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# A generalization of Cramér large deviations for martingales

Xiequan Fan<sup>a,b</sup> Ion Grama<sup>a</sup> Quansheng Liu<sup>a</sup>

<sup>a</sup>*Univ. Bretagne-Sud, UMR 6205, LMBA, F-56000 Vannes, France*

<sup>b</sup>*Regularity Team, Inria and MAS Laboratory, Ecole Centrale Paris - Grande Voie des Vignes,  
92295 Châtenay-Malabry, France*

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## Abstract

In this note, we give a generalization of Cramér's large deviations for martingales, which can be regarded as a supplement of Fan, Grama and Liu (Stochastic Process. Appl., 2013). Our method is based on the change of probability measure developed by Grama and Haeusler (Stochastic Process. Appl., 2000). *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## Résumé

**Une généralisation des grandes déviations de Cramér pour les martingales.** Dans cette note, nous donnons une généralisation des grandes déviations de Cramér pour les martingales, qui peut être considéré comme un supplément de Fan, Grama et Liu (Stochastic Process. Appl., 2013). Notre méthode est basée sur le changement de mesure de probabilité développé par Grama et Haeusler (Stochastic Process. Appl., 2000). *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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## 1. Introduction

Assume that  $\eta_1, \dots, \eta_n$  is a sequence of independent and identically distributed (i.i.d.) centered real valued random variables satisfying the following Cramér condition :  $\mathbb{E} \exp\{c_0|\eta_1|\} < \infty$  for some  $c_0 > 0$ . Denote by  $\sigma^2 = \mathbb{E}\eta_1^2$ ,  $\xi_i = \eta_i/(\sqrt{n}\sigma)$  and  $X_n = \sum_{i=1}^n \xi_i$ . Cramér [1] has established the following asymptotic expansion of the tail probabilities of  $X_n$ , for all  $0 \leq x = o(n^{1/6})$  as  $n \rightarrow \infty$ ,

$$\mathbb{P}(X_n > x) = \left(1 - \Phi(x)\right) \left(1 + o(1)\right), \quad (1)$$

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*Email addresses:* fanxiequan@hotmail.com (Xiequan Fan), ion.grama@univ-ubs.fr (Ion Grama), quansheng.liu@univ-ubs.fr (Quansheng Liu).

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{t^2}{2}\right\} dt$$

is the standard normal distribution function. More precise results can be found in Feller [4], Petrov [10,11], Saulis and Statulevičius [15], Sakhanenko [14] and [2] among others.

Let  $(\xi_i, \mathcal{F}_i)_{i=0, \dots, n}$  be a sequence of martingale differences defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\xi_0 = 0$  and  $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_n \subseteq \mathcal{F}$  are increasing  $\sigma$ -fields. Set

$$X_0 = 0, \quad X_k = \sum_{i=1}^k \xi_i, \quad k = 1, \dots, n. \quad (2)$$

Denote by  $\langle X \rangle$  the quadratic characteristic of the martingale  $X = (X_k, \mathcal{F}_k)_{k=0, \dots, n}$  :

$$\langle X \rangle_0 = 0, \quad \langle X \rangle_k = \sum_{i=1}^k \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}), \quad k = 1, \dots, n. \quad (3)$$

Consider the stationary case for simplicity. For the martingale differences having a  $(2+p)$ th moment, i.e.  $\|\xi_i\|_{2+p} < \infty$  for some  $p \in (0, 1]$ , expansions of the type (1) in the range  $0 \leq x = o(\sqrt{\log n})$  have been obtained by Haeusler and Joos [8], Grama [5] and Grama and Haeusler [7]. If the martingale differences are bounded  $|\xi_i| \leq C/\sqrt{n}$  and satisfy  $\|\langle X \rangle_n - 1\|_\infty \leq L^2/n$  a.s. for two positive constants  $C$  and  $L$ , expansion (1) has been firstly established by Račkauskas [12,13] in the range  $0 \leq x = o(n^{1/6})$ , and then this range has been extended to a larger one  $0 \leq x = o(n^{1/4})$  by Grama and Haeusler [6] with a method based on change of probability measure. Recently, Fan et al. [3] have generalized the result of Grama and Haeusler [6] to a much larger range  $0 \leq x = o(n^{1/2})$  for  $\xi_i$  satisfying the following conditional Bernstein condition : for a positive constant  $C$ ,

$$|\mathbb{E}(\xi_i^k | \mathcal{F}_{i-1})| \leq \frac{1}{2} k! \left(\frac{C}{\sqrt{n}}\right)^{k-2} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) \quad \text{for all } k \geq 2 \text{ and all } 1 \leq i \leq n. \quad (4)$$

It is worth noting that the conditional Bernstein condition does not imply that  $\xi_i$ 's are bounded.

