A^{(V')}(\eta') \subset \mathcal{P}_A^{(V')}(\eta')$. The opposite inclusion follows analogously. $\qquad \square$

5.4 Smallness of $\mathcal{P}(\mu)$

We express the idea that for *most* configurations η, the collection of sets $s(\rho), \rho \in \mathcal{P}^{(V)}(\eta)$ looks like a sparse collection of isolated islands in V. To be closer to the application that we have in mind, we however formulate this for the collection of sets $S(\rho) \cap V, \rho \in \mathcal{P}^{(V)}(\eta)$. Since we need to count configurations η, it is useful to introduce the counting probability measure $\mathbb{P}^{(M)}$ on $\Omega_V^{(M)}$. We abbreviate $\mathbb{P} = \mathbb{P}^{(M)}$. The following definition depends on a constant c_1 that will be chosen to be small enough in Lemma 6.1 below.

Definition 5.4 (Good partitioning sets). *A $\mu \in \mathcal{F}^{(V)}$ is 'good' if the collection of subsets of V*

$$\{S(\rho) \cap V : \rho \in \mathcal{P}_V'(\eta) \text{ for some } \eta \in \mu\} \tag{5.40}$$

can be covered by $c_1 r$ sets such that each of those sets has diameter r. We let $\mathcal{F}_{\mathrm{g}}^{(V)} \subset \mathcal{F}^{(V)}$ be the collection of good μ.

By Proposition 5.1, we now deduce that configurations η such that $\mu(\eta)$ is not 'good', have small probability.

Lemma 5.8.
$$\mathbb{P}(\mu \notin \mathcal{F}_{\mathrm{g}}^{(V)}) \le C(r) M^{-cr} \tag{5.41}$$

Proof. Define, analogously to $\mathcal{P}_A'(\eta)$,

$$\mathcal{P}_A''(\eta) := \bigcup_{\eta': |\eta - \eta'| \le 2C_0(r) M^{\gamma_2}} \{\rho : s(\rho) \subset A, \exists \eta'' : \eta' \underset{\rho}{\sim} \eta''\} \tag{5.42}$$

Then, by Proposition 5.1,

$$\bigcup_{\eta' \in \mu(\eta)} \mathcal{P}_A'(\eta') \subset \mathcal{P}_A''(\eta), \qquad \text{for any } A \subset V. \tag{5.43}$$

First, note for any given $\rho \in \mathcal{P}^{(V)}$,

$$\mathbb{P}(\rho \in \mathcal{P}_V''(\eta)) \le C(2C_0(r) M^{\gamma_2})^{|s(\rho)|} \mathbb{P}(|E_{s(\rho)}(\eta + \rho) - E_{s(\rho)}(\eta)| \le M^{1+\gamma_1}) \tag{5.44}$$

The second factor on the right hand side can be bounded by $C(r) M^{-c|s(\rho)|}$, as follows from Lemma 5.6. Then, choosing γ_2 small enough, we get

$$\mathbb{P}(\rho \in \mathcal{P}_V''(\eta)) \le C(r) M^{-c} \tag{5.45}$$

If the collection (5.40) cannot be covered by n sets with diameter r, then, for any $\eta \in \mu$, there are at least $m = cn$ moves $\rho^1, \dots, \rho^m \in \mathcal{P}_V''(\eta)$ with mutually disjoint supports, i.e. $s(\rho^i) \cap s(\rho^j) = \emptyset$ for any $\rho^i \ne \rho^j$. We will now prove that

$$\mathbb{P}(\rho^1, \dots, \rho^m \in \mathcal{P}_V''(\eta)) = \prod_{i=1}^m \mathbb{P}(\rho^i \in \mathcal{P}''(\eta)) \le C(r) M^{-cm} \tag{5.46}$$

First, because of the locality of the definition of $\mathcal{P}_V''(\eta)$ and the fact that $\mathbb{P}$ is a product measure, the events

$$\rho^i \in \mathcal{P}_V''(\eta), \qquad \rho^j \in \mathcal{P}_V''(\eta), \qquad \text{for } s(\rho^i) \cap s(\rho^j) = \emptyset \tag{5.47}$$

are $\mathbb{P}$-independent. This is the equality in (5.46). The inequality follows directly from (5.45). The claim (5.41) now follows from (5.46) by noting that

1. for any η such that $\mu(\eta) \notin \mathcal{F}_{\mathrm{g}}^{(V)}$, $\rho^1, \ldots, \rho^m \in \mathcal{P}_V''(\eta)$ must hold for some collection $\rho^1, \ldots, \rho^m$ with mutually disjoint supports and $m = cr$.

2. the number of collections of ρ's with disjoint supports is $C(r)$ since the cardinality of the set $\mathcal{P} = \mathcal{P}^{(V)}$ is bounded by $C(r)$.

$\square$

6 Analysis of the resonant Hamiltonian: Left-Right splitting

As announced, we split the Hamiltonian Z into a left and a right part, Z_{L} and Z_{R}, such that these parts have a sparse commutator. The main result is in Proposition 6.2.

6.1 Preliminary definitions

Recall the (restricted) hyperplane $\mathbb{H}_a = \{x : x_1 = a\}$ and the strip

$$\mathbb{S} = \mathbb{S}_{a,r^2} = \{x \in \Lambda, |x_1 - a| < r^2\}. \tag{6.1}$$

For convenience we gather $y = (x_2, x_3, \ldots, x_d) \in \mathbb{Z}^{d-1}$. Sums over y are understood to range over the set $\{y : (a, y) \in \Lambda\}$, and we define the regions ($\tilde{B}_y$ will be used only in Section 7)

$$B_y := \{x : |x - (a, y)| \leq 2r^2\} \cap \mathbb{S}_{a,r^2}.$$

$$\tilde{B}_y := \{x : |x - (a, y)| \leq (2r)^{2d}\} \cap \mathbb{S}_{a,r^2} \tag{6.2}$$

We also abbreviate $\mathcal{P}^{(y)} = \mathcal{P}^{(B_y)}$ and $\mathcal{F}^{(y)} = \mathcal{F}^{(B_y)}$.

First, we define a procedure that assigns to any $\mu \in \mathcal{F}^{(y)}$ a decomposition of B_y into a left and right region $\mathrm{L}(\mu)$ and $\mathrm{R}(\mu)$:

Definition 6.1 (Left-right decomposition). *Fix y and $\mu \in \mathcal{F}^{(y)}$. Let $K_1^{(y)}, K_2^{(y)}, \ldots$ be the connected components of the collection*

$$\{(S(\rho) \cap B_y) : \rho \in \mathcal{P}^{(y)}(\mu)\}, \tag{6.3}$$

i.e.

$$\cup_j K_j^{(y)} = \cup_{\rho \in \mathcal{P}^{(y)}(\mu)}(S(\rho) \cap B_y), \qquad and \quad j \neq j' \Rightarrow \quad K_j^{(y)} \cap K_{j'}^{(y)} = \emptyset \tag{6.4}$$

Then, the left, resp. right region is

$$\mathrm{L}(\mu) := \{x \in B_y : x_1 \leq a\} \bigcup_{j : K_j \cap \{x_1 \leq a\} \neq \emptyset} K_j^{(y)}, \qquad \mathrm{R}(\mu) := B_y \setminus \mathrm{L}(\mu) \tag{6.5}$$

Note that, if $\mathcal{P}^{(y)}(\mu) = \emptyset$, then $\mathrm{L}(\mu) = \{x \in B_y : x_1 \leq a\}$, i.e. the left-right splitting is the most obvious one. The intuition is that for 'good' μ, the $\mathrm{L}(\mu)$ deviates from $\{x \in B_y : x_1 \leq a\}$ only in a few places, and in particular, $\mathrm{L}(\mu)$ can be determined locally. This is established in the following lemma. The reader might find it helpful to consult Figure 3, even though the latter is not meant to illustrate the full generality of Lemma 6.1

Lemma 6.1. *Let r be large enough and the constant c_1 in Definition 5.4 small enough. Fix y, y'. Let the triple $\{F_0, F_1, F_2\}$ form a partition of $B_y \cup B_{y'}$ such that*

$$\mathrm{dist}(F_1, F_2) \geq r^2/4, \qquad and \qquad B_y \setminus F_2 = B_{y'} \setminus F_2 =: F_{01} \tag{6.6}$$

and hence $F_{01} = F_0 \cup F_1 \subset (B_y \cap B_{y'})$. Choose $\mu \in \mathcal{F}^{(y)}, \mu' \in \mathcal{F}^{(y')}$ such that at least one of them is good, i.e. $\mu \in \mathcal{F}_{\mathrm{g}}^{(y)}$ or $\mu' \in \mathcal{F}_{\mathrm{g}}^{(y')}$ and such that

$$\eta_{F_{01}} = \eta'_{F_{01}} \quad for some \eta \in \mu, \eta' \in \mu' \tag{6.7}$$

Then,

$$\mathcal{P}_{F_1}^{(y)}(\mu) = \mathcal{P}_{F_1}^{(y')}(\mu') \tag{6.8}$$

and

$$\mathrm{L}(\mu) \cap F_1 = \mathrm{L}(\mu') \cap F_1 \tag{6.9}$$

Proof. For concreteness, let us assume that μ is good. From straightforward geometric considerations, using that μ is good, that $\text{dist}(F_1, F_2) \geq r^2/4$ and that r is chosen large enough and c_1 small enough, we can construct a partition $(\tilde{F}_0, \tilde{F}_1, \tilde{F}_2)$ of $B_y \cup B_{y'}$ such that

1.

$$\tilde{F}_0 \subset F_0, \qquad F_1 \subset \tilde{F}_1, \qquad F_2 \subset \tilde{F}_2 \tag{6.10}$$

2.

$$\tilde{F}_0 \cap S(\rho) = \emptyset \qquad \text{for any } \rho \in \underset{\eta \in \mu}{\cup} \mathcal{P}'_{B_y}(\eta). \tag{6.11}$$

3.

$$\text{dist}(\tilde{F}_1, \tilde{F}_2) > 2r. \tag{6.12}$$

We put

$$\tilde{F}_{01} := B_y \setminus \tilde{F}_2 = B_{y'} \setminus \tilde{F}_2 \tag{6.13}$$

(and hence $\tilde{F}_{01} = \tilde{F}_0 \cup \tilde{F}_1$). From 1) we get $\tilde{F}_{01} \subset F_{01}$ and from 3), we get $\partial_{2r}\tilde{F}_{01} \subset \tilde{F}_0 = \tilde{F}_0 \cap B_{y''}$ with $y'' = y, y'$. Therefore, 2) implies

$$\mathcal{P}'_{\partial_{2r}\tilde{F}_{01}}(\eta) = \emptyset \quad \text{for any } \eta \in \mu \tag{6.14}$$

Take now η, η' as in (6.7), i.e. in particular $\eta_{\tilde{F}_{01}} = \eta'_{\tilde{F}_{01}}$. For these η, η', we can apply Lemma 5.7 with $V = B_y, V' = B_{y'}, A = \tilde{F}_{01}$ to conclude that

$$\mathcal{P}^{(y)}_{\tilde{F}_{01}}(\mu) = \mathcal{P}^{(y')}_{\tilde{F}_{01}}(\mu') \tag{6.15}$$

We now show (6.9). Let us consider the connected components $K_j^{(y)}, K_i^{(y')}$ from Definition (6.1) for y, y', respectively. From (6.11) we get (by (5.32)) that

$$S(\rho) \cap \tilde{F}_0 = \emptyset, \qquad \text{for any } \rho \in \mathcal{P}^{(y)}(\mu). \tag{6.16}$$

Hence, $\tilde{F}_0$ does not intersect any of the components $K_j^{(y)}$ and therefore any one of them is either contained in $\tilde{F}_1$ or in $\tilde{F}_2$. This need not be true for the components $K_i^{(y')}$, but nevertheless, we can still deduce that no $K_i^{(y')}$ can intersect both $\tilde{F}_1$ and $\tilde{F}_2$. Indeed, if a given $K_i^{(y')}$ would intersect $\tilde{F}_1$ and $\tilde{F}_2$, than, since $\text{diam}(S(\rho)) \leq 2r$ for any ρ and $\text{dist}(\tilde{F}_1, \tilde{F}_2) > 2r$, we conclude that there must be a $\rho \in \mathcal{P}^{y'}(\mu')$ with $S(\rho) \in \tilde{F}_{01}$. However, this is in contradiciition with (6.15-6.16). Now we conclude by (6.15) that the connected components $K_j^{(y)}$ contained in $\tilde{F}_{01}$ coincide with the connected components $K_i^{(y')}$ contained in $\tilde{F}_{01}$ (and moreover, these components are in fact contained in $\tilde{F}_1$). This implies (6.9).

$\square$

6.2 Definition of Left-right splitting

For notational reasons, we associate, in an arbitrary way, to any subset $A \subset \mathbb{S}$ with $|A| \leq r$ a unique coordinate $y = y(A) \in \mathbb{Z}^{d-1}$ such that

$$y(A) \in \text{proj}_{2,\ldots,d-1}A \tag{6.17}$$

where we used the coordinate projections: if $x = (x_I, x_{I^c})$ with I a subset of $\{1, \ldots, d\}$, then $\text{proj}_I x = x_I$.

The definition of the left-right splitting is

$$Z_{\text{L},y} := \sum_{\mu \in \mathcal{F}^{(y)}} P_\mu \left(\sum_{\substack{A \subset \mathbb{S}\,:\, y(A) = y \\ A \subset \text{L}(\mu)}} \mathscr{D}(A) + \sum_{\substack{\rho \in \mathcal{P}^{(\mathbb{S})}\,:\, y(s(\rho)) = y \\ s(\rho) \subset \text{L}(\mu)}} W_\rho \right) \tag{6.18}$$

$$Z_{\text{R},y} := \sum_{\mu \in \mathcal{F}^{(y)}} P_\mu \left(\sum_{\substack{A \subset \mathbb{S}\,:\, y(A) = y \\ A \not\subset \text{L}(\mu)}} \mathscr{D}(A) + \sum_{\substack{\rho \in \mathcal{P}^{(\mathbb{S})}\,:\, y(s(\rho)) = y \\ s(\rho) \not\subset \text{L}(\mu)}} W_\rho \right) \tag{6.19}$$

It immediately follows that

$$Z_{\mathrm{L},y} + Z_{\mathrm{R},y} = \sum_{A:y(A)=y} \mathscr{D}(A) + \sum_{\rho \in \mathcal{P}^{(y)}:y(s(\rho))=y} W_\rho \tag{6.20}$$

Since the region B_y is 'broader' than the strip $\mathbb{S}$, any $\rho \in \mathcal{P}^{(\mathbb{S})}$ satisfies $\rho \in \mathcal{P}^{(y(s(\rho)))}$ and hence ρ appears exactly once in the sum on the right hand side. Therefore, we have indeed defined a splitting of Z:

$$Z = \sum_{y \in \mathbb{Z}^{d-1}:(a,y)\in\Lambda} Z_y, \qquad Z_y := Z_{\mathrm{L},y} + Z_{\mathrm{R},y} \tag{6.21}$$

We will now describe the good properties of this splitting. To translate the sparseness of a collection of configurations η into a bound on operators, we introduce the normalized trace of operators O that are restricted to low-energy, i.e. $\mathcal{P}_{\leq M}(O) = O$, by

$$\mathrm{tr}_A^{(M)}(O) := \frac{\mathrm{Tr}_A(O)}{\mathrm{Tr}_A(\mathbb{1}_{\leq M})}, \qquad \text{whenever } s(O) \subset A \tag{6.22}$$

where we used $\mathbb{1}_{\leq M} = \otimes_{x \in A}\chi(N_x \leq M)$. We note that the right-hand side does not depend on the set A, provided that $s(O) \subset A$ and so we can write $\mathrm{tr}^{(M)}(O)$ without ambiguity. For example, we will use the projections onto good configurations:

$$P_{\mathcal{F}_{\mathrm{g}}^{(y)}} = \sum_{\mu \in \mathcal{F}^{(y)}} P_\mu, \qquad \bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}} = \mathbb{1} - P_{\mathcal{F}_{\mathrm{g}}^{(y)}} \tag{6.23}$$

acting on $\mathcal{H}_{B_y}$, then, with the normalized trace, we can restate (5.41) as

$$\mathrm{tr}^{(M)}(\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}}) = \mathbb{P}(\mu(\eta) \notin \mathcal{F}^{(y)}) \leq C(r)M^{-cr} \tag{6.24}$$

We will also need the associated Hilbert-schmidt norm

$$\|O\|_{\mathrm{tr}^{(M)}}^2 := \mathrm{tr}^{(M)}(O^*O) \tag{6.25}$$

FIGURE

Figure 3: Spatial structure of the sets F_0, F_1, F_2 as used in Case C of the proof of Proposition 6.2. The shaded area is the set $\tilde{F}_0 = \tilde{F}_0 \cap B_y = \tilde{F}_0 \cap B_{y'}$. The small spirals indicate the sets $S(\rho)$ with $\rho \in \mathcal{P}'(\eta)$ or $\rho \in \mathcal{P}'(\eta')$.

