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► **To cite this version:**

Roland Badeau, Antoine Liutkus. Proof of Wiener-like linear regression of isotropic complex symmetric alpha-stable random variables. 2014. hal-01069612

**HAL Id: hal-01069612**

**<https://hal.science/hal-01069612>**

Submitted on 29 Sep 2014

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# PROOF OF WIENER-LIKE LINEAR REGRESSION OF ISOTROPIC COMPLEX SYMMETRIC ALPHA-STABLE RANDOM VARIABLES

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This document features supplementary materials to the reference paper [1]. It provides the proof of equation (8) in [1]. This proof concerns a particular regression property of complex isotropic symmetric  $\alpha$ -stable random variables (S $\alpha$ S, see [2]). In [1], this property is shown paramount in building efficient filters for separating S $\alpha$ S processes. Such processes exhibit very large dynamic ranges while being locally stationary, and have been shown appropriate for audio modeling.

**Proposition 1** (Wiener-like linear regression of isotropic complex S $\alpha$ S random variables). *Let  $\alpha \in ]0, 2]$ . Let  $s_1$  and  $s_2$  be two independent isotropic complex S $\alpha$ S random variables of scale parameters  $\sigma_1$  and  $\sigma_2$ , respectively. Let  $x = s_1 + s_2$ . Then the conditional expectation of  $s_1$  given  $x$  is expressed as follows:*

$$\mathbb{E}[s_1|x] = \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} x. \quad (1)$$

*Proof.* Let  $s_1^r = \text{Re}(s_1)$ ,  $s_1^i = \text{Im}(s_1)$ ,  $s_2^r = \text{Re}(s_2)$ ,  $s_2^i = \text{Im}(s_2)$ ,  $x^r = \text{Re}(x)$ , and  $x^i = \text{Im}(x)$ , where  $\text{Re}(\cdot)$  and  $\text{Im}(\cdot)$  denote the real and imaginary parts of complex numbers, respectively. For  $j \in \{1, 2\}$ , the characteristic function of the isotropic complex S $\alpha$ S random variable  $s_j$  is  $\phi_{s_j}(\theta_x^r, \theta_x^i) = e^{-\sigma_j^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}}$  [2]. Therefore the characteristic function of the random vector  $(s_1^r, x^r, x^i) \in \mathbb{R}^3$  is

$$\begin{aligned} \phi_{(s_1^r, x^r, x^i)}(\theta_1^r, \theta_x^r, \theta_x^i) &= \mathbb{E}[e^{i(\theta_1^r s_1^r + \theta_x^r x^r + \theta_x^i x^i)}] \\ &= \mathbb{E}[e^{i(\theta_1^r s_1^r + \theta_x^r (s_1^r + s_2^r) + \theta_x^i (s_1^i + s_2^i))}] \\ &= \mathbb{E}[e^{i((\theta_1^r + \theta_x^r) s_1^r + \theta_x^i s_1^i)}] \mathbb{E}[e^{i(\theta_x^r s_2^r + \theta_x^i s_2^i)}] \\ &= \phi_{s_1}(\theta_1^r + \theta_x^r, \theta_x^i) \phi_{s_2}(\theta_x^r, \theta_x^i) \\ &= e^{-(\sigma_1^\alpha (|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}})}. \end{aligned}$$

Using [2, eq. (5.1.7) p. 226], the conditional characteristic function of  $s_1^r$  given  $x^r$  and  $x^i \in \mathbb{R}$  is expressed as

$$\begin{aligned} \phi_{(s_1^r|x^r, x^i)}(\theta_1^r) &= \frac{\int_{\mathbb{R}^2} \phi_{(s_1^r, x^r, x^i)}(\theta_1^r, \theta_x^r, \theta_x^i) e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2} \phi_{(s_1^r, x^r, x^i)}(0, \theta_x^r, \theta_x^i) e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i} \\ &= \frac{\int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha (|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}})} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}. \end{aligned} \quad (2)$$