The aim of this note is to extend the expansion of Fan et al. [3] to the case of martingale differences satisfying the following conditional Cramér condition considered in Liu and Watbled [9] :

$$\sup_i \mathbb{E}(\exp\{C_0 \sqrt{n} |\xi_i|\} | \mathcal{F}_{i-1}) \leq C_1, \quad (5)$$

where  $C_0$  and  $C_1$  are two positive constants. It is worth noting that, in general, condition (5) does not imply the conditional Bernstein condition (4), unless  $n\mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1})$  are all bounded from below by a positive constant. Thus our result is not a consequence of Fan et al. [3].

Throughout this paper,  $c$  and  $c_\alpha$ , probably supplied with some indices, denote respectively a generic positive absolute constant and a generic positive constant depending only on  $\alpha$ . Moreover,  $\theta$  stands for any value satisfying  $|\theta| \leq 1$ .

## 2. Main Results

The following theorem is our main result, which can be regarded as a parallel result of Fan et al. [3] under the conditional Cramér condition :

$$\mathbf{(A1)} \quad \sup_{1 \leq i \leq n} \mathbb{E}(\exp\{c_0 n^{1/2} |\xi_i|\} | \mathcal{F}_{i-1}) \leq c_1 ;$$

(A2)  $\|\langle X \rangle_n - 1\|_\infty \leq \delta^2$  a.s., where  $\delta$  is nonnegative and usually depends on  $n$ .

**Theorem 2.1** Assume conditions (A1) and (A2). Then there exists a positive absolute constant  $\alpha_0$ , such that for all  $0 \leq x \leq \alpha_0 n^{1/2}$  and  $\delta \leq \alpha_0$ , the following equalities hold

$$\frac{\mathbb{P}(X_n > x)}{1 - \Phi(x)} = \exp \left\{ \theta c_{\alpha_0} \left( \frac{x^3}{\sqrt{n}} + x^2 \delta^2 + (1+x) \left( \frac{\log n}{\sqrt{n}} + \delta \right) \right) \right\} \quad (6)$$

and

$$\frac{\mathbb{P}(X_n < -x)}{\Phi(-x)} = \exp \left\{ \theta c_{\alpha_0} \left( \frac{x^3}{\sqrt{n}} + x^2 \delta^2 + (1+x) \left( \frac{\log n}{\sqrt{n}} + \delta \right) \right) \right\}, \quad (7)$$

where  $|\theta| \leq 1$ . In particular, for all  $0 \leq x = o(\min\{n^{1/6}, \delta^{-1}\})$  as  $\min\{n, \delta^{-1}\} \rightarrow \infty$ ,

$$\mathbb{P}(X_n \geq x) = (1 - \Phi(x)) (1 + o(1)). \quad (8)$$

From (6), we find that there is an absolute constant  $\alpha_0 > 0$  such that for all  $0 \leq x \leq \alpha_0 n^{1/2}$  and  $\delta \leq \alpha_0$ ,

$$\left| \log \frac{\mathbb{P}(X_n > x)}{1 - \Phi(x)} \right| \leq c_{\alpha_0} \left( \frac{x^3}{\sqrt{n}} + x^2 \delta^2 + (1+x) \left( \frac{\log n}{\sqrt{n}} + \delta \right) \right). \quad (9)$$

Note that this result can be regarded as a refinement of the moderate deviation principle (MDP) under conditions (A1) and (A2). Let  $a_n$  be any sequence of real numbers satisfying  $a_n \rightarrow \infty$  and  $a_n n^{-1/2} \rightarrow 0$  as  $n \rightarrow \infty$ . If  $\delta \rightarrow 0$  as  $n \rightarrow \infty$ , then inequality (9) implies the MDP for  $X_n$  with the speed  $a_n$  and good rate function  $x^2/2$ ; for each Borel set  $B$ ,

$$-\inf_{x \in B^o} \frac{x^2}{2} \leq \liminf_{n \rightarrow \infty} \frac{1}{a_n^2} \log \mathbb{P} \left( \frac{1}{a_n} X_n \in B \right) \leq \limsup_{n \rightarrow \infty} \frac{1}{a_n^2} \log \mathbb{P} \left( \frac{1}{a_n} X_n \in B \right) \leq -\inf_{x \in \bar{B}} \frac{x^2}{2},$$

where  $B^o$  and  $\bar{B}$  denote the interior and the closure of  $B$ , respectively (see Fan et al. [3] for details).