Proposition 6.2. *Let $Z_{\mathrm{L},y}, Z_{\mathrm{R},y}$ be as described above. If r is chosen large enough, the following properties hold:*

1. *The supports satisfy $s(Z_{\mathrm{L},y}), s(Z_{\mathrm{R},y}) \subset B_y$ and the operators are bounded as*

$$\|Z_{\mathrm{L},y}\| \leq C(r)M^C, \qquad \|Z_{\mathrm{R},y}\| \leq C(r)M^C \tag{6.26}$$

2. *The 'left' and 'right' operators commute up to a sparse term:*

$$\|[Z_{\mathrm{L},y}, Z_{\mathrm{R},y'}]\|_{\mathrm{tr}^{(M)}} \leq C(r)M^{C-cr} \tag{6.27}$$

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3. *The 'left' part does not go too far to the right: for any operator O_{R} such that $\mathrm{proj}_1(s(O_{\mathrm{R}})) > a + r^2/2$,*

$$\|[Z_{\mathrm{L},y}, O_{\mathrm{R}}]\|_{\mathrm{tr}^{(M)}} \leq C(r) M^{C-cr} \|O_{\mathrm{R}}\| \tag{6.28}$$

Analogously, for any O_{L} such that $\mathrm{proj}_1(s(O_{\mathrm{L}})) < a - r^2/2$,

$$\|[Z_{\mathrm{R},y}, O_{\mathrm{L}}]\|_{\mathrm{tr}^{(M)}} \leq C(r) M^{C-cr} \|O_{\mathrm{L}}\| \tag{6.29}$$

Proof of Propostion 6.2. Let us start with some easy remarks.

a) If $\mu \in \mathcal{F}^{(y)}, \mu' \in \mathcal{F}^{(y')}$, then $[P_\mu, P_{\mu'}] = 0$ because both projections are diagonal in the same basis. For the same reason $[P_\mu, \mathscr{D}(A)] = 0$.

b) For ρ such that $s(\rho) \subset B_y$, $[P_\mu, W_\rho] = 0$ for any $\mu \in \mathcal{F}^{(y)}$, by the definition of $\mathcal{F}^{(y)}$, i.e. Definition 5.2.

c) If $s(\rho) \cap S(\rho') = \emptyset$ and $s(\rho') \cap S(\rho) = \emptyset$, then $[W_\rho, W_{\rho'}] = 0$.

The bounds in 1) follow trivially from the bounds following Definition 5.1. We now prove point 2).

To estimate the commutator $[Z_{\mathrm{L},y}, Z_{\mathrm{R},y'}]$, we first consider the term

$$[\sum_{\mu \in \mathcal{F}^{(y)}} P_\mu \sum_{\substack{\rho \in \mathcal{P}^{(y)} : y(s(\rho)) = y \\ s(\rho) \subset \mathrm{L}_y(\mu)}} W_\rho, \sum_{\mu' \in \mathcal{F}^{(y')}} P_{\mu'} \sum_{\substack{\rho' \in \mathcal{P}^{(y')} : y(s(\rho')) = y' \\ s(\rho') \not\subset \mathrm{L}_{y'}(\mu')}} W_{\rho'}] \tag{6.30}$$

where we added primes to the variables in the second term of the commutator for clarity. We consider four cases:

Case A: $y = y'$

If $\mu \neq \mu'$, then (6.30) vanishes by b) above and $P_\mu P_{\mu'} = P_\mu \delta_{\mu,\mu'}$. To see that

$$[P_\mu W_\rho, P_\mu W_{\rho'}] = 0, \qquad \text{if } s(\rho) \subset \mathrm{L}(\mu), s(\rho') \not\subset \mathrm{L}(\mu) \tag{6.31}$$

we use that, by the definition of the Left-Right decomposition, $S(\rho) \cap S(\rho') = \emptyset$ and therefore $[W_\rho, W_{\rho'}] = 0$, by c) above.

Case B: $3r < |y - y'| \leq 2r^2 - r$

We have that $s(\rho) \subset B_{y'}$ and $s(\rho') \subset B_y$. Therefore, by a), b) above, all commutators involving a projection P_μ vanish, just as in **case A**, and hence it suffices to consider the commutator

$$[W_\rho, W_{\rho'}] \tag{6.32}$$

Because $|y - y'| > 3r$, this trivially vanishes by c) above.

Case C: $0 < |y - y'| \leq 3r$

By the same reasoning as in **Case B**, it suffices to consider the commutator (6.32). Define

$$F_0 := \{x \in B_y \cup B_{y'} : r^2/4 < |y(x) - \frac{y + y'}{2}| \leq r^2/2\}, \tag{6.33}$$

$$F_1 := \{x \in B_y \cup B_{y'} : |y(x) - \frac{y + y'}{2}| \leq r^2/4\}, \tag{6.34}$$

$$F_2 := (B_y \cup B_{y'}) \setminus (F_0 \cup F_1) \tag{6.35}$$

If either μ or μ' is good, then the conditions of Lemma 6.1 are satisfied. Moreover, $S(\rho), S(\rho') \subset F_1$. If $S(\rho) \cap S(\rho') = \emptyset$, then there is nothing to prove because (6.32) vanishing trivially. If $S(\rho) \cap S(\rho') \neq \emptyset$, then Lemma 6.1 tells us that either both $S(\rho), S(\rho')$ are included in the 'left' L-set, or both are not included in the L-set. Therefore, their commutator does not appear in (6.30).

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So we conclude that the only non-vanishing contribution to (6.30) originates from pairs (μ, μ') such that none of them is good. We recast the sum over such pairs, for fixed ρ, ρ', as

$$\sum_{\mu \in \mathcal{F}^{(y)} \setminus \mathcal{F}_{\mathrm{g}}^{(y)}} \sum_{\mu' \in \mathcal{F}^{(y')} \setminus \mathcal{F}_{\mathrm{g}}^{(y')}} [P_\mu W_\rho, P_{\mu'} W_{\rho'}] = [\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}} W_\rho, \bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y')}} W_{\rho'}] \tag{6.36}$$

with the projections $\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}}$ as in (6.23). Then we bound

$$\| [\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}} W_\rho, \bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y')}} W_{\rho'}] \|_{\mathrm{tr}^{(M)}} \le 2 \|W_{\rho'}\| \|W_\rho\| \|\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y')}}\| \, \mathrm{tr}^{(M)}(\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}}) \tag{6.37}$$

The norms on the RHS are estimated as $\|W_\rho\| \le C(r) M^C$ by the properties stated following Definition 5.1, $\|\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y')}}\| \le 1$ using that $\bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y')}}$ is a projection, and the trace is bounded by $C(r) M^{-cr}$, see (6.24). Finally the sum over pairs ρ, ρ' is bounded by $C(r)$, the number of terms in the sum.

Case D: $2r^2 - r \le |y - y'| \le 2r^2$

In this case there is no reason for $[W_\rho, P_{\mu'}]$ to be small since it can happen that $s(\rho) \not\subset B_{y'}$. Instead we recast (6.30) as

$$\sum_{\rho, \rho': y(s(\rho)) = y, y(s(\rho')) = y'} [P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})} W_\rho, \bar{P}_{\mathcal{F}^{(y')}(\rho' \to \mathrm{L})} W_{\rho'}] \tag{6.38}$$

with

$$P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})} := \sum_{\mu \in \mathcal{F}^{(y)}: s(\rho) \subset \mathrm{L}(\mu)} P_\mu, \qquad \bar{P}_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})} := \mathbb{1} - P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})} \tag{6.39}$$

Since $S(\rho) \cap S(\rho') = \emptyset$, (6.38) is rewritten as a sum over $W_\rho [P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})}, W_{\rho'}] + [W_\rho, P_{\mathcal{F}^{(y')}(\rho' \to \mathrm{R})}] W_{\rho'}$. Let us look more generally at an expression of the form

$$[P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})}, O], \qquad \mathrm{dist}(s(O), s(\rho)) \ge r^2/2 \tag{6.40}$$

Then we claim

$$[P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})}, P_{\mathcal{F}_{\mathrm{g}}^{(y)}} O P_{\mathcal{F}_{\mathrm{g}}^{(y)}}] = 0. \tag{6.41}$$

To check this, we first write

$$P_{\mathcal{F}_{\mathrm{g}}^{(y)}} O P_{\mathcal{F}_{\mathrm{g}}^{(y)}} = \sum_{\eta, \eta': \mu(\eta), \mu(\eta') \in \mathcal{F}_{\mathrm{g}}^{(y)}} P_\eta O P_{\eta'}, \tag{6.42}$$

and, using the formula

$$P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})} P_\eta = P_\eta P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})} = \chi\big(\mu(\eta) \in \mathcal{F}^{(y)} : s(\rho) \subset \mathrm{L}(\mu)\big) P_\eta, \tag{6.43}$$

we note that any nonzero contribution to (6.41) has to come from pairs (η, η') in (6.42) such that

1. $\eta_{s(O)^c} = \eta'_{s(O)^c}$.

2. $\mu(\eta), \mu'(\eta')$ are good.

3. $s(\rho) \subset \mathrm{L}(\mu(\eta)), s(\rho) \not\subset \mathrm{L}(\mu(\eta'))$ or vice versa (see (6.43)).

Since $\mathrm{dist}(s(O), s(\rho)) > r^2/2$, we can construct a partition $\{F_0, F_1, F_2\}$ as in Lemma 6.1 (applied with $y' = y$) such that $S(\rho) \cap B_y \subset F_1$ and $s(O) \subset F_2$. Then Lemma 6.1 implies that there are no pairs that can contribute to the commutator in (6.41) and hence (6.41) holds. Therefore,

$$[P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})}, O] = [P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})}, \bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}} O P_{\mathcal{F}_{\mathrm{g}}^{(y)}} + P_{\mathcal{F}_{\mathrm{g}}^{(y)}} O \bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}} + \bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}} O \bar{P}_{\mathcal{F}_{\mathrm{g}}^{(y)}}] \tag{6.44}$$

and proceeding as in (6.37), this is bounded by $\|O\| M^{-cr}$. Plugging in $O = W_\rho$, we obtain again the bound $C(r) M^{-cr}$.

Case E: $|y - y'| > 2r^2$

In this case, the commutator vanishes obviously since the two terms act on disjoint regions.

Finally, if we consider instead of the commutator (6.30), one of the terms containing $\mathscr{D}(A)$, then we can repeat the above reasoning (though with many simplifications) and obtain the same result. This proves point 2).

And now to point 3): The estimates (6.28) and (6.29) are of course analogous and we consider only the former. Again we first consider only the terms W_ρ in $Z_{\mathrm{L},y}$. Let us first assume that $\mathrm{proj}_1(s(\rho)) \geq a + r^2/8$. Since $s(\rho) \subset \mathrm{L}(\mu)$, this means that the left $\mathrm{L}(\mu)$ set extends far to the right, and hence, if c_1 in Definition 5.4 is small enough, this implies that μ cannot be good. Just as in the estimates above, non-good μ give a contribution that is sufficiently small in $\|\cdot\|_{\mathrm{tr}^{(M)}}$ (see (6.37)) and hence we can disregard them. Then, we are left with ρ such that $\mathrm{proj}_1(s(\rho)) < a + r^2/8$. In that case $[W_\rho, O_\mathrm{R}] = 0$ because of disjoint supports and we simply have to estimate

$$[P_{\mathcal{F}^{(y)}(\rho \to \mathrm{L})}, O_\mathrm{R}], \qquad \mathrm{dist}(s(\rho), s(O_\mathrm{R})) \geq 3r^2/8 \tag{6.45}$$

The same expression was considered in (6.40) in **Case D** above, except that there we had the distance $r^2/4$ instead of $3r^2/8$. However, we easily see that the reasoning applies with $3r^2/8$ as well, provided that r is large enough.

The terms originating from $\mathscr{D}(A)$ are treated analogously. $\qquad\square$

7 Proof of Theorems 2.2 and 2.1

The crux of the proof of Theorem 2.2 is a left-right splitting of the Hamiltonian in the strip $\mathbb{S}$. This will be achieved below following Proposition 7.1 which itself relies heavily on Proposition 6.2. This left-right decomposition will allow us to study the local currents as sums of operators with small variance. We do this in Section 7.3. The proof of Theorem 2.1 is then a simple consequence, and it is given in the final Section 7.5.