We know that  $\mathbb{E}[s_1^r|x^r, x^i]$  is defined if and only if  $\phi_{(s_1^r|x^r, x^i)}(\theta_1^r)$  is differentiable at  $\theta_1^r = 0$ , and that in that case  $\frac{d\phi_{(s_1^r|x^r, x^i)}}{d\theta_1^r}(0) = i\mathbb{E}[s_1^r|x^r, x^i]$ . Firstly, it is easy to prove that the first order derivative of  $\phi_{(s_1^r|x^r, x^i)}$  is well-defined and has the following expression:

$$\frac{d\phi_{(s_1^r|x^r, x^i)}}{d\theta_1^r}(\theta_1^r) = \frac{\int_{\mathbb{R}^2} -\alpha\sigma_1^\alpha (\theta_1^r + \theta_x^r) (|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}-1} e^{-(\sigma_1^\alpha (|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}})} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}. \quad (3)$$

Indeed, differentiating under the  $\int$  sign in the numerator of the last member of (2) is allowed, because the term following the  $\int$  sign in the numerator of (3) can be upper bounded by an integrable function independently of  $\theta_1^r \in \mathbb{R}$ :

$$\begin{aligned} & \left| -\alpha\sigma_1^\alpha(\theta_1^r + \theta_x^r)(|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}-1} e^{-\left(\sigma_1^\alpha(|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}\right)} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} \right| \\ & \leq \alpha\sigma_1^\alpha(|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha-1}{2}} e^{-\left(\sigma_1^\alpha(|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}} + \sigma_2^\alpha(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}\right)} \\ & \leq g(|\theta_1^r + \theta_x^r|^2 + |\theta_x^i|^2) h(\theta_x^r, \theta_x^i) \\ & \leq \|g\|_\infty h(\theta_x^r, \theta_x^i), \end{aligned}$$

where the nonnegative functions  $g \in L^\infty(\mathbb{R})$  and  $h \in L^1(\mathbb{R}^2)$  are defined according to the value of  $\alpha$ :

- if  $1 \leq \alpha \leq 2$ ,  $g(t) = \alpha\sigma_1^\alpha |t|^{\frac{\alpha-1}{2}} e^{-\sigma_1^\alpha |t|^{\frac{\alpha}{2}}}$  and  $h(\theta_x^r, \theta_x^i) = e^{-\sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}}$ ;
- if  $0 < \alpha < 1$ ,  $g(t) = \alpha\sigma_1^\alpha e^{-\sigma_1^\alpha |t|^{\frac{\alpha}{2}}}$  and  $h(\theta_x^r, \theta_x^i) = \frac{e^{-\sigma_2^\alpha (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}}}{|\theta_x^i|^{1-\alpha}}$ .

Then applying equation (3) to  $\theta_1^r = 0$  yields

$$\begin{aligned} \frac{d\phi_{(s_1^r|x^r, x^i)}(0)}{d\theta_1^r} &= \frac{\int_{\mathbb{R}^2} -\alpha\sigma_1^\alpha \theta_x^r (|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}-1} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i} \\ &= \frac{\sigma_1^\alpha \int_{\mathbb{R}^2} \frac{de^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}}}{d\theta_x^r} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i}{\sigma_1^\alpha + \sigma_2^\alpha \int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i} \\ &= -\frac{\sigma_1^\alpha \int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} \frac{de^{-i(\theta_x^r x^r + \theta_x^i x^i)}}{d\theta_x^r} d\theta_x^r d\theta_x^i}{\sigma_1^\alpha + \sigma_2^\alpha \int_{\mathbb{R}^2} e^{-(\sigma_1^\alpha + \sigma_2^\alpha)(|\theta_x^r|^2 + |\theta_x^i|^2)^{\frac{\alpha}{2}}} e^{-i(\theta_x^r x^r + \theta_x^i x^i)} d\theta_x^r d\theta_x^i} \\ &= i \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} x^r. \end{aligned}$$

This proves that  $\mathbb{E}[s_1^r|x^r, x^i] = \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} x^r$ . In exactly the same way, it is proved that  $\mathbb{E}[s_1^i|x^r, x^i] = \frac{\sigma_1^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha} x^i$ , which finally proves equation (1).  $\square$

## 1. REFERENCES

- [1] A. Liutkus and R. Badeau, *Generalized Wiener filtering with fractional power spectrograms*, submitted to IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP), Brisbane, Australia, April 2015.
- [2] G. Samoradnitsky and M. Taqqu, *Stable non-Gaussian random processes: stochastic models with infinite variance*, vol. 1, CRC Press, 1994.