### 3. Sketch of the proof

Let  $(\xi_i, \mathcal{F}_i)_{i=0, \dots, n}$  be a martingale differences satisfying the condition (A1). For any real number  $\lambda$  with  $|\lambda| \leq c_0 n^{1/2}$ , define

$$Z_k(\lambda) = \prod_{i=1}^k \frac{e^{\lambda \xi_i}}{\mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1})}, \quad k = 1, \dots, n, \quad Z_0(\lambda) = 1.$$

Then  $Z(\lambda) = (Z_k(\lambda), \mathcal{F}_k)_{k=0, \dots, n}$  is a positive martingale and for each real number  $\lambda$  with  $|\lambda| \leq c_0 n^{1/2}$  and each  $k = 1, \dots, n$ , the random variable  $Z_k(\lambda)$  is a probability density on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Thus we can define the *conjugate probability measure*  $\mathbb{P}_\lambda$  on  $(\Omega, \mathcal{F})$ , where

$$d\mathbb{P}_\lambda = Z_n(\lambda) d\mathbb{P}. \quad (10)$$

Denote by  $\mathbb{E}_\lambda$  the expectation with respect to  $\mathbb{P}_\lambda$ . Setting

$$b_i(\lambda) = \mathbb{E}_\lambda(\xi_i | \mathcal{F}_{i-1}) \quad \text{and} \quad \eta_i(\lambda) = \xi_i - b_i(\lambda) \quad \text{for } i = 1, \dots, n,$$

we obtain the decomposition of  $X_n$  similar to that of Grama and Haeusler [6] :

$$X_n = B_n(\lambda) + Y_n(\lambda), \quad (11)$$

where

$$B_n(\lambda) = \sum_{i=1}^n b_i(\lambda) \quad \text{and} \quad Y_n(\lambda) = \sum_{i=1}^n \eta_i(\lambda).$$

Note that  $(Y_k(\lambda), \mathcal{F}_k)_{k=1, \dots, n}$  is also a sequence of martingale differences w.r.t.  $\mathbb{P}_\lambda$ .

In the sequel, we establish some auxiliary lemmas which will be used in the proof of Theorem 2.1. We first give upper bounds for the conditional moments.

**Lemma 3.1** *Assume condition (A1). Then*

$$\mathbb{E}(|\xi_i|^k | \mathcal{F}_{i-1}) \leq k! (c_0 n^{1/2})^{-k} c_1, \quad k \geq 3.$$

*Proof.* Applying the elementary inequality  $x^k/k! \leq e^x$  to  $x = c_0 |n^{1/2} \xi_i|$ , we have, for  $k \geq 3$ ,

$$|\xi_i|^k \leq k! (c_0 n^{1/2})^{-k} \exp\{c_0 |n^{1/2} \xi_i|\}. \quad (12)$$

Taking conditional expectations on both sides of the last inequality, by condition (A1), we obtain the desired inequality.  $\square$

*Remark 1* *It is worth noting that both condition (A1) and the conditional Bernstein condition (4) imply the following hypothesis.*

**(A1')** *There exists  $\epsilon > 0$ , usually depends on  $n$ , such that*

$$\mathbb{E}(|\xi_i|^k | \mathcal{F}_{i-1}) \leq c_1 k! \epsilon^k \quad \text{for all } k \geq 2 \text{ and all } 1 \leq i \leq n.$$

When  $\epsilon = c_2/\sqrt{n}$ , condition (A1'), together (A2), yields Theorem 2.1.

Using Lemma 3.1, we obtain the following two technical lemmas. Their proofs are similar to the arguments of Lemmas 4.2 and 4.3 of Fan et al. [3].