7.1 States and Hilbert-Schmidt norms

We recall/introduce the states

$$\omega_{\beta,\Lambda}(O) := \frac{\mathrm{Tr}\, e^{-\beta H_\Lambda} O}{\mathrm{Tr}\, e^{-\beta H_\Lambda}}, \qquad \omega_{\beta,\Lambda,0}(O) := \frac{\mathrm{Tr}\, e^{-\beta H_\Lambda^{(0)}} O}{\mathrm{Tr}\, e^{-\beta H_\Lambda^{(0)}}}, \qquad O \in \mathcal{B}(\mathcal{H}_\Lambda) \tag{7.1}$$

In what follows, we suppress the dependence on β, Λ since all of our estimates will be uniform in Λ and in β whenever β is small enough, hence we simply write $\omega(\cdot), \omega_0(\cdot)$. We define the associated Hilbert-Schmidt norms

$$\|O\|_\omega := \omega(O^*O)^{1/2}, \qquad \|O\|_{\omega_0} := \omega_0(O^*O)^{1/2}, \tag{7.2}$$

completely analogous to the the norm (6.25) that was the Hilbert-Schmidt norm associated to the state $\mathrm{tr}^{(M)}(\cdot)$. We denote the covariance of observables by

$$\omega(O_1; O_2) := \omega(O_1 O_2) - \omega(O_1)\omega(O_2) = \omega((O_1 - \omega(O_1))(O_2 - \omega(O_2))) \tag{7.3}$$

7.2 Decomposition of the Hamiltonian in a strip

First we split the Hamiltonian $H = H_\Lambda$ as

$$H = H_{\mathbb{S}^c}^{(\mathrm{L})} + H_{\mathbb{S}^c}^{(\mathrm{R})} + H_{\mathbb{S}} \tag{7.4}$$

where

$$H_{\mathbb{S}^c}^{(\mathrm{L})} = \sum_{x:x_1 \leq a-r^2} H_x, \qquad H_{\mathbb{S}^c}^{(\mathrm{R})} = \sum_{x:x_1 \geq a+r^2} H_x \tag{7.5}$$

and

$$H_{\mathbb{S}} = \sum_{x:a-r^2 < x_1 < a+r^2} H_x \tag{7.6}$$

From the technical point of view, the main step in our analysis is the splitting of the Hamiltonian $H_{\mathbb{S}}$ as a sum of three terms. It is accomplished in the next proposition, which relies crucially on Proposition 4.3 and Proposition 6.2. Recall the regions $\tilde{B}_y$ introduced in (6.2).

Proposition 7.1 (Splitting of Hamiltonian). *Let $M = \beta^{-(1+c(r))/q}$. For sufficiently large $r \in \mathbb{N}$, there is a decomposition*

$$H_{\mathbb{S}} = \sum_{x \in \mathbb{S}} (1 - \mathcal{P}_{\leq M}) H_x + X_{\mathbb{S}}(\mathscr{A}) + \sum_{y \in \mathbb{Z}^{d-1} : (a,y) \in \Lambda} \tilde{Z}_y \tag{7.7}$$

such that

1. *The potentail $\mathscr{A}$ consists only of low-energy terms, $\mathcal{P}_{\leq M}(\mathscr{A}) = \mathscr{A}$ and*

$$\||\mathscr{A}\||_{C\beta^{-c},1} \leq C(r)\beta^{-C+cr} \tag{7.8}$$

2. *The operators $\tilde{Z}_y$ can be split as*

$$\tilde{Z}_y = \tilde{Z}_{\mathrm{L},y} + \tilde{Z}_{\mathrm{R},y} \tag{7.9}$$

   such that

$$s(\tilde{Z}_{\mathrm{L},y}) \subset \tilde{B}_y, \qquad \mathcal{P}_{\leq M}(\tilde{Z}_{\mathrm{L},y}) = \tilde{Z}_{\mathrm{L},y}, \qquad \|\tilde{Z}_{\mathrm{L},y}\| \leq C(r)\beta^{-C}, \tag{7.10}$$

   idem for $\tilde{Z}_{\mathrm{R},y}$, and

$$\||[\tilde{Z}_{\mathrm{L},y}, \tilde{Z}_{\mathrm{R},y'}]\||_{\omega_0} \leq C(r)\beta^{cr-C} \tag{7.11}$$

   Furthermore, the left/right part is supported on the left/right side of the strip $\mathbb{S}$ in the following sense:

$$\||[\tilde{Z}_{\mathrm{L},y}, O_{\mathrm{R}}]\||_{\omega_0} \leq C(r)\beta^{cr-C}\|O_{\mathrm{R}}\| \tag{7.12}$$

$$\||[\tilde{Z}_{\mathrm{R},y}, O_{\mathrm{L}}]\||_{\omega_0} \leq C(r)\beta^{cr-C}\|O_{\mathrm{L}}\| \tag{7.13}$$

   for any operators $O_{\mathrm{R}}, O_{\mathrm{L}}$ satisfying $|s(O_{\mathrm{R}})|, |s(O_{\mathrm{L}})| \leq C$ and

$$a + (3/4)r^2 < \mathrm{proj}_1(s(O_{\mathrm{R}})) < a + r^2, \qquad a - (3/4)r^2 > \mathrm{proj}_1(s(O_{\mathrm{L}})) > a - r^2 \tag{7.14}$$

Proof. We recall the operator $Z_y = Z_{\mathrm{L},y} + Z_{\mathrm{R},y}$ as defined in (6.21) and we also use the notation Z_y to denote the potential $\mathscr{A}_{Z_y}$ defined by

$$\mathscr{A}_{Z_y}(A) := \begin{cases} Z_y & A = B_y \text{ for some } y \\ 0 & \text{otherwise} \end{cases} \tag{7.15}$$

We claim that

$$H_{\mathbb{S}} = \sum_{x \in \mathbb{S}} (1 - \mathcal{P}_{\leq M}) H_x + X_{\mathbb{S}}(\mathcal{K}(\hat{\mathscr{F}})) + \sum_y X_{\mathbb{S}}(\mathcal{K}(Z_y)) \tag{7.16}$$

where (also below) sums over y are understood to range over the set $\{y \in \mathbb{Z}^{d-1} : (a,y) \in \Lambda\}$. By the decomposition in (4.39) and the relation $\mathcal{K}\mathcal{K}^{-1} = 1$ from Section 4.5,

$$H_{\mathbb{S}} - \sum_{x \in \mathbb{S}} (1 - \mathcal{P}_{\leq M})(H_x) = X_{\mathbb{S}}(\mathscr{E}) = X_{\mathbb{S}}(\mathcal{K}(\hat{\mathscr{F}} + \hat{\mathscr{G}} + \mathscr{D})) \tag{7.17}$$

Furthermore,

$$X_{\mathbb{S}}(\mathcal{K}(\hat{\mathscr{G}} + \mathscr{D})) = U_{\mathbb{S}} X_{\mathbb{S}}(\hat{\mathscr{G}} + \mathscr{D}) U_{\mathbb{S}}^* = U_{\mathbb{S}} Z U_{\mathbb{S}}^* = \sum_y U_{\mathbb{S}} Z_y U_{\mathbb{S}}^* = \sum_y X_{\mathbb{S}}(\mathcal{K}(Z_y)) \tag{7.18}$$

The first equality is (4.64), the second is the definition of Z (see (5.1)), the third is $Z = \sum_y Z_y$ (see (6.21)) and the fourth is (4.66) with $O = Z_y$. Hence, (7.16) follows by the equality of the first and last expression in (7.18) combined with (7.17).

We now split, according to (4.68),

$$H_{\mathbb{S}}(\mathcal{K}(Z_{\mathrm{L},y})) = U_{\tilde{B}_y} Z_{\mathrm{L},y} U_{\tilde{B}_y}^* + H_{\mathbb{S}}(\mathcal{I}_{\tilde{B}_y^c} \mathcal{K}(Z_{\mathrm{L},y})) \tag{7.19}$$

with $\mathcal{I}_{B^c}$ for a set B the restriction of potentials introduced in Section 4.5. Since $\mathrm{dist}(\tilde{B}_y^c, s(Z_{\mathrm{L},y})) \geq c|s(Z_{\mathrm{L},y})|$, we get from (4.69) that

$$\||\mathcal{I}_{\tilde{B}_y^c} \mathcal{K}(Z_{\mathrm{L},y})\||_{M^c,1} \leq C(r)M^{-cr^2+C} \tag{7.20}$$

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This estimate was in fact the reason to choose $(2r)^{2d}$ as the radius of the new ball. Finally we put

$$\mathscr{A} := \mathcal{K}(\hat{\mathscr{F}}) + \sum_y \mathcal{I}_{\tilde{B}_y^c} \mathcal{K}(Z_{\mathrm{L},y}) + \sum_y \mathcal{I}_{\tilde{B}_y^c} \mathcal{K}(Z_{\mathrm{R},y}) \tag{7.21}$$

and by the bounds (4.69) and (4.62), we then get the claimed bound on $\mathscr{A}$. From the above, it is already clear that we have chosen to define, with $V = U_{\tilde{B}_y}$,

$$\tilde{Z}_{\mathrm{L},y} = V Z_{\mathrm{L},y} V^*, \qquad \tilde{Z}_{\mathrm{L},y} = V Z_{\mathrm{L},y} V^* \tag{7.22}$$

which implies by the unitarity of V,

$$\|[\tilde{Z}_{\mathrm{L},y}, \tilde{Z}_{\mathrm{R},y'}]\|_{\mathrm{tr}^{(M)}} = \|[Z_{\mathrm{L},y}, Z_{\mathrm{R},y'}]\|_{\mathrm{tr}^{(M)}} \tag{7.23}$$

We can bound this by Proposition 6.2 2) and we thus obtain the bound (7.11), but in the norm $\|\cdot\|_{\mathrm{tr}^{(M)}}$ rather than $\|\cdot\|_{\omega_0}$. In Lemma 7.2 below, we explain how to relate these norms.

Then, we also have to control the commutators of $\tilde{Z}_{\mathrm{L},y}$ with O_{R}. Let us set

$$W := U_{B_y'}, \qquad \text{with } B_y' = \tilde{B}_y \cap \{x : \mathrm{dist}(x, s(O_{\mathrm{R}})) \le r\}. \tag{7.24}$$

and calculate

$$[\tilde{Z}_{\mathrm{L},y}, O_{\mathrm{R}}] = V[Z_{\mathrm{L},y}, V^* O_{\mathrm{R}} V] V^*$$
$$= V[Z_{\mathrm{L},y}, W^* O_{\mathrm{R}} W] V^* + V[Z_{\mathrm{L},y}, (V^* O_{\mathrm{R}} V - W^* O_{\mathrm{R}} W)] V^* \tag{7.25}$$

By Lemma 4.6, we get

$$\|V^* O_{\mathrm{R}} V - W^* O_{\mathrm{R}} W\| \le C(r) M^{-cr} \|O_{\mathrm{R}}\| \tag{7.26}$$

By the properties of $s(O_{\mathrm{R}})$ and B_y', the operator $W^* O_{\mathrm{R}} W$ is supported to the right of $a + r^2 - r - C$, and hence Proposition 6.2 3) gives, for large enough r,

$$\|[Z_{\mathrm{L},y}, W^* O_{\mathrm{R}} W]\|_{\mathrm{tr}^{(M)}} \le C(r) M^{C-cr} \|W^* O_{\mathrm{R}} W\|. \tag{7.27}$$

Combining (7.25), (7.26) and (7.27) and using the unitarity of V, W, we get

$$\|[\tilde{Z}_{\mathrm{L},y}, O_{\mathrm{R}}]\|_{\mathrm{tr}^{(M)}} \le C(r) M^{C-cr} \|O_{\mathrm{R}}\| \tag{7.28}$$

and analogously for $[\tilde{Z}_{\mathrm{R},y}, O_{\mathrm{L}}]$.

It remains to argue that the bounds on O' in $\|\cdot\|_{\mathrm{tr}^{(M)}}$ can be converted to bounds in the norm $\|\cdot\|_{\omega_0}$.

Lemma 7.2. *For any operator O with $\mathcal{P}_{\le M}(O) = O$ and $M = \beta^{-(1+\gamma_c)/q}$, with $\gamma_c > 0$,*

$$\|O\|_{\omega_0} \le (C\beta^{-\gamma_c/q})^{|s(O)|} \|O\|_{\mathrm{tr}^{(M)}}, \tag{7.29}$$

Proof. Since the density matrices of the state ω_0 and $\mathrm{tr}^{(M)}$ are both product and diagonal in the same basis, this boils down to the estimate

$$M\left(\sum_{\eta \ge 0} e^{-\beta \eta^q}\right)^{-1} \le CM\beta^{1/q} \le C\beta^{-\gamma_c/q}. \tag{7.30}$$

$\square$

Hence, to get the bounds (7.11, 7.12, 7.13) from the corresponding bounds with $\|\cdot\|_{\mathrm{tr}^M}$ and Lemma 7.2, we have to choose the exponent γ_c/q such that $cr - (\gamma_c/q)r^{2d} > c'r$ (for some $c' > 0$ depending on c). This is achieved by decreasing the cut-off exponent $\gamma_c = \gamma_c(r)$ sufficiently fast as r grows. $\square$

Finally, we define a left-right decomposition of the full strip Hamiltonian

$$H_{\mathbb{S}} = \tilde{H}_{\mathbb{S}}^{(\mathrm{L})} + \tilde{H}_{\mathbb{S}}^{(\mathrm{R})} \tag{7.31}$$

Since only the splitting of the $\tilde{Z}_y$-terms in the Hamiltonian matters in the end, we can simply assign all other terms to, say, the right part, and define by setting simply

$$\tilde{H}_{\mathbb{S}}^{(\mathrm{L})} := \sum_{y \in \mathbb{Z}^{d-1}, (a,y) \in \Lambda} \tilde{Z}_{\mathrm{L},y}, \qquad \tilde{H}_{\mathbb{S}}^{(\mathrm{R})} = H_{\mathbb{S}} - \tilde{H}_{\mathbb{S}}^{(\mathrm{L})} \tag{7.32}$$

The tildes in this expression serve to distinguish this splitting from the 'naive' left-right decomposition $H_{\mathbb{S}} = H_{\mathbb{S}}^{(\mathrm{L})} + H_{\mathbb{S}}^{(\mathrm{R})}$ with

$$H_{\mathbb{S}}^{(\mathrm{L})} := \sum_{x:a-r^2 < x_1 \leq a} H_x, \qquad H_{\mathbb{S}}^{(\mathrm{R})} := \sum_{x:a < x_1 < r^2 + a} H_x \tag{7.33}$$

corresponding to the 'naive' left-right decomposition of the total Hamiltonian $H = H^{(\mathrm{L})} + H^{(\mathrm{R})}$

$$H^{(\mathrm{L})} = H_{\mathbb{S}}^{(\mathrm{L})} + H_{\mathbb{S}^c}^{(\mathrm{L})}, \qquad H^{(\mathrm{R})} = H_{\mathbb{S}}^{(\mathrm{R})} + H_{\mathbb{S}^c}^{(\mathrm{R})} \tag{7.34}$$

which was already introduced in Section 2.4 to define the current.

7.3 Decomposition of the current

Our goal is to estimate

$$J = J_{\mathbb{H}} = \mathrm{i}[H, H^{(\mathrm{L})}] \tag{7.35}$$

where $H = H^{(\mathrm{L})} + H^{(\mathrm{R})}$ by using the left-right decomposition of the Hamiltonian constructed in Section 7.2. Namely, we set

$$\tilde{H}^{(\mathrm{L})} = \tilde{H}_{\mathbb{S}}^{(\mathrm{L})} + H_{\mathbb{S}^c}^{(\mathrm{L})}, \qquad \tilde{H}^{(\mathrm{R})} = \tilde{H}_{\mathbb{S}}^{(\mathrm{R})} + H_{\mathbb{S}^c}^{(\mathrm{R})} \tag{7.36}$$

such that

$$H = \tilde{H}^{(\mathrm{L})} + \tilde{H}^{(\mathrm{R})}. \tag{7.37}$$

Hence by splitting

$$H^{(\mathrm{L})} = \tilde{O}_{\mathbb{S}} + \tilde{H}^{(\mathrm{L})}, \qquad \tilde{O}_{\mathbb{S}} := (H_{\mathbb{S}}^{(\mathrm{L})} - \tilde{H}_{\mathbb{S}}^{(\mathrm{L})}) \tag{7.38}$$

we get

$$-\mathrm{i}J = -\mathrm{i}J^{(1)} - \mathrm{i}J^{(2)} := [H, \tilde{O}_{\mathbb{S}}] + [H, \tilde{H}^{(\mathrm{L})}] \tag{7.39}$$

$$= [H, \tilde{O}_{\mathbb{S}}] + [\tilde{H}^{(\mathrm{R})}, \tilde{H}^{(\mathrm{L})}] \tag{7.40}$$

$$= [H, \tilde{O}_{\mathbb{S}}] + [H_{\mathbb{S}^c}^{(\mathrm{R})}, \tilde{H}_{\mathbb{S}}^{(\mathrm{L})}] + [\tilde{H}_{\mathbb{S}}^{(\mathrm{R})}, \tilde{H}_{\mathbb{S}}^{(\mathrm{L})}] + [\tilde{H}_{\mathbb{S}}^{(\mathrm{R})}, H_{\mathbb{S}^c}^{(\mathrm{L})}] \tag{7.41}$$

For future use, note that $\omega(J) = \omega(J^{(1)}) = 0$ by stationarity, and therefore also $\omega(J^{(2)}) = 0$. We recognize the expression for J in Theorem 2.2 since the commutators on the right-hand side are sums of local operators. For further discussion, let us make this explicit by defining

$$V_A^{(\mathrm{R},j)} := \begin{cases} (1 - \mathcal{P}_{\leq M})(H_x) & \text{if } A = \{x' : |x' - x| \leq 1\} & j = 1 \\ \mathscr{A}(A) & & j = 2 \\ \tilde{Z}_{\mathrm{R},y} & \text{if } A = \tilde{B}_y & j = 3 \\ \mathcal{P}_{\leq M}(H_x) & \text{if } A = \{x' : |x' - x| \leq 1\} \text{ with } x_1 - a = r^2 \text{ or } r^2 + 1 & j = 4 \\ (1 - \mathcal{P}_{\leq M})(H_x) & \text{if } A = \{x' : |x' - x| \leq 1\} \text{ with } x_1 - a = r^2 \text{ or } r^2 + 1 & j = 5 \end{cases}$$

for some x, y and $V_A^{(\mathrm{R},j)} = 0$ in all other cases. Similarly,

$$V_A^{(\mathrm{L},j)} := \begin{cases} \tilde{Z}_{\mathrm{L},y} & \text{if } A = \tilde{B}_y & j = 3 \\ \mathcal{P}_{\leq M}(H_x) & \text{if } A = \{x' : |x' - x| \leq 1\} \text{ with } -(x_1 - a) = r^2 \text{ or } r^2 + 1 & j = 4 \\ (1 - \mathcal{P}_{\leq M})(H_x) & \text{if } A = \{x' : |x' - x| \leq 1\} \text{ with } -(x_1 - a) = r^2 \text{ or } r^2 + 1 & j = 5 \end{cases}$$

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for some x, y and $V_A^{(\mathrm{L},j)} = 0$ in all other cases. The assymetry between R and L in these formulas is due to the arbitrary choice, made following (7.31), to assign all nonessential terms to the R part. Next, we set

$$-i\tilde{I}_{A,A'}^{(j,j')} := [V_A^{(\mathrm{R},j)}, V_{A'}^{(\mathrm{L},j')}] \tag{7.42}$$

such that, from the decomposition of Proposition 7.1 and the definition of $J^{(2)}$ above, we indeed have

$$J^{(2)} = \sum_{A,A':A\cap A'\neq\emptyset} \sum_{j,j'=1,\ldots,5} \tilde{I}_{A,A'}^{(j,j')} \tag{7.43}$$

The tilde on $\tilde{I}_{A,A'}^{(j,j')}$ is to distinguish it from the centered operators $I_{A,A'}^{(j,j')} := \tilde{I}_{A,A'}^{(j,j')} - \omega(I_{A,A'}^{(j,j')})$ that will be used later.

Next, we have to establish the desired properties of these local terms.

7.4 Classification of current operators and proof of Theorem 2.2

We classify the 'current' operators $\tilde{I}_{A,A'}^{(j,j')}$ introduced above.

Lemma 7.3. *For any j, j' and A, A', the operator $\tilde{I}_{A,A'}^{(j,j')}$ can be written as*

$$\tilde{I}_{A,A'}^{(j,j')} = \sum_{i=1}^{C} K_{A,A'}^{(j,j',i)} \tag{7.44}$$

where each of the operators $K_{A,A'}^{(j,j',i)}$ is of the K-type introduced in Section A.1 (Appendix), such that, for any of these operators $K = K_{A,A'}^{(j,j',i)}$, we have $s(K) \subset A \cup A'$ and

$$w(K) \leq C(r)\beta^{-C+cr+c(r)|A\cup A'|}. \tag{7.45}$$

with $w(K)$ as defined in (A.7) (Appendix). Similarly, for any pair of the operators $\tilde{I}_{A,A'}^{(j,j')}, \tilde{I}_{A'',A'''}^{(j'',j''')}$ (not necessarily distinct) with $(A \cup A') \cap (A'' \cup A''') \neq \emptyset$, the product

$$(\tilde{I}_{A,A'}^{(j,j')})^* \tilde{I}_{A'',A'''}^{(j'',j''')} \tag{7.46}$$

is a sum of C operators of the K-type satisfying $s(K) \subset A \cup A' \cup A'' \cup A'''$ and

$$w(K) \leq C(r)\beta^{-C+cr+c(r)|A\cup A'\cup A''\cup A'''|}. \tag{7.47}$$

Before proceeding with the proof, let us try to clarify the meaning of this lemma. Let us choose one term $\tilde{I} = \tilde{I}_{A,A'}^{(j,j')}$ contributing to $J^{(2)}$. Then, the bounds (7.46) and (7.45) tell us that, for some operators $K_1, \ldots, K_C$

$$\omega(\tilde{I}^*\tilde{I}) = \omega(K_1) + \ldots + \omega(K_C) \leq w(K_1) + \ldots + w(K_C) \leq C(r)\beta^{-C+cr+c(r)|A\cup A'|} \tag{7.48}$$

where the first inequality follows from Theorem A.1. Hence, $\tilde{I}$ is small in the $\|\cdot\|_\omega$ norm. This fact is of course a crucial ingredient of the intuition that correlations of the current are small in the thermal state. Furthermore, the lemma stresses the fact that these operators K_i are of the K-type and this takes most of the effort in the proof. This is important because operators of K-type are the operators for which we can prove spatial decorrelation estimates and estimate the $\|\cdot\|_\omega$. The philosophy of estimating $\|K\|_\omega$ in Theorem A.1 consists in essence in relating $\|K\|_\omega$ to $\|K\|_{\omega_0}$.

Then, let us give the main (quite simple intuition) why $\|\tilde{I}\|_{\omega_0} = \omega_0(\tilde{I}^*\tilde{I})$ is small, taking for granted that this can then be translated to the $\|\cdot\|_\omega$ norm. Recall that $\tilde{I}$ is a commutator of the form $\tilde{I} = [V^{\mathrm{R}}, V^{\mathrm{L}}]$ for some operators $V^{\mathrm{R}}, V^{\mathrm{L}}$. The most simple-minded bound is (by Cauchy-Schwarz)

$$|\omega_0([V^{\mathrm{R}}, V^{\mathrm{L}}])| \leq |\omega_0(V^{\mathrm{R}}V^{\mathrm{L}})| + |\omega_0(V^{\mathrm{L}}V^{\mathrm{R}})| \leq 2\|V^{\mathrm{R}}\|_{\omega_0}\|V^{\mathrm{L}}\|_{\omega_0}, \tag{7.49}$$

hence it suffices to show that at least one of the V-operators is small in the $\|\cdot\|_{\omega_0}$ and the other not too big. Of course, if also suffices if this is true in the $\|\cdot\|$-norm since

$$\|V\|_{\omega_0} \leq \|V\|. \tag{7.50}$$

Let us now apply this to the problem at hand. The operators $V^{\mathrm{L},j}, V^{\mathrm{R},j}$ are small in $\|\cdot\|_{\omega_0}$ in the cases $j = 1, 2, 5$, and not too big in all other cases $j = 3, 4$, meaning that the norm is bounded by $C(r)M^C$. Hence any commutator involving $j = 1, 2, 5$ is obviously small. That leaves the commutators $[V^{\mathrm{L},3}, V^{\mathrm{R},3}]$, $[V^{\mathrm{L},3}, V^{\mathrm{R},4}]$ and $[V^{\mathrm{L},4}, V^{\mathrm{R},3}]$. Those commutators cannot be controlled by the simple bound (7.49). Instead, the first of these commutators is small by the bound (7.11) in Proposition 7.1 (This was the main result achieved in the previous sections) and the second and third are small because of (7.12) and (7.13) in Proposition 7.1. The reason that these bounds apply is that $V^{\mathrm{R},4}$ is situated 'far to the right' and $V^{\mathrm{L},4}$ 'far to the left', because they are terms of the Hamiltonian that are situated at the boundary of the strip $\mathbb{S}$.

Proof. We consider the cases for j, j' separately and we give the proof of (7.44),(7.45) for some exemplary cases, the others being simplifications of the former. The bounds on (7.46) are then also obtained analogously and therefore we skip them entirely.

The case $j = 3, j' = 3$. This is the most intuitive case. The bound (7.11) in Proposition 7.1 gives immediately, with $O := \tilde{I}_{A,A'}^{(3,3)}$

$$\|O\|_{\omega_0} \leq C(r)\beta^{cr-C}, \qquad \mathcal{P}_{\leq 2M}(O) = O \tag{7.51}$$

and hence O is of K-type, and the bound on $w(K)$ follows since $|A \cup A'| = C(r)$.

In what follows, we let h_x stand for one of the following three operators a_x, a_x^*, N_x^q.
The case $j = 3, j' = 4$. Here, $\tilde{I}_{A,A'}^{(3,4)}$ is a sum of terms of the form

$$[Z_{\mathrm{R},y}, E_{x_1}] \otimes E_{x_2}, \qquad E_{x_i} = \mathcal{P}_{\leq M}(h_{x_i}), \quad x_1 \in A, x_2 \notin A'. \tag{7.52}$$

Since x_1 is necessarily on the left boundary of the strip $\mathbb{S}_a$, the bound (7.12) in Proposition 7.1 yields

$$\|O\|_{\omega_0} \leq C(r)\beta^{-C+cr}, \qquad \text{with } O := [Z_{\mathrm{R},y}, E_{x_1}] \tag{7.53}$$

Then, $O' = O \otimes E_{x_2}$ is of K-type. By the Cauchy-Schwarz inequality

$$\|O_1 O_2\|_{\omega_0}^2 = \omega_0(O_1 O_2 O_2^* O_1^*) \leq \|O_1\|_{\omega_0} \|O_2 O_2^* O_1^*\|_{\omega_0} \leq \|O_1\|_{\omega_0} \|O_2\|^2 \|O_1\| \tag{7.54}$$

applied to $O_1 = O', O_2 = E_{x_2}$, we then get the desired bound on $w(K)$, because $|A \cup A'| = C(r)$.

The case $j = 2, j' = 5$. Then $O := V^{(\mathrm{R},2)} = \mathscr{A}_2(A)$ satisfies $\mathcal{P}_{\leq M}(O) = O$ and $V^{(\mathrm{L},5)}$ is (a sum of) operators of the form $(1 - \mathcal{P}_{\leq M})(h_x)$ or $(1 - \mathcal{P}_{\leq M})(h_{x_1} h_{x_2})$ for some x, x_1, x_2. Let us first do the simpler case $(1 - \mathcal{P}_{\leq M})(h_x)$. If $x \notin s(O)$ then the commutator vanishes so we assume $x \in s(O)$. We split

$$(1 - \mathcal{P}_{\leq M})(h_x) = \mathcal{P}_{>2M}(h_x) + E_x, \qquad \mathcal{P}_{\leq 2M}(E_x) = (E_x), \qquad \|E_x\| \leq M^C \tag{7.55}$$

Obviously, $O\mathcal{P}_{>2M}(B) = \mathcal{P}_{>2M}(B)O = 0$ for any B so it suffices to consider the E_x-term. We set

$$O' := [O, E_x], \qquad \mathcal{P}_{\leq 2M}(O') = O', \tag{7.56}$$

such that $K := O'$ is of K-type, and we estimate, using the information on O from Theorem 7.1 1),

$$\|O'\| \leq 2\|E_x\|\|O\| \leq C(r)\beta^{-C+cr+c|A|} \tag{7.57}$$

Since $\|O'\|_{\omega_0} \leq \|O'\|$ and $|A \cup A'| \geq c|A|$, the desired estimate (7.45) holds.

Next, let us consider the case $(1 - \mathcal{P}_{\leq M})(h_{x_1} h_{x_2})$ and we again consider $x_1, x_2 \in s(O)$. We can split

$$(1 - \mathcal{P}_{\leq M})(h_{x_1} h_{x_2}) = (1 - \mathcal{P}_{\leq M})(h_{x_1})(1 - \mathcal{P}_{\leq M})(h_{x_2}) \tag{7.58}$$

$$+ (1 - \mathcal{P}_{\leq M})(h_{x_1})\mathcal{P}_{\leq M}(h_{x_2}) \tag{7.59}$$

$$+ \mathcal{P}_{\leq M}(h_{x_1})(1 - \mathcal{P}_{\leq M})(h_{x_2}) \tag{7.60}$$

and then

$$(1 - \mathcal{P}_{\leq M})(h_{x_i}) = \mathcal{P}_{>2M}(h_{x_i}) + E_{x_i} \tag{7.61}$$

with E_{x_i} the same properties as in (7.55). Terms with $\mathcal{P}_{>2M}(h_{x_i})$ vanish again such that all non-vanishing terms consist of operators invariant under $\mathcal{P}_{\leq 2M}$ whose norm is estimated as in (7.57), so also in this case we get operators of K-type with the desired estimate on $w(K)$.

In the case where $x_1 \in s(O), x_2 \notin s(O)$, we define $O' := [O, E_{x_1}]$. We split $h_{x_2} = \mathcal{P}_{>M}(h_{x_2}) + (1 - \mathcal{P}_{>M})(h_{x_2})$. Taking the first term, we obtain the operator

$$O' \otimes \mathcal{P}_{>M}(h_{x_2}) \tag{7.62}$$

which is of K-type, and the desired bound on $w(K)$ follows by the bounds on O' above. For the second term, we now set $E_{x_2} := (1 - \mathcal{P}_{>M})(h_{x_2})$ and we have again $\mathcal{P}_{\leq 2M}(E_{x_2}) = E_{x_2}$ so that we obtain terms of the type

$$O'' = O' \otimes E_{x_2}, \qquad \mathcal{P}_{\leq 2M}(O'') = O'' \tag{7.63}$$

which is of K-type, and the bound on $w(K)$ follows by $\|O''\|_{\omega_0} \leq \|O''\| \leq \|O'\|\|E_{x_2}\| \leq \beta^{-C+cr+c|A|}$ and, again $|A \cup A'| \geq c|A|$.