**Lemma 3.2** *Assume conditions (A1) and (A2). Then, for all  $0 \leq \lambda \leq \frac{1}{4} c_0 n^{1/2}$ ,*

$$|B_n(\lambda) - \lambda| \leq c(\lambda \delta^2 + \lambda^2 n^{-1/2}). \quad (13)$$

**Lemma 3.3** *Assume conditions (A1) and (A2). Then, for all  $0 \leq \lambda \leq \frac{1}{4} c_0 n^{1/2}$ ,*

$$\left| \Psi_n(\lambda) - \frac{\lambda^2}{2} \right| \leq c(\lambda^2 \delta^2 + \lambda^3 n^{-1/2}),$$

where

$$\Psi_n(\lambda) = \sum_{i=1}^n \log \mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}).$$

The following lemma gives the rate of convergence in the central limit theorem for the conjugate martingale  $(Y_i(\lambda), \mathcal{F}_i)$  under the probability measure  $\mathbb{P}_\lambda$ . Its proof is similar to that of Lemma 3.1 of Fan et al. [3] by noting the fact that  $\mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) \leq c/n$ .

**Lemma 3.4** *Assume conditions (A1) and (A2). Then, for all  $0 \leq \lambda \leq \frac{1}{4} c_0 n^{1/2}$ ,*

$$\sup_x \left| \mathbb{P}_\lambda(Y_n(\lambda) \leq x) - \Phi(x) \right| \leq c \left( \lambda \frac{1}{\sqrt{n}} + \frac{\log n}{\sqrt{n}} + \delta \right).$$

*Proof of Theorem 2.1.* The proof of Theorem 2.1 is similar to the arguments of Theorems 2.1 and 2.2 in Fan et al. [3] with  $\epsilon = \frac{c_0}{4\sqrt{n}}$ . However, instead of using Lemmas 4.2, 4.3 and 3.1 of [3], we shall make use of Lemmas 3.2, 3.3 and 3.4 respectively.  $\square$

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The proofs of Lemmas 3.2 and 3.3 are given below.

*Proof of Lemma 3.2.* Recall that  $0 \leq \lambda \leq \frac{1}{4} c_0 n^{1/2}$ . By the relation between  $\mathbb{E}$  and  $\mathbb{E}_\lambda$  on  $\mathcal{F}_i$ , we have

$$b_i(\lambda) = \frac{\mathbb{E}(\xi_i e^{\lambda \xi_i} | \mathcal{F}_{i-1})}{\mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1})}, \quad i = 1, \dots, n.$$

Jensen's inequality and  $\mathbb{E}(\xi_i | \mathcal{F}_{i-1}) = 0$  imply that  $\mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) \geq 1$ . Since

$$\mathbb{E}(\xi_i e^{\lambda \xi_i} | \mathcal{F}_{i-1}) = \mathbb{E}(\xi_i (e^{\lambda \xi_i} - 1) | \mathcal{F}_{i-1}) \geq 0,$$

by Taylor's expansion for  $e^x$ , we find that

$$B_n(\lambda) \leq \sum_{i=1}^n \mathbb{E}(\xi_i e^{\lambda \xi_i} | \mathcal{F}_{i-1}) = \lambda \langle X \rangle_n + \sum_{i=1}^n \sum_{k=2}^{+\infty} \mathbb{E} \left( \frac{\xi_i (\lambda \xi_i)^k}{k!} \middle| \mathcal{F}_{i-1} \right). \quad (14)$$

Using Lemma 3.1, we obtain

$$\begin{aligned} \sum_{i=1}^n \sum_{k=2}^{+\infty} \left| \mathbb{E} \left( \frac{\xi_i (\lambda \xi_i)^k}{k!} \middle| \mathcal{F}_{i-1} \right) \right| &\leq \sum_{i=1}^n \sum_{k=2}^{+\infty} |\mathbb{E}(\xi_i^{k+1} | \mathcal{F}_{i-1})| \frac{\lambda^k}{k!} \\ &\leq \sum_{i=1}^n \sum_{k=2}^{+\infty} c_1 (k+1) \lambda^k (c_0 n^{1/2})^{-k-1} \\ &\leq c_2 \lambda^2 n^{-1/2}. \end{aligned} \quad (15)$$