As already indicated above, the other cases follow analogously. $\qquad\square$

Analogously to Lemma 7.3, we have to check

Lemma 7.4. *The operators $\tilde{O}_{\mathbb{S}}$ introduced in Section 7.3 can be written as*

$$\tilde{O}_{\mathbb{S}} = \sum_{A \subset \mathbb{S}} \tilde{O}_A, \qquad \tilde{O}_A = \sum_{i=1}^{C} K_A^i \tag{7.64}$$

where each of the operators K_A^i is of the K-type introduced in Section A.1 (Appendix) such that, for any $K = K_A^i$, we have $s(K) \subset A$ and

$$w(K) \leq C(r)\beta^{-C+c(r)|s(K)|} \tag{7.65}$$

Similarly, for any pair of the operators $\tilde{O}_A, \tilde{O}_{A'}$ (not necessarily distinct) with $A \cap A' \neq \emptyset$, the product $\tilde{O}_A \tilde{O}_{A'}^*$ is a sum of C operators of K-type satisfying (7.65).

This is proven using the same ideas as in Lemma 7.3, though there are much less terms to consider. Therefore, we skip the proof.

Proof of Theorem 2.2. We put

$$\tilde{I}_A = \sum_{A_1, A_2 : A_1 \cup A_2} \tilde{I}_{A_1, A_2}^{(j,j')} \tag{7.66}$$

and from Lemma 7.3 and Theorem A.1 1), and noting that the number of terms in the above sum is bounded by $C^{|A|}$, we get that (for $A \cap A' \neq \emptyset$)

$$\omega(\tilde{I}_A) \leq C(r)\beta^{-C+cr+c(r)|A|} \tag{7.67}$$

$$\omega(\tilde{I}_A^* \tilde{I}_{A'}) \leq C(r)\beta^{-C+cr+c(r)|A \cup A'|} \tag{7.68}$$

We now put $I_A := \tilde{I}_A - \omega(\tilde{I}_A)$. Since $\omega(J^{(2)}) = 0$, we get that

$$J^{(2)} = \sum_A \tilde{I}_A \qquad \Rightarrow \qquad J^{(2)} = \sum_A I_A \tag{7.69}$$

Moreover, (7.67), (7.68) are still valid for $I_A, I_{A'}$ replacing $\tilde{I}_A, \tilde{I}_{A'}$. Hence we have shown that the operators I_A have all properties claimed in Theorem 2.2

Analogously, from Lemma 7.4 and Theorem A.1 1), we get (for $A \cap A' \neq \emptyset$)

$$\omega(\tilde{O}_A) \leq C(r)\beta^{-C+c(r)|A|} \tag{7.70}$$

$$\omega(\tilde{O}_A^* \tilde{O}_{A'}) \leq C(r)\beta^{-C+c(r)|A \cup A'|} \tag{7.71}$$

we put $O_A := \tilde{O}_A - \omega(\tilde{O}_A)$, then we have

$$[H, \sum_A \tilde{O}_A] = [H, \sum_A O_A] \tag{7.72}$$

and the bounds (7.70), (7.71) are still valid for $O_A, O_{A'}$ replacing $\tilde{O}_A, \tilde{O}_{A'}$. $\qquad\square$

7.5 Proof of Theorem 2.1

At this point, we reinstate the dependence on the hyperplane position a, writing $J_{\mathbb{H}_a}^{(j)}$. Let us now define

$$\mathcal{J}_\tau^{(j)} = \frac{1}{\sqrt{\tau|\Lambda|}} \int_0^\tau \mathrm{d}t \sum_a J_{\mathbb{H}_a}^{(j)}(t) \tag{7.73}$$

and recall from (7.39) that $\mathcal{J}_\tau = \mathcal{J}_\tau^{(1)} + \mathcal{J}_\tau^{(2)}$. By Cauchy-Schwarz,

$$\omega(\mathcal{J}_\tau \mathcal{J}_\tau) \le 2 \sum_{j=1,2} \omega(\mathcal{J}_\tau^{(j)} \mathcal{J}_\tau^{(j)}) \tag{7.74}$$

And hence we can estimate $j = 1, 2$ separately.

7.5.1 The current $\mathcal{J}_\tau^{(1)}$

For $j = 1$, we use, with $O_{\mathbb{S}_a} = \tilde{O}_{\mathbb{S}_a} - \omega(\tilde{O}_{\mathbb{S}_a})$,

$$\mathcal{J}_\tau^{(1)} = \frac{1}{\sqrt{\tau|\Lambda|}} \sum_a (O_{\mathbb{S}_a}(t) - O_{\mathbb{S}_a}(0)) \tag{7.75}$$

and hence, by using again Cauchy-Schwarz and the invariance of ω under the dynamics,

$$\omega(\mathcal{J}_\tau^{(1)} \mathcal{J}_\tau^{(1)}) \le \frac{4}{\tau|\Lambda|} \sum_{a,a'} \omega(O_{\mathbb{S}_a} O_{\mathbb{S}_{a'}})$$

$$= \frac{4}{\tau|\Lambda|} \sum_{a,a'} \omega(\tilde{O}_{\mathbb{S}_a}; \tilde{O}_{\mathbb{S}_{a'}})$$

$$= \frac{4}{\tau|\Lambda|} \sum_{a,a',A,A'} \omega(\tilde{O}_{A,a}; \tilde{O}_{A',a'}) \chi(A \subset \mathbb{S}_{a,r^2}) \chi(A' \subset \mathbb{S}_{a',r^2}). \tag{7.76}$$

Using Theorem A.1 2), Lemma 7.4, and particular the fact that all our estimates are uniform in the hyperplane position a, we see that $\omega(\tilde{O}_{A,a}; \tilde{O}_{A',a'})$ decays exponentially in $\mathrm{dist}(A, A')$. and we bound (7.76) by β^{-C}/τ.

7.5.2 The current $\mathcal{J}_\tau^{(2)}$

For $j = 2$, we proceed somewhat differently; by Cauchy-Schwarz,

$$\omega(\mathcal{J}_\tau^{(2)} \mathcal{J}_\tau^{(2)}) = \frac{1}{|\Lambda|} \sum_{a,a'} \int_0^\tau \mathrm{d}t \int_0^\tau \mathrm{d}t' \omega(J_{\mathbb{H}_a}^{(2)}(t) J_{\mathbb{H}_{a'}}^{(2)}(t')) \le \frac{\tau}{|\Lambda|} \sum_{a,a'} \omega(J_{\mathbb{H}_a}^{(2)}(t) J_{\mathbb{H}_{a'}}^{(2)}(t)) \tag{7.77}$$

By time-translation invariance we can drop t in the argument. Since $\omega(J_{\mathbb{H}_a}^{(2)}) = 0$, the last expression equals a connected correlation function

$$\frac{\tau}{|\Lambda|} \sum_{a,a'} \omega(J_{\mathbb{H}_a}^{(2)}; J_{\mathbb{H}_{a'}}^{(2)}) = \frac{\tau}{|\Lambda|} \sum_{a,a'} \sum_{A,A'} \omega(\tilde{I}_{A,a}; \tilde{I}_{A',a'}) \chi(A \subset \mathbb{S}_{a,r^2+2}) \chi(A' \subset \mathbb{S}_{a',r^2+2}) \tag{7.78}$$

Using Theorem A.1 2), Lemmas 7.3, and particular the fact that all our estimates are uniform in the hyperplace position a, we see that $\omega(\tilde{I}_{A,a}; \tilde{I}_{A',a'})$ decays exponentially in $\mathrm{dist}(A, A')$, and in particular, we bound (7.77) by

$$\tau \beta^{-C+cr} \tag{7.79}$$

7.5.3 Bound on $\kappa_\tau(\beta)$

Combining the conclusions from Sections 7.5.1 and 7.5.2

$$\kappa_\tau(\beta) = \beta^2 \omega(\mathcal{J}_\tau \mathcal{J}_\tau) \leq 2\beta^2 \sum_{j=1,2} \omega(\mathcal{J}_\tau^{(j)} \mathcal{J}_\tau^{(j)}) \leq \frac{\beta^{-C}}{\tau} + \tau \beta^{-C+cr} \tag{7.80}$$

Taking now $\tau = \beta^{-m}$, we get Theorem 2.1.

A   Appendix: Decay of correlations

In this section, we prove some clustering properties of the high-temperature states in our model. Recall the states $\omega_{\beta,\Lambda}(\cdot)$ and $\omega_{\beta,\Lambda,0}(\cdot)$ defined in (7.1). In what follows, we again suppress the dependence on β, Λ since all of our estimates will be uniform in Λ and in β whenever β is small enough, hence we simply write $\omega(\cdot), \omega_0(\cdot)$.

A.1   Result

Recall the projection operators $\mathcal{P}_{\leq M}$ and $\mathcal{P}_{>M}$ acting on potentials and operators. Throughout this section, we set

$$M = \beta^{-(1+\gamma_c)/q}, \qquad \text{for some } 0 < \gamma_c < q/(q-1) - 1 \tag{A.1}$$

We specify two classes of observables. The first class consists of low-energy operators O, satisfying

$$O = \mathcal{P}_{\leq 2M}(O), \qquad \text{and} \qquad |s(O)| < \infty \tag{A.2}$$

The second class of observables is defined starting from monomials Y in creation/annihilation operators

$$Y = a_{x_m}^\sharp \dots a_{x_2}^\sharp a_{x_1}^\sharp, \tag{A.3}$$

for some $x_1, x_2, \dots, x_m \in \mathbb{Z}^d$, $m \in \mathbb{N}$ and $a_x^\sharp$ either a_x^* or a_x. Moreover, we assume the polynomial to be normal-ordered, i.e. all a_x appear to the right of a_x^*. We let $\deg(Y) := m$, i.e. the degree of Y and, for any $x \in s(Y) = \{x_1, \dots, x_m\}$, we define $\deg_x(Y)$ as the number of $j \in \{1, \dots, m\}$ such that $x_j = x$. Then

$$\sum_{x \in s(Y)} \deg_x(Y) = \deg(Y). \tag{A.4}$$

Given a low-energy observable O and a monomial Y as above, with $s(O) \cap s(Y) = \emptyset$, we consider

$$K = O \otimes \mathcal{P}_{>M}(Y) \tag{A.5}$$

allowing that $O = \mathbb{1}$ or $Y = \mathbb{1}$, corresponding to $s(O) = \emptyset$ and $\deg(Y) = 0$. We will refer to operators of the form (A.5) 'observables of K-type'. This class of operators is chosen so that it matches our needs as closely as possible, but it is of course in no sense the minimal one for which a result like the upcoming theorem can be proven.

Theorem A.1 (Correlation decay at high temperature). *Assume that $q > 1$ and fix a parameter α such that $0 < 2\alpha < 1 - 1/q$. Let us abbreviate, for monomials Y as above,*

$$v(Y) := \beta^{-\deg(Y)/2} \left(e^{-\beta^{-\gamma_c/2}}\right)^{|s(Y)|} \prod_{x \in s(Y)} \deg_x(Y)! \tag{A.6}$$

provided that $Y \neq \mathbb{1}$ and $v(Y) = 1$ if $Y = \mathbb{1}$. There is a $\beta_c > 0$ such that for $\beta < \beta_c$, the following hold, for all observables K, K' of K-type with O, Y as in (A.5),

1)
$$|\omega(K)| \leq C^{|s(O)|+\deg(Y)} \|O\|_{\omega_0} v(Y) =: w(K) \tag{A.7}$$

2)
$$|\omega(K; K')| \leq w(K) w(K') \sum_{x \in s(K), x' \in s(K')} \beta^{\alpha|x-x'|}, \qquad \text{for } s(K) \cap s(K') = \emptyset \tag{A.8}$$

The constant C in (A.7) depends only on α, γ_c, the exponent q and the spatial dimension d.

From the estimate in (A.8) and inspecting the range of values for α, we could guess the behaviour of the correlation length as a function of q, β

$$\xi_{\text{corr.}} \propto \frac{q}{(q-1)} |\ln \beta|^{-1}, \qquad \text{for small } \beta. \tag{A.9}$$

We see that $\xi_{\text{corr.}}$ diverges as $q \to 1$. This is consistent with the fact that for $q = 1$, the system is harmonic and the correlation length is seen to be independent of β. In contrast, as the above formula shows, for $q > 1$, our upper bound for the correlation length decreases with decreasing β.

There is an extensive literature on exponential decay of correlations at high temperature, i.e. results like Theorem A.1. However, we did not find any existing result that fits our needs. This is due to 1) the fact that our one-site space is unbounded and 2) the necessity to have bounds in terms of the Hilbert-Schmidt norm (or some other norm that can capture the sparseness) of the observables, as we have on the right hand side of the inequality (A.7) and hence the right hand side of (A.8). The work [21] addresses the first point, in that it treats unbounded spin systems, and [22] gets close to addressing the second point, but we have not found any combination of these results. In classical spin systems, the approach to decay of correlations via the logarithmic Sobolev inequality or Poincare inequality provides just the type of bounds we need, see e.g. [23], but, as far as we know, this approach has not been fully adapted to the quantum case yet.

Therefore, we set up a cluster expansion to prove Theorem A.1, following to some extent [24]. This is organized as follows. In Section A.2, we give the general setup which is not specific to our model and which contains some basic results and philosophy from cluster expanions. In Section A.3, we prove bounds on so-called polymer weights needed to carry through the cluster expansion. It is this part where we deal with the unboundedness of the on-site Hilbert space, and, more generally, where we need the observables to be of K-type. In the short Section A.3.1, we combine the bounds of Section A.3 with the machinery of Section A.2 to give the proof of Theorem A.1. We should stress that the material in Sections A.2 and A.3.1 is completely standard, therefore we present proofs in those sections in a compact way.

A.2 Polymer representations and cluster expansion

To decompose the Hamiltonian, we will use 'plaquettes' B. Each plaquette is defined to consist of a finite set $s(B) \subset \mathbb{Z}^d$ and, for each $x \in s(B)$, a pair of variables $(\sigma_{x,+}, \sigma_{x,-}) \in \mathbb{N} \times \mathbb{N}$. To such a plaquette B, we associate the operator

$$V_B = \prod_{x \in s(B)} (a_x^*)^{\sigma_{x,+}} (a_x)^{\sigma_{x,-}} \tag{A.10}$$

Moreover, we restrict ourselves to the case where, $s(B) = \{x\}$ or $s(B) = \{x, x'\}$ for some x, x' with $|x - x'| = 1$, and

$$\sum_{x \in s(B)} (\sigma_{x,+} + \sigma_{x,-}) \leq 2.$$

If needed, we indicate that $\sigma_{x,\pm}$ are associated to a plaquette by writing $\sigma_{x,\pm}(B)$.

Then, our Hamiltonian can be written as

$$H_\Lambda = H_\Lambda^{(0)} + \sum_{B : s(B) \subset \Lambda} g(B) V_B \tag{A.11}$$

where the sum is over plaquettes B, and $g(B)$ is a coupling constant satisfying $|g(B)| \leq 1$. We consider finite collections Γ of pairs (B, τ) with $\tau \in [0, \beta]$ and we write them as ordered sequences

$$\Gamma = ((B_1, \tau_1), \ldots, (B_n, \tau_n)), \qquad \text{with } n = |\Gamma| \tag{A.12}$$

such that $(\tau_1, \ldots, \tau_n)$ is in the simplex

$$\Delta_n(\beta) = \{0 \leq \tau_1 \leq \tau_2 \ldots \leq \tau_n \leq \beta\} \tag{A.13}$$

The ambiguity in (A.12) that occurs when $\tau_j = \tau_{j+1}$ will be irrelevant as we will mostly integrate $\tau_1, \ldots, \tau_n$ with the Lesbegue measure. For convenience, we also define the collection of plaquettes appearing in (A.12);

$$\mathcal{B}(\Gamma) := \{B_j : j = 1, \ldots, n\}, \tag{A.14}$$

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and n_B the multiplicity with which a plaquette B appears, such that

$$\sum_{B \in \mathcal{B}(\Gamma)} n_B = n \tag{A.15}$$

A.2.1 Polymer representation of the partition function

For a sequence Γ as in (A.12), we set, for $\Gamma \neq \emptyset$,

$$R(\Gamma) := V_{B_n}(\tau_n) \ldots V_{B_2}(\tau_2) V_{B_1}(\tau_1), \qquad \text{with } V_B(\tau) = e^{\tau H_\Lambda^{(0)}} V_B e^{-\tau H_\Lambda^{(0)}} \tag{A.16}$$

and $R(\emptyset) := \mathbb{1}$. Then we can represent the partition function

$$Z_\Lambda = Z_\Lambda(\beta) = \operatorname{Tr} e^{-\beta H_\Lambda} \tag{A.17}$$

as a series;

$$\frac{Z_\Lambda}{Z_{\Lambda,0}} = \int d\Gamma \, \omega_0(R(\Gamma)), \tag{A.18}$$

where we used the shorthand

$$\int d\Gamma \, \ldots = \sum_{n \geq 0} \sum_{\substack{B_1, \ldots, B_n \\ s(B_j) \subset \Lambda}} \int_{\Delta_n(\beta)} d\tau_1 \ldots d\tau_n \quad \ldots \tag{A.19}$$

and it is understood that for $n = 0$, the sums/integrals are absent. For example, the right hand side of (A.18) starts with the term $\omega_0(R(\emptyset)) = \omega_0(\mathbb{1}) = 1$. Formally, the identity (A.18) follows readily by the Duhamel expansion. To establish this rigorously, one first checks that the series on the right hand side is absolutely convergent, uniformly for $g(B) \in \{z \in \mathbb{C} : |z| \leq 1\}$. This is not explicitly proven here but one can easily deduce it from the bounds derived in Section A.3. Therefore, the right hand side of (A.18) is the Taylor series of an analytic function in $g(B)$. By explicit calculation, one checks that it coincides with the Taylor series of the left hand side.

For two finite sets $S, S' \subset \mathbb{Z}^d$, we define the adjacency relation

$$S \sim S' \quad \Leftrightarrow \quad S \cap S' \neq \emptyset \tag{A.20}$$

and we call a collection $\mathcal{S}$ of sets S connected if the collection is connected by the adjacency relation $\sim$. A connected collection will below also be called a cluster. We say that Γ is connected iff. the collection $\mathcal{S}(\Gamma) := \{s(B) : B \in \mathcal{B}(\Gamma)\}$ is connected. If Γ is not connected, then we can decompose $\mathcal{S}(\Gamma)$ in a unique way into maximally connected components, and this induces a decomposition $\Gamma_1, \ldots, \Gamma_m$, such that

$$\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m, \tag{A.21}$$

We then obtain the factorization

$$\omega_0(R(\Gamma)) = \prod_{j=1}^{m} \omega_0(R(\Gamma_j)) \tag{A.22}$$

because ω_0 is a product state, i.e. $\omega_0(O_1 O_2) = \omega_0(O_1)\omega_0(O_2)$ whenever $s(O_1) \cap s(O_2) = \emptyset$. It is now advantageous to reorganize the expansion (A.18) by collecting the contributions of connected Γ corresponding to the same domain $s(\Gamma) := \cup_{B \in \mathcal{B}(\Gamma)} s(B)$.

To that end, we define, for a finite, nonempty S,

$$\varrho(S) := \int_{\substack{s(\Gamma)=S \\ \Gamma \text{ connected}}} d\Gamma \, \omega_0(R(\Gamma)) \tag{A.23}$$

Let us denote by $\mathfrak{B}_\Lambda$ the set of all finite collections $\mathcal{S}$ of sets $S \subset \Lambda$ and we call such a collection $\mathcal{S} \in \mathfrak{B}_\Lambda$ admissible iff., for any two different $S, S' \in \mathcal{S}$, $S \nsim S'$. Then our polymer representation for the partition function reads

$$\frac{Z_\Lambda}{Z_{\Lambda,0}} = \sum_{\substack{\mathcal{S} \in \mathfrak{B}_\Lambda \\ \mathcal{S} \text{ admissible}}} \prod_{S \in \mathcal{S}} \varrho(S) \tag{A.24}$$

where the term with $\mathcal{S} = \emptyset$ is defined to be 1. To check (A.24), one relies on (A.22) and a similar factorization property for the sums/integrals abbreviated by $\int d\Gamma$.

A.2.2 Abstract cluster expansion

In this section, it is convenient to take an abstract point of view. Consider complex weights $\varpi(S) \in \mathbb{C}$ for finite sets $S \subset \mathbb{Z}^d$. Define

$$\Upsilon_\Lambda := \sum_{\substack{\mathcal{S} \in \mathfrak{B}_\Lambda \\ \mathcal{S} \text{ admissible}}} \prod_{S \in \mathcal{S}} \varpi(S) \tag{A.25}$$

For a collection $\mathcal{S}$, we introduce 'truncated weights'

$$\varpi^T(\mathcal{S}) = \sum_{\mathscr{G} \in \mathfrak{G}^c(\mathcal{S})} (-1)^{|\mathscr{E}(\mathscr{G})|} \prod_{\{S,S'\} \in \mathscr{E}(\mathscr{G})} 1_{[S \sim S']} \prod_{S'' \in \mathcal{S}} \varpi(S'') \tag{A.26}$$

where the sum runs over $\mathfrak{G}^c(\mathcal{S})$, the set of connected graphs with vertex set $\mathcal{S}$, $\mathscr{E}(\mathscr{G})$ is the edge set of the graph $\mathscr{G}$ (there are no self-edges), and the first product runs over the edge set $\mathscr{E}(\mathscr{G})$. Note that if $\mathcal{S}$ is not a cluster, then $\varpi^T(\mathcal{S}) = 0$.

Next, we state the basic result of cluster expansions, cfr. (eq. 4) in [25].

Theorem A.2. *Assume there is $a > 0$ such that, for any x,*

$$\sum_{S \subset \Lambda : S \sim \{x\}} \mathrm{e}^{a|S|} |\varpi(S)| \le a. \tag{A.27}$$

Then $\Upsilon_\Lambda \ne 0$,

$$\log \Upsilon_\Lambda = \sum_{\mathcal{S} \in \mathfrak{B}_\Lambda} \varpi^T(\mathcal{S}), \tag{A.28}$$

and, for any x,

$$\sum_{\mathcal{S} \in \mathfrak{B}_\Lambda : \mathcal{S} \sim \{x\}} \left| \varpi^T(\mathcal{S}) \right| \le a \tag{A.29}$$

where the condition $\mathcal{S} \sim S'$ means that there is a $S \in \mathcal{S}$ such that $S \sim S'$.

In what follows, we use the notation $\varrho^T(\cdot)$, defined from weights $\varrho(\cdot)$, as in the abstract case above.

A.2.3 Expansion for observables and correlations

We have already defined the notion of connectedness for sequences Γ as connectedness for the collection $\mathcal{S}(\Gamma)$. Given a nonempty set A, we say that Γ is A-connected if the collection $\mathcal{S}(\Gamma) \cup_j \{A_j\}$ is connected, with A_j the connected components of A.

Consider an operator K with $|s(K)| < \infty$. For a finite set $S \subset s(K)^c$, we define formally (because we do not address here the convergence of the series on the right hand side)

$$\varrho_K(S) := \int_{\substack{\Gamma \ s(K)\text{-connected} \\ s(\Gamma) \cap s(K)^c = S}} \mathrm{d}\Gamma \, \omega_0(R(\Gamma)K) \tag{A.30}$$

The contribution to the right hand side from $\Gamma = \emptyset$ is $\omega_0(K)$ whenever $s(K)$ is connected, and 0 whenever $s(K)$ is not connected. Note that for $S = \emptyset$, the constraint in (A.30) reads simply $s(\Gamma) \subset s(K)$ whenever $s(K)$ is connected, and then $\varrho_K(\emptyset)$ does in general not vanish, whereas $\varrho_K(\emptyset) = 0$ whenever $s(K)$ is not connected. Note also that, if $\varrho_K(S) \ne 0$ and $S \ne \emptyset$, then S has distance 1 to any of the connected components of $s(K)$. Let us for the time being, until the end of Section A.2.3, assume that $s(K), s(K')$ are connected. By mimicking the steps leading to (A.24), we then obtain the following polymer representation for $\omega(K)$

$$\omega(K) = \frac{Z_0}{Z} \sum_{S_0 \subset \Lambda} \varrho_K(S_0) \sum_{\substack{\mathcal{S} \in \mathfrak{B}_{\Lambda \setminus (S_0 \cup s(K))} \\ \mathcal{S} \text{ admissible}}} \prod_{S \in \mathcal{S}} \varrho(S) \tag{A.31}$$

$$= \frac{Z_0}{Z} \sum_{S_0 \subset \Lambda} \varrho_K(S_0) \sum_{\substack{\mathcal{S} \in \mathfrak{B}_\Lambda \\ \mathcal{S} \text{ admissible}}} \prod_{S \in \mathcal{S}} \varrho(S) \chi(S \nsim (S_0 \cup s(K))) \tag{A.32}$$

Let us now assume that the criterion (A.27) of Theorem A.2 is satisfied for some a, then we can apply Theorem A.2 both to the quotient of partition functions in (A.24) and to each term in the S_0-sum in (A.32) to obtain

$$\log \frac{Z}{Z_0} = \sum_{\mathcal{S} \in \mathfrak{B}_\Lambda} \varrho^T(\mathcal{S}) \tag{A.33}$$

$$\log \sum_{\substack{\mathcal{S} \in \mathfrak{B}_\Lambda \\ \mathcal{S} \text{ admissible}}} \prod_{S \in \mathcal{S}} \varrho(S) \chi(S \nsim (S_0 \cup s(K))) = \sum_{\mathcal{S} \in \mathfrak{B}_\Lambda} \chi[\mathcal{S} \nsim (S_0 \cup s(K))] \varrho^T(\mathcal{S}) \tag{A.34}$$

Therefore, we can write.

$$\omega(K) = \sum_{S_0} \varrho_K(S_0) \, \mathrm{e}^{-\sum_{\mathcal{S} \in \mathfrak{B}_\Lambda} \varrho^T(\mathcal{S})} \mathrm{e}^{\sum_{\mathcal{S} \in \mathfrak{B}_\Lambda} \chi[\mathcal{S} \nsim (S_0 \cup s(K))] \varrho^T(\mathcal{S})}$$

$$= \sum_{S} \varrho_K(S) \, \mathrm{e}^{-f(S \cup s(K))} \tag{A.35}$$

where it is understood (also below) that S, S_0 range over subsets of Λ and we used the shorthand (up to now only with $m = 1$)

$$f(A_1, A_2, \ldots, A_m) := \sum_{\mathcal{S} \in \mathfrak{B}_\Lambda} \chi(\mathcal{S} \sim A_1, \mathcal{S} \sim A_2, \ldots, \mathcal{S} \sim A_m) \varrho^T(\mathcal{S}) \tag{A.36}$$

Take now K, K' such that $\mathrm{dist}(s(K), s(K')) > 1$ and both $s(K), s(K')$ are connected. Then $s(KK') = s(K) \cup s(K')$ is not connected. Mimicking again all the above steps, and using the definition of $\varrho_{KK'}(\cdot)$, we can then derive

$$\omega(KK') = \sum_{\substack{S, S' \\ (S \cup s(K)) \nsim (S' \cup s(K'))}} \varrho_K(S) \varrho_K(S') \, \mathrm{e}^{-f(S \cup S' \cup s(KK'))} + \sum_S \varrho_{KK'}(S) \, \mathrm{e}^{-f(S \cup s(KK'))} \tag{A.37}$$

such that, after some algebra involving in particular the identity

$$f(A_1 \cup A_2) = f(A_1) + f(A_2) - f(A_1, A_2) \tag{A.38}$$

we obtain

$$\omega(K; K') = \sum_{S,S'} \varrho_K(S) \varrho_{K'}(S') \mathrm{e}^{-f(S \cup S' \cup s(KK'))} \left(\mathrm{e}^{f(S \cup s(K), S' \cup s(K'))} - 1 \right)$$

$$- \sum_{\substack{S,S' \\ (S \cup s(K)) \sim (S' \cup s(K'))}} \varrho_K(S) \varrho_{K'}(S') \, \mathrm{e}^{-f(S \cup S' \cup s(KK'))}$$

$$+ \sum_S \varrho_{KK'}(S) \, \mathrm{e}^{-f(S \cup s(KK'))} \tag{A.39}$$

where we used the shorthand $\omega(K; K') = \omega(KK') - \omega(K)\omega(K')$. This formula can be used to exhibit some decay of the correlation $\omega(K; K')$ in $\mathrm{dist}(s(K), s(K'))$, as we explain now.

Lemma A.3. *Assume that the criterion* (A.27) *is satisfied for the weights* $\varrho_\theta(S) := \theta^{-|S|} \varrho(S)$ *for some* $a > 0$ *and* $0 < \theta < 1$. *Then*

$$|f(A)| \leq a\theta|A|, \qquad |f(A, A')| \leq a \sum_{x \in A, x' \in A'} \theta^{|x-x'|} \tag{A.40}$$

Let K, K' *be observables such that* $s(K), s(K')$ *is connected, but* $\mathrm{dist}(s(K), s(K')) > 1$. *For an observable* $\tilde{K}$, *let* $\#_c(\tilde{K})$ *be the number of connected components of the set* $s(\tilde{K})$ *and let*

$$b_{\tilde{\theta}}(\tilde{K}) := |\varrho_{\tilde{K}}(\emptyset)| + \sum_{S: S \neq \emptyset} \tilde{\theta}^{-(|S| + \#_c(\tilde{K}))} |S| |\varrho_{\tilde{K}}(S)|, \qquad 0 < \tilde{\theta} < 1. \tag{A.41}$$

Then

$$|\omega(K)| \le \theta_1^{-|s(K)|} b_{\theta_1}(K). \tag{A.42}$$

$$|\omega(K;K')| \le C(1+a)\theta_1^{-|s(K)|-|s(K')|}\left(b_{\theta\theta_1}(K)b_{\theta\theta_1}(K') + b_{\theta\theta_1}(KK')\right) \sum_{x\in s(K), x'\in s(K')} \theta^{|x-x'|}. \tag{A.43}$$

with $\theta_1 = \mathrm{e}^{-2a\theta}$ and with the constant C independent of θ, a.