Condition (A2) together with (14) and (15) imply the upper bound of  $B_n(\lambda)$ :

$$B_n(\lambda) \leq \lambda + \lambda \delta^2 + c_2 \lambda^2 n^{-1/2}.$$

Using Lemma 3.1, we have

$$\begin{aligned} \mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) &\leq 1 + \sum_{k=2}^{+\infty} \left| \mathbb{E} \left( \frac{(\lambda \xi_i)^k}{k!} \middle| \mathcal{F}_{i-1} \right) \right| \\ &\leq 1 + \sum_{k=2}^{+\infty} c_1 \lambda^k (c_0 n^{1/2})^{-k} \\ &\leq 1 + c_3 \lambda^2 n^{-1}. \end{aligned} \quad (16)$$

This inequality together with (15) and condition (A2) imply the lower bound of  $B_n(\lambda)$ :

$$\begin{aligned} B_n(\lambda) &\geq \left( \sum_{i=1}^n \mathbb{E}(\xi_i e^{\lambda \xi_i} | \mathcal{F}_{i-1}) \right) \left( 1 + c_3 \lambda^2 n^{-1} \right)^{-1} \\ &\geq \left( \lambda \langle X \rangle_n - \sum_{i=1}^n \sum_{k=2}^{+\infty} \left| \mathbb{E} \left( \frac{\xi_i (\lambda \xi_i)^k}{k!} \middle| \mathcal{F}_{i-1} \right) \right| \right) \left( 1 + c_3 \lambda^2 n^{-1} \right)^{-1} \\ &\geq \left( \lambda - \lambda \delta^2 - c_2 \lambda^2 n^{-1/2} \right) \left( 1 + c_3 \lambda^2 n^{-1} \right)^{-1} \\ &\geq \lambda - \lambda \delta^2 - c_4 \lambda^2 n^{-1/2}. \end{aligned}$$

The proof of Lemma 3.2 is finished.  $\square$

*Proof of Lemma 3.3.* Recall that  $0 \leq \lambda \leq \frac{1}{4} c_0 n^{1/2}$ . Since  $\mathbb{E}(\xi_i | \mathcal{F}_{i-1}) = 0$ , it is easy to see that

$$\Psi_n(\lambda) = \sum_{i=1}^n \left( \log \mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) - \lambda \mathbb{E}(\xi_i | \mathcal{F}_{i-1}) - \frac{\lambda^2}{2} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) \right) + \frac{\lambda^2}{2} \langle X \rangle_n.$$

Using a two-term Taylor's expansion of  $\log(1+x)$ ,  $x \geq 0$ , we obtain

$$\begin{aligned} \Psi_n(\lambda) - \frac{\lambda^2}{2} \langle X \rangle_n &= \sum_{i=1}^n \left( \mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) - 1 - \lambda \mathbb{E}(\xi_i | \mathcal{F}_{i-1}) - \frac{\lambda^2}{2} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) \right) \\ &\quad - \frac{1}{2 \left( 1 + |\theta| (\mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) - 1) \right)^2} \sum_{i=1}^n \left( \mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) - 1 \right)^2. \end{aligned}$$

Since  $\mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) \geq 1$ , we find that

$$\begin{aligned} \left| \Psi_n(\lambda) - \frac{\lambda^2}{2} \langle X \rangle_n \right| &\leq \sum_{i=1}^n \left| \mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) - 1 - \lambda \mathbb{E}(\xi_i | \mathcal{F}_{i-1}) - \frac{\lambda^2}{2} \mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) \right| \\ &\quad + \frac{1}{2} \sum_{i=1}^n \left( \mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) - 1 \right)^2 \\ &\leq \sum_{i=1}^n \sum_{k=3}^{+\infty} \frac{\lambda^k}{k!} |\mathbb{E}(\xi_i^k | \mathcal{F}_{i-1})| + \frac{1}{2} \sum_{i=1}^n \left( \sum_{k=2}^{+\infty} \frac{\lambda^k}{k!} |\mathbb{E}(\xi_i^k | \mathcal{F}_{i-1})| \right)^2. \end{aligned}$$

In the same way as in the proof of Lemma 3.2, by Lemma 3.1, we have

$$\left| \Psi_n(\lambda) - \frac{\lambda^2}{2} \langle X \rangle_n \right| \leq c_3 \lambda^3 n^{-1/2}.$$

Combining this inequality with condition (A2), we obtain the desired inequality.  $\square$