Proof. Note first, by inspection of (A.26), that

$$\varrho_\theta^T(\mathcal{S}) = \theta^{-\sum_{S\in\mathcal{S}}|S|}\varrho^T(\mathcal{S}). \tag{A.44}$$

Therefore the estimate

$$\sum_{\mathcal{S}} \chi(\mathcal{S}\sim\{x\}, \mathcal{S}\sim\{x'\})|\varrho^T(\mathcal{S})| \le \theta^{|x-x'|}\sum_{\mathcal{S}}\chi(\mathcal{S}\sim\{x\})|\varrho_\theta^T(\mathcal{S})| \le a\theta^{|x-x'|}. \tag{A.45}$$

follows from the simple observation $\inf_{\mathcal{S}:\mathcal{S}\sim\{x\}, \mathcal{S}\sim\{x'\}} \theta^{\sum_{S\in\mathcal{S}}|S|} \le \theta^{|x-x'|}$ and Theorem A.2 applied with $\varpi = \varrho_\theta$. Summing over $x\in A, x'\in A'$, this yields the second claim in (A.40), whereas the first one follows more directly from Theorem A.2.

With the estimates (A.40) in hand, the proof of (A.43) is a lengthy but straightforward calculation starting from (A.39). The stated bound is proven for the three terms on the right hand side of (A.39). Let us do the first term which is the most complicated one. Using the bounds $|\mathrm{e}^z - 1| \le |z|\mathrm{e}^{|z|}$ for $z\in\mathbb{C}$, and the bounds in (A.40), we get

$$\left|\sum_{S,S'} \varrho_K(S)\varrho_K(S')\mathrm{e}^{-f(S\cup S'\cup s(KK'))}\left(\mathrm{e}^{f(S\cup s(K), S'\cup s(K'))} - 1\right)\right|$$

$$\le a\sum_{S,S'}|\varrho_K(S)||\varrho_K(S')|\,\mathrm{e}^{2a\theta(|S|+|S'|+|s(K)|+|s(K')|)}\times$$

$$\left(\sum_{x\in S, x'\in S'} + \sum_{x\in S, x'\in s(K')} + \sum_{x\in s(K), x'\in S'} + \sum_{x\in s(K), x'\in s(K')}\right)\left(\theta^{|x-x'|}\right) \tag{A.46}$$

We again split this into four terms corrsponding to the four sums in the last expression. The fourth sum gives, upon summing S, S'

$$a\mathrm{e}^{2a\theta(s(K)+s(K'))}b_{\theta_1}(K)b_{\theta_1}(K') \sum_{x\in s(K), x'\in s(K')} \theta^{|x-x'|}. \tag{A.47}$$

The third sum gives, upon summing S,

$$a\mathrm{e}^{2a\theta(s(K)+s(K'))}b_{\theta_1}(K)\sum_{S'}\sum_{x\in s(K), x'\in S'} \theta^{|x-x'|}|\varrho_K(S')|\theta^{(|S'|+1)}(\theta\theta_1)^{-(|S'|+1)} \tag{A.48}$$

$$\le a\mathrm{e}^{2a\theta(s(K)+s(K'))}b_{\theta_1}(K)b_{\theta_1\theta}(K')\sum_{x\in s(K)} \theta^{\mathrm{dist}(x, s(K'))} \tag{A.49}$$

where we used the triangle inequality $|S'| + 1 + |x - x'| \ge \mathrm{dist}(x, s(K'))$ to get the last inequality. The first and second sums in (A.46) are similar. Hence, since $b_{\theta'}(K) \le b_{\theta''}(K)$ for $\theta' \ge \theta''$, we have obtained the desired bound on (A.46), namely (A.43). In the two remaining terms of (A.39), we always estimate $\mathrm{e}^{f(A)}$ by $\mathrm{e}^{a\theta|A|}$. The smallness comes then from the constraint on S, S'. The bound on (A.42) is obtained analogously, but simpler. $\square$

A.3   Bounds on polymer weights

The following lemma contains estimates on the weights ϱ, from which Theorem A.1 will easily follow. Throughout this section, we assume that β is taken small enough and we not repeat this at every step.

Lemma A.4. *Fix a parameter $\alpha = \alpha(q)$ satisfying $0 < 2\alpha < 1 - 1/q$. Recall the weights $\varrho(S), \varrho_K(S)$ from Section A.2 and let $S \subset \mathbb{Z}^d$ be finite. Then*

1.

$$|\varrho(S)| \leq (C\beta)^{\alpha|S|} \tag{A.50}$$

2. Consider an observable $K = O \otimes \mathcal{P}_{\geq M}(Y)$ 'of K-type', as defined in Section A.1. Then,

$$|\varrho_K(S)| \leq w(K) \times \begin{cases} (C\beta)^{\alpha(|S|+\#_c(K))} & S \neq \emptyset \\ 1 & S = \emptyset \end{cases}, \tag{A.51}$$

with $w(K)$ as defined in Theorem A.1 and $\#_c(K)$ the number of connected components of $s(K)$.

uniformly in Λ, provided that $S, s(K) \subset \Lambda$.

Note that replacing $(C\beta)$ by β in the above lemma yields an equivalent claim upon adjusting α. The same will be true often in the proof, below in Section A.3.2, but we prefer to keep the constants to avoid repeated readjusting of exponents. However, we do need to readjust constants, in particular the constant in the definition of $w(K)$. Before giving the lengthy proof of Lemma A.4, let us first use it to give the

A.3.1 Proof of Theorem A.1

We give the proof in the case where the sets $s(K), s(K')$ are connected (because Lemma A.3 is restricted to this case). The general case follows by the same reasoning.

Step 1 For any α' satisfying $0 < 2\alpha' < 1 - 1/q$, the criterion (A.27) is satisfied for the weights $\varrho_\theta(S) := \theta^{-|S|}\varrho(S)$ with $a = 1$ and $\theta = \beta^{\alpha'}$. To see this, we combine Lemma A.4 1) for some $\alpha'' > \alpha'$ with the geometrical fact

$$\sum_{S:S \ni x} \chi(S \text{ connected}) c^{|S|} \leq 1, \qquad \text{for small enough } c \tag{A.52}$$

Step 2 For any α' satisfying $0 < 2\alpha' < 1 - 1/q$ and observable K of K-type, we establish

$$b_\theta(K) \leq w(K), \qquad \text{with } \theta = \beta^{\alpha'}. \tag{A.53}$$

This is a straightforward consequence of Lemma A.4 2) for some $\alpha'' > \alpha'$, using again the geometrical fact (A.52) (and keeping in mind that $\text{dist}(S, s(K)) = 1$ whenever $\varrho_K(S) \neq 0$ and $S \neq \emptyset$).

Step 3 The two claims of Theorem A.1 follow by the results (A.42) and (A.43) of Lemma A.3, using Steps 1 and 2 above with $\alpha' > \alpha$ and noting that, for $\theta = \beta^\alpha$, the quantity $\theta_1 = e^{-2a\theta}$ in Lemma A.3 can made arbitrarily close to 1 by taking β large enough, and that $w(KK') = w(K)w(K')$ whenever $s(K) \cap s(K') = \emptyset$.

A.3.2 Proof of Lemma A.4

Let us first fix some additional notation. For a given Γ, we set

$$\sigma_x(B) := \sigma_{x,+}(B) + \sigma_{x,-}(B), \qquad n(x) := \sum_{B_j:s(B_j) \ni x} \sigma_x(B_j),$$

and

$$\mathcal{N}(\Gamma) := \prod_{x \in s(\Gamma)} n_x! \tag{A.54}$$

We introduce a 'cut-off state'

$$\omega_{0,2M}(O) := \omega_0(\mathcal{P}_{\leq 2M}(O)) \tag{A.55}$$

The following lemma is a purely combinatorial bound. Recall the quantity

$$v(Y) = \beta^{-\deg(Y)/2} (e^{-\beta^{-\gamma_c/2}})^{|s(Y)|} \prod_{x \in s(Y)} \deg_x(Y)! \tag{A.56}$$

for a monomial Y (introduced in Theorem A.1).

Lemma A.5. *Fix a parameter κ such that $1 > \kappa > 1/q$. Then, for any Γ and monomial Y,*

1.

$$|\omega_0(R(\Gamma))| \leq \mathcal{N}(\Gamma)^{1/2} \prod_{x \in s(\Gamma)} C^{n(x)} \beta^{-n(x)\kappa/2} \tag{A.57}$$

2.

$$|\omega_{0,2M}(R(\Gamma)R^*(\Gamma))| \leq \mathcal{N}(\Gamma) \prod_{x \in s(\Gamma)} C^{n(x)} \beta^{-n(x)\kappa} \tag{A.58}$$

3.

$$|\omega_0(R(\Gamma)\mathcal{P}_{\geq M}(Y))| \leq \mathcal{N}(\Gamma)^{1/2} C^{\deg(Y)} v(Y) \prod_{x \in s(\Gamma)} C^{n(x)} \beta^{-n(x)\kappa/2} \tag{A.59}$$

Proof. Consider a sequence $\eta_0, \eta_1, \ldots, \eta_n$ in $\Omega_{s(\Gamma)}$ with $n = |\Gamma|$. We note, by inserting decompositions of identity and using cyclity of the trace, that

$$|\omega_0(R(\Gamma))| \leq \sum_{\substack{\eta_0, \eta_1, \ldots, \eta_{n-1} \\ \eta_n = \eta_0}} e^{\sum_{j=1}^n \tau_j(E(\eta_j) - E(\eta_{j-1}))} \prod_{j=1}^n |\langle \eta_j, V_{B_j} \eta_{j-1} \rangle| \tag{A.60}$$

Since $0 \leq \tau_1 \leq \ldots \leq \tau_n \leq \beta$, we can bound the exponent as

$$\sum_{j=1}^n \tau_j(E(\eta_j) - E(\eta_{j-1})) = \int_0^\beta d\tau\, \tau \frac{\partial e}{\partial \tau} = \int_0^\beta d\tau \frac{\partial(\tau e)}{\partial \tau} - \int_0^\beta d\tau\, e$$

$$\leq \beta e(\beta) - \beta \inf_\tau e(\tau) = \beta E(\eta_n) - \min_j \beta E(\eta_j) = \beta E(\eta_0) - \min_j \beta E(\eta_j)$$

$$\leq \beta \sum_x \left(E(\eta_0(x)) - \min_j E(\eta_j(x)) \right)$$

$$\leq \beta \sum_x \left(E(\eta_0(x)) - E(\eta_0(x) - n(x)/2) \right) \tag{A.61}$$

where we let the function $e(\tau)$ on $[0, \beta]$ be the linear interpolation of $\tau_j \mapsto E(\eta_j)$ with $e(0) = E(\eta_0)$ and we adopted the convention that $E(\xi) = \xi^q$ for $\xi > 0$ and $E(\xi) = 0$ for $\xi \leq 0$. The last inequality follows by using that $n(x)$ is the number of field a_x / a_x^* operators appearing on site x, and $\eta_0(x) = \eta_n(x)$. Combining (A.60) and (A.61), using the basic bound $|\langle \eta(x), a_x^*(\eta(x) - 1) \rangle| \leq \sqrt{\eta(x)}$ and abbreviating

$$Z_0(\beta) = \sum_{\xi \in \mathbb{N}} e^{-\beta E(\xi)}, \tag{A.62}$$

we get

$$|\omega_0(R(\Gamma))| \leq \sum_{\eta_0 \in \Omega_{s(\Gamma)}} \omega_0(P_{\eta_0}) \prod_{x \in s(\Gamma)} \frac{\lfloor \eta_0(x) + n(x)/2 \rfloor!}{\eta_0(x)!} e^{\beta(E(\eta_0(x)) - E(\eta_0(x) - n(x)/2))} \tag{A.63}$$

$$\leq \prod_{x \in s(\Gamma)} \sum_{\eta(x)} \frac{\lfloor \eta(x) + n(x)/2 \rfloor!}{(\eta(x))!} \times \frac{e^{-\beta E(\eta(x) - n(x)/2)}}{Z_0(\beta)} \tag{A.64}$$

For any $0 \leq z \leq 1$, we can use the bound $\frac{m!}{p!(m-p)!} \leq z^{-p}(1-z)^{-(m-p)}$ to get

$$\frac{1}{\prod_x \lfloor n_x/2 \rfloor!} |\omega_0(R(\Gamma))| \leq \prod_{x \in s(\Gamma)} \sum_{\eta(x)} (1-z)^{-n(x)/2} z^{-\eta(x)} \times \frac{e^{-\beta E(\eta(x) - n(x)/2)}}{Z_0(\beta)} \tag{A.65}$$

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Similarly (take $z = 1/2$), we have

$$\prod_{x \in s(\Gamma)} C^{n_x} \lfloor n_x/2 \rfloor! \geq \mathcal{N}(\Gamma)^{1/2} \tag{A.66}$$

Let us now choose $z = e^{-\beta^\kappa}$. For sufficiently small β, we can then estimate

$$(1 - z)^{-n} \leq 2^n \beta^{-n\kappa} \tag{A.67}$$

and we obtain

$$\mathcal{N}(\Gamma)^{-1/2} |\omega_0(R(\Gamma)| \leq \prod_{x \in s(\Gamma)} \sum_{\eta(x)} C^{n(x)} \beta^{-n(x)\kappa} \times \frac{e^{-\beta E(\eta(x) - n(x)/2) + \beta^\kappa \eta(x)}}{Z_0(\beta)} \tag{A.68}$$

Since $\kappa \geq 1/q$, we can bound

$$\sum_{\eta(x)} \frac{e^{-\beta E(\eta(x) - n(x)/2) + \beta^\kappa \eta(x)}}{Z_0(\beta)} \leq C^{n(x)} \tag{A.69}$$

by using the explicit form of $E(\cdot)$. Hence

$$\mathcal{N}(\Gamma)^{-1/2} |\omega_0(R(\Gamma)| \leq \prod_x C^{n(x)} \beta^{-n(x)\kappa/2} \tag{A.70}$$

The claim of 1) now follows since $n(x) \geq 1$ for any $x \in s(\Gamma)$.

To get 3), we first restrict ourselves to the case $s(Y) \subset s(\Gamma)$. We view for notational convenience the a_x/a_x^*-operators in $Y = a_{x_m}^\sharp \ldots a_{x_2}^\sharp a_{x_1}^\sharp$ as additional plaquettes $B_{i=1,\ldots,m}$ with $s(B_i) = \{x_i\}$, $\sigma_{x_i,+/-} = 1/0$ if $a_{x_i}^\sharp = a_{x_i}^*$ and $\sigma_{x_i,+/-} = 0/1$ if $a_{x_i}^\sharp = a_{x_i}$. We define the ordered sequence

$$\Gamma' = \big((B_1, \tau_1), \ldots, (B_m, \tau_m), (B_{m+1}, \tau_{m+1}), \ldots (B_{m+n}, \tau_{m+n})\big) \tag{A.71}$$

where $\tau_{1,\ldots,m} = 0$ and $(B_{m+1}, \tau_{m+1}), \ldots (B_{m+n}, \tau_{m+n})$ are the (ordered) elements of Γ with renamed indices. Now we apply the same reasoning as in the proof of 1) to get the analogue of (A.68), which now reads

$$\mathcal{N}(\Gamma')^{-1/2} |\omega_0(R(\Gamma) \mathcal{P}_{\geq M}(Y))| \leq \prod_{x \in s(\Gamma)} C^{n'(x)} \beta^{-n'(x)\kappa} \sum_{\substack{\eta(x) \geq 0 \text{ for } x \notin s(Y) \\ \eta(x) \geq M \text{ for } x \in s(Y)}} \frac{e^{-\beta E(\eta(x) - n'(x)/2) + \beta^\kappa \eta(x)}}{Z_0(\beta)} \tag{A.72}$$

with $n'(x)$ corresponding to Γ'. For $x \notin s(Y)$, we bound the the sum over $\eta(x)$ by $C^{n'(x)}$, as in 1), and, for $x \in s(Y)$, we bound it as

$$\sum_{\eta(x) \geq M} \frac{e^{-\beta E(\eta(x) - n'(x)/2) + \beta^\kappa \eta(x)}}{Z_0(\beta)} \leq C^{n'(x)} e^{-\beta^{-\gamma_c/2}} \tag{A.73}$$

Hence, altogether, we bound (A.72) as

$$|\omega_0(R(\Gamma) \mathcal{P}_{\geq M}(Y))| \leq \mathcal{N}(\Gamma')^{1/2} (e^{-\beta^{-\gamma_c/2}})^{|s(Y)|} \prod_{x \in s(\Gamma)} C^{n'(x)} \beta^{-n'(x)\kappa} \tag{A.74}$$

Since $n'(x) = n(x) + \deg_x(Y)$, we can bound

$$\mathcal{N}(\Gamma') \leq \mathcal{N}(\Gamma) 2^{\deg(Y)} \prod_{x \in s(Y)} \deg_x(Y)! \tag{A.75}$$

and we get the claim of point 3) for the restricted case $s(Y) \subset s(\Gamma)$. In the general case, we split $Y = Y_1 Y_2$ such that $s(Y_1) \subset s(\Gamma), s(Y_2) \cap s(\Gamma) = \emptyset$, and we use the fact that ω_0 is a product state:

$$\omega_0(R(\Gamma) \mathcal{P}_{\geq M}(Y)) = \omega_0(R(\Gamma) \mathcal{P}_{\geq M}(Y_1)) \, \omega_0(\mathcal{P}_{\geq M}(Y_2)) \tag{A.76}$$

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For the first factor we use the bound above (for the restricted case). For the second factor, we show, by analogous but simpler reasoning, that it is bounded by $v(Y_2)$. Since $v(Y) = v(Y_1)v(Y_2)$, this proves the full claim of 3).

For 2), we mimick the derivation of (A.63) to arrive at

$$|\omega_{0,2M}(R(\Gamma)R^*(\Gamma))| \leq \prod_{x \in s(\Gamma)} \sum_{\eta(x) \leq 2M} \frac{e^{-\beta E(\eta(x))}}{Z_0(\beta)} \frac{(\eta(x) + n(x))!}{(\eta(x))!} e^{2\beta(E(\eta(x)) - E(\eta_0(x) - n(x)))} \tag{A.77}$$

The main difference with the argument in 1) is that every perturbation term appears twice now (therefore we have now $n(x)$ instead of $n(x)/2$ in the argument of the factorial) and that we had to apply the bounds of (A.61) twice. Proceeding as in (A.64) and (A.68), we bound (A.77) by

$$\prod_{x \in s(\Gamma)} \sum_{\eta(x) \leq 2M} \frac{e^{-\beta E(\eta(x))}}{Z_0(\beta)} C^{n(x)} \beta^{-n(x)\kappa_1} e^{\beta^{\kappa_1} \eta(x)} e^{2\beta(E(\eta(x)) - E(\eta(x) - n(x)))} \tag{A.78}$$

for any $0 < \kappa_1 < 1$. To deal with the right-most exponential, we note that

$$\sup_{0 < 2\beta < 1} \sup_{0 \leq \tilde{\xi} \leq \xi \leq 2M} e^{2\beta(\xi^q - \tilde{\xi}^q)} (2\beta)^{\kappa_2(\xi - \tilde{\xi})} \leq C(\kappa_2) < \infty, \tag{A.79}$$

for any $\kappa_2 > 0$. Indeed, for $0 < 2\beta < 1$, the function $\xi \mapsto f(\xi) = e^{2\beta\xi^q}(2\beta)^{\kappa_2\xi}$ is decreasing on the interval $[0, (\frac{\kappa_2|\ln 2\beta|}{2q\beta})^{\frac{1}{q-1}}]$ and, since $M = \beta^{-(1+\gamma_c)/q}$ with $0 < \gamma_c < q/(q-1) - 1$, we see that $2M$ lies in this interval. We use (A.79) with $\xi = \eta(x), \tilde{\xi} = \eta(x) - n(x)$ to obtain, for $\kappa_1 - 1/q \geq 0$,

$$\mathcal{N}(\Gamma)^{-1} |\omega_{0,2M}(R(\Gamma)R^*(\Gamma))| \leq \prod_{x \in s(\Gamma)} \sum_{\eta(x) \leq 2M} \frac{e^{-\beta E(\eta(x)) + \beta^{\kappa_1} \eta(x)}}{Z_0(\beta)} C^{n(x)} \beta^{-n(x)\kappa_1} \beta^{-2\kappa_2 n(x)} \tag{A.80}$$

$$\leq \prod_{x \in s(\Gamma)} C^{n(x)} \beta^{-n(x)(\kappa_1 + 2\kappa_2)} \tag{A.81}$$

where the last inequality uses the explicit form of $E(\eta(x))$ to perform the sums over $\eta(x)$. The claim of 2) follows by choosing κ_1, κ_2 such that $\kappa_1 + 2\kappa_2 = \kappa$, taking advantage of the fact that κ_2 can be chosen arbitrarily small. $\qquad \square$

To perform the sum/integral over the sequences Γ, we will need to exploit the smallness of the Lesbegue mass over the simplex $\Delta_m(\beta)$ for large m

Lemma A.6. *For any $\kappa' < 1/2$*

$$\int_{s(\Gamma) = S} d\Gamma \, (c\beta)^{-\kappa' \sum_x n_x} \mathcal{N}(\Gamma)^{1/2} \leq (C\beta)^{|S|(1/2 - \kappa')} \tag{A.82}$$

Proof. We first establish, for any Γ,

$$\Big(\prod_{x \in s(\Gamma)} \sqrt{n_x!} \Big) \leq C^{\sum_x n_x} \Big(\prod_{B \in \mathcal{B}(\Gamma)} n_B! \Big) \tag{A.83}$$

In the remainder of the proof, it is understood (unless mentioned otherwise) that x ranges over $s(\Gamma)$ and B over $\mathcal{B}(\Gamma)$. To prove (A.83), note that

$$n_x = \sum_{B : s(B) \ni x} \sigma_x(B) n_B \tag{A.84}$$

where the maximal number of nonzero terms on the right hand side is $C = C(d)$, hence

$$n_x! \leq C^{n_x} \prod_{B : s(B) \ni x} (\sigma_x(B) n_B)! \tag{A.85}$$

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and

$$\prod_x n_x! \le C^{\sum_x n_x} \Big(\prod_{B:|s(B)|=1} \sigma_x(B) n_B)! \Big) \Big(\prod_{B:|s(B)|=2} n_B! \Big)^2 \tag{A.86}$$

$$\le C^{\sum_x n_x} 2^{\sum_{B:|s(B)|=1} 2n_B} \Big(\prod_B n_B! \Big)^2 \tag{A.87}$$

where the first inequality follows because $\sigma_x(B) \le 1$ whenever $|s(B)| = 2$ and because every factor with $|s(B)| = 2$ appears twice in the product in (A.85). The second inequality follows from $\sigma_x(B) \le 2$. Observing that (for now, we use only the first inequality)

$$\sum_x n_x \ge \sum_B n_B \ge (1/2) \sum_x n_x, \tag{A.88}$$

we get (A.83) from (A.87).

Since $\mathcal{N}(\Gamma)$ does not depend on the τ-variables, we can perform all $d\tau$-integrals on the LHS of (A.82). This gives the products of the Lesbegue measure of the simplices;

$$\prod_B \frac{\beta^{n_B}}{n_B!} \tag{A.89}$$

Using this bound, the definition of $\mathcal{N}(\Gamma)$, (A.83) and (A.88), we bound the LHS of (A.82) by

$$\sum_{\mathcal{B}:\, \bigcup_{B\in\mathcal{B}} s(B)=S} \sum_{\substack{n_B \ge 1 : B \in \mathcal{B} \\ n_{\mathcal{B}} \ge |S|/2}} (C\beta)^{(1-2\kappa')n_{\mathcal{B}}}, \qquad \text{with } n_{\mathcal{B}} := \sum_{B\in\mathcal{B}} n_B \tag{A.90}$$

where the sum is now over collections $\mathcal{B}$ of plaquettes. Using that the number of terms in the leftmost sum is bounded by $C^{|S|}$, the claim follows by straightforward combinatorics. $\square$

Proof of Lemma A.4. To prove 1), we write for any $1 > \kappa > 1/q$,

$$|\varrho(S)| \le \int_{\substack{s(\Gamma)=S \\ \Gamma \text{ connected}}} d\Gamma \, |\omega_0(R(\Gamma))| \tag{A.91}$$

$$\le \int_{s(\Gamma)=S} d\Gamma \, |\omega_0(R(\Gamma))| \tag{A.92}$$

$$\le \int_{s(\Gamma)=S} d\Gamma \, \mathcal{N}(\Gamma)^{1/2} \prod_x C^{n(x)} \beta^{-n(x)\kappa/2} \tag{A.93}$$

$$\le \beta^{|S|(1/2-\kappa/2)} C^{|S|} \tag{A.94}$$

where the third inequality follows from Lemma A.5 1) and the fourth from Lemma A.6 with $\kappa' = \kappa/2$. The claim follows by setting $\kappa = 1 - 2\alpha$.

Next, we prove 2): For any set S and plaquette B, we can consider the reduced plaquette B_S with $s(B_S) := s(B) \cap S$ and $\sigma_{x,\pm}(B_S) := \sigma_{x,\pm}(B)$ whenever $x \in s(B_S)$. Then given a collection Γ, we define the collection Γ_S

$$\Gamma_S := \{(B_S, \tau) : (B, \tau) \in \Gamma \text{ and } s(B_S) \ne \emptyset\} \tag{A.95}$$

Note that

$$s(\Gamma) = s(\Gamma_S) \cup s(\Gamma_{S^c}), \qquad s(\Gamma_S) \cap s(\Gamma_{S^c}) = \emptyset. \tag{A.96}$$

and, for $x \in S$, $n(x)$ as defined from Γ_S equals $n(x)$ as defined from Γ. In particular, we have

$$\mathcal{N}(\Gamma_S)\mathcal{N}(\Gamma_{S^c}) = \mathcal{N}(\Gamma). \tag{A.97}$$

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We apply this below with $S = s(O), s(O)^c$. We have

$$\omega_0(R(\Gamma)K) = \omega_0(R(\Gamma_{s(O)})O)\omega_0(R(\Gamma_{s(O)^c})\mathcal{P}_{\geq M}(Y)) \tag{A.98}$$

$$= \omega_0(\mathcal{P}_{\leq 2M}(R(\Gamma_{s(O)}))\mathcal{P}_{\leq 2M}(O)) \times \omega_0(R(\Gamma_{s(O)^c})\mathcal{P}_{\geq M}(Y)) \tag{A.99}$$

where the second equality follows because $\mathcal{P}_{\leq 2M}(O) = O$ and the density matrix of ω_0 is diagonal in the N_x-basis. Using Cauchy-Schwartz, positivity of the projector $\otimes_{x \in A}\chi(N_x \leq 2M)$ for any A, and $(\mathcal{P}_{\leq 2M}(O))^* = \mathcal{P}_{\leq 2M}(O^*)$, the first factor can be bounded as

$$|\omega_0(\mathcal{P}_{\leq 2M}(R(\Gamma_{s(O)})))\mathcal{P}_{\leq 2M}(O))|^2 \leq \omega_0(\mathcal{P}_{\leq 2M}(O^*)\mathcal{P}_{\leq 2M}(O)) \times \omega_0(\mathcal{P}_{\leq 2M}(R(\Gamma_{s(O)}))\mathcal{P}_{\leq 2M}(R^*(\Gamma_{s(O)}))) \tag{A.100}$$

$$\leq \omega_{0,2M}(O^*O) \times \omega_{0,2M}(R(\Gamma_{s(O)})R^*(\Gamma_{s(O)})) \tag{A.101}$$

We recall the definition of $\varrho_K(S)$ in (A.30) and we abbreviate

$$\int_K d\Gamma \ldots := \int_{\substack{\Gamma \ s(K)\text{-connected} \\ s(\Gamma) \cap s(K)^c = S}} d\Gamma \ldots \tag{A.102}$$

We estimate, for any $1 > \kappa > 1/q$,

$$|\varrho_K(S)| \leq \int_K d\Gamma \, |\omega_0(R(\Gamma)K)| \tag{A.103}$$

$$\leq |\omega_{0,2M}(O^*O)|^{1/2} \int_K d\Gamma \, |\omega_{0,2M}(R(\Gamma_{s(O)})R^*(\Gamma_{s(O)}))|^{1/2} \, |\omega_0(R(\Gamma_{s(O)^c})\mathcal{P}_{\geq M}(Y))| \tag{A.104}$$

$$\leq |\omega_{0,2M}(O^*O)|^{1/2} C^{\deg(Y)} v(Y) \int_K d\Gamma |\mathcal{N}(\Gamma_{s(O)})\mathcal{N}(\Gamma_{s(O^c)})|^{1/2} \prod_{x \in s(\Gamma)} (c\beta)^{-\kappa n(x)/2} \tag{A.105}$$

The first inequality follows from (A.99) and (A.101) and the second from Lemma A.5 2),3), using (A.96). Let us now first take $S \neq \emptyset$. Starting from (A.97), we rewrite and bound the $d\Gamma$-integral in (A.105) as (recall that $\#_c(K)$ is the number of connected components of $s(K)$),

$$\int_K d\Gamma \, \mathcal{N}(\Gamma)^{1/2} \prod_{x \in s(\Gamma)} (c\beta)^{-\kappa n(x)/2} \tag{A.106}$$

$$\leq \sum_{\substack{S': S' \cap s(K)^c = S \\ |S' \cap s(K)| \geq \#_c(K)}} \int_{s(\Gamma)=S'} d\Gamma \, \mathcal{N}(\Gamma)^{1/2} \prod_{x \in s(\Gamma)} (c\beta)^{-\kappa n(x)/2} \tag{A.107}$$

$$\leq \sum_{\substack{S': S' \cap s(K)^c = S \\ |S' \cap s(K)| \geq \#_c(K)}} (C\beta)^{\alpha|S'|} \leq C^{|s(K)|}(C\beta)^{\alpha(|S|+\#_c(K))} \tag{A.108}$$

where we used Lemma A.6 with $\kappa' = \kappa/2$ and we set $\kappa = 1 - 2\alpha$. Plugging this into (A.105) and recalling the definition of $w(K)$ yields the desired claim. For $S = \emptyset$, the above proof still applies if we drop the constraint $|S' \cap s(K)| \geq \#_c(K)$ in the last lines. Then the resulting bound on the right hand side of (A.108) is simply $C^{|s(K)|}$, and we can again conclude by plugging into (A.105).

$\square$